

Worksheet 4.5 Polynomials

Section 1 INTRODUCTION TO POLYNOMIALS

A polynomial is an expression of the form

$$p(x) = p_0 + p_1x + p_2x^2 + \cdots + p_nx^n, \quad (n \in \mathbb{N})$$

where p_0, p_1, \dots, p_n are constants and x is a variable.

Example 1 : $f(x) = 3 + 5x + 7x^2 - x^4$ ($p_0 = 3, p_1 = 5, p_2 = 7, p_3 = 0, p_4 = -1$)

Example 2 : $g(x) = 2x^3 + 3x$ ($p_0 = 0, p_1 = 3, p_2 = 0, p_3 = 2$)

- The constants p_0, \dots, p_n are called coefficients. In Example 1 the coefficient of x is 5 and the coefficient of x^4 is -1 . The term which is independent of x is called the constant term. In Example 1 the constant term of $f(x)$ is 3; in Example 2 the constant term of $g(x)$ is 0.
- A polynomial $p_0 + p_1x + \cdots + p_nx^n$ is said to have degree n , denoted $\deg n$, if $p_n \neq 0$ and x^n is the highest power of x which appears. In Example 1 the degree of $f(x)$ is 4; in Example 2 the degree of $g(x)$ is 3.
- A zero polynomial is a polynomial whose coefficients are all 0, i.e. $p_0 = p_1 = \cdots = p_n = 0$.
- Two polynomials are equal if all the coefficients of the corresponding powers of x are equal.

Exercises:

1. Find (i) the constant term, (ii) the coefficient of x^4 and (iii) the degree of the following polynomials.
 - (a) $x^4 + x^3 + x^2 + x + 1$
 - (b) $9 - 3x^2 + 7x^3$
 - (c) $x - 2$
 - (d) $10x^5 - 3x^4 + 5x + 6$
 - (e) $3x^6 + 7x^4 + 2x$
 - (f) $7 + 5x + x^2 - 6x^4$
2. Suppose $f(x) = 2ax^3 - 3x^2 - b^2x - 7$ and $g(x) = cx^4 + 10x^3 - (d+1)x^2 - 4x + e$. Find values for constants a, b, c, d and e given that $f(x) = g(x)$.

Section 2 OPERATIONS ON POLYNOMIALS

Suppose

$$h(x) = 3x^2 + 4x + 5 \quad \text{and} \quad k(x) = 7x^2 - 3x.$$

- *Addition/Subtraction:* To add/subtract polynomials we combine like terms.

Example 1 : We have

$$h(x) + k(x) = 10x^2 + x + 5$$

- *Multiplication:* To multiply polynomials we expand and then simplify their product.

Example 2 : We have

$$\begin{aligned} h(x) \cdot k(x) &= (3x^2 + 4x + 5)(7x^2 - 3x) \\ &= 21x^4 - 9x^3 + 28x^3 - 12x^2 + 35x^2 - 15x \\ &= 21x^4 + 19x^3 + 23x^2 - 15x \end{aligned}$$

- *Substitution:* We can substitute in different values of x to find the value of our polynomial at this point.

Example 3 : We have

$$\begin{aligned} h(1) &= 3(1)^2 + 4(1) + 5 = 12 \\ k(-2) &= 7(-2)^2 - 3(-2) = 34 \end{aligned}$$

- *Division:* To divide one polynomial by another we use the method of long division.

Example 4 : Suppose we wanted to divide $3x^3 - 2x^2 + 4x + 7$ by $x^2 + 2x$.

$$\begin{array}{r} \overline{3x - 8} \\ x^2 + 2x \overline{) 3x^3 - 2x^2 + 4x + 7} \\ \underline{3x^3 + 6x^2} \\ -8x^2 + 4x \\ \underline{-8x^2 - 16x} \\ 20x + 7 \end{array}$$

$$\text{So } 3x^3 - 2x^2 + 4x + 7 = (x^2 + 2x)(3x - 8) + (20x + 7).$$

More formally, suppose $p(x)$ and $f(x)$ are polynomials where $\deg p(x) \geq \deg f(x)$. Then dividing $p(x)$ by $f(x)$ gives us the identity

$$p(x) = f(x)q(x) + r(x),$$

where $q(x)$ is the quotient, $r(x)$ is the remainder and $\deg r(x) < \deg f(x)$.

