Quantum Groups: an entrée to modern algebra

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This document is one chapter of a book to be made available electronically on the Internet[1]. It has been prepared in TEX, LATEX and LATEX2HTML by Ross Moore, Sam Williams and Ross Talent (now deceased).

The diagrams have been typeset using the macro package XY-pic for graphics and technical diagrams, written by Kristoffer Rose[2] and Ross Moore[3,4]. All of the effects used here can be specified in a reasonably simple, intuitive manner using XY-pic, currently available as version 3.2. Other effects are continually being developed, to become available as extra modules for use with XY-pic.

Individual chapters of this book, in PostScript format, can be obtained electronically[5]. Consult one of the XY-pic home-page[6] or "down-under"[7], for complete information concerning XY-pic. At either site there is access to a detailed User Guide, prepared by Kristoffer Rose, using LATEX2HTML.

[1] http://www.mpce.mq.edu.au/~ross/maths/Quantum/Quantum.html

[2] http://www.diku.dk/users/kris

[3] http://www.mpce.mq.edu.au/~ross

[4] work supported by a Macquarie University Research Grant(MURG), the Australian Research Council(ARC) and a research agreement with the Digital Equipment Corporation(DEC).

[5] ftp://ftp.mpce.mq.edu.au/pub/maths/TeX/Quantum

[6] http://www.diku.dk/users/kris/Xy-pic.html

[7] http://www.mpce.mq.edu.au/~ross/Xy-pic.html

Contents

1
Revision of basic structures

The *cartesian product* of n sets $X_1, \ldots, X_n$ is the set

$$X_1 \times \cdots \times X_n \;=\; \{(x_1, \ldots, x_n) \mid x_i \in X_i\}\;.$$

There is a *canonical bijection*

$$(X_1 \times \cdots \times X_m) \times (X_{m+1} \times \cdots \times X_n) \;\cong\; X_1 \times \cdots \times X_n$$

given by deleting the inside brackets. The *diagonal* function

$$\delta : X \longrightarrow X \times \cdots \times X$$

is given by $\delta(x) = (x, \ldots, x)$.

The cartesian product of no sets is the special set 1, with precisely one element, which should technically be denoted by empty parentheses $(\ )$. Particular cases of the canonical bijections are

$$X \times 1 \;\cong\; X \;\cong\; 1 \times X\;.$$

The diagonal $X \longrightarrow 1$ will be denoted by ε rather than δ; it is the *only* function $X \longrightarrow 1$. Functions $f_1 : X_1 \longrightarrow Y_1\,,\ \ldots\,,\ f_n : X_n \longrightarrow Y_n$ induce a function

$$f_1 \times \cdots \times f_n : X_1 \times \cdots \times X_n \longrightarrow Y_1 \times \cdots \times Y_n$$

given by $\big(f_1 \times \cdots \times f_n\big)(x_1, \ldots, x_n) = \big(f_1(x_1), \ldots, f_n(x_n)\big)$.

The *identity* function $1_X : X \longrightarrow X$ on a set X is given by $1_X(x) = x$. We noted that $\varepsilon : X \longrightarrow 1$ is uniquely determined. Similarly the diagonal $\delta : X \longrightarrow X \times X$ is unique, determined by commutativity of the diagram

(Identity)

$$\begin{tikzcd} & X \arrow[dl, "\cong"'] \arrow[d, "\delta"] \arrow[dr, "\cong"] & \\ 1 \times X & X \times X \arrow[l, "\varepsilon \times 1_X"] \arrow[r, "1_X \times \varepsilon"'] & X \times 1 \end{tikzcd}\;.$$

Furthermore, the following diagram commutes

(Associativity)
$$\begin{tikzcd} X \arrow[r, "\delta"] & X \times X \arrow[r, shift left, "\delta \times 1_X"] \arrow[r, shift right, "1_X \times \delta"'] & X \times X \times X \end{tikzcd} .$$

The function $X \longrightarrow X \times X \times X$ so determined is none other than the ternary diagonal.

A *monoid* is a set M together with special purpose functions $\eta : 1 \longrightarrow M$, $\mu : M \times M \longrightarrow M$ such that the following diagrams commute.

(Id)
$$\begin{tikzcd} & M & \\ 1 \times M \arrow[ur, "\cong"] \arrow[r, "\eta \times 1_M"'] & M \times M \arrow[u, "\mu"'] & M \times 1 \arrow[ul, "\cong"'] \arrow[l, "1_M \times \eta"] \end{tikzcd}$$

(Assoc)
$$\begin{tikzcd} M & M \times M \arrow[l, "\mu"'] & M \times M \times M \arrow[l, shift right, "\mu \times 1_M"'] \arrow[l, shift left, "1_M \times \mu"] \end{tikzcd}$$

If we write 1 for the value of η at the only element of 1 and we write $x\,y$ for $\mu(x\,,y)$ then the above diagrams translate to the equations

$$\begin{aligned} 1\,x &= x = x\,1 \\ (x\,y)\,z &= x\,(y\,z) \end{aligned} \qquad \text{for all} \quad x\,,y\,,z \in M\ .$$

This time functions η and μ are not uniquely determined by the set M. However given μ, condition (Id) uniquely determines η while the condition (Assoc) gives an unambiguous ternary operation $\mu : M \times M \times M \longrightarrow M$ which we write as $\mu(x\,,y\,,z) = x\,y\,z$. Generally there is an unambiguous multiple product function $\mu : M \times \cdots \times M \longrightarrow M$ determined by the binary μ.

An element $x \in M$ is called *invertible* when there exist $y\,,z \in M$ such that $y\,x = 1$ and $x\,z = 1$. Notice that

$$y = y\,1 = y(x\,z) = (y\,x)z = 1\,z = z$$

so each invertible element x determines uniquely an element, denoted x^{-1}, satisfying $x^{-1}x = 1 = x\,x^{-1}$.

A *group* is a monoid in which each element is invertible. Then we have a function $\iota : M \longrightarrow M$ such that this next diagram commutes.

(Invertibility)
$$\begin{tikzcd} M \arrow[r, "\delta"] \arrow[drr, "\varepsilon"'] & M \times M \arrow[r, shift left, "\iota \times 1_M"] \arrow[r, shift right, "1_M \times \iota"'] & M \times M \arrow[r, "\mu"] & M \\ & & I \arrow[ur, "\eta"'] & \end{tikzcd}$$

Note carefully the dependence of this axiom on the diagonal structure of cartesian product.

For a set X, the n-fold cartesian product $X \times \cdots \times X$ is denoted by X^n. Each permutation ξ on $\{1, \ldots , n\}$ induces a bijection

$$\sigma_\xi : X^n \longrightarrow X^n$$

given by $\sigma_\xi(x_1, \ldots , x_n) = (x_{\xi(1)}, \ldots , x_{\xi(n)})$. In particular, we have the *switch* coming from the non-identity permutation of $\{1 , 2\}$:

$$\sigma : X \times X \longrightarrow X \times X \quad , \quad \sigma(x , y) = (y , x) .$$

Each σ_ξ is a composite of bijections of the form $1_X \times \cdots \times \sigma \times \cdots \times 1_X$. Notice that the following diagram commutes.

