Surprising relationships connecting ploughing a field, mathematical trees, permutations, and trigonometry

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Introduction

- This talk is about a little adventure my students and I had 15 years ago when a colleague told us of his interest in certain kinds of trees.
- He wanted to count the number J_n of these trees with a given number n of nodes.
- A hunt was conducted to find where these numbers J_n occurred elsewhere in mathematics.
- The Macquarie University Mathematics Department offers four-week Summer Vacation Scholarships to bright undergraduate students.
- The two Scholars I supervised at the beginning of 2001 were Ryan Crompton and Tam Pham.
  Their project was to work on the topic.
  This talk will explain the amazing results of their research.

History in more detail

- In late 2000, my postdoctoral research associate William Joyce gave a seminar on an application of Category Theory to Physics.
- He made use of a particular kind of mathematical TREE and produced a formula for iteratively calculating the number J_n of these trees with a given number n of nodes.
- As we allow the number n of nodes to increase we obtain an increasing sequence $J_1, J_2, J_3, \ldots$ of numbers.
- There is a Web Page: `<https://oeis.org/>` by N.J.A. Sloane. It tells, from typing the first few terms of a sequence, whether that sequence has occurred somewhere else in Mathematics.
- Postgraduate student Daniel Steffen traced this down and found, to our surprise, that the sequence was related to the tangent function $\tan x$.
- Ryan and Tam searched out what was known about this connection and discovered some apparently new results. We all found this a lot of fun and I hope you will too.

Permutations and combinations

- A permutation of a list of distinct numbers is a list of the same numbers in a different order; this second list must involve all the numbers of the original list without repetition.
- For example, one permutation of the list 23679 is 63927.
- The total number of permutations of any list $a_1 a_2 \dots a_n$ of length n is

$$n! = n \times (n-1) \times (n-2) \dots 2 \times 1 \; ;$$

  the symbol "!" is called *factorial* in this context.
- For us, the original list will be in increasing order $a_1 < a_2 < \cdots < a_n$.
- A choice of *some* of the numbers $a_1 a_2 \dots a_n$ can always be written in increasing order too: such a choice is called a *combination* from the given list.
- The number of combinations of length r from an original list of length n is the *binomial coefficient*

$$^nC_r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!} \; .$$

Pascal's triangle

- The binomial coefficients can be constructed by a triangular process noticed by Blaise Pascal (1623–1662).
- The formula explaining the triangle is

$$^{n+1}C_r = {}^{n}C_{r-1} + {}^{n}C_r \ .$$

- Each row has a 1 on both ends and the other entries are the sum of the two entries above it. The r-th entry in row n is ${}^{n}C_r$; (we start counting at $n = 0$ and $r = 0$).

n=0								1							
1							1		1						
2						1		2		1					
3					1		3		3		1				
4				1		4		6		4		1			
5			1		5		10		10		5		1		
6		1		6		15		20		15		6		1	
7	1		7		21		35		35		21		7		1

Figure 1

Polynomial approximations

- If we know that a function $f(x)$ is well behaved and we know the values $f(0), f'(0), f''(0), \ldots$ of the function and its repeated derivatives at 0, we know a lot about the function.
- For example, suppose that we suspect our function is a cubic

$$f(x) = a + bx + cx^2 + dx^3$$

then

$$f'(x) = b + 2cx + 3dx^2, f''(x) = 2c + 6dx, f'''(x) = 6d\ .$$

- So we find that

$$a = f(0),\ b = f'(0),\ c = \frac{1}{2!}f''(0),\ d = \frac{1}{3!}f'''(0)\ .$$

Polynomials for $\sin x$

- Recall that the function $\sin x$ is well approximated by the polynomial x for small x (radians, not degrees!).
- For $f(x) = \sin x$ we can easily calculate that

$$f(0) = \sin 0 = 0,\ f'(0) = \cos 0 = 1,$$

$$f''(0) = -\sin 0 = 0,\ f'''(0) = -\cos 0 = -1\ .$$

- Of course, we do not really expect $\sin x$ to be a cubic, but, if we use the formulas above for a, b, c, d, we obtain the cubic

$$x - \frac{1}{3!}x^3\ .$$

Indeed, you can check on your computer that this gives a better approximation to $\sin x$ than x for fairly small x.

