Update on the efficient reals

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The description of the real numbers that we are concerned with here I learnt from Steve Schanuel. An "efficient" real number is an equivalence class of "quasihomomorphisms" on the integers **Z** ; that is, endofunctions on the integers **Z** which preserve addition up to an "additivity constant". The efficient reals are isomorphic to the "usual" reals. Peter Johnstone remembered a talk at the University of Sussex by Richard Lewis with the same description. Convinced of its need for wider circulation, I wrote the note [St] outlining the construction and the proof that it formed a complete ordered field. Almost immediately Steve Schanuel noticed a problem with my suggested proof of completeness. I have learnt too that Richard Lewis did not develop a direct proof of completeness. I had a suggestion for an approach but nothing was written until January 2002 when I set the topic as part of a Vacation Scholar project for two bright undergraduate students Ben Odgers and Nguyen Hanh Vo.

My suggestion was to first prove a lemma that every efficient real number could be represented by an endofunction whose additivity constant was no bigger than 3, say. I provided the Vacation Scholars with a handwritten proof (see Lemma 2 below) which is described in their report [OV] as part of their account of completeness. My proof of this lemma was based on operations assigning to each (positive) quasihomomorphism f integers $\lceil f \rceil$ and $\lfloor f \rfloor$; see page 13 of [OV].

Over the years other people have expessed interest in the construction to the point of working seriously on it; some independently rediscovering it, others aware of [St]. For example, in January 2003, I was informed of the appearance of [A'C]; the author was unaware of [St]. An important feature of his completeness proof is the dependence on the lemma I had suggested to Odgers and Vo. What is more, he has a vastly simpler proof and obtains an additivity constant of 1 (which we certainly knew was possible from the identification of our reals with the "usual" reals).

This month (on 18 September 2003) Rob Arthan (who does refer to [St]) kindly sent me his preprint [Ar] with an interesting new slant on the topic. The fundamental tool in his proof of completeness is the operation taking f to $\lfloor f \rfloor$. It had been my intention to look back at [OV] and tidy up some of the arguments but I had not. I now see that there are gaps in the argument appearing in [OV]. I console myself that the techniques I suggested to Odgers and Vo are used in both the proofs of completeness by [A'C] and [Ar].

What follows is some background to these events. The proof of Lemma 1 is essentially due to Odgers and Vo [OV].

A function $f : M \longrightarrow \mathbf{Z}$ from a commutative monoid M to the ring **Z** of integers is called a *quasihomomorphism* (or *qhm*) [St] when there exists a natural number k such that

$$|f(m+n)-f(m)-f(n)| \le k$$

for all m and n in M. We call k the *additivity constant* for the qhm.

Two qhms f and g are *equivalent* when there is a natural number h such that

$$|f(n)-g(n)| \le h$$

for all n in M. Clearly every qhm f is equivalent to one with $f(0)=0$.

A qhm $f:\mathbf{Z}\longrightarrow\mathbf{Z}$ is called *positive* (in the non-strict sense) when there exists an integer a such that $f(n)\ge a$ for all natural numbers n. It is called *negative* when there exists an integer b such that $f(n)\le b$ for all natural numbers n. Clearly these concepts are invariant under equivalence. Equally clearly, we can always replace a positive [respectively, negative] qhm by an equivalent one with $a=0$ [respectively, $b=0$].

Lemma 1 *Every quasihomomorphism* $f:\mathbf{Z}\longrightarrow\mathbf{Z}$ *is either positive or negative.*

Proof Assume f is a qhm with $f(0)=0$ and additivity constant k. Assume f is neither positive nor negative. Let r be the smallest natural number with $f(r)>k$ and let s be the smallest natural number with $f(s)<-k$. Since $f(0)=0$, both r and s are strictly positive; clearly also $r\ne s$. If $r>s$ then $0<r-s<r$, so the minimality of r implies $f(r-s)\le k$; but then

$$f(r)-f(r-s)-f(s) > k-k+k = k$$

contrary to the additivity constant property of k. Similarly, if $s<r$ then $0<s-r<s$, so the minimality of s implies $f(s-r)\ge -k$; but then

$$f(s)-f(s-r)-f(r) < -k+k-k = -k$$

contrary to the additivity constant property of k. **QED**

Let $f:\mathbf{N}\longrightarrow\mathbf{Z}$ be a qhm from the additive monoid $\mathbf{N}$. The *extension* $\overleftarrow{f}:\mathbf{Z}\longrightarrow\mathbf{Z}$ *of* f, defined by