(Commutativity)

$$\begin{tikzcd} & X \arrow[dl, "\delta"'] \arrow[dr, "\delta"] & \\ X \times X \arrow[rr, "\sigma"] & & X \times X \end{tikzcd}$$

A monoid (M , η , μ) is called *commutative* when the following diagram commutes.

$$\begin{tikzcd} & M & \\ M \times M \arrow[ur, "\mu"] & & M \times M \arrow[ul, "\mu"'] \arrow[ll, "\sigma"'] \end{tikzcd}$$

It follows that the composite $M^n \xrightarrow{\sigma_\xi} M^n \xrightarrow{\mu} M$ is independent of the permutation ξ.

Suppose M and N are monoids. A *monoid morphism* (or *homomorphism of monoids*) is a function $f : M \longrightarrow N$ such that the following diagrams commute.

$$\begin{tikzcd} & 1 \arrow[dl, "\eta"'] \arrow[dr, "\eta"] & \\ M \arrow[rr, "f"] & & N \end{tikzcd} \qquad \begin{tikzcd} M \times M \arrow[r, "f \times f"] \arrow[d, "\mu"'] & N \times N \arrow[d, "\mu"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Expressed in terms of elements, these diagrams merely say that $f(1) = 1$ and $f(x y) = f(x)f(y)$. If N has cancellation (e.g. if N is a group) then $f(1) = 1$ is redundant.

Monoid morphisms preserve invertibility: if $x \in M$ is invertible, $f(x^{-1}) = f(x)^{-1}$. So for groups M and N we have commutativity of the square

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\iota"'] & N \arrow[d, "\iota"] \\ M \arrow[r, "f"] & N \end{tikzcd} .$$

A *rig* is a set R enriched with two monoid structures, a commutative one written additively and the other written multiplicatively, such that the following equations hold:

$$a\,0 \;=\; 0 \;=\; 0\,a$$

(Distributive) $$a(b+c) \;=\; a\,b + a\,c \quad , \quad (a+b)c \;=\; a\,c + b\,c\,.$$

The natural numbers $\mathbb{N} = \{0\,,1\,,2\,,\ldots\}$ provide an example of a rig.

A *ring* is a rig for which the additive monoid is a group. The integers $\mathbb{Z}$ provide an example.

A rig is *commutative* when the multiplicative monoid is commutative.
A *field* is a commutative ring for which each element is either 0 or has a multiplicative inverse.

For rigs R and S a *rig morphism* $f : R \longrightarrow S$ is a function which is a monoid morphism for both the additive and multiplicative structures.

Let $\mathbf{k}$ denote a field. A $\mathbf{k}$-*algebra* is a ring A together with a ring morphism $\eta : \mathbf{k} \longrightarrow A$. Notice that either A is trivial $(1 = 0)$, or that η is injective $[\kappa \neq \kappa' \;\Rightarrow\; \kappa - \kappa' \neq 0 \;\Rightarrow\; \kappa - \kappa'$ invertible $\Rightarrow\; \eta(\kappa - \kappa')$ invertible $\overset{1\neq 0}{\Longrightarrow}\; \eta(\kappa - \kappa') \neq 0 \;\Rightarrow\; \eta(\kappa) \neq \eta(\kappa')]$. We can define *scalar multiplication* $\mathbf{k} \times A \longrightarrow A$ by $\kappa\,a = \eta(\kappa)\,a$.

For $\mathbf{k}$-algebras A and B, a $\mathbf{k}$-*algebra morphism* $f : A \longrightarrow B$ is a ring morphism such that the next diagram commutes

$$\begin{tikzcd} & \mathbf{k} \arrow[dl, "\eta"'] \arrow[dr, "\eta"] & \\ A \arrow[rr, "f"] & & B \end{tikzcd}$$

that is $f(\kappa\,a) = \kappa\,f(a)$. We write $\mathbf{Alg_k}(A\,,B)$ for the set of $\mathbf{k}$-algebra morphisms $f : A \longrightarrow B$.

An *isomorphism* is a bijective morphism; automatically its inverse function is also a morphism.

2

Duality between geometry and algebra

The purpose of this section is to convince you that commutative algebras are really *spaces* seen from the other side of your brain.

For a compact hausdorff space X, we have the algebra $C(X)$ of continuous, complex-valued functions $a : X \longrightarrow \mathbb{C}$. The addition and multiplication are obtained pointwise from $\mathbb{C}$.

A continuous function $f : X \longrightarrow Y$ gives rise to an algebra morphism $C(f) : C(Y) \longrightarrow C(X)$ (note the reversal of direction!) via $C(f)(b) = a$, where $a(x) = b(f(x))$. In particular, the unique $X \longrightarrow 1$ gives the algebra morphism $\eta : \mathbb{C} = C(1) \longrightarrow C(X)$, while each *point* $x : 1 \longrightarrow X$ of the space gives an algebra morphism $C(X) \longrightarrow \mathbb{C}$.

Actually $C(X)$ is more than just a $\mathbb{C}$-algebra; it is what is called a *commutative* C^**-algebra* (there is a norm and an involution $(\_)^*$ coming from conjugation). With this extra structure the duality becomes precise:

> *Each commutative C^*-algebra A is isomorphic to $C(X)$ for some compact hausdorff space X; each C^*-algebra morphism $C(Y) \longrightarrow C(X)$ has the form $C(f)$ for a* unique *continuous function $f : X \longrightarrow Y$.*

This result is commonly referred to as *Gelfand duality.*

Algebraic geometry is the study of spaces called *varieties*: the solutions to polynomial equations in several variables. In studying the variety given by $x^2 + 2y^3 = z^4$ over the field $\mathbf{k}$, we pass to the $\mathbf{k}$-algebra

$$A \;=\; \mathbf{k}[\,x\,,y\,,z\,]\,/\,(x^2 + 2y^3 = z^4)\,.$$

By $\mathbf{k}[\,x\,,y\,,z\,]$ we mean the $\mathbf{k}$-algebra of polynomials in three commuting indeterminates $x\,,y\,,z$; the elements are expressions

$$\sum_{i,j,k} \alpha_{ijk}\, x^i\, y^j\, z^k$$

where $\alpha_{ijk} \in \mathbf{k}$ and $(i\,,j\,,k)$ runs over a finite subset of $\mathbb{N}^3$. The quotient algebra A is obtained from $\mathbf{k}[\,x\,,y\,,z\,]$ by identifying elements when they may be transformed one into another by means of the equation $x^2+2y^3 = z^4$ and the algebra axioms.

Given a $\mathbf{k}$-algebra B , a $\mathbf{k}$-*algebra morphism* $f : \mathbf{k}[\,x\,,y\,,z\,] \longrightarrow B$ is uniquely determined by its values on $x\,,y\,,z$. In fact we have a bijection

$$\mathbf{Alg_k}\big(\mathbf{k}[\,x\,,y\,,z\,]\,,B\big) \;\cong\; B^3 \;.$$

Similarly, we have a bijection

$$\mathbf{Alg_k}\big(A\,,B\big) \;\cong\; \big\{(u\,,v\,,w) \in B^3 \mid u^2 + 2v^3 = w^4\big\}$$

where A is as above. Again we see that a $\mathbf{k}$-algebra morphism $A \longrightarrow B$ corresponds to a map of varieties in the reverse direction.

