A SKEW-MONOIDAL REFLECTION THEOREM

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Contents

1. Introduction

We squeeze some more results out of Brian Day's PhD thesis [1]. While his thesis was about monoidal categories, we can, without even modifying the biggest diagrams, adapt the results to skew monoidal categories. Elsewhere [4, 5] we have discussed convolution [2]. Here we will provide the skew version of the Day Reflection Theorem [1, 3]. The beauty of this variant is further evidence that the direction choices involved in the skew notion are important for organizing, and adding depth to, certain mathematical phenomena.

2. Skew monoidal reflection

Recall from [6, 4, 5] the notion of *(left) skew monoidal structure* on a category $\mathscr{X}$. It involves a functor $\otimes : \mathscr{X} \times \mathscr{X} \longrightarrow \mathscr{X}$, an object $I \in \mathscr{X}$, and natural families of (not necessarily invertible) morphisms

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C), \qquad \lambda_A : I \otimes A \to A, \qquad \rho_A : I \to A \otimes I,$$

satisfying five coherence conditions. Suppose $(\mathscr{X}, \otimes, I, \alpha, \lambda, \rho)$ and $(\mathscr{A}, \bar{\otimes}, \bar{I}, \bar{\alpha}, \bar{\lambda}, \bar{\rho})$ are skew monoidal categories.

Recall, also from these references, that an *opmonoidal structure* on a functor $L\colon \mathscr{X} \to \mathscr{A}$ consists of a natural family of morphisms

$$\psi_{X,Y}\colon L(X \otimes Y) \to LX \bar{\otimes} LY$$

and a morphism $\psi_0\colon LI \to \bar{I}$ satisfying three axioms. We say the opmonoidal functor is *normal* when ψ_0 is invertible. We say the opmonoidal functor is *strong* when ψ_0 and all $\psi_{X,Y}$ are invertible. However, in this paper, a limited amount of such strength is important.

Date: August 30, 2014.

Both authors gratefully acknowledge the support of the Australian Research Council Discovery Grant DP130101969; Lack acknowledges with equal gratitude the support of an Australian Research Council Future Fellowship.

Theorem 2.1. *Suppose $L \dashv N\colon \mathscr{A} \to \mathscr{X}$ is an adjunction with unit $\eta\colon 1_{\mathscr{X}} \Rightarrow NL$ and invertible counit $\varepsilon\colon LN \Rightarrow 1_{\mathscr{A}}$. Suppose $\mathscr{X}$ is skew monoidal. There exists a skew monoidal structure on $\mathscr{A}$ for which $L\colon \mathscr{X} \to \mathscr{A}$ is normal opmonoidal with each $\psi_{X,NB}$ invertible if and only if, for all $X \in \mathscr{X}$ and $B \in \mathscr{A}$, the morphism*

$$L(\eta_X \otimes 1_{NB})\colon L(X \otimes NB) \to L(NLX \otimes NB) \tag{2.1}$$

is invertible. In that case, the skew monoidal structure on $\mathscr{A}$ is unique up to isomorphism.

Proof. Suppose $\mathscr{A}$ has a skew monoidal structure $(\bar\otimes, \bar I, \bar\alpha, \bar\lambda, \bar\rho)$ for which L is normal opmonoidal with the $\psi_{X,NB}$ invertible. We have the commutative square

$$\begin{array}{ccc}
LX\bar\otimes LNB & \xrightarrow{L\eta_X\bar\otimes 1} & LNLX\bar\otimes LNB \\
{\scriptstyle \psi^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle \psi^{-1}} \\
L(X\otimes NB) & \xrightarrow{L(\eta_X\otimes 1)} & L(NLX\otimes NB)
\end{array}$$

in which the vertical arrows are invertible. The top arrow is invertible with inverse $\varepsilon_{LX}\bar\otimes 1$. So the bottom arrow is invertible.

