

# Reconstruction Notes

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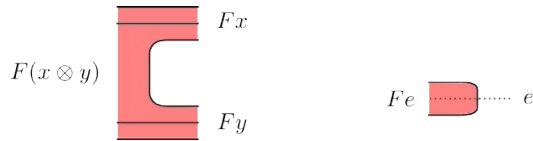
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We discuss Tannaka reconstruction in general categories, not necessarily vector spaces. For an excellent introduction, see [4].

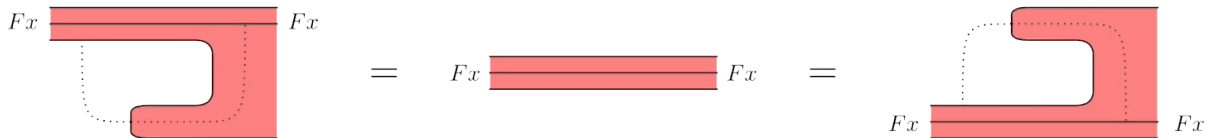
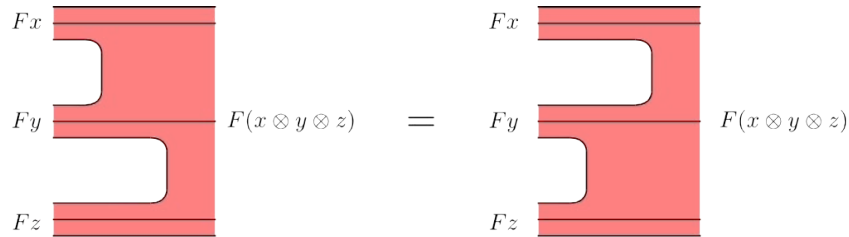
Before that, though, an introduction to my notation for monoidal and comonoidal functors. The original notion for graphically depicting monoidal functors as transparent boxes in string diagrams is due to Cockett and Seely[1], and has recently been revived and popularized by Mellies[5] with prettier graphics and an excellent pair of example calculations which nicely show the worth of the notation. However, a small modification improves the notation considerably. For a monoidal functor  $F : A \rightarrow B$ , we have a pair of maps,  $Fx \otimes Fy \rightarrow F(x \otimes y)$  and  $e \rightarrow Fe$ , which we notate as follows:



Similarly, for a comonoidal  $F$ , we have maps  $F(x \otimes y) \rightarrow Fx \otimes Fy$  and  $Fe \rightarrow e$  which we notate in the obvious dual way, as follows:



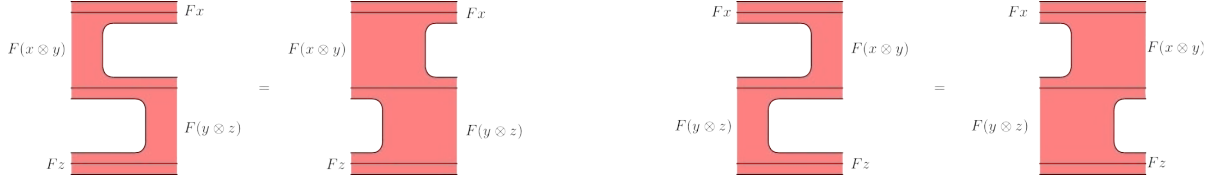
Graphically, the axioms for a monoidal functor are depicted as follows:



where, once again, the similar constraints for a comonoidal functor are exactly the above with composition read right-to-left instead of left-to-right.

It is curious and pleasing that the two unit axioms bear a superficial resemblance to triangle-identities being applied along the boundary of the “ $F$ -region”.

The above axioms seem to indicate some sort of “invariance under continuous deformation of  $F$ -regions”. For a functor which is both monoidal and comonoidal, pursuing this line of thinking leads one to consider the following pair of axioms:



A functor with these properties has been called a Frobenius monoidal functor by Brian and Craig.

Let  $F : A \rightarrow B$  be a (mere) functor between monoidal categories, where  $B$  is assumed to be left closed. Then define

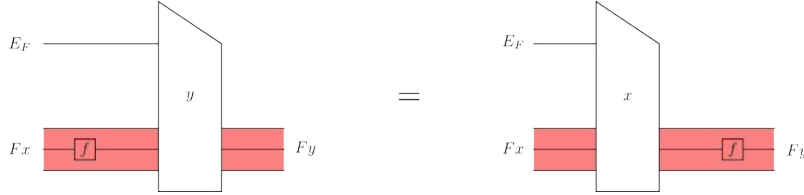
$$E_F = \int_{a \in A} [Fa, Fa]$$

where I assume that  $A$  and  $B$  are such that the indicated end exists. As Richard Garner imitabily asked at the 2006 PSSL in Nice, “Have you considered enriching everything?”. I do not discuss the matter here, but it has been considered by Brian Day[2].

There is a canonical action of  $E_F$  on  $Fx$  for each object  $x$  in  $A$ , which we denote as  $\alpha = \alpha_x : E_F \otimes Fx \rightarrow Fx$ . This is defined as:

$$E_F \otimes Fx = \left( \int_{a \in A} [Fa, Fa] \right) \otimes Fx \xrightarrow{\pi_x \otimes Fx} [Fx, Fx] \otimes Fx \xrightarrow{\text{ev}} Fx$$

using the  $x$ 'th projection from the end followed by the evaluation of the monoidal closed structure of  $B$ . The dinaturality of the end in  $a$  gives rise to the naturality of the action on  $Fa$  in  $a$ , which we notate as:



Let us now assume that the closed structure of  $B$  is given by left duals, that is,  $[a, b] = b \otimes La$ . If we also assume that  $B$  is braided and that the tensor product coheres with the ends in  $B$ , then we obtain canonical actions of  $E_F^n$  on  $Fx_1 \otimes \dots \otimes Fx_n$ , written  $\alpha_x^n$ . Taking  $\alpha^1 = \alpha$ , we define  $\alpha^n$  recursively as follows:

$$\left( \bigotimes_{i=1}^n E_F \right) \otimes \left( \bigotimes_{j=1}^n Fx_j \right) = \left( \bigotimes_{i=1}^{n-1} E_F \right) \otimes E_F \otimes \left( \bigotimes_{j=1}^{n-1} Fx_j \right) \otimes Fx_n \xrightarrow{\text{braid}} \left( \bigotimes_{i=1}^{n-1} E_F \right) \otimes \left( \bigotimes_{j=1}^{n-1} Fx_j \right) \otimes E_F \otimes Fx_n \xrightarrow{\alpha^{n-1} \otimes \alpha^1} \bigotimes_{i=1}^n Fx_i$$

For any map  $f : X \rightarrow E_F^n$ , we may paste together  $f$  with  $\alpha^n$  along  $\bigotimes_{i=1}^n Fx_i$  to obtain a “discharged form” of  $f$ :

$$X \otimes Fx_1 \otimes \dots \otimes Fx_n \rightarrow Fx_1 \otimes Fx_1 \otimes \dots \otimes Fx_n$$

Two maps are equal if and only if they have the same discharged forms.