For general $\mathbf{k}$-algebras A and B, it is suggestive to call a morphism $f : A \longrightarrow B$ a B-*point* of A . A *point* of (the space corresponding to) A is a $\mathbf{k}$-point, not to be confused with an element of the algebra A itself.

$$\begin{tikzcd}
\begin{pmatrix} \text{commutative} \\ \mathbf{k}\text{-algebras} \end{pmatrix}^{\mathrm{op}} \arrow[r, bend left, "\text{spectrum}"] & \big(\text{spaces}\big) \arrow[l, bend left, "\text{coordinate algebra}"]
\end{tikzcd}$$

Let $\mathcal{X}$ denote a category. I am thinking of the *objects* of $\mathcal{X}$ as spaces X and Y say, and the *arrows* $X \longrightarrow Y$ as the maps appropriate to that kind of space. Write $\mathcal{X}\big(X\,,Y\big)$ for the set of arrows from X to Y in $\mathcal{X}$.

Let $X_1\,,\ldots,\,X_n$ be arbitrary objects of $\mathcal{X}$. A *product* for this list of objects consists of an object $X_1 \times \cdots \times X_n$ together with arrows

$$p_i : X_1 \times \cdots \times X_n \longrightarrow X_i \qquad \text{for } \; i = 1,\ldots,n$$

such that, given any other object K and arrows

$$f : K \longrightarrow X_i \qquad \text{for } \; i = 1,\ldots,n$$

there exists a unique arrow $f : K \longrightarrow X_1 \times \cdots \times X_n$ with $p_i \circ f = f_i$.

$$\begin{tikzcd}
X_1 \times \cdots \times X_n \arrow[r, "p_i"] & X_i \\
K \arrow[u, dashed, "f"] \arrow[ur, "f_i"'] &
\end{tikzcd}$$

This means we have a bijection

$$\mathcal{X}\big(K\,,\,X_1 \times \cdots \times X_n\big) \;\cong\; \mathcal{X}\big(K,X_1\big) \times \cdots \times \mathcal{X}\big(K,X_n\big) \;.$$

In particular, the empty product is called a *terminal object,* denoted by 1. We have

$$\mathcal{X}(K,1) \;\cong\; 1 \,.$$

Products are unique up to isomorphism (if they exist).

The *diagonal* $\delta : X \longrightarrow X \times \cdots \times X$ is defined by $p_i \circ \delta = 1_X$ for all i. The canonical isomorphisms $f_1 \times \cdots \times f_n$ and isomorphisms σ_ξ can be defined as for sets.

The diagrammatic definition of *monoid* and *group* can be carried into the category $\mathcal{X}$ (provided the products exist; 1 and $M \times M$ are enough). If M is a monoid (group) in $\mathcal{X}$ then each $\mathcal{X}(K,M)$ becomes a monoid (group) using the multiplication $*$ given by

$$f * g \;=\; \mu \circ (f \times g) \circ \delta$$

$$K \xrightarrow{\;\delta\;} K \times K \xrightarrow{\;f\times g\;} M \times M \xrightarrow{\;\mu\;} M \,.$$

A group in the category of topological spaces and continuous maps is called a *topological group.* A group in the category of smooth manifolds and smooth maps is called a *Lie group.*

We are more interested here in groups in the category $(\mathbf{Comm\,Alg_k})^{\mathrm{op}}$ of commutative $\mathbf{k}$-algebras and reversed morphisms; these are called *affine groups* over $\mathbf{k}$. This is the variety point of view. On the algebraic side they are called *commutative Hopf algebras* over $\mathbf{k}$. Product of varieties becomes tensor product $A \otimes_{\mathbf{k}} B$ of $\mathbf{k}$-algebras (more on this later). A commutative Hopf algebra H thus has structure given by the $\mathbf{k}$-algebra morphisms

$$\varepsilon : H \longrightarrow \mathbf{k}\;, \qquad \delta : H \longrightarrow H \otimes_{\mathbf{k}} H\;, \qquad \nu : H \longrightarrow H$$

called *counit, comultiplication, antipode* (corresponding respectively to the unit, multiplication, inversion for the group). Now for each commutative $\mathbf{k}$-algebra A, we obtain a group $\mathbf{Alg_k}(H,A)$ of A-points in H.

It will also be necessary to consider the algebraic version of *affine monoids* over $\mathbf{k}$. These are called *commutative bialgebras* over $\mathbf{k}$. They have a counit and comultiplication, but generally no antipode.

Example 2.1 *Let $M(2)$ denote $\mathbf{k}[\,a\,,b\,,c\,,d\,]$ as a commutative $\mathbf{k}$-algebra. A counit $\varepsilon : M(2) \longrightarrow \mathbf{k}$ is defined by $\varepsilon(a) = \varepsilon(d) = 1$, $\varepsilon(b) = \varepsilon(c) = 0$. Clearly*

$$\mathbf{k}[\,a\,,b\,,c\,,d\,] \otimes_{\mathbf{k}} \mathbf{k}[\,a\,,b\,,c\,,d\,] \;\cong\; \mathbf{k}[\,a\,,b\,,c\,,d\,,a',b',c',d'\,]$$

with the coprojections

$$\begin{array}{ccccc} \mathbf{k}[\,a\,,b\,,c\,,d\,] & \longrightarrow & \mathbf{k}[\,a\,,b\,,c\,,d\,,a',b',c',d'\,] & \longleftarrow & \mathbf{k}[\,a\,,b\,,c\,,d\,] \\ a\,,b\,,c\,,d \longmapsto a\,,b\,,c\,,d & & \text{and} & & a',b',c',d' \longleftarrow\!\shortmid\; a\,,b\,,c\,,d\;. \end{array}$$

The comultiplication $\delta : M(2) \longrightarrow M(2) \otimes_{\mathbf{k}} M(2)$ *is given by*

$$a\,,b\,,c\,,d \;\longmapsto\; a\,a' + b\,c',\; a\,b' + b\,d',\; c\,a' + d\,c',\; c\,b' + d\,d'\;.$$

This makes $M(2)$ *into a commutative* $\mathbf{k}$*-bialgebra. Notice that we have a monoid isomorphism*

$$\mathbf{Alg}_{\mathbf{k}}\big(M(2)\,,A\big) \;\cong\; \mathbf{Mat}\big(2\,,A\big)$$

where on the right we have the multiplicative monoid of 2×2 *matrices with entries in* A*. Thus* $M(2)$ *is the* coordinate $\mathbf{k}$-algebra *of the variety of* 2×2 *matrices.*

To obtain the coordinate $\mathbf{k}$*-algebra of the* general linear group*, we take*

$$\mathrm{GL}\,(2) \;=\; \mathbf{k}[\,a\,,b\,,c\,,d\,,x\,]\,/\,(x\,a\,d - x\,b\,c = 1)\;.$$

There is an epimorphic $\mathbf{k}$*-algebra morphism* $M(2) \longrightarrow \mathrm{GL}\,(2)$ *which induces a bialgebra structure on* $\mathrm{GL}\,(2)$ *from that on* $M(2)$*. The antipode*

$$\begin{array}{rcl} \nu \;:\; \mathrm{GL}\,(2) & \longrightarrow & \mathrm{GL}\,(2) \\ a\,,b\,,c\,,d\,,x & \longmapsto & x\,d\,,-x\,b\,,-x\,c\,,x\,a\,,a\,d - b\,c \end{array}$$

makes $\mathrm{GL}\,(2)$ *into a commutative Hopf algebra.*

3

The quantum general linear group

The passage from quantum to classical mechanics is quite well defined by taking the limit as Planck's constant $\hbar$ tends to 0. The passage in the other direction is not so clear cut, and may not be uniquely determined. On the algebraic side, "quantization" involves deforming commutative algebras to non-commutative ones:

$$\text{e.g.} \qquad x\,y \;=\; y\,x \qquad \text{becomes} \qquad x\,y \;=\; e^{\hbar}\,y\,x\ .$$

Usually we deal with $q = e^{\hbar}$ rather than $\hbar$, so classical results correspond to the case $q = 1$. Quantum spaces correspond to more general $\mathbf{k}$-algebras, not necessarily commutative.