MacLaurin-Taylor polynomials

- Taking this further, we obtain a polynomial

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots (-1)^n \frac{1}{(2n+1)!}x^{2n+1}$$

  which can be used to approximate $\sin x$ for all x by taking n large enough.
- Polynomials obtained from functions in this way, by using the values of the derivatives at 0, are called *MacLaurin or Taylor polynomials*.
- In the case of $y = \sin x$, finding the derivatives at 0 is made easy by the fact that the derivatives start repeating after four steps: $y'''' = \sin x$. Even before that we see that $y'' = -y$.
- Whenever we have an expression for a higher derivative of a function in terms of its earlier derivatives (what is called a *differential equation*), we can use it to find the Taylor polynomials.

Differential equation for $\tan x$

- We focus our attention on the tangent function $y = \tan x$. We can record the *initial condition* $y(0) = 0$.
- We have $y' = \sec^2 x = 1 + \tan^2 x = 1 + y^2$ yielding $y'(0) = 1$ and the differential equation
$$y' = 1 + y^2 \ .$$
- Differentiating again we obtain another differential equation $y'' = 2yy'$, yielding $y''(0) = 0$.
- Continuing, we see that $y''' = 2y'y' + 2yy''$, $y^{(4)} = 6y'y'' + 2yy'''$, $y^{(5)} = 6y''y'' + 8y'y''' + 2yy^{(4)}$, $y^{(6)} = 20y''y''' + 10y'y^{(4)} + 2yy^{(5)}$, $y^{(7)} = 20y'''y''' + 30y''y^{(4)} + 12y'y^{(5)} + 2yy^{(6)}$, so
$$y'''(0) = 2,\ y^{(4)}(0) = 0,\ y^{(5)}(0) = 16,\ y^{(6)}(0) = 0,\ y^{(7)}(0) = 272\ .$$

Taylor polynomials for $\tan x$

- Our calculation of seven derivatives of $y = \tan x$ at 0 leads us to the Taylor polynomial

$$x + 2\frac{x^3}{3!} + 16\frac{x^5}{5!} + 272\frac{x^7}{7!}$$

which is quite a good approximation to $\tan x$.

- You can obtain a better approximation by finding higher $y^{(n)}(0)$.
- It is fairly obvious that $y^{(n)}(0) = 0$ for all even n .
- The pattern for $y^{(n)}(0)$ with n odd is nowhere near so obvious.

Trees

- Graphs of various kinds are used throughout Mathematics and its applications. For example, graphs can be used to organize both computer hardware design and computer programs. Graphs have nodes (represented by points) and edges (represented by curves – straight lines if possible) connecting various nodes.
- A *tree* is a graph which is connected (you can get from one node to any other by a path of edges) and loop free (you cannot get back to the same node once you set out on a path of edges, without backtracking).
- We are only interested in trees (see Figure 2) for which a particular node is selected and called the root. We then draw trees on a plane piece of paper with the *root* at the bottom, thought of as at level 0. We wish to record the fact that certain nodes are at the same level.