Example 5 : Dividing $p(x) = x^3 - 7x^2 + 4$ by $f(x) = x - 1$ we obtain the following result:

$$\begin{array}{r} x^2 \quad -6x \quad -6 \\ x-1 \overline{) x^3 \quad -7x^2 \quad +0x \quad +4} \\ \underline{x^3 \quad -x^2} \\ -6x^2 \quad +0x \\ \underline{-6x^2 \quad +6x} \\ -6x \quad +4 \\ \underline{-6x \quad +6} \\ -2 \end{array}$$

Here the quotient is $q(x) = x^2 - 6x - 6$ and the remainder is $r = -2$. *Note:* As we can see, division doesn't always produce a polynomial answer – sometimes there's just a constant remainder.

Exercises:

1. Perform the following operations and find the degree of the result.
 - (a) $(2x - 4x^2 + 7) + (3x^2 - 12x - 7)$
 - (b) $(x^2 + 3x)(4x^3 - 3x - 1)$
 - (c) $(x^2 + 2x + 1)^2$
 - (d) $(5x^4 - 7x^3 + 2x + 1) - (6x^4 + 8x^3 - 2x - 3)$
2. Let $p(x) = 3x^4 + 7x^2 - 10x + 4$. Find $p(1)$, $p(0)$ and $p(-2)$.
3. Carry out the following divisions and write your answer in the form $p(x) = f(x)q(x) + r(x)$.
 - (a) $(3x^3 - x^2 + 4x + 7) \div (x + 2)$
 - (b) $(3x^3 - x^2 + 4x + 7) \div (x^2 + 2)$
 - (c) $(x^4 - 3x^2 - 2x + 4) \div (x - 1)$
 - (d) $(5x^4 + 30x^3 - 6x^2 + 8x) \div (x^2 - 3x + 1)$

$$(e) (3x^4 + x) \div (x^2 + 4x)$$

4. Find the quotient and remainder of the following divisions.

$$(a) (2x^4 - 2x^2 - 1) \div (2x^3 - x - 1)$$

$$(b) (x^3 + 2x^2 - 5x - 3) \div (x + 1)(x - 2)$$

$$(c) (5x^4 - 3x^2 + 2x + 1) \div (x^2 - 2)$$

$$(d) (x^4 - x^2 - x) \div (x + 2)^2$$

$$(e) (x^4 + 1) \div (x + 1)$$

Section 3 REMAINDER THEOREM

We have seen in Section 2 that if a polynomial $p(x)$ is divided by polynomial $f(x)$, where $\deg p(x) \geq \deg f(x)$, we obtain the expression $p(x) = f(x)q(x) + r(x)$, where $q(x)$ is the quotient, $r(x)$ is the remainder and $\deg r(x) < \deg f(x)$, or $r = 0$.

Now suppose $f(x) = x - a$, where $a \in \mathbb{R}$. Then

$$\begin{aligned} p(x) &= (x - a)q(x) + r(x) \\ \text{i.e. } p(x) &= (x - a)q(x) + r, \quad \text{since } \deg r < \deg f. \end{aligned}$$

So the remainder when $p(x)$ is divided by $x - a$ is $p(a)$. This important result is known as the remainder theorem.

Remainder Theorem: *If a polynomial $p(x)$ is divided by $(x - a)$, then the remainder is $p(a)$.*

Example 1 : Find the remainder when $x^3 - 7x^2 + 4$ is divided by $x - 1$.

Instead of going through the long division process to find the remainder, we can now use the remainder theorem. The remainder when $p(x) = x^3 - 7x^2 + 4$ is divided by $x - 1$ is

$$p(1) = (1)^3 - 7(1)^2 + 4 = -2.$$

Note: Checking this using long division will give the same remainder of -2 see Example 5 from Section 2).

Exercises:

1. Use the remainder theorem to find the remainder of the following divisions and then check your answers by long division.

(a) $(4x^3 - x^2 + 2x + 1) \div (x - 5)$

(b) $(3x^2 + 12x + 1) \div (x - 1)$

2. Use the remainder theorem to find the remainder of the following divisions.

(a) $(x^3 - 5x + 6) \div (x - 3)$

(b) $(3x^4 - 5x^2 - 20x - 8) \div (x + 1)$

(c) $(x^4 - 7x^3 + x^2 - x - 1) \div (x + 2)$

(d) $(2x^3 - 2x^2 + 3x - 2) \div (x - 2)$

Section 4 THE FACTOR THEOREM AND ROOTS OF POLYNOMIALS

The remainder theorem told us that if $p(x)$ is divided by $(x - a)$ then the remainder is $p(a)$. Notice that if the remainder $p(a) = 0$ then $(x - a)$ fully divides into $p(x)$, i.e. $(x - a)$ is a factor of $p(x)$. This is known as the factor theorem.

Factor Theorem: *Suppose $p(x)$ is a polynomial and $p(a) = 0$. Then $(x - a)$ is a factor of $p(x)$ and we can write $p(x) = (x - a)q(x)$ for some polynomial $q(x)$.*

Note: If $p(a) = 0$ we call $x = a$ a root of $p(x)$.