Let $\mathbf{k}$ be a fixed field and fix $q \in \mathbf{k}$ with $q \neq 0$. Write $\mathbf{k}\langle x_1, \ldots, x_n\rangle$ for the $\mathbf{k}$-algebra of polynomials in *non-commuting* indeterminates $x_1, \ldots, x_n$. As a vector space over $\mathbf{k}$, a basis is given by those elements

$$x_{\xi(1)}^{m_1}\, x_{\xi(2)}^{m_2} \cdots x_{\xi(r)}^{m_r}$$

for which $r \in \mathbb{N}$ and $m_1, \ldots, m_r \in \mathbb{Z}^+$ and $\xi : \{1, \ldots, r\} \longrightarrow \{1, \ldots, n\}$ is any function. Notice that

$$\mathbf{k}[\,x\,,y\,] = \mathbf{k}\langle x\,,y\,\rangle/(\,x\,y = y\,x\,)\ .$$

The coordinate algebra of the space of *quantum* 2×2 *matrices* is defined by

$$M_q(2) \;=\; \mathbf{k}\langle a\,,b\,,c\,,d\,\rangle/R$$

where R is the system of equations

$$ab \;=\; q^{-1}ba\ ,\quad ac \;=\; q^{-1}ca\ ,\quad cd \;=\; q^{-1}dc\ ,\quad bd \;=\; q^{-1}db$$
$$bc \;=\; cb\ ,\quad ad - da \;=\; (q^{-1} - q)\,bc\ .$$

(mnemonic)

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d] \arrow[dl, leftrightarrow] \\ c \arrow[r] & d \end{tikzcd}$$

The monomials $a^{m_1}\, b^{m_2}\, c^{m_3}\, d^{m_4}$ form a basis for the algebra, as a vector space over $\mathbf{k}$.

$$\mathbf{Alg_k}\big(M_q(2)\,,A\big) \;\cong\; \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{Mat}\,(2\,,A) \mid R \text{ holds} \right\}$$

Theorem 3.1 *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ *be two* A*-points of* $M_q(2)$ *such that each entry of the first commutes with each entry of the second.*
(i) *The product* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ *(as matrices) is an* A*-point of* $M_q(2)$.
(ii) *The "q-determinant"* $\det_q\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - q^{-1}bc)$ *commutes with each of* $a\,,b\,,c\,,d$ *and*

$$\det_q\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) \;=\; \det_q\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \det_q\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} .$$

(iii) *If* $\det_q\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is invertible in* A *then*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \;=\; \left(\det_q\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^{-1}\begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}$$

is an A*-point of* $M_{q^{-1}}(2)$.

The above result can be proved by direct calculation, but this gives little insight into the special nature of the relations R. Examples such as this arose in work of L.D. Faddeev [FRT87] and his school on the *quantum inverse scattering transform* (QIST) method. The version I present here comes from some lectures of Yu Manin [Man88] given at Université de Montréal in June 1988. The following "explanation" of Theorem 3.1 is due to Yu Kobyzev (Moscow, winter 1986–87).

Introduce the *quantum plane*, as defined by the $\mathbf{k}$-algebra

$$\mathbb{A}_q^{2|0} \;=\; \mathbf{k}\langle x\,,y\,\rangle/(xy = q^{-1}\,yx) .$$

The monomials $x^m\, y^n$ with $m\,,n \in \mathbb{N}$ form a basis for this as a vector space. We also need to consider a quantized version of a Grassmannian algebra in two variables:

$$\mathbb{A}_q^{0|2} \;=\; \mathbf{k}\langle \xi\,,\eta\,\rangle/(\xi^2 = \eta^2 = 0 = \xi\eta + q\,\eta\xi) .$$

The monomials $\xi^m\, \eta^n$ with $m\,,n \in \{0,1\}$ form a basis for this algebra. The reason for the funny superscripts $2|0$ and $0|2$ comes from "supergeometry" where dimensions are represented by pairs $d\,|\,d'$ of numbers. This $\mathbb{A}_q^{0|2}$ is a *quantum superplane.*

An A-point of B is called *generic* when the algebra morphism $B \longrightarrow A$ is injective.

Theorem 3.2 *Suppose* (x,y) *is a generic* A*-point of* $\mathbb{A}_q^{2|0}$ *and* (ξ,η) *is a generic* A*-point of* $\mathbb{A}_q^{0|2}$. *Suppose* $a,b,c,d \in A$ *all commute with* x,y,ξ,η. *Put*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \;,\; \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \;,\; \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} .$$

If $q^2 \neq -1$, *the following conditions are equivalent* :

(i) (x',y') *and* (x'',y'') *are points of* $\mathbb{A}_q^{2|0}$;

(ii) (x',y') *is a point of* $\mathbb{A}_q^{2|0}$ *and* (ξ',η') *is a point of* $\mathbb{A}_q^{0|2}$;

(iii) $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ *is a point of* $M_q(2)$.

[*For* $q^2 = -1$ *we only have* (iii) $\Rightarrow$ (i) & (ii).]

Proof. (i) $\Leftrightarrow$ (iii). (x',y') is a point of $\mathbb{A}_q^{2|0}$ iff $x'y' = q^{-1}y'x'$; that is, iff $(a\,x + b\,y)(c\,x + d\,y) = q^{-1}(c\,x + d\,y)(a\,x + b\,y)$. Multiply out the products using the fact that a,b,c,d each commute with x and y; since (x,y) is generic, we can equate coefficients of $x^2, y^2, x\,y$. So the single equation is in fact equivalent to the following set of three equations:

$$(*) \qquad ac = q^{-1}ca \quad , \quad bd = q^{-1}db \quad , \quad ad - da = q^{-1}cb - q\,bc\,.$$

Interchanging b and c we see that (x'',y'') is a point of $\mathbb{A}_q^{2|0}$ iff

$$(**) \qquad ab = q^{-1}ba \quad , \quad cd = q^{-1}dc \quad , \quad ad - da = q^{-1}bc - q\,cb\,.$$

Taking the last equations in $(*)$ & $(**)$ we get $q^{-1}cb - q\,bc = q^{-1}bc - q\,cb$; that is, $(q + q^{-1})(bc - cb) = 0$ hence $bc = cb$, provided $q^2 \neq -1$.
So (iii) $\Leftrightarrow$ $(*)$ & $(**)$, which together are equivalent to (i).
(ii) $\Leftrightarrow$ (iii). (ξ',η') is a point of $\mathbb{A}_q^{0|2}$ iff $0 = (a\,\xi + b\,\eta)^2 = (c\,\xi + d\,\eta)^2 = (a\,\xi + b\,\eta)(c\,\xi + d\,\eta) + q\,(c\,\xi + d\,\eta)(a\,\xi + b\,\eta)$. Using $\xi^2 = \eta^2 = 0$ these become $ab\xi\eta + ba\,\eta\xi = 0$ and $cd\,\xi\eta + dc\,\eta\xi = 0$ and $ab\,\xi\eta + bc\,\eta\xi + q\,(cb\,\xi\eta + da\,\eta\xi) = 0$. Using $\xi\eta = -q\,\eta\xi$ and the linear independence of η and ξ in A, we get that $-q\,ab + ba = 0$ and that $-q\,cd + dc = 0$ and also $-q\,(ad + q\,cb) + bc + q\,da = 0$. These are equivalent to $(**)$. So (ii) $\Leftrightarrow$ $(*)$ and $(**)$ $\Leftrightarrow$ (i).