Conversely, suppose each $L(\eta_X \otimes 1_{NB})$ is invertible. Wishing L to become opmonoidal with the limited strength, we are forced (up to isomorphism) to put

$$A\bar\otimes B = L(NA \otimes NB) \ \text{ and } \ \bar I = LI \ ,$$

and to define the constraints $\bar\alpha, \bar\lambda, \bar\rho$ by commutativity in the following diagrams.

$$\begin{array}{ccc}
L((NA\otimes NB)\otimes NC) & \xrightarrow{L(\eta\otimes 1)} & L(NL(NA\otimes NB)\otimes NC) \\
{\scriptstyle L\alpha}\big\downarrow & & \big\downarrow{\scriptstyle \bar\alpha} \\
L(NA\otimes(NA\otimes NC)) & \xrightarrow[L(1\otimes\eta)]{} & L(NA\otimes NL(NB\otimes NC))
\end{array}$$

$$\begin{array}{ccc}
L(I\otimes NA) & \xrightarrow{L(\eta_I\otimes 1)} & L(NLI\otimes NA) \\
{\scriptstyle L\lambda}\big\downarrow & & \big\downarrow{\scriptstyle \bar\lambda} \\
LNA & \xrightarrow[\varepsilon_A]{} & A
\end{array}
\qquad
\begin{array}{ccc}
LNA & \xrightarrow{\varepsilon_A} & A \\
{\scriptstyle L\rho}\big\downarrow & & \big\downarrow{\scriptstyle \bar\rho} \\
L(NA\otimes I) & \xrightarrow[L(1\otimes\eta_I)]{} & L(NA\otimes NLI)
\end{array}$$

The definitions make sense because the top arrows of the squares are invertible (while the bottom arrows may not be). Now we need to verify the five axioms. The proofs all proceed by preceding the desired diagram of barred morphisms by suitable invertible morphisms involving only ε_A, $L\eta_X$, η_{NA}, or $L(\eta_X \otimes 1_{NB})$, then manipulating until one can make use of the corresponding unbarred diagram.

The biggest diagram for this is the proof of the pentagon for $\bar\alpha$. Fortunately, the proof in Brian Day's thesis [1] of the corresponding result for closed monoidal categories has the necessary Diagram 4.1.3 on page 94 written without any inverse isomorphisms, so saves us rewriting it here. (The notation is a little different with ψ in place of N and with some of the simplifications we also use below.)

It remains to verify the other four axioms. The simplest of these is

$$\begin{aligned}\bar{\lambda}_{LI}\bar{\rho}_{LI} &= \bar{\lambda}_{LI}\bar{\rho}_{LI}\varepsilon_{LI}L\eta_I \\ &= \bar{\lambda}_{LI}L(1\otimes\eta_I)L\rho_{NLI}L\eta_I \\ &= \bar{\lambda}_{LI}L(1\otimes\eta_I)L(\eta_I\otimes I)L\rho_I \\ &= \bar{\lambda}_{LI}L(\eta_I\otimes I)L(1\otimes\eta_I)L\rho_I \\ &= \varepsilon_{LI}L\lambda_{NLI}L(1\otimes\eta_I)L\rho_I \\ &= \varepsilon_{LI}L\eta_I L\lambda_I L\rho_I \\ &= 1_{LI}L(\lambda_I\rho_I) \\ &= 1_{LI}\ .\end{aligned}$$

For the other three, to simplify the notation (but to perhaps complicate the reading), we write as if N were an inclusion of a full subcategory, choose L so that the counit is an identity, and write XY for $X\otimes Y$. Then we have

$$\begin{aligned}\bar{\lambda}_{B\bar{\otimes}C}\bar{\alpha}_{LI,B,C}L(\eta_{(LI)B}1_C)L((\eta_I 1_B)1_C) &= \bar{\lambda}_{B\bar{\otimes}C}L(1\eta_{BC})L\alpha_{LI,B,C}L((\eta_I 1_B)1_C) \\ &= \bar{\lambda}_{B\bar{\otimes}C}L(1_{LI}\eta_{BC})L(\eta_I 1_{BC})L\alpha_{I,B,C} \\ &= \bar{\lambda}_{B\bar{\otimes}C}L(\eta_I 1_{BC})L(1_I\eta_{BC})L\alpha_{I,B,C} \\ &= L\lambda_{BC}L\alpha_{I,B,C} \\ &= L(\lambda_B 1_C) \\ &= (\bar{\lambda}_B\bar{\otimes}1_C)L(\eta_{(LI)B}1_C)L((\eta_I 1_B)1_C)\end{aligned}$$

yielding the axiom $\bar{\lambda}_{B\bar{\otimes}C}\bar{\alpha}_{LI,B,C} = \bar{\lambda}_B\bar{\otimes}1_C$ on right cancellation.