We can use trial and error to find solutions of a polynomial $p(x)$ by finding a number a where $p(a) = 0$. If we can find such a number a then we know $(x - a)$ is a factor of $p(x)$, and then we can use long division to find the remaining factors of $p(x)$.

Example 1 :

a) Find all the factors of $p(x) = 6x^3 - 17x^2 + 11x - 2$.

b) Hence find all the solutions to $6x^3 - 17x^2 + 11x - 2 = 0$.

Solution a). By trial and error notice that

$$p(2) = 48 - 66 + 22 - 2 = 0$$

i.e. 2 is a root of $p(x)$.

So $(x - 2)$ is a factor of $p(x)$.

To find all the other factors we'll divide $p(x)$ by $(x - 2)$.

$$\begin{array}{r}
 6x^2 \quad -5x \quad +1 \\
 x - 2 \overline{) 6x^3 \quad -17x^2 \quad +11x \quad -2} \\
 \underline{6x^3 \quad -12x^2} \\
 -5x^2 \quad +11x \\
 \underline{-5x^2 \quad +10x} \\
 x \quad -2 \\
 \underline{x \quad -2} \\
 0
 \end{array}$$

So $p(x) = (x - 2)(6x^2 - 5x + 1)$. Now notice

$$\begin{aligned}
 6x^2 - 5x + 1 &= 6x^2 - 3x - 2x + 1 \\
 &= 3x(2x - 1) - (2x - 1) \\
 &= (3x - 1)(2x - 1)
 \end{aligned}$$

So $p(x) = (x - 2)(3x - 1)(2x - 1)$ and its factors are $(x - 2)$, $(3x - 1)$ and $(2x - 1)$.

Solution b). The solutions to $p(x) = 0$ occur when

$$x - 2 = 0, \quad 3x - 1 = 0, \quad 2x - 1 = 0.$$

That is

$$x = 2, \quad x = \frac{1}{3}, \quad x = \frac{1}{2}.$$

Exercises:

- For each of the following polynomials find (i) its factors; (ii) its roots.
 - $x^3 - 3x^2 + 5x - 6$
 - $x^3 + 3x^2 - 9x + 5$
 - $6x^3 - x^2 - 2x$
 - $4x^3 - 7x^2 - 14x - 3$
- Given that $x - 2$ is a factor of the polynomial $x^3 - kx^2 - 24x + 28$, find k and the roots of this polynomial.
- Find the quadratic whose roots are -1 and $\frac{1}{3}$ and whose value at $x = 2$ is 10.

4. Find the polynomial of degree 3 which has a root at -1 , a double root at 3 and whose value at $x = 2$ is 12 .
5. (a) Explain why the polynomial $p(x) = 3x^3 + 11x^2 + 8x - 4$ has at least one root in the interval from $x = 0$ to $x = 1$.
(b) Find all roots of this polynomial.

Exercises for Worksheet 4.5

1. Find the quotient and remainder of the following divisions.

(a) $(x^3 - x^2 + 8x - 5) \div (x^2 - 7)$

(b) $(x^3 + 5x^2 + 15) \div (x + 3)$

(c) $(2x^3 - 6x^2 - x + 6) \div (x - 3)$

(d) $(x^4 + 3x^3 - x^2 - 2x - 7) \div (x^2 + 3x + 1)$

2. Find the factors of the following polynomials.

(a) $3x^3 - 8x^2 - 5x + 6$

(b) $x^3 - 4x^2 + 6x - 4$

(c) $2x^3 + 5x^2 - 3x$

(d) $x^3 + 6x^2 + 12x + 8$

3. Solve the following equations.

(a) $x^3 - 3x^2 + x + 2 = 0$

(b) $5x^3 + 23x^2 + 10x - 8 = 0$

(c) $x^3 - 8x^2 + 21x - 18 = 0$

(d) $x^3 - 2x^2 + 5x - 4 = 0$

(e) $x^3 + 5x^2 - 4x - 20 = 0$

4. Consider the polynomial $p(x) = x^3 - 4x^2 + ax - 3$.

(a) Find a if, when $p(x)$ is divided by $x + 1$, the remainder is -12 .

(b) Find all the factors of $p(x)$.

5. Consider the polynomial $h(x) = 3x^3 - kx^2 - 6x + 8$.

(a) Given that $x - 4$ is a factor of $h(x)$, find k and find the other factors of $h(x)$.

(b) Hence find all the roots of $h(x)$.