In other words, the relations R are precisely what is needed for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its transpose to both transform the quantum plane into itself; or for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to transform both the plane and superplane into themselves.

Proof of Theorem 3.1. (i) Let B be the free $\mathbf{k}$-algebra containing the indeterminates a,b,c,d, a',b',c',d', x,y subject to the relations on these variables in the hypotheses of Theorems 3.1 and 3.2. Then (x,y) is generic;

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ are B-points of $M_q(2)$. By Theorem 3.2, we have that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ are B-points of $\mathbb{A}_q^{2|0}$. Each coordinate in the first of these commutes with all of a', b', c', d' while coordinates in the second commute with a, b, c, d. Also $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ is generic since when composed with $B \longrightarrow \mathbb{A}_q^{2|0}$ for which $(a, b, c, d, x, y) \longmapsto (1, 0, 0, 1, x, y)$ we get (x, y), which is generic. Similarly $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ is generic. So by Theorem 3.2 we have $\begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ both being B-points of $\mathbb{A}_q^{2|0}$. Again by Theorem 3.2, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ is a B-point of $M_q(2)$.

To obtain the result for the given A apply the morphism $B \longrightarrow A$ for which $(a, b, \ldots, d', x, y) \longmapsto (a, b, \ldots, d', 0, 0)$.

(ii) We now get a natural definition of the quantum determinant which immediately yields its multiplicativity: in the notation of Theorem 3.2

$$\xi' \eta' \;=\; (a\,\xi + b\,\eta)(c\,\xi + d\,\eta) \;=\; \det_q\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xi\,\eta\,.$$

(iii) This is left as an exercise for the reader.

The *quantum general linear group* is defined from 2×2 matrices by inverting the determinant:

$$\mathrm{GL}_{\,q}(2) \;=\; M_q(2)[t]/(t\,a = a\,t\,,\; t\,b = b\,t\,,\; t\,c = c\,t\,,\; d\,t = t\,d\,,\; t\det_q = 1)\,.$$

Similarly, the *quantum special linear group* is defined by requiring that the determinant be equal to 1:

$$\mathrm{SL}_{\,q}(2) \;=\; M_q(2)/(\det_q = 1)\,.$$

Theorem 3.2 describes the representations of these "groups" on quantum spaces $\mathbb{A}_q^{2|0}$ and $\mathbb{A}_q^{0|2}$.

Exercises

1. Give a direct proof of Theorem 3.1 on quantum 2×2-matrices.

4

Modules and tensor products

Let R be a ring (not necessarily commutative). We write R^{op} for the ring with opposite multiplication

$$R \times R \xrightarrow{\sigma} R \times R \xrightarrow{\mu} R\,.$$

(To say R is commutative is to say $R^{\mathrm{op}} = R$.)

A *left R-module* is an abelian group M (written additively) together with a function

$$R \times M \longrightarrow M \quad \text{whereby} \quad (r\,, m) \longmapsto r\,m$$

called *scalar multiplication*, such that

$$\begin{aligned} 1m \;&=\; m \quad , \quad (r\,s)\,m \;=\; r\,(s\,m) \\ (r + r')\,m \;&=\; r\,m + r'\,m \quad , \quad r\,(m + m') \;=\; r\,m + r\,m'\,. \end{aligned}$$

A *right R-module* is defined similarly, with multiplication $M \times R \longrightarrow M$.

A left R^{op}-module structure on an abelian group M "is the same" as a right R-module structure. More precisely, $\mu : R \times M \longrightarrow M$ is a scalar multiplication for a left R^{op}-module iff $M \times R \xrightarrow{\sigma} R \times M \xrightarrow{\mu} M$ is one for a right R-module. In this way, we can deal only with left R-modules and omit "left", unless we explicitly stipulate otherwise.

If R is commutative, $R = R^{\mathrm{op}}$ and there is no need to distinguish left and right modules. If R is a field, an R-module is precisely a *vector space* over R. Furthermore, $\mathbb{Z}$-modules are precisely abelian groups since each abelian group A admits a unique $\mathbb{Z}$-scalar multiplication given by $n\,a = a + \cdots + a$ (n terms) for $n \geq 0$ and $n\,a = -((-n)a)$ for $n < 0$.

A subset X of an R-module M is said to *generate* M (or *span* M) when, for each $m \in M$, there exist $r_1, \ldots, r_n \in R$ and $x_1, \ldots, x_n \in X$ such that

$$(*) \qquad m \;=\; r_1\,x_1 + \cdots + r_n\,x_n\,.$$

Call M *finitely generated* when it is generated by some finite subset.

A (not necessarily finite) subset X of M is *linearly independent* when for $x_1, \ldots, x_n \in X$ distinct elements, having a relation of the form $r_1 x_1 + \cdots + r_n x_n = 0$ with $r_1, \ldots, r_n \in R$ implies that $r_1 = \cdots = r_n = 0$. Then each expression $(*)$ is unique up to order of factors (with $x_1, \ldots, x_n$ distinct).

An R-module F is said to be *free* when it is generated by some linearly independent subset. Every vector space is free, but this is peculiar to R being a field. It is easy to see that $\mathbb{Z}/(2)$ is not a free abelian group.

Each set X determines an R-module

$$\mathcal{F}_R(X) = \{r_1 x_1 + \cdots + r_n x_n \mid r_i \in R, x_i \in X, n \in \mathbb{N}\}$$

with addition and scalar multiplication defined in the obvious way. We can identify $x \in X$ with $1x \in \mathcal{F}_R(X)$ and see easily that X is linearly independent and generates $\mathcal{F}_R(X)$. So $\mathcal{F}_R(X)$ is free.

For R-modules M and N, a function $f : M \longrightarrow N$ is *(left)R-linear* (or an *R-module morphism*) when $f(m+m') = f(m)+f(m')$ and $f(r\,m) = r\,f(m)$ for all $m, m' \in M$ and $r \in R$. Write $\mathrm{Hom}_R(M, N)$ for the abelian group of R-linear functions $f : M \longrightarrow N$; the addition is given by $(f + g)(m) = f(m) + g(m)$.

Warning: You may think $\mathrm{Hom}_R(M, N)$ becomes an R-module by defining $(rf)(m) = r\,f(m)$. But this rf does not preserve scalar multiplication when R is non-commutative.

For sets X and Y, write Y^X for the set of all functions $f : X \longrightarrow Y$. An R-linear function $f : \mathcal{F}_R(X) \longrightarrow M$ is uniquely determined by its restriction to X. Indeed, this gives an isomorphism of abelian groups

$$\mathrm{Hom}_R(\mathcal{F}_R(X), M) \cong M^X$$

where the addition on M^X is pointwise.