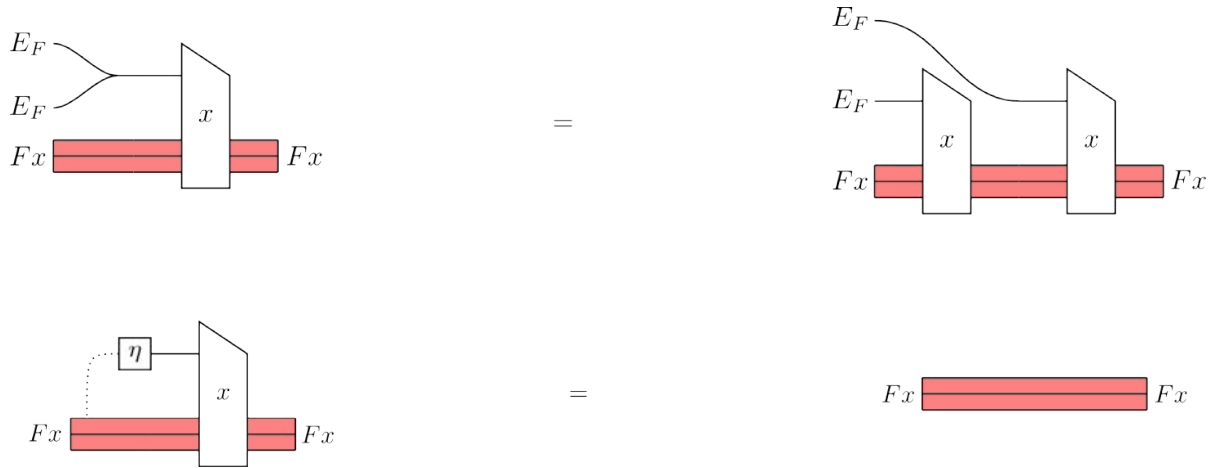
Many treatments instead consider

$$E^F = \int^{a \in A} [Fa, Fa]$$

Under dual assumptions to the ones above – namely, that the coend exists and coheres with the tensor product in  $B$  – we obtain canonical coactions of  $E^F$  on  $Fx$  for each  $x \in A$ , and iterated coactions, &c. This

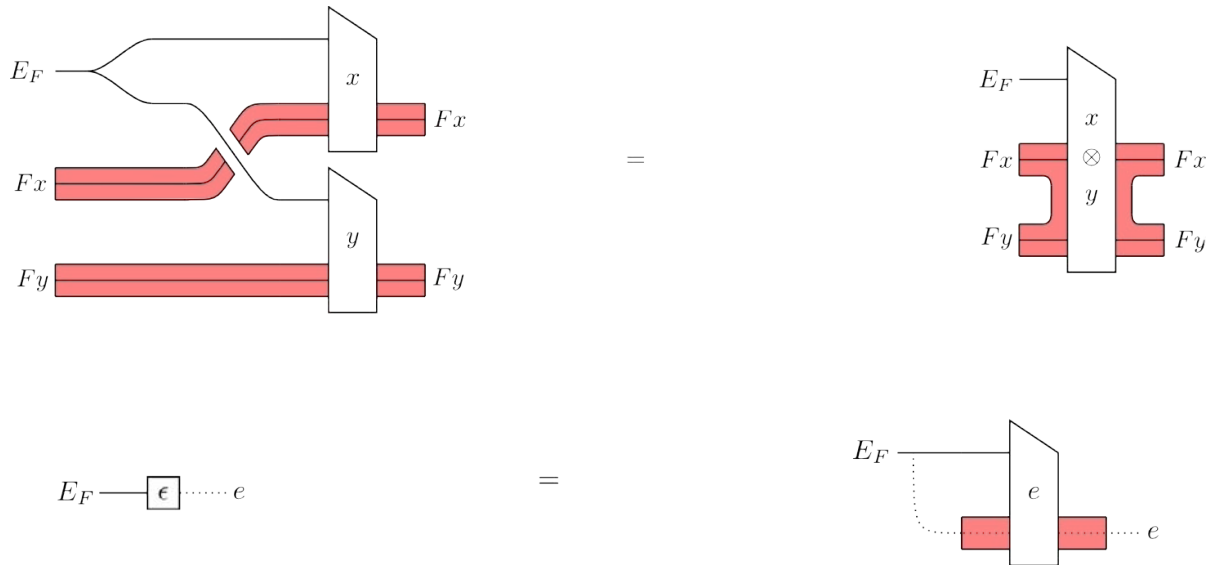
approach has certain technical benefits; among others, that the tensor product in  $B = \text{Vect}$  coheres with coends but not with ends. However, the notation we use covers *both* cases,  $E_F$  and  $E^F$ . For the former, one must read composition left-to-right, and for the latter, from right-to-left. We write  $E_F$  as a label for convenience, preferring for convenience to read in the conventional English way, but it is a crucial feature of the notation that in fact no choice is made.

Without assuming that  $F$  bears a monoidal structure, one can define a monoid structure on  $E_F$ , as follows:

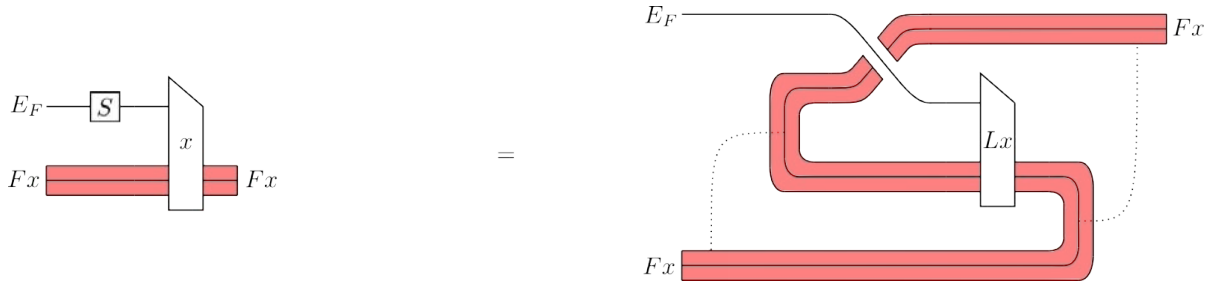


Note that this monoidal structure is associative and unital, without any assumption on  $F$ .

Furthermore, if  $F$  is known to be (lax) monoidal and comonoidal (without at the moment assuming any coherence between these structures) we can define a comonoid structure on  $E_F$ .