$$\overleftarrow{f}(n)=\begin{cases} f(n) & \text{for } n\ge 0 \\ -f(-n) & \text{for } n<0 \end{cases},$$

is a qhm having the same additivity constant as f and satisfying

$$\overleftarrow{f}(-n)=-\overleftarrow{f}(n)$$

for all integers n. Clearly every qhm is equivalent to the extension of its restriction to $\mathbf{N}$. Since every positive qhm $\mathbf{Z}\longrightarrow\mathbf{Z}$ is equivalent to one mapping $\mathbf{N}$ into $\mathbf{N}$, it is also equivalent to the extension of a qhm $\mathbf{N}\longrightarrow\mathbf{N}$. Similarly, every negative qhm $\mathbf{Z}\longrightarrow\mathbf{Z}$ is equivalent to minus the extension of a qhm $\mathbf{N}\longrightarrow\mathbf{N}$.

Lemma 2 *Every quasihomomorphism* $\mathbf{N} \longrightarrow \mathbf{N}$ *is equivalent to one with additivity constant* 3.

Proof Suppose $f : \mathbf{N} \longrightarrow \mathbf{N}$ is a qhm with additivity constant k and $f(0) = 0$. By induction on m we can prove that

(1) $$|f(mn) - mf(n)| \le mk$$

for all natural numbers m and n. Consequently,

(2) $$|nf(m) - mf(n)| \le |f(mn) - mf(n)| + |nf(m) - f(mn)| \le mk + nk = (m+n)k.$$

From (1) we also have $f(m) - mf(1) \le mk$ so that

(3) $$f(m) \le (f(1)+k)m \quad \text{for all natural numbers } m.$$

So $f(1)+k$ is an element of the set

$$S_f = \{u \in \mathbf{N} \mid f(m) \le um \text{ for almost all } m \in \mathbf{N}\}.$$

[Here "almost all" means "all but a finite number of".] Since $\mathbf{N}$ is well ordered, S_f has a first element $\lceil f \rceil$.

For $r \in \mathbf{N}$, we define the qhm $rf : \mathbf{N} \longrightarrow \mathbf{N}$ in the usual way by $(rf)(m) = rf(m)$. Then define

(4) $$f^+ : \mathbf{N} \longrightarrow \mathbf{N} \quad \text{by} \quad f^+(r) = \lceil rf \rceil.$$

This means that

(5) $rf(m) \le f^+(r)m$ for almost all $m \in \mathbf{N}$, and

(6) $rf(m) \le um$ for almost all $m \in \mathbf{N}$ implies $f^+(r) \le u$.

By (5) and (2) we have

$$f^+(r)m \ge rf(m) \ge mf(r) - (r+m)k.$$

So $m(f^+(r) - f(r)) \ge -(r+m)k$ for almost all m. By taking m large enough, it follows that

(7) $$f^+(r) - f(r) \ge -k.$$

From (2) we see that

$$rf(m) \le mf(r) + (m+r)k \le (f(r)+k)m + rk$$

and hence, by taking $m > 2kr$, we obtain $rf(m) \le (f(r)+k)m$. By (6) it follows that

(8) $$f^+(r) \le f(r) + k.$$

It follows from (7) and (8) that f and f^+ are equivalent; but why is f^+ a qhm? Well, by (5),

$$(r+s)f(m) = rf(m) + sf(m) \le f^+(r)m + f^+(s)m;$$

so, by (6),

(9) $$f^+(r+s) \le f^+(r) + f^+(s).$$

On the other hand, by (5) and (8),

$$(f^+(r+s) - f^+(s))m \ \geq\ (r+s)f(m) - f^+(s)m \ \geq\ (r+s)f(m) - (f(s)+k)m = rf(m) - km$$

for almost all natural numbers m. By (6) we deduce

(10) $\qquad f^+(r) \ \leq\ f^+(r+s) - f^+(s) + k$.

Combining (9) and (10), we get

(11) $\qquad 0 \ \leq\ f^+(r) + f^+(s) - f^+(r+s) \leq k$.

So f^+ is indeed a qhm.

Similarly we can consider the set

$$T_f = \{v \in \mathbf{N} \mid vm \leq f(m) \text{ for almost all } m \in \mathbf{N}\}.$$

For each v in T_f, using (3) we see that $vm \leq f(m) \leq (f(1)+k)m$ for almost all m; so certainly $v \leq f(1)+k$. So T_f is finite and so has a last element $\lfloor f \rfloor$. Then define a function

(12) $\qquad f^- : \mathbf{N} \longrightarrow \mathbf{N} \quad$ by $\quad f^-(r) \ =\ \lfloor rf \rfloor$.