A *submodule* H of an R-module M is a subset which is closed under addition and scalar multiplication. This gives an equivalence relation $\equiv_H$ on M whereby

$$m \equiv_H m' \qquad \text{if and only if} \qquad m - m' \in H\,.$$

The equivalence class containing $m \in M$ is $m + H = \{m + h \mid h \in H\}$, called the *$H$-coset* containing m. The set M/H of H-cosets becomes an R-module via

$$(m + H) + (n + H) = (m + n) + H \quad , \quad r(m + H) = r\,m + H\,.$$

We have a surjective R-linear function $\rho : M \longrightarrow M/H$ for which $\rho(m) = m + H$. For each R-linear $g : M \longrightarrow N$ with $g(m) = 0$ for all $m \in H$, there exists a unique R-linear $\hat{g} : M/H \longrightarrow N$ with $\hat{g} \circ \rho = g$. The kernel $\ker f = \{m \in M \mid f(m) = 0\}$ of any R-linear $f : M \longrightarrow N$ is a submodule of M; we have a commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\rho"'] & N \\ M/\ker f \arrow[r, "\cong"] & im\, f \arrow[u, hook] \end{tikzcd}$$

of R-modules, where $im\ f = \{f(m) \mid m \in M\}$ is the image of f, the bottom arrow is an R-linear isomorphism, and the right arrow is an inclusion of a submodule.

The submodule (X) *generated* by a subset X of an R-module M is the smallest submodule of M which contains X. As such it is the image of the R-linear function $\mathcal{F}_R(X) \longrightarrow M$ whose restriction to X is the inclusion $X \hookrightarrow M$. Of course (X) is generated by X, but in general not freely.

Suppose that M is a right R-module and N is a left R-module. A function $f : M \times N \longrightarrow A$ into an abelian group A is *R-bilinear* when it satisfies

$$\begin{aligned} f(m, n+n') &= f(m,n) + f(m,n') \\ f(m+m', n) &= f(m,n) + f(m',n) \\ f(mr, n) &= f(m, rn) . \end{aligned}$$

Write $\mathbf{Bil}_R(M, N; A)$ for the abelian group, which is a subgroup of $A^{M\times N}$, of R-bilinear functions $f : M \times N \longrightarrow A$. Our main goal is to construct a "universal" bilinear function $\lambda : M \times N \longrightarrow M \otimes_R N$.

Let B denote the subset of the abelian group $\mathcal{F}_{\mathbb{Z}}(M \times N)$ consisting of all elements of the form

$$\begin{gathered} (m+m', n) - (m, n) - (m', n) , \\ (m, n+n') - (m, n) - (m, n') , \\ (mr, n) - (m, rn) \end{gathered}$$

for $m, m' \in M$ with $n, n' \in N$ and $r \in R$. Put

$$M \otimes_R N = \mathcal{F}_{\mathbb{Z}}(M \times N)/(B) .$$

Then we have abelian group isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R N, A) &= \mathrm{Hom}_{\mathbb{Z}}(\mathcal{F}_{\mathbb{Z}}(M \times N)/(B), A) \\ &\cong \{g \in \mathrm{Hom}_{\mathbb{Z}}(\mathcal{F}_{\mathbb{Z}}(M \times N), A) \mid f \text{ is zero on } B\} \\ &\cong \{f \in A^{M\times N} \mid f \text{ is } R\text{-bilinear}\} \\ &= \mathbf{Bil}_R(M, N; A) . \end{aligned}$$

In particular by taking $A = M \otimes_R N$ we get the identity morphism $A \longrightarrow A$ corresponding, under the composite of the above string of isomorphisms, to a bilinear morphism $\lambda : M \times N \longrightarrow M \otimes_R N$. Then we easily see that each R-bilinear $f : M \times N \longrightarrow A$ uniquely determines an abelian group morphism $g : M \otimes_R N \longrightarrow A$ with $g \circ \lambda = f$.

For $(m, n) \in M \times N$, we put $m \otimes n = \lambda(m, n)$. A typical element of $M \otimes_R N$ then has the form

$$\sum_{i=1}^{k} m_i \otimes n_i$$

where $m_1, \ldots, m_k \in M$ and $n_1, \ldots, n_k \in N$. These elements satisfy

$$\begin{aligned} (m + m') \otimes n &= m \otimes n + m' \otimes n \\ m \otimes (n + n') &= m \otimes n + m \otimes n' \\ m\, r \otimes n &= m \otimes r\, n\ . \end{aligned}$$

With R and S rings, a *module M from R to S*, written $M : R \nrightarrow S$, is an abelian group M enriched with a left R-module structure and a right S-module structure related by

$$(r\, m) s = r (m\, s)$$

for all $r \in R$, $m \in M$ and $s \in S$. (In the literature this structure is also known as a *left R-/right S-bimodule*.) In this notation, tensor product can be viewed as a kind of "composition of modules".

$$\begin{tikzcd} & S \arrow[dr, "N", "|" marking] & \\ R \arrow[ur, "M", "|" marking] \arrow[rr, "M \otimes_S N"', "|" marking] & & T \end{tikzcd}$$

For M and N as above, $M \otimes_S N$ becomes a module from R to T by defining

$$r (m \otimes n) t = (r\, m) \otimes (n\, t)\ .$$

This composition of modules is not strictly associative, but is associative up to canonical isomorphisms much like cartesian product of sets. This can be seen by defining a *multiple tensor product* as we now proceed to do.

For rings R and S and any set X, there is a *free module from R to S generated by X*. It is denoted by $\mathcal{F}_R^S(X)$ and its elements have the form

$$r_1 x_1 s_1 + \cdots + r_n x_n s_n \qquad \text{for} \quad r_i \in R\ ,\ s_i \in S\ ,\ x_i \in X\ ,\ n \in \mathbb{N}\ .$$

For each module $M : R \nrightarrow S$ we have

$$\mathrm{Hom}_R^S(\mathcal{F}_R^S(X)\,, M) \;\cong\; M^X$$

where $\mathrm{Hom}_R^S(N, M)$ is the abelian group which has as elements the left R-/right S-module morphisms $N \longrightarrow M$.

Exercise 4.1 *For rings R , S , T and any sets X , Y prove that*

$$\begin{array}{ccc} \mathcal{F}_R^T(X \times Y) & \cong & \mathcal{F}_R^S(X) \otimes_S \mathcal{F}_S^T(Y) \\ (x\,, y\,) & \longmapsto & x \otimes y \ . \end{array}$$

Hint: *Look at left R-/right T-module morphisms into $M : R \nrightarrow T$.*

Given rings and modules as in the diagram

$$\begin{tikzcd}
 & & R_2 \arrow[r, "M_3"] & \cdots \arrow[ddr, "M_n"] & \\
 & R_1 \arrow[ur, "M_2"] & & & \\
R_0 \arrow[ur, "M_1"] \arrow[rrrr, "L"] & & & & R_n
\end{tikzcd}$$

a function $f : M_1 \times \cdots \times M_n \longrightarrow L$ is called *multilinear* when it satisfies the equations

$$\begin{array}{rcl} f(m_1, \ldots, m_i + m_i', \ldots, m_n) & = & f(m_1, \ldots, m_i\,, \ldots, m_n) \\ & & \quad + f(m_1, \ldots, m_i', \ldots, m_n) \\ r_0\, f(m_1, \ldots, m_n) & = & f(r_0\, m_1\,, m_2, \ldots, m_n) \\ f(m_1, \ldots, m_i\, r_i\,, m_{i+1}, \ldots, m_n) & = & f(m_1, \ldots, m_i, \, r_i\, m_{i+1}, \ldots, m_n) \\ f(m_1, \ldots, m_n)\, r_n & = & f(m_1, \ldots, m_{n-1}, \, m_n\, r_n) \end{array}$$

for $r_i \in R_i$ and $m_i\,, m_i' \in M_i$. Write

$$\mathbf{Mult}\,(M_1, \, \ldots \,, M_n\,; L)$$

for the abelian group of such functions f . It should now be clear how to construct a module

$$M_1 \otimes_{R_1} M_2 \otimes_{R_2} \cdots \otimes_{R_{n-1}} M_n : R_0 \nrightarrow R_n$$

and multilinear function

$$\lambda : M_1 \times \, \cdots \, \times M_n \longrightarrow M_1 \otimes_{R_1} \cdots \, \otimes_{R_{n-1}} M_n$$

having the universal property that, for each multilinear function $f : M_1 \times \cdots \times M_n \longrightarrow L$, there exists a unique left R_0-/right R_n-module morphism