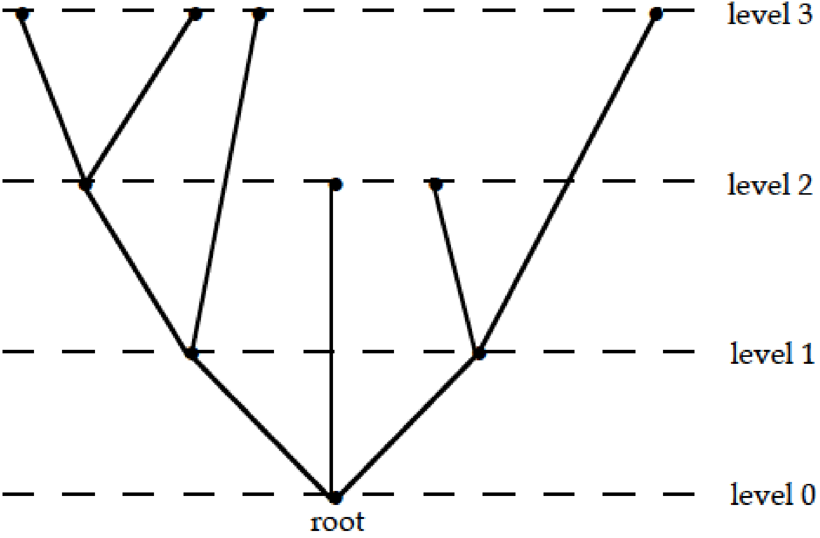


Figure 2

Joyce trees

- In Figure 2, there are two nodes at level 1, three at level 2, and four at level 3.
- Nodes with no edges connected above them are called *leaves*. The tree in Figure 2 has six leaves.
- A tree is called *binary* when each node is either a leaf or has precisely two edges above connected to it. Figure 2 is not a binary tree since the root has three edges connected above it.
- A *Joyce tree* is a binary tree for which no two nodes have the same level and all levels, up to that of the top leaf, have a node. An example is provided in Figure 3.

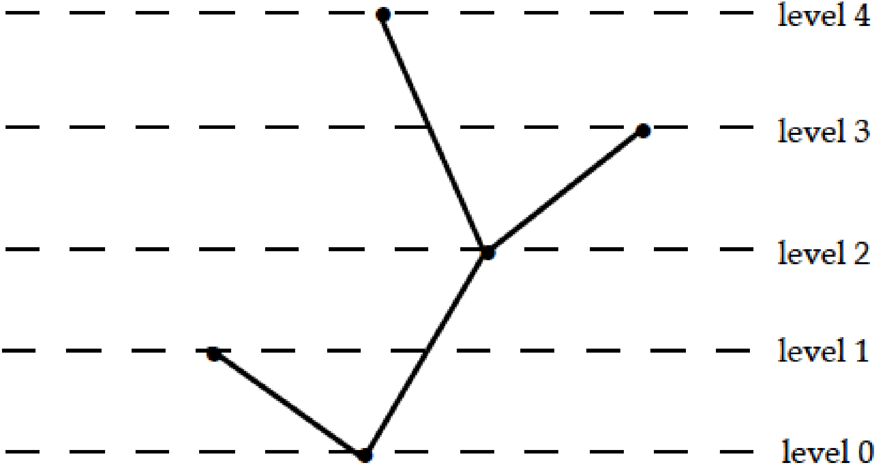


Figure 3

Counting Joyce trees

- A fact we need to observe about a Joyce tree is that there is a simple relationship between the number of leaves and the number of nodes: if there are m leaves then there are $2m - 1$ nodes altogether. In Figure 3, there are $m = 3$ leaves and $2.3 - 1 = 5$ nodes.
- To prove this in general notice that, if we have a Joyce tree with more than one leaf, look at the highest level where there is a node which is not a leaf. The two edges above and connected to this node must connect the node to two leaves. Removal of those two edges and the two leaves creates a new leaf. So we have reduced the number of leaves by one and the number of nodes by two. What remains is still a Joyce tree. The process continues until we have a single node. The fact we want now follows by induction.
- Let J_n be the number of Joyce trees with n nodes. One thing we have seen is that $J_n = 0$ for n even.

Tremolo permutations

- A permutation of an increasing list of numbers is called *tremolo* when consecutive differences between consecutive numbers in the permuted list have opposite sign.

- For example, the permutation 635241 of 123456 is tremolo since $6-3$ is positive, $3-5$ is negative, $5-2$ is positive, $2-4$ is negative, $4-1$ is positive. It reminds me of playing tremolo on a mandolin: down-up-down-up-down.

- There is a one-to-one correspondence between Joyce trees with n nodes and tremolo permutations of $0123 \ldots n\overline{n+1}$ which begin with 1 and end with 0.