6. Find the quadratic whose roots are -3 and $\frac{1}{5}$ and whose value at $x = 0$ is -3 .

7. Find the quadratic which has a remainder of -6 when divided by $x - 1$, a remainder of -4 when divided by $x - 3$ and no remainder when divided by $x + 1$.

8. Find the polynomial of degree 3 which has roots at $x = 1$, $x = 1 + \sqrt{2}$ and $x = 1 - \sqrt{2}$, and whose value at $x = 2$ is -2 .

Answers for Worksheet 4.5

Section 1

- | | | | | | |
|------------|--------|---------|-----------|---------|---------|
| (a) (i) 1 | (ii) 1 | (iii) 4 | (d) (i) 6 | (ii) -3 | (iii) 5 |
| (b) (i) 9 | (ii) 0 | (iii) 3 | (e) (i) 0 | (ii) 7 | (iii) 6 |
| (c) (i) -2 | (ii) 0 | (iii) 1 | (f) (i) 7 | (ii) -6 | (iii) 4 |
- $a = 5, \quad b = \pm 2, \quad c = 0, \quad d = 2, \quad e = -7.$

Section 2

- | | |
|---|------------------|
| (a) $-x^2 - 10x,$ | $\text{deg} = 2$ |
| (b) $4x^5 + 12x^4 - 3x^3 = 10x^2 - 3x,$ | $\text{deg} = 5$ |
| (c) $x^4 + 4x^3 + 6x^2 + 4x + 1,$ | $\text{deg} = 4$ |
| (d) $-x^4 - 15x^3 + 4x + 4,$ | $\text{deg} = 4$ |
- $p(1) = 4, \quad p(0) = 4, \quad p(-2) = 100$
- | |
|--|
| (a) $3x^3 - x^2 + 4x + 7 = (x + 2)(3x^2 - 7x + 18) - 29$ |
| (b) $3x^3 - x^2 + 4x + 7 = (x^2 + 2)(3x - 1) + (-2x + 9)$ |
| (c) $x^4 - 3x^2 - 2x + 4 = (x - 1)(x^3 + x^2 - 2x - 4)$ |
| (d) $5x^4 + 30x^3 - 6x^2 + 8x = (x^2 - 3x + 1)(5x^2 + 45x + 124) + (335x - 124)$ |
| (e) $3x^4 + x = (x^2 + 4x)(3x^2 - 12x + 48) - 191x$ |
- | | |
|---------------------------------|-----------------------|
| (a) $q(x) = x,$ | $r(x) = -x^2 + x - 1$ |
| (b) $q(x) = x + 3,$ | $r(x) = 3$ |
| (c) $q(x) = 5x^2 + 7,$ | $r(x) = 2x + 15$ |
| (d) $q(x) = x^2 - 4x + 11,$ | $r(x) = -29x - 44$ |
| (e) $q(x) = x^3 - x^2 + x - 1,$ | $r(x) = 2$ |

Section 3

- | | |
|---------|--------|
| (a) 486 | (b) 16 |
|---------|--------|

2. (a) 18 (b) 10 (c) 77 (d) 12

Section 4

1. (a) $(x - 2)(x^2 - x + 3)$ (c) $x(2x + 1)(3x - 2)$
 (b) $(x - 1)^2(x + 5)$ (d) $(4x + 1)(x - 3)(x + 1)$
2. $k = -3$, roots are: 2 (double) and -7
3. $2(x + 1)(x - 1/3)$
4. $4x^3 - 20x^2 + 12x + 36$
5. (b) $\frac{1}{3}, -2$

Exercises 4.5

1. (a) $q(x) = x - 1$, $r(x) = 15x - 12$ (c) $q(x) = 2x^2 - 1$, $r(x) = 3$
 (b) $q(x) = x^2 + 2x - 6$, $r(x) = 33$ (d) $q(x) = x^2 - 2$, $r(x) = 4x - 5$
2. (a) $(3x - 2)(x + 1)(x - 3)$
 (b) $(x - 2)(x^2 - 2x + 2)$
 (c) $x(2x - 1)(x + 3)$
 (d) $(x + 2)^3$
3. (a) 2, $\frac{-1+\sqrt{5}}{2}$, $\frac{-1-\sqrt{5}}{2}$
 (b) $\frac{2}{5}$, -1 , -4
 (c) 3, 2
 (d) 1
 (e) 2, -2 , -5
4. (a) $a = 4$
 (b) Factors are $(x - 3)$ and $(x^2 - x + 1)$
5. (a) $k = 11$, factors are $(x - 4)$, $(x + 1)$ and $(3x - 2)$
 (b) Roots are 4, -1 and $\frac{2}{3}$
6. $5x^2 + 14x - 3$
7. $x^2 - 3x - 4$
8. $2x^3 - 6x^2 + 2x + 2$