$g : M_1 \otimes_{R_1} \cdots \otimes_{R_{n-1}} M_n \longrightarrow L$ for which $g \circ \lambda = f$. This describes an abelian group isomorphism

$$\mathbf{Mult}\,(M_1, \ldots, M_n; L) \;\cong\; \mathrm{Hom}_{R_0}^{R_n}(M_1 \otimes_{R_1} \cdots \otimes_{R_{n-1}} M_n\,, L)$$

(where $\mathrm{Hom}_R^S(M, N) = \mathbf{Mult}\,(M, N)$ is the abelian group of left R-/right S-module morphisms $M \longrightarrow N$). When there is no ambiguity about the rings, we write $M_1 \otimes \cdots \otimes M_n$ instead of $M_1 \otimes_{R_1} \cdots \otimes_{R_{n-1}} M_n$. As with cartesian product we have canonical isomorphisms

$$(M_1 \otimes \cdots \otimes M_k) \otimes (M_{k+1} \otimes \cdots \otimes M_n) \;\cong\; M_1 \otimes \cdots \otimes M_n\ .$$

However, the diagonal $M \longrightarrow M \otimes M$ in which $m \longmapsto m \otimes m$, does not preserve addition. The empty tensor product $M_1 \otimes \cdots \otimes M_n$ for $n = 0$ is just R_0 as a module $R_0 \,{\to\!\!\!\!\!+}\, R_0$, using multiplication in R as scalar multiplication on both sides. We have canonical isomorphisms

$$R \otimes_R M \;\cong\; M \;\cong\; M \otimes_S S\ .$$

Given $M, M' : R \,{\to\!\!\!\!\!+}\, S$, we write

$$f : M \Rightarrow M' : R \,{\to\!\!\!\!\!+}\, S \qquad \text{or} \qquad \begin{tikzcd} R \arrow[r, bend left, "M", "\shortmid"{marking}] \arrow[r, bend right, "M'"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow f"] & S \end{tikzcd}$$

to mean $f : M \longrightarrow M'$ is a left R- and right S-module morphism. Given the data

$$\begin{tikzcd} R_0 \arrow[r, bend left, "M_1", "\shortmid"{marking}] \arrow[r, bend right, "M_1'"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow f_1"] & R_1 \arrow[r, bend left, "M_2", "\shortmid"{marking}] \arrow[r, bend right, "M_2'"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow f_2"] & R_2 \arrow[r, bend left, "\shortmid"{marking}] \arrow[r, bend right, "\shortmid"{marking}] & \cdots \arrow[r, bend left, "\shortmid"{marking}] \arrow[r, bend right, "\shortmid"{marking}] & R_{n-1} \arrow[r, bend left, "M_n", "\shortmid"{marking}] \arrow[r, bend right, "M_n'"', "\shortmid"{marking}] \arrow[r, phantom, "\Downarrow f_n"] & R_n \end{tikzcd}$$

we obtain $f_1 \otimes \cdots \otimes f_n : M_1 \otimes_{R_1} \cdots \otimes_{R_{n-1}} M_n \Rightarrow M_1' \otimes_{R_1} \cdots \otimes_{R_{n-1}} M_n' :$ $R_0 \,{\to\!\!\!\!\!+}\, R_n$ given by $(f_1 \otimes \cdots \otimes f_n) \circ \lambda = \lambda \circ (f_1 \times \cdots \times f_n)$.

We have seen that tensor products allow us to represent bilinear functions as module morphisms. Another way of doing this uses Hom instead of tensor. Given a triangle of modules

$$\begin{tikzcd} R \arrow[rr, "M", "\shortmid"{marking}] \arrow[dr, "L"', "\shortmid"{marking}] & & S \arrow[dl, "N", "\shortmid"{marking}] \\ & T & \end{tikzcd}$$

we can enrich the abelian group $\mathrm{Hom}_R(M,L)$ (resp. $\mathrm{Hom}^T(N,L)$) of left R- (resp. right T-) module morphisms with a module structure

$$\mathrm{Hom}_R(M,L) : S \twoheadrightarrow T$$
$$(\text{resp.}\quad \mathrm{Hom}^T(N,L) : R \twoheadrightarrow S\ )$$

using the scalar multiplications

$$(s\,f\,t)(m) = f(m\,s)\,t \qquad (\text{resp. } (r\,g\,s)(n) = r\,g(s\,n)\ )\ .$$

We then have abelian group isomorphisms

$$\begin{aligned}\mathrm{Hom}_S^T(N,\mathrm{Hom}_R(M,L)) &\cong \mathbf{Mult}\,(M,N\,;L)\\ &\cong \mathrm{Hom}_R^S(M,\mathrm{Hom}^T(N,L))\end{aligned}$$

induced by the canonical bijections

$$\left(L^M\right)^N \cong L^{M\times N} \cong \left(L^N\right)^M .$$

Combining these with the earlier results, we have

$$\begin{aligned}\mathrm{Hom}_S^T(N,\mathrm{Hom}_R(M,L)) &\cong \mathrm{Hom}_R^T(M\otimes_S N,L)\\ &\cong \mathrm{Hom}_R^S(M,\mathrm{Hom}^T(N,L))\ .\end{aligned}$$

These isomorphisms are determined by the *evaluation morphisms*

$$\begin{aligned} ev_M &: M\otimes_S \mathrm{Hom}_R(M,L) \longrightarrow L \quad , \quad m\otimes f \longmapsto f(m)\\ ev^N &: \mathrm{Hom}^T(N,L)\otimes_S N \longrightarrow L \quad , \quad g\otimes n \longmapsto g(n)\ .\end{aligned}$$

Explicitly, the first isomorphism takes any $u : N \longrightarrow \mathrm{Hom}_R(M,L)$ to the composite

$$M\otimes_S N \xrightarrow{1_M\otimes u} M\otimes_S \mathrm{Hom}_R(M,L) \xrightarrow{ev_M} L\ .$$

Exercises

1. Describe $\mathbb{Z}/(2)\otimes_{\mathbb{Z}}\mathbb{Z}/(5)$.

2. (a) If R,S are rings, describe a canonical ring structure on $R\otimes_{\mathbb{Z}}S$.
   (b) Is the function from R to $R\otimes_{\mathbb{Z}}S$ taking R to $r\otimes 1$ a ring morphism? Why?
   (c) Show that $R\otimes_{\mathbb{Z}}S$ is the coproduct of R,S in the category of commutative rings.