Finally, if  $A$  is known to have (left, say) duals, we can define a canonical map  $S : E_F \rightarrow E_F$  which we think of as a candidate for an antipode.



Notice in particular how the monoidal and comonoidal structures on  $F$  permit one to consider the application of  $F$  as not merely “boxes” but more like a flexible sheath.

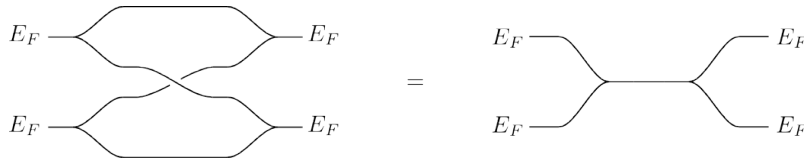
Now, the above is the raw data for two different structures, namely, Hopf algebras and weak Hopf algebras, which differ only in axioms. It has been remarked before that requiring  $F$  to be strong (that is, in our treatment, demanding that the monoidal and comonoidal structures be mutually inverse) makes the above data into a Hopf algebra. Before we discuss the Hopf data (that is, the antipode), let us first consider the bialgebra data. A bialgebra in a braided monoidal category satisfies the following four axioms. First, the unit followed by the counit must be the identity:

$$e \cdots \boxed{\eta} \text{---} \boxed{\epsilon} \cdots e \quad = \quad e \cdots \cdots \cdots e$$

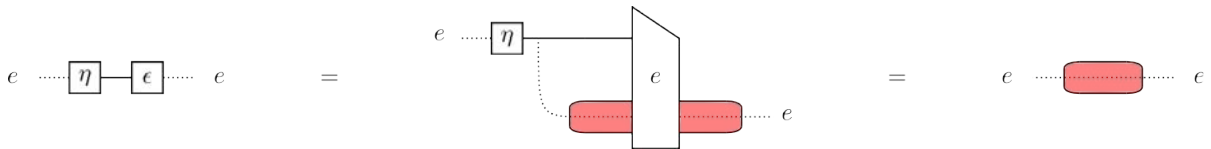
Second and third, the unit and counit must respect the comultiplication and multiplication, respectively:



Fourthly, the multiplication must cohere with the comultiplication, with the help of the braiding:

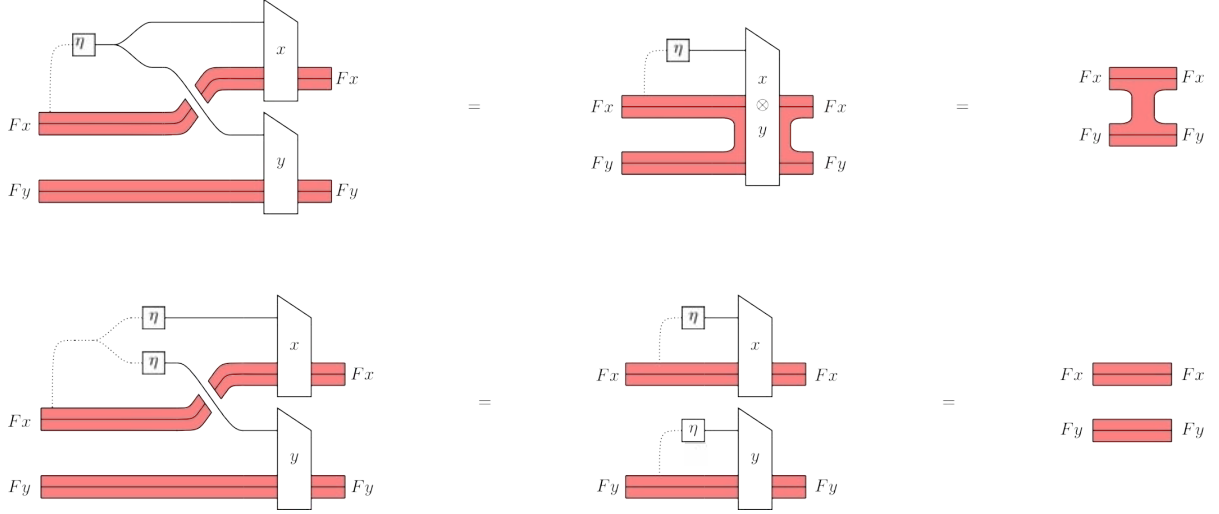


Some easy calculations show how the strength of  $F$  features crucially in showing all four of these axioms. For the first of these, we calculate:



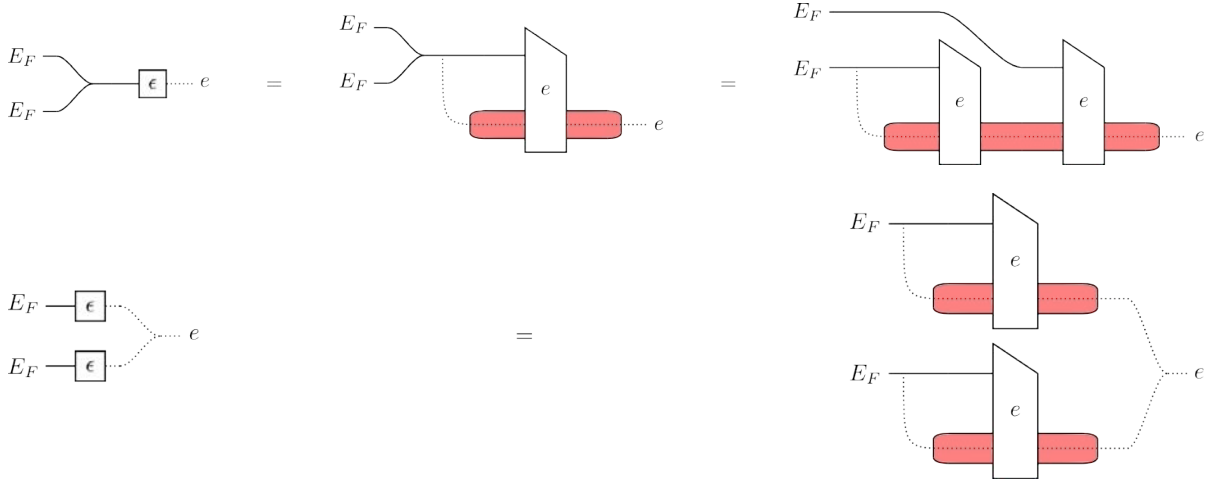
and we see that this composite is the identity on  $e$  precisely when  $e \longrightarrow Fe \longrightarrow e$  is the identity.

For the second bialgebra axiom, we have the following two calculations:



and so we see that these two are equal precisely when  $Fx \otimes Fy \rightarrow F(x \otimes y) \rightarrow Fx \otimes Fy$  is the identity.