For the proof of the axiom $(1_A\bar{\otimes}\bar{\lambda}_C)\bar{\alpha}_{A,LI,C}(\bar{\rho}_A\bar{\otimes}1_C) = 1_{A\bar{\otimes}C}$, we can look at Diagram 4.1.2 on page 93 of [1]. The required commutativities are all there once we reverse the direction of the right unit constraint which Day calls r instead of ρ.

For the final axiom, we have

$$\begin{aligned}\bar{\alpha}_{A,B,LI}\bar{\rho}_{A\bar{\otimes}B} &= \bar{\alpha}_{A,B,LI}L(\eta_{AB}1_{LI})L(1_{AB}\eta_I)L\rho_{AB} \\ &= L(1_A\eta_{BLI})L\alpha_{A,B,LI}L(1_{AB}\eta_I)L\rho_{AB} \\ &= L(1_A\eta_{BLI})L(1_A(1_B\eta_I))L\alpha_{A,B,I}L\rho_{AB} \\ &= L(1_A\eta_{BLI})L(1_A(1_B\eta_I))L(1_A\rho_B) \\ &= 1_A\bar{\otimes}\bar{\rho}_B\ .\end{aligned}$$

The desired opmonoidal structure on L is defined by $\psi_0 = 1\colon LI \to \bar{I}$ and $\psi_{X,Y} = L(\eta_X\otimes\eta_Y)\colon L(X\otimes Y)\to L(NLX\otimes NLY)$. The three axioms for opmonoidality are easily checked and we have each $\psi_{X,NB} = L(1_{NLX}\otimes\eta_{NB})L(\eta_X\otimes 1_{NB})$ invertible. □

3. A reflective lemma

Assume we have an adjunction $L \dashv N\colon \mathscr{A}\to\mathscr{X}$ with unit $\eta\colon 1_{\mathscr{X}}\Rightarrow NL$ and counit $\varepsilon\colon LN\Rightarrow 1_{\mathscr{A}}$. Assume N is fully faithful; that is, equivalently, the counit ε is invertible.

Lemma 3.1. *For $Z\in\mathscr{X}$, the following conditions are equivalent:*

(i) there exists $A\in\mathscr{A}$ and $Z\cong NA$;

(ii) for all $X\in\mathscr{X}$, the function $\mathscr{X}(\eta_X,1)\colon\mathscr{X}(NLX,Z)\to\mathscr{X}(X,Z)$ is surjective;

(iii) the morphism $\eta_Z \colon Z \to NLZ$ is a coretraction (split monomorphism);
(iv) the morphism $\eta_Z \colon Z \to NLZ$ is invertible;
(v) for all $X \in \mathscr{X}$, the function $\mathscr{X}(\eta_X, 1) \colon \mathscr{X}(NLX, Z) \to \mathscr{X}(X, Z)$ is invertible.

Proof. $(i) \Rightarrow (ii)$

$$\begin{array}{ccccc} \mathscr{X}(X,Z) & \xrightarrow{\cong} & \mathscr{X}(X,NA) & \xrightarrow{\cong} & \mathscr{A}(LX,A) \\ {\scriptstyle 1}\downarrow & & & & \downarrow{\scriptstyle N} \\ \mathscr{X}(X,Z) & \xleftarrow{\mathscr{X}(\eta_X,1)} & \mathscr{X}(NLX,Z) & \xleftarrow{\cong} & \mathscr{X}(NLX,NA) \end{array}$$

$(ii) \Rightarrow (iii)$ Take $X = Z$ and obtain $\nu \colon NLZ \to Z$ with $\mathscr{X}(\eta_Z, 1)\nu = 1_Z$.