Proof of the correspondence

Label the nodes of the tree with numbers obtained by adding 2 to the level of the node.
Now we "read" the tree from left to right and from top to bottom taking note of the numbers labelling the nodes; this gives a tremolo permutation of $234...\overline{n n+1}$.
Put the 1 at the front and 0 at the end.
For example, for the tree in Figure 3 we obtain the labelling of nodes as in Figure 4 on next slide.
Reading the tree gives the list 32645. So the desired tremolo permutation of 0123456 is 1326450. Notice that the numbers in even positions (the second, fourth, sixth, . . .) in the permutation always correspond to leaves.
Notice too that our insistence on a 1 at the start and 0 at the end means there can be no tremolo permutations of $0123...\overline{n n+1}$ for n even.

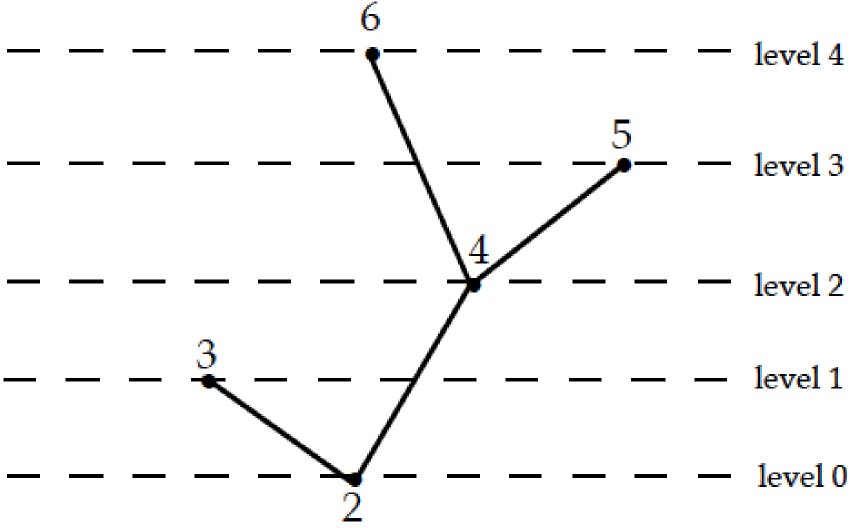


Figure 4

Towards a recursive formula for J_n

- This shows is that J_n is equal to the number of tremolo permutations of any list of $n+2$ increasing numbers which have the smallest number at the end and the second smallest at the beginning.

- Equally, J_n is the number of tremolo permutations of any list of $n+2$ increasing numbers which have the smallest number at the beginning and second smallest at the end (just read the permutation backwards).

- We shall now find a formula for recursively determining the numbers J_n ; we shall express J_{n+1} in terms of the J_m with $m \leq n$.

Analysis of tremolo permutations

- Consider any tremolo permutation of $0123\ldots n\overline{n+2}$ beginning with 1 and ending with 0.

$$1\overbrace{*****************}^{n+1}0$$

- How is such a permutation made up? First we must choose one of the n starred positions to place the number 2; let us say we choose the $(m+1)$-th star.

$$1\overbrace{******}^{m}2\overbrace{***********}^{n-m}0$$

- Now we need to choose m numbers from the list $345\ldots n+2$ of n remaining numbers; this choice can be made in ${}^nC_m$ ways. Using the chosen numbers to insert between 1 and 2, we need to choose a tremolo permutation starting at 1 and ending at 2; this choice can be made in J_m ways.

The recursion for J_n

- Using the remaining $n - m$ numbers, we need to choose a tremolo permutation starting at 2 and ending at 0; this can be done in J_{n-m} ways. This means, once we fix the position of 2 at the $(m+1)$-th star, there are ${}^nC_m J_m J_{n-m}$ possibilities.
- It follows that

$$J_{n+1} = \sum_{m=0}^{n} {}^nC_m \ J_m \ J_{n-m} \ .$$

- Given that $J_0 = 0$ and $J_1 = 1$, we can begin to calculate successive J_n.
- $J_2 = {}^1C_0 J_0 J_1 + {}^1C_1 J_1 J_0 = 0$,
  $J_3 = {}^2C_0 J_0 J_2 + {}^2C_1 J_1 J_1 + {}^2C_2 J_2 J_0 = 0 + 2 + 0 = 2$
- At this point it becomes clear why $J_n = 0$ for n even and why we need only worry about m odd in the summation.
- $J_5 = {}^4C_1 J_1 J_3 + {}^4C_3 J_3 J_1 = 4 \times 2 + 4 \times 2 = 16$