For the third bialgebra axiom, we have the following two calculations:



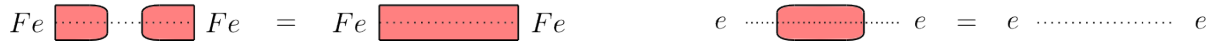
and we see that for these two to be equal, it suffices to have  $Fe \rightarrow e \rightarrow Fe$  be the identity, the use of which between the two actions in the first calculation gives the result.

Finally, for the final bialgebra axiom, the calculations shown in figure (1) compute the discharged forms as

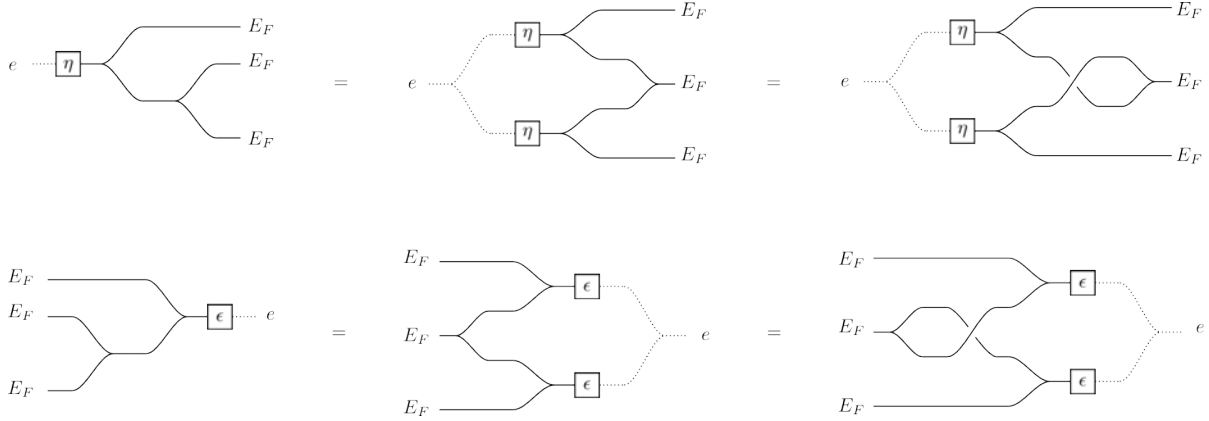


which shows that it suffices to request that  $F(x \otimes y) \rightarrow Fx \otimes Fy \rightarrow F(x \otimes y)$  should be the identity.

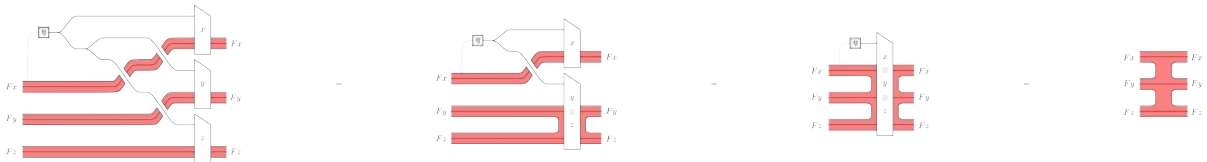
Demanding that  $F$  be strong imposes the following four conditions on the monoidal/comonoidal structure:



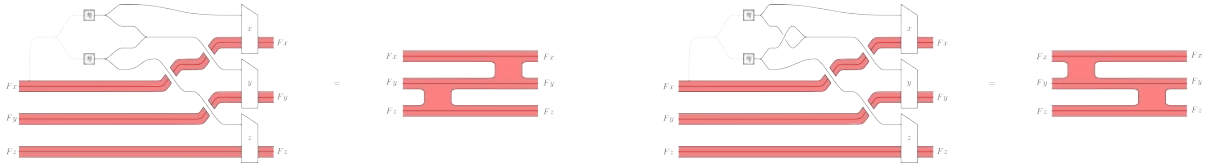
Notice that precisely one of them preserves the number of connected components of  $F$ , namely, the one which is used in the proof of the coherence of the multiplication with the comultiplication. A monoidal functor satisfying this axiom has been called “separable” by some. To move from a (strong) bialgebra to a weak one, this axiom is the only one which is retained. The coherence of the unit with the counit is discarded entirely, and the second and third axioms are replaced with the following four axioms:



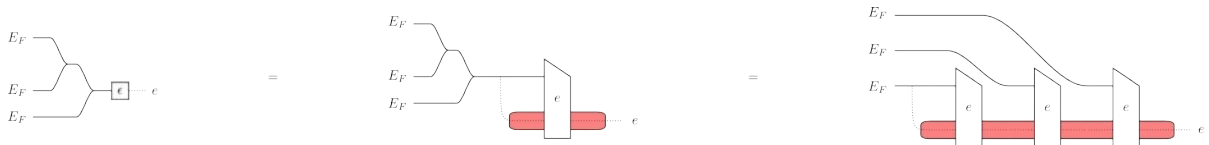
We first examine the unit axioms. In discharged form, the first unit expression is calculated as:



The calculations in figure (2) show that the second and third unit expressions have the following discharged forms:



For these unit axioms, we see that it suffices to assume that  $F$  is Frobenius. As for the counit axioms, the discharged form of the first of these is easily calculated:



The discharged forms of the second and third counit expression are computed in figure (3) and are, of course the same. Examining this figure shows that the counit axioms follow merely from  $F$  being both monoidal and comonoidal, without requiring Frobenius or separable.

This asymmetry (between unit and counit axioms) results from defining  $E_F$  using an end, had we instead used a coend, the situation would be reversed.

As for the antipode axioms, we can also consider the pair of strong antipode axioms or the trio of weak antipode axioms. The strong antipode axioms request the following two equations:

$$E_F \text{ --- } \left( \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \text{---} \right) \text{---} E_F = E_F \text{ --- } \boxed{\epsilon} \text{---} \boxed{\eta} \text{---} E_F = E_F \text{ --- } \left( \text{---} \begin{array}{c} \boxed{S} \\ \text{---} \\ \text{---} \end{array} \text{---} \right) \text{---} E_F$$