$(iii) \Rightarrow (iv)$ If $\nu\eta_Z = 1$ then $(\eta_Z\nu)\eta_Z = 1\eta_Z$, so, by the universal property of η_Z, we have $\eta_Z\nu = 1$.

$(iv) \Rightarrow (v)$ The non-horizontal arrows in the commutative diagram

$$\begin{array}{ccccc} \mathscr{X}(NLX,Z) & & \xrightarrow{\mathscr{X}(\eta_X,1)} & & \mathscr{X}(X,Z) \\ {\scriptstyle \mathscr{X}(1,\eta_Z)}\downarrow & & & & \downarrow{\scriptstyle \mathscr{X}(1,\eta_Z)} \\ \mathscr{X}(NLX,NLZ) & & \xrightarrow{\mathscr{X}(\eta_X,1)} & & \mathscr{X}(X,NLZ) \\ & {\scriptstyle N}\nwarrow & & \nearrow{\scriptstyle \cong} & \\ & & \mathscr{A}(LX,LZ) & & \end{array}$$

are all invertible, so the horizontal arrows are invertible too.

$(v) \Rightarrow (i)$ Clearly $(v) \Rightarrow (ii)$ and we already have $(ii) \Rightarrow (iii) \Rightarrow (iv)$, so take $A = LZ$ and the invertible η_Z. ∎

4. Skew closed reflection

Recall from Section 8 of [5], if $- \otimes Y$ has a right adjoint

$$\mathscr{X}(X \otimes Y, Z) \cong \mathscr{X}(X, [Y, Z])$$

in the skew monoidal category $\mathscr{X}$ then $\mathscr{X}$ becomes left skew closed via (what we here call) the *left internal hom* $[Y, Z]$; but this may exist for only certain objects Z.

Theorem 4.1. *Suppose $L \dashv N \colon \mathscr{A} \to \mathscr{X}$ is an adjunction with unit $\eta \colon 1_{\mathscr{X}} \Rightarrow NL$ and invertible counit $\varepsilon \colon LN \Rightarrow 1_{\mathscr{A}}$. Suppose $\mathscr{X}$ is skew monoidal and left internal homs of the form $[NB, NC]$ exist for all $B, C \in \mathscr{A}$. The morphisms (2.1) are invertible for all $X \in \mathscr{X}$ and $B \in \mathscr{A}$ if and only if the morphisms*

$$\eta_{[NB,NC]} \colon [NB, NC] \to NL[NB, NC] \tag{4.2}$$

are invertible for all $B, C \in \mathscr{A}$. In that case, the skew monoidal structure abiding on $\mathscr{A}$, as seen from Theorem 2.1, is left closed. Also, the functor N is left closed.

Proof. Consider the following commutative diagram.

$$\begin{array}{ccc} \mathscr{A}(L(NLX \otimes NB), C) & \xrightarrow{\mathscr{A}(L(\eta\otimes 1),1)} & \mathscr{A}(L(X \otimes NB), C) \\ \cong \downarrow & & \downarrow \cong \\ \mathscr{X}(NLX \otimes NB, NC) & & \mathscr{X}(X \otimes NB, NC) \\ \cong \downarrow & & \downarrow \cong \\ \mathscr{X}(NLX, [NB, NC]) & \xrightarrow[\mathscr{X}(\eta_X,1)]{} & \mathscr{X}(X, [NB, NC]) \end{array}$$

Invertibility of the arrows (2.1) is equivalent to the invertibility of the top horizontal arrows. This is equivalent to invertibility of the bottom horizontal arrows. By Lemma 3.1, this is equivalent to invertibility of the arrows (4.2).