Further values of J_n

- $J_7 = 272, J_9 = 7936, J_{11} = 353792, J_{13} = 22368256,$
- $J_{15} = 1903757312, J_{17} = 209865342976, J_{19} = 29088885112832,$
- $J_{21} = 4951498053124096, J_{23} = 1015423886506852352,$
- $J_{25} = 246921480190207983616, J_{27} = 70251601603943959887872,$
- $J_{29} = 23119184187809597841473536, \dots \ .$
- We therefore have empirical evidence that the J_n are the coefficients in the Taylor polynomials of $\tan x$. Later, we shall show how to prove it completely.

Back to $\tan x$

- Recall that $y = \tan x$ satisfies the differential equation $y' = 1 + y^2$.
- Let us see whether we can find numbers t_n such that

$$y = \frac{t_0}{0!}x^0 + \frac{t_1}{1!}x^1 + \frac{t_2}{2!}x^2 + \frac{t_3}{3!}x^3 + \dots \tag{1}$$

  satisfies that differential equation.
- Differentiate (**??**).

$$y' = \frac{t_1}{0!}x^0 + \frac{t_2}{1!}x^1 + \frac{t_3}{2!}x^2 + \frac{t_4}{3!}x^3 + \dots$$

- Square (**??**).

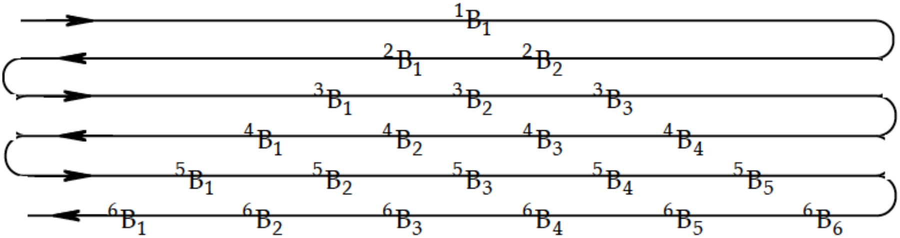
$$y^2 = \frac{t_0\,t_0}{0!0!}x^0 + \left(\frac{t_0\,t_1}{0!1!} + \frac{t_1\,t_0}{1!0!}\right)x^1 + \left(\frac{t_0\,t_2}{0!2!} + \frac{t_1\,t_1}{1!1!} + \frac{t_2\,t_0}{2!0!}\right)x^2$$
$$+ \left(\frac{t_0\,t_3}{0!3!} + \frac{t_1\,t_2}{1!2!} + \frac{t_2\,t_1}{2!1!} + \frac{t_3\,t_0}{3!0!}\right)x^3 + \dots$$

Recursion for the t_n

- We can equate coefficients using $y' = 1 + y^2$.
- $$\frac{t_1}{0!} = 1 + \frac{t_0 t_0}{0!0!},\ \frac{t_2}{1!} = \frac{t_0 t_1}{0!1!} + \frac{t_1 t_0}{1!0!},\ \frac{t_3}{2!} = \frac{t_0 t_2}{0!2!} + \frac{t_1 t_1}{1!1!} + \frac{t_2 t_0}{2!0!},$$
$$\frac{t_4}{3!} = \frac{t_0 t_3}{0!3!} + \frac{t_1 t_2}{1!2!} + \frac{t_2 t_1}{2!1!} + \frac{t_3 t_0}{3!0!},\ \ldots$$
- The general term gives
$$t_{n+1} = \sum_{m=0}^{n} {}^nC_m\ t_m\ t_{n-m}\ ,$$
the same formula satisfied by the J_n.
- Since this formula together with $t_0 = 0$ and $t_1 = 1$ determine the t_n, it follows that
$$J_n = t_n$$
for all $n \geq 0$.