On the other hand, the weak antipode axioms request the following three equations:

$$\begin{aligned} E_F \text{ --- } \left( \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \text{---} \right) \text{---} E_F &= E_F \text{ --- } \left( \text{---} \begin{array}{c} \boxed{\eta} \\ \text{---} \\ \text{---} \end{array} \text{---} \right) \text{---} \left( \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \text{---} \left( \text{---} \begin{array}{c} \boxed{\epsilon} \\ \text{---} \\ \text{---} \end{array} \right) \text{---} E_F \\ E_F \text{ --- } \left( \text{---} \begin{array}{c} \boxed{S} \\ \text{---} \\ \text{---} \end{array} \text{---} \right) \text{---} E_F &= E_F \text{ --- } \left( \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \text{---} \left( \text{---} \begin{array}{c} \boxed{\eta} \\ \text{---} \\ \text{---} \end{array} \right) \text{---} \left( \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \text{---} \left( \text{---} \begin{array}{c} \boxed{\epsilon} \\ \text{---} \\ \text{---} \end{array} \right) \text{---} E_F \\ E_F \text{ --- } \left( \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \text{---} \left( \text{---} \begin{array}{c} \boxed{S} \\ \text{---} \\ \text{---} \end{array} \right) \text{---} \left( \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \text{---} E_F &= E_F \text{ --- } \boxed{S} \text{---} E_F \end{aligned}$$

Both sets of axioms involve the two convolutions of the antipode with the (compositional) identity, and so we calculate these two quantities explicitly. The pair of calculations in figure (4) show that the discharged forms of  $S \star E_F$  and  $E_F \star S$  are the following:

$$E_F \text{ --- } \left( \text{---} \begin{array}{c} \boxed{S} \\ \text{---} \\ \text{---} \end{array} \text{---} \right) \text{---} \left( \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \text{---} Fx = E_F \text{ --- } \left( \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \text{---} \left( \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \text{---} Fx$$

$$E_F \text{ --- } \left( \text{---} \begin{array}{c} \boxed{S} \\ \text{---} \\ \text{---} \end{array} \text{---} \right) \text{---} \left( \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \text{---} Fx = E_F \text{ --- } \left( \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \text{---} \left( \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \text{---} Fx$$

In the (strong) case, both of these convolutions are supposed to equal the composite  $E_F \xrightarrow{\epsilon} e \xrightarrow{\eta} E_F$ , the discharged form of which we compute:

$$E_F \text{ --- } \boxed{\epsilon} \text{---} \boxed{\eta} \text{---} \left( \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \text{---} Fx = E_F \text{ --- } \boxed{\epsilon} \text{---} \left( \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \text{---} Fx = E_F \text{ --- } \left( \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \text{---} \left( \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \text{---} Fx$$

Therefore, we see that, in the usual Hopf algebra case, it suffices to take  $Fe \rightarrow e \rightarrow Fe$  equal to the identity. For the weak case, these convolutions are instead set equal to the expressions which are computed in figure (5). So we see that for these two axioms it suffices to take  $F$  to be monoidal and comonoidal.

There is one additional antipode axiom which is imposed for a weak Hopf algebra, namely, that the convolution  $S \star E_F \star S$  should equal  $S$ . For this, we compute the left hand side in figure (6), the last diagram of which is the definition of  $S$ , as desired.

Furthermore, if  $A$  is known to be braided, then we can define a quasitriangular structure on  $E_F$ .

## References

- [1] Cockett, J.R.B., and R.A.G. Seely, *Linearly Distributive Functors*, JPAA v143 (1999), p155-203, available from the website of the second author.
- [2] Day, Brian J., *Enriched Tannaka reconstruction*, JPAA v108-1 (1996), p17-22
- [3] Etingof, Pavel, and Olivier Schiffman, *Lectures on Quantum Groups*, International Press, 1998.
- [4] Joyal, Andre, and Ross Street, *An Introduction to Tannaka Duality and Quantum Groups*, SLNM 1488 (1991), p411-492, available from the website of the second author.
- [5] Mellies, Paul-Andre, *Functorial Boxes in String Diagrams*, SLNCS 4207, available from the website of the author.