For the penultimate sentence of the Theorem, we now have the natural isomorphisms:

$$\begin{aligned} \mathscr{A}(A\bar{\otimes}B, C) &\cong \mathscr{X}(NA \otimes NB, NC) \\ &\cong \mathscr{X}(NA, [NB, NC]) \\ &\cong \mathscr{X}(NA, NL[NB, NC]) \\ &\cong \mathscr{A}(A, L[NB, NC]) \end{aligned}$$

yielding the left internal hom $[B, C] = L[NB, NC]$ for $\mathscr{A}$. For the last sentence, we have $N[B, C] = NL[NB, NC] \cong [NB, NC]$. □

Our notation for a right adjoint to $X \otimes -$ is

$$\mathscr{X}(X \otimes Y, Z) \cong \mathscr{X}(Y, \langle X, Z\rangle)\ .$$

The *right internal hom* $\langle X, Z\rangle$ may exist for only certain objects Z. In general, the existence of right homs in a left skew monoidal category does not give a left or right skew closed structure. However, in its presence, we can reinterpret a stronger form of the invertibility condition (2.1) of Theorem 2.1.

Theorem 4.2. *Suppose $L \dashv N : \mathscr{A} \to \mathscr{X}$ is an adjunction with unit $\eta : 1_{\mathscr{X}} \Rightarrow NL$ and invertible counit $\varepsilon : LN \Rightarrow 1_{\mathscr{A}}$. Suppose $\mathscr{X}$ is skew monoidal, and left internal homs of the form $[Y, NC]$ and right internal homs of the form $\langle X, NC\rangle$ exist. The invertibility of one of the following three natural transformations implies invertibility of the other two:*

$$L(\eta_X \otimes 1_Y) : L(X \otimes Y) \to L(NLX \otimes Y)\ ; \tag{4.3}$$

$$\eta_{[Y,NC]} : [Y, NC] \to NL[Y, NC]\ ; \tag{4.4}$$

$$\langle \eta_X, NC\rangle : \langle NLX, NC\rangle \to \langle X, NC\rangle\ . \tag{4.5}$$

Proof. Consider the commutative diagram (4.6). Invertibility of any one of the horizontal families in the diagram implies that of the other two. Invertibility of the arrows (4.3) is equivalent to the invertibility of the top horizontal family. By Lemma 3.1, invertibility of the middle horizontal family is equivalent to invertibility of the arrows (4.2). By the Yoneda Lemma, invertibility of the bottom horizontal family is equivalent to invertibility of the arrows (4.5). □

$$\begin{array}{ccc}
\mathscr{A}(L(NLX\otimes Y),C) & \xrightarrow{\mathscr{A}(L(\eta\otimes 1),1)} & \mathscr{A}(L(X\otimes Y),C) \\
\cong\downarrow & & \downarrow\cong \\
\mathscr{X}(NLX\otimes Y,NC) & & \mathscr{X}(X\otimes Y,NC) \\
\cong\downarrow & & \downarrow\cong \\
\mathscr{X}(NLX,[Y,NC]) & \xrightarrow{\mathscr{X}(\eta_X,1)} & \mathscr{X}(X,[Y,NC]) \\
\cong\downarrow & & \downarrow\cong \\
\mathscr{X}(Y,\langle NLX,NC\rangle) & \xrightarrow{\mathscr{X}(1,\langle\eta_X,1\rangle)} & \mathscr{X}(Y,\langle X,NC\rangle)
\end{array} \tag{4.6}$$

5. An example

This is an example of the opposite (dual) of the above theory. Instead of a reflection we have a coreflection. Instead of left skew monoidal categories we have right skew monoidal categories.

Consider an injective function $\mu\colon U\to O$. We have an adjunction

$$N \dashv R\colon \mathrm{Set}/O \to \mathrm{Set}/U$$

defined by $(NA)_i = \sum_{\mu(u)=i} A_u$ and $(RX)_u = X_{\mu(u)}$ with invertible unit. The ith component of the counit $\varepsilon_X\colon NRX \to X$ is the function $\sum_{\mu(u)=i} X_{\mu(u)} \to X_i$ which is the identity of X_i when i is in the image of μ.