Proceeding as before, we see that f^- is a qhm equivalent to f and satisfying

(13) $\qquad -k \ \leq\ f^-(r) + f^-(s) - f^-(r+s) \ \leq\ 0$.

Define a function

(14) $\qquad \bar{f} : \mathbf{N} \longrightarrow \mathbf{N} \quad$ by $\quad \bar{f}(m)$ = the integer part of $\dfrac{f^-(m)+f^+(m)}{2}$.

Using (11) and (13), we see that $\bar{f}$ is a qhm with additivity constant[1] no greater than $\dfrac{k+3}{2}$.

Clearly also $\bar{f}$ is equivalent to f. Now $\dfrac{k+3}{2} < k$ for $k > 3$. So we can use this process to reduce the additivity constant until it is ≤ 3 . **QED**

Now here is the stronger result and short proof due to Norbert A'Campo [A'C].

Lemma 3 *Every quasihomomorphism* $\mathbf{Z} \longrightarrow \mathbf{Z}$ *is equivalent to one with additivity constant* 1.

Proof For integers p and q with $q \neq 0$, write $\langle p{:}q \rangle$ for a choice of integer satisfying

$$\left| \langle p{:}q \rangle - \frac{p}{q} \right| \ \leq\ \frac{1}{2} .$$

For any qhm $f : \mathbf{Z} \longrightarrow \mathbf{Z}$ with additivity constant k, define $f' : \mathbf{Z} \longrightarrow \mathbf{Z}$ by

$$f'(n) \ =\ \langle f(3kn) : 3k \rangle.$$

Then $\left| f'(n) - f(n) \right| \ \leq\ \left| f'(n) - \dfrac{f(3kn)}{3k} \right| + \left| \dfrac{f(3kn)}{3k} - f(n) \right| \ \leq\ \dfrac{1}{2} + k$ using (1) above. So f and f' are equivalent. Moreover,

[1] Actually with only a little more care we can see that (k+2)/2 will work, so we can replace 3 by 2 in Lemma 2.

$$\left|f'(m+n)-f'(m)-f'(n)\right|$$

$$\leq \left|f'(m+n)-\frac{f(3k(m+n))}{3k}\right| + \left|\frac{f(3km)}{3k}-f'(m)\right| + \left|\frac{f(3kn)}{3k}-f'(n)\right| + \left|\frac{f(3k(m+n))}{3k}-\frac{f(3km)}{3k}-\frac{f(3kn)}{3k}\right|$$

$$\leq \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{k}{3k} \leq \frac{11}{6} < 2 .$$

So f' is a qhm with additivity constant 1. **QED**

Let $\mathbf{R}_{\text{eff}}$ denote the set of equivalence classes $[f]$ of qhms $f:\mathbf{Z}\longrightarrow\mathbf{Z}$. Addition is induced by pointwise addition of representative qhms. Multiplication is defined by composition of representative qhms. For qhms f and g, define $[f]\leq[g]$ when the qhm $g-f$ is positive. The techniques of [St] and [OV] show that $\mathbf{R}_{\text{eff}}$ becomes an Archimedean ordered field.

Theorem 3 $\mathbf{R}_{\text{eff}}$ *is a complete ordered field.*

For the proof of completeness see [A'C] or [Ar]. However, it seems that the construction of the infimum by [Ar] can be expressed as follows. Let S be a non-empty subset of positive elements of $\mathbf{R}_{\text{eff}}$ with no least element. Then, for each natural number n, the set $\left\{ f^{+}(n) \,\middle|\, f\in S \right\}$ has a first element $s(n)$. This defines a qhm $s:\mathbf{N}\longrightarrow\mathbf{N}$ whose extension $\bar{s}:\mathbf{Z}\longrightarrow\mathbf{Z}$ is the infimum of S.

References

[St] Ross Street, An efficient construction of the reeal numbers, *Gazette of the Australian Math. Soc.* **12** (1985) 57-58.

[OV] Ben Odgers and Nguyen Hanh Vo, Analysis of an efficient construction of the reals (Vacation Scholar Project, January 2002); see <www.math.mq.edu.au/~street/efficient.pdf>.

[A'C] Norbert A'Campo, A natural construction for the real numbers, arXiv:math.GN/0301015 v1 (3 Jan 2003).

[Ar] R.D. Arthan, The Eudoxus real numbers (preprint, September 2003).

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