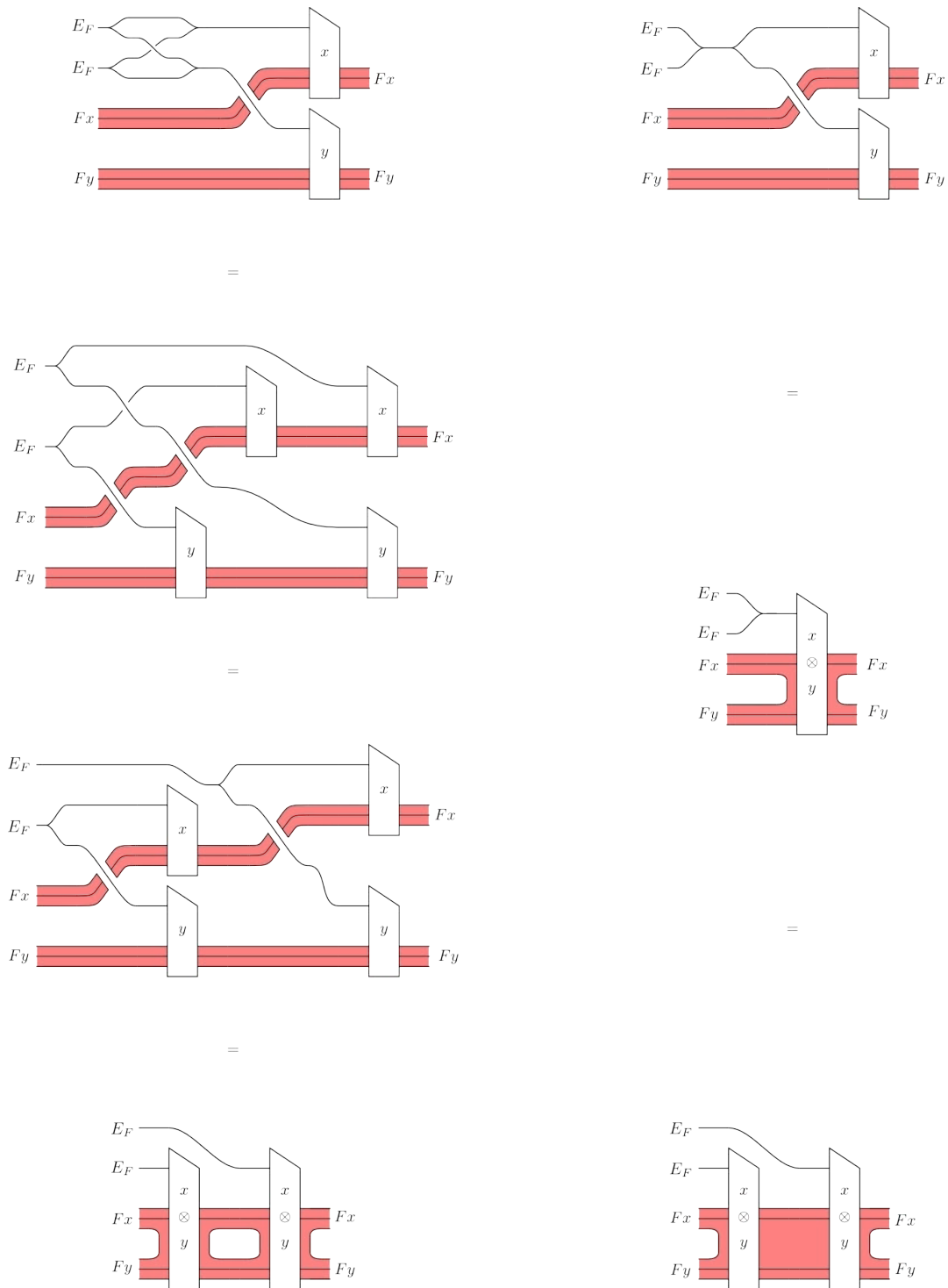


Figure 1: Coherence of the multiplication with the comultiplication

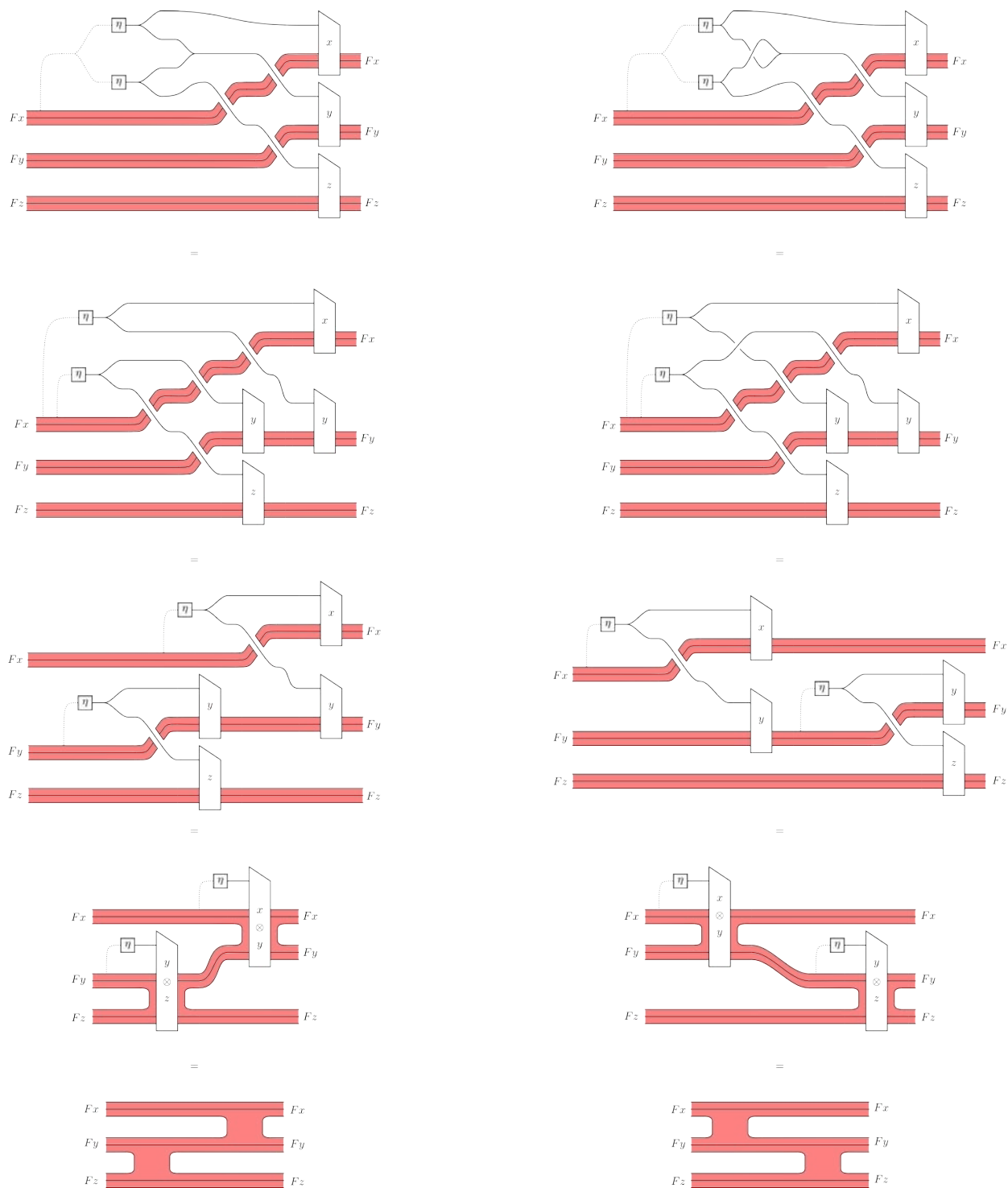


Figure 2: Weak unit calculations