Let $\mathscr{C}$ be a category with $\mathrm{ob}\mathscr{C} = O$. Then Set/O becomes right skew monoidal on defining the tensor $X\otimes Y$ by

$$(X\otimes Y)_i = \sum_j X_i \times \mathscr{C}(i,j) \times Y_j$$

and the (skew) unit I by $I_i = 1$. The associativity constraint $\alpha\colon X\otimes(Y\otimes Z) \to (X\otimes Y)\otimes Z$ is defined by the component functions

$$\sum_{j,k} X_i\times\mathscr{C}(i,j)\times Y_j\times\mathscr{C}(j,k)\times Z_k \to \sum_{j,k} X_i\times\mathscr{C}(i,j)\times Y_j\times\mathscr{C}(i,k)\times Z_k$$

induced by the functions

$$\mathscr{C}(i,j)\times\mathscr{C}(j,k)\to\mathscr{C}(i,j)\times\mathscr{C}(i,k)$$

taking $(a\colon i\to j, b\colon j\to k)$ to $(a\colon i\to j, b\circ a\colon i\to k)$. Define $\lambda_Y\colon I\otimes Y\to Y$ to have components $Y_i \to \sum_j \mathscr{C}(i,j)\times Y_j$ using the ith injection and $1_i\colon i\to i$. Define $\rho_X\colon X\otimes I\to X$ to have components $\sum_j X_i\times\mathscr{C}(i,j)\to X_i$ whose restriction to the jth injection is the first projection onto X_i.

The reason this provides an example of Theorem 2.1 is that the dual version of the stronger invertibility condition (4.3) holds. To see that

$$R(\varepsilon_X\otimes Y)\colon R(NRX\otimes Y)\to R(X\otimes Y)$$

is invertible, we have:

$$\begin{aligned} R(NRX \otimes Y)_u &= (NRX \otimes Y)_{\mu(u)} \\ &= \sum_j (NRX)_{\mu(u)} \times \mathscr{C}(\mu(u), j) \times Y_j \\ &= \sum_j \sum_{\mu(v)=\mu(u)} (RX)_v \times \mathscr{C}(\mu(u), j) \times Y_j \\ &\cong \sum_j X_{\mu(u)} \times \mathscr{C}(\mu(u), j) \times Y_j \\ &= (X \otimes Y)_{\mu(u)} \\ &= R(X \otimes Y)_u \ . \end{aligned}$$

The resultant right skew structure on Set/U has tensor product

$$\begin{aligned} (A \bar{\otimes} B)_u &= R(NA \otimes NB)_u \\ &= (NA \otimes NB)_{\mu(u)} \\ &= \sum_j (NA)_{\mu(u)} \times \mathscr{C}(\mu(u), j) \times (NB)_j \\ &\cong \sum_v A_u \times \mathscr{C}(\mu(u), \mu(v)) \times B_v \ . \end{aligned}$$

Of course we can see that this is merely the right skew structure on Set/U arising from the category whose objects are the elements $u \in U$ and whose morphisms $u \to v$ are morphisms $\mu(u) \to \mu(v)$ in $\mathscr{C}$; that is, the category arising as the full image of the functor $\mu \colon U \to \mathscr{C}$.

As an easy exercise the reader might like to calculate the monoidal structure

$$\varphi_{X,Y} \colon RX \bar{\otimes} RY \to R(X \otimes Y)$$

on R and check that $\varphi_{X,Y}$ is not invertible in general while, of course, it is for $Y = NB$.

References

[1] Brian J. Day, "Construction of Biclosed Categories" (PhD Thesis, University of New South Wales, 1970) http://www.math.mq.edu.au/~street/DayPhD.pdf. 1, 2, 3

[2] Brian J. Day, On closed categories of functors, *Lecture Notes in Mathematics* **137** (Springer-Verlag, 1970)1–38. 1

[3] Brian J. Day, A reflection theorem for closed categories, *J. Pure Appl. Algebra* **2** (1972) 1–11. 1

[4] Stephen Lack and Ross Street, Skew monoidales, skew warpings and quantum categories, *Theory and Applications of Categories* **26** (2012) 385–402. 1

[5] Ross Street, Skew-closed categories, *Journal of Pure and Applied Algebra 217* (June 2013) 973–988. 1, 4

[6] Kornel Szlachányi, Skew-monoidal categories and bialgebroids, *Advances in Math. 231* (2012) 1694–1730. 1

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