Ox ploughing

- Something like a Pascal triangle can be used to generate the numbers J_n. The triangle involves other numbers besides the J_n.
- Write ${}^nB_m$ for the m-th entry of the n-th row.
- In traversing this triangle we move as if we were ploughing rows in a field with an ox; this is called the *Boustrophedon order*.



$^1B_1$
$^2B_1$ $^2B_2$
$^3B_1$ $^3B_2$ $^3B_3$
$^4B_1$ $^4B_2$ $^4B_3$ $^4B_4$
$^5B_1$ $^5B_2$ $^5B_3$ $^5B_4$ $^5B_5$
$^6B_1$ $^6B_2$ $^6B_3$ $^6B_4$ $^6B_5$ $^6B_6$

Figure 5

The boustrophedon triangle

- Using this order, we start with ${}^1B_1 = 1$, however, we begin each new row with 0, so that

$$ {}^2B_2 = {}^3B_1 = {}^4B_4 = {}^5B_1 = \cdots = 0 \ , $$

  and we obtain all other ${}^nB_m$ by adding the one before it to the one above and between those two.

- To be precise, we have

$$ {}^1B_1 = 1, \ {}^{2k}B_{2k} = {}^{2k+1}B_1 = 0, $$

$$ {}^{2k}B_m = {}^{2k}B_{m+1} + {}^{2k-1}B_m, \ {}^{2k+1}B_m = {}^{2k+1}B_{m-1} + {}^{2k}B_{m-1} \ . $$

- This process leads to the Boustrophedon triangle shown in Figure 6.

1

1 0

0 1 1

2 2 1 0

0 2 4 5 5

16 16 14 10 5 0

0 16 32 46 56 61 61

272 272 256 224 178 122 61 0

0 272 544 800 1024 1202 1324 1385 1385

7936 7936 7664 7120 6320 5296 4094 2770 1385 0

Boustrophedon triangle

Figure 6

The boustrophedon triangle (continued)

- We can easily see from this construction that each ${}^{n}B_m$ with n even is obtained by adding all the entries to the right of it in the row above, and that each ${}^{n}B_m$ with n odd is obtained by adding all the entries to the left of it in the row above; more formally:

$$ {}^{2k}B_m = \sum_{r=m}^{2k-1} {}^{2k-1}B_r \ , \ \ {}^{2k+1}B_m = \sum_{r=1}^{m-1} {}^{2k}B_r \ . $$

- The thing to notice about Figure 6 is that the left hand side of the triangle again gives the numbers J_n. This empirically verifies the simple identity

$$ {}^{n+1}B_1 = J_n \ \text{ for } n > 0 \ . $$

  This identity can actually be proved rigorously.
- It follows from a much nicer observation of Ms Tam Pham which identifies all the entries in the Boustrophedon triangle.

The theorem of Tam Pham

Theorem

The entry $^{n}B_{m}$ in the Boustrophedon triangle is the number of tremolo permutations of $01\ldots n$ that begin with m and end with 0.

Proof We shall verify that, with this interpretation of $^{n}B_{m}$ in terms of tremolos, the Boustrophedon construction is satisfied.
The idea is fully understandable by looking at the special case

$$^{8}B_{4} = {}^{7}B_{4} + {}^{7}B_{5} + {}^{7}B_{6} + {}^{7}B_{7} \ .$$

So consider a tremolo permutation

$$4 * * * * * * * 0$$

of 012345678.

Proof (continued)

Because 8 is even, in order to tremelo to 0 in 8 steps, the first star $*$ must be greater than 4.

So we can leave out the 4 and see that the number of such tremelos is equal to the number of tremelos of 01235678 which start with a number greater than 4 and end with 0.

By renumbering the 5678 as 4567, we see that this is the same as the number of tremelos of 01234567 which start with a number greater than 3 and end with 0.

This is the same as the number of tremolos of 01234567 ending in 0 and starting with 4 or 5 or 6 or 7.

This gives the identity ${}^{8}B_4 = {}^{7}B_4 + {}^{7}B_5 + {}^{7}B_6 + {}^{7}B_7$. Q.E.D.

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