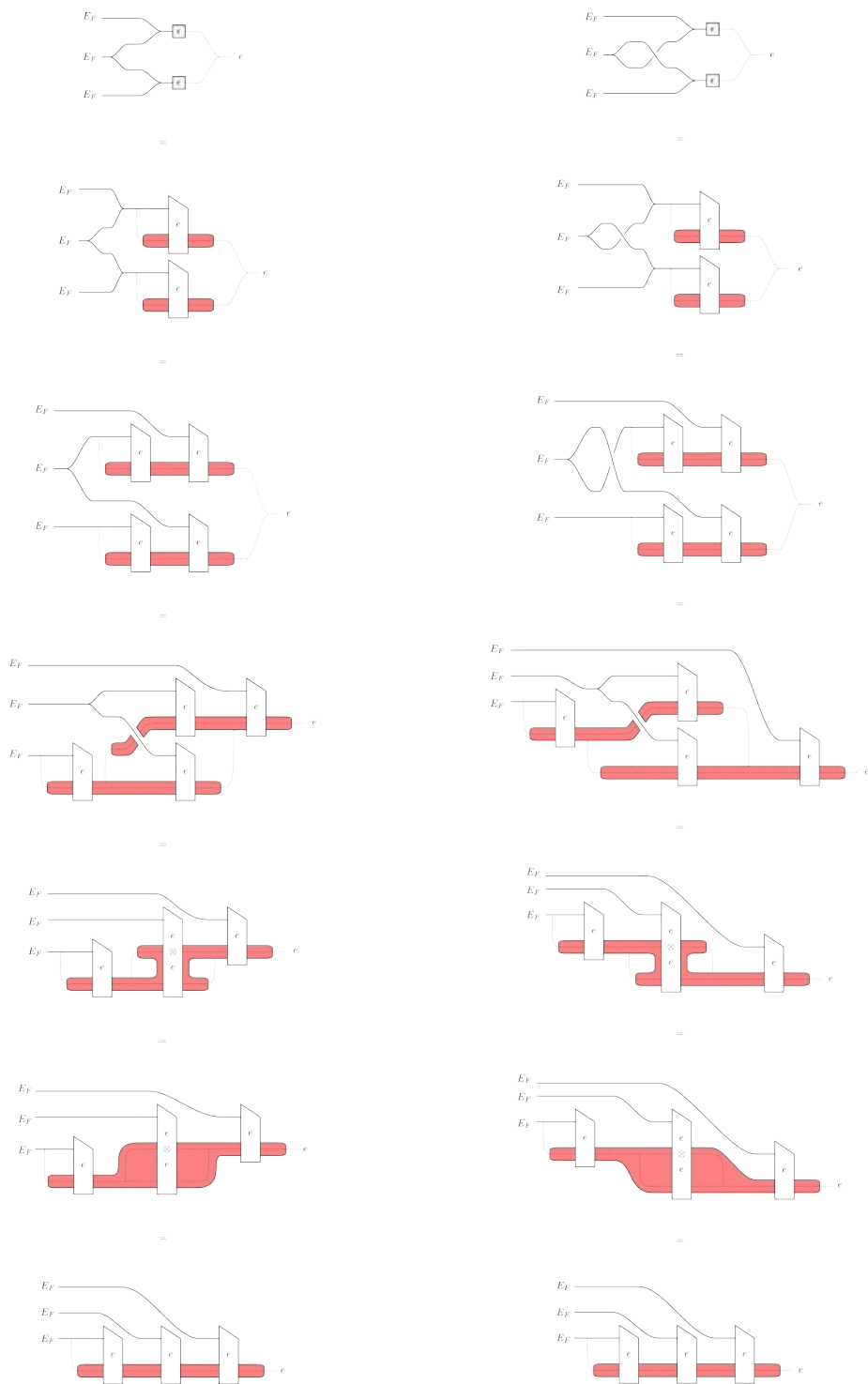


Figure 3: Weak counit calculations

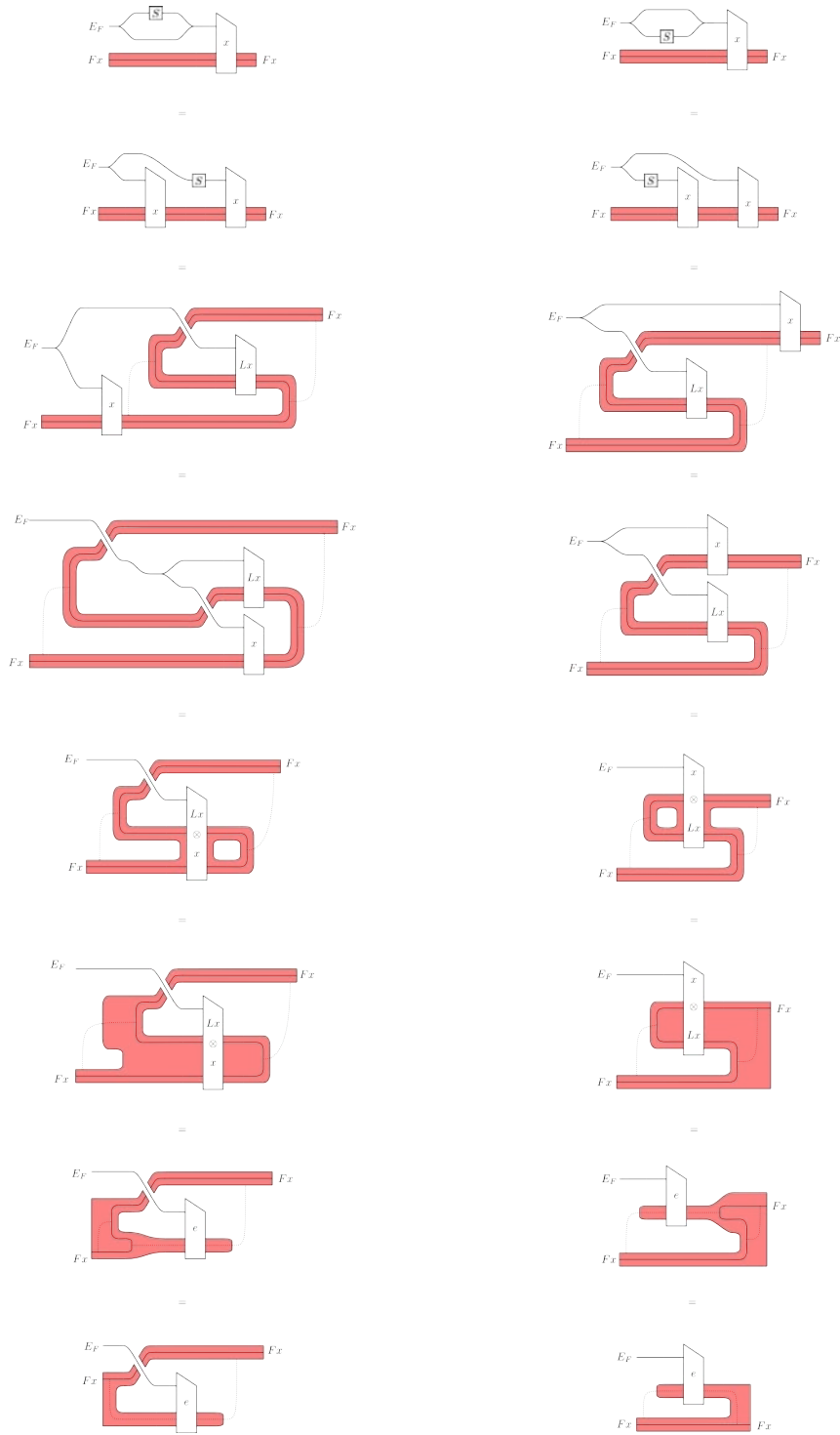


Figure 4: Calculations of  $S \star E_F$  and  $E_F \star S$

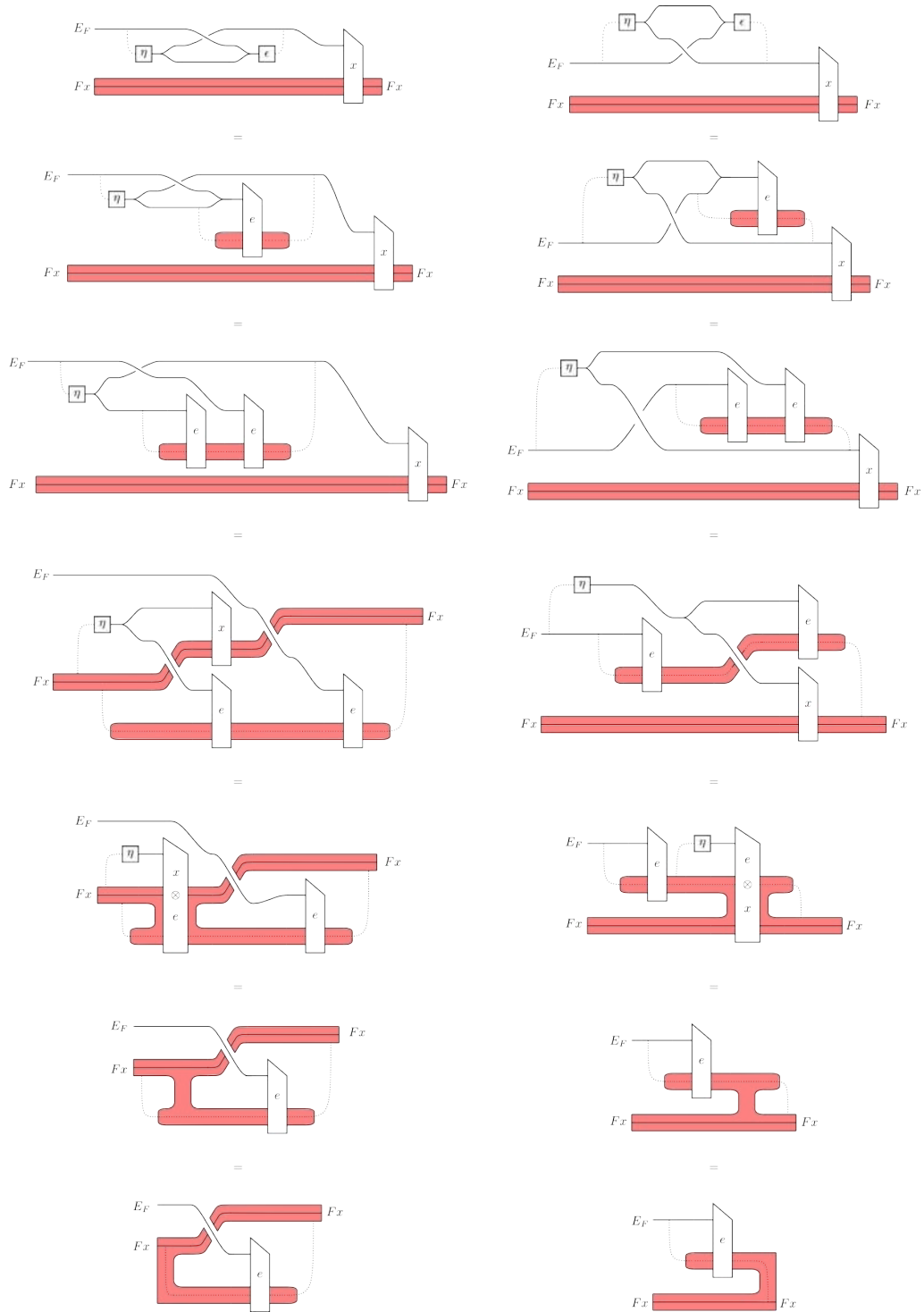


Figure 5: “Source” and “Target” maps.

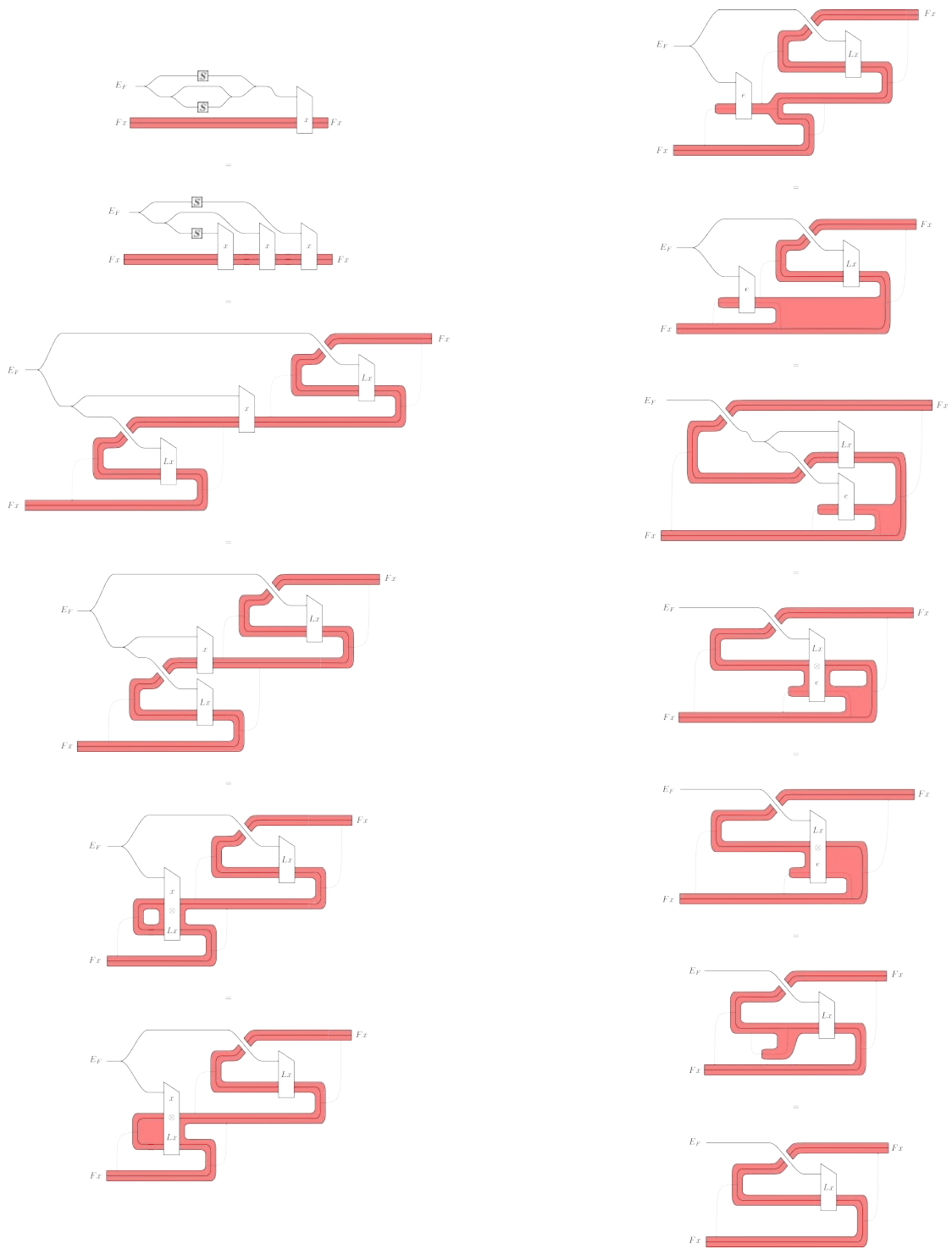


Figure 6: The calculation showing  $S \star E_F \star S = S$