3. Show that a module M from R to S amounts to the same thing as a left $R\otimes_{\mathbb{Z}}S^{\mathrm{op}}$-module.

4. Describe explicitly the construction of $M\otimes_S N\otimes_T L$.

5
Cauchy modules

A module $M : R \mathrel{\mapstochar\mkern-3mu\rightarrow} S$ gives rise to a module $M^* = \mathrm{Hom}_R(M, R) : S \mathrel{\mapstochar\mkern-3mu\rightarrow} R$ called the *left dual* of M. There is a canonical module morphism

$$\rho_L^M : M^* \otimes_R L \longrightarrow \mathrm{Hom}_R(M, L)$$

given by $\rho_L^M(u \otimes l)(m) = u(m)l$, for each left R-module L.

Call an $M : R \mathrel{\mapstochar\mkern-3mu\rightarrow} S$ *cauchy* when ρ_L^M is an isomorphism for all left R-modules L. Our goal in this section is to characterize cauchy modules more intrinsically.

A module P is called *projective* when, for all surjective module morphisms $e : L \longrightarrow L'$ and all module morphisms $f : P \longrightarrow L'$, there exists some module morphism $g : P \longrightarrow L$ with $f = e \circ g$.

$$\begin{tikzcd} & P \arrow[dl, dashed, "g"'] \arrow[dr, "f"] & \\ L \arrow[rr, two heads, "e"] & & L' \end{tikzcd}$$

A morphism $r : M \longrightarrow N$ is said to be a *retraction* when there exists a morphism $i : N \longrightarrow M$ with $r \circ i = 1_N$. When a retraction exists from M to N, we call N a *retract* of M.

Proposition 5.1 *A module P is projective iff P is a retract of some free module F.*

Proof. 1) *A retract Q of a projective P is projective.* To see this take $i : Q \longrightarrow P$ and $r : P \longrightarrow Q$ with $r \circ i = 1_Q$. Suppose $e : L \twoheadrightarrow L'$ is a surjective morphism and $f : Q \longrightarrow L'$. Then $f \circ r : P \longrightarrow L'$, and since P is projective, there is a morphism $h : P \longrightarrow L$ with $e \circ h = f \circ r$. But then $e \circ (h \circ i) = (e \circ h) \circ i = f \circ r \circ i = f \circ 1_Q = f$, so $g = h \circ i$ has $e \circ g = f$.

2) *Free modules $\mathcal{F}(X)$ are projective.* Take $e : L \longrightarrow L'$ surjective and $f : \mathcal{F}(X) \longrightarrow L'$. Then we can choose (using the axiom of choice) an

element $g(x) \in L$ for each $x \in X$ such that $e(g(x)) = f(x)$. Since $\mathcal{F}(X)$ is free, we can extend g uniquely to a morphism $g : \mathcal{F}(X) \longrightarrow L$; and furthermore $e \circ g = f$ since they agree on X.

3) *For each module M there is a free module F and a surjective morphism* $e : F \longrightarrow M$. Just take F to be the free module $\mathcal{F}(M)$ on the underlying set of M. To give a morphism $e : F \longrightarrow M$ we only have to give it on M, so we take $e(m) = m$. Clearly this e is surjective.

4) *If $e : F \longrightarrow P$ is surjective and P projective then e is a retraction.* For we have i as in:

$$\begin{tikzcd} & P \arrow[dl, dashed, "i"'] \arrow[dr, "1_P"] & \\ F \arrow[rr, two heads, "e"] & & P \end{tikzcd} .$$

Exercise 5.2 *Show that a module P is finitely generated and projective if and only if P is a retract of a free module on a finite set.*
Hint*: In (3) we did not need $\mathcal{F}(M)$; only $\mathcal{F}(X)$ for any X generating M.*

This brings us to the *fundamental theorem of "Morita theory"*.

Theorem 5.3 *The following conditions on a module $M : R \nrightarrow S$ are equivalent:*
(i) *M is cauchy.*
(ii) *there exists a morphism $d \colon S \Rightarrow M^* \otimes_R M : S \nrightarrow S$ such that both the following two composites are identity morphisms*

$$M \;\cong\; M \otimes_S S \xrightarrow{\;1_M \otimes d\;} M \otimes_S M^* \otimes_R M \xrightarrow{\;ev_M \otimes 1_M\;} R \otimes_R M \;\cong\; M$$
$$M^* \;\cong\; S \otimes_S M^* \xrightarrow{\;d \otimes 1_{M^*}\;} M^* \otimes_R M \otimes_S M^* \xrightarrow{\;1_{M^*} \otimes ev_M\;} M^* \otimes_R R \;\cong\; M^* .$$

(iii) *there exists a module $N : S \nrightarrow R$ and morphisms*

$$e : M \otimes_S N \longrightarrow R \quad , \quad d : S \longrightarrow N \otimes_R M$$

such that the following composite is the identity morphism

$$M \;\cong\; M \otimes_S S \xrightarrow{\;1_M \otimes d\;} M \otimes_S N \otimes_R M \xrightarrow{\;e \otimes 1_M\;} R \otimes_R M \;\cong\; M .$$

(iv) *M is a finitely generated projective left R-module.*

Proof. (i) $\Rightarrow$ (ii). Since ρ^M_M is an isomorphism, there is an element of $M^*\otimes_R M$ taken by ρ^M_M to $1_M : M \longrightarrow M$. This element of $M^*\otimes_R M$ now determines a unique morphism $d : S \longrightarrow M^*\otimes_R M$ whose value at $1 \in S$ is the element. Write $d(1) = \sum_i u_i \otimes m_i$. The condition $\rho^M_M(d(1))(m) = m$ becomes $\sum_i u_i(m)\, m_i = m$ for all $m \in M$. This immediately gives that the first composite of (ii) takes m to m. To see that the second takes $u \in M^*$ to itself we use $u(m) = u(\sum u_i(m)\, m_i) = \sum u_i(m)\, u(m_i)$.

(ii) $\Rightarrow$ (iii). Just take $N = M^*$, $e = ev_M$ and d as in (ii).

(iii) $\Rightarrow$ (iv). Just put $d(1) = \sum_{i=1}^k n_i \otimes m_i \in N \otimes_R M$. From the fact that the composite in (iii) is the identity, we have $\sum_i e(m \otimes n_i)\, m_i = m$ for all $m \in M$. So M is generated by $m_1, \ldots m_k$. It remains to see that M is projective. Take $s : L \longrightarrow L'$ surjective and $f : M \longrightarrow L'$. Then we can choose $l_1, \ldots, l_k \in L$ with $s(l_i) = f(m_i)$. Define $g : M \longrightarrow L$ by $g(m) = \sum_i e(m \otimes n_i)\, l_i$ and we get $s(g(m)) = \sum_i e(m \otimes n_i)\, s(l_i) = \sum_i e(m \otimes n_i)\, f(m_i) = f(\sum_i e(m \otimes n_i) m_i) = f(m)$, as required.

(iv) $\Rightarrow$ (i). It is easy to see that a retract of a cauchy module is cauchy (exercise). So it suffices to show that $M = \mathcal{F}_R(X)$ is cauchy for X a finite set $\{x_1, \ldots, x_k\}$. But then $M^* = \mathrm{Hom}_R(\mathcal{F}(X), R) \cong R^k$ and $\mathrm{Hom}_R(M, L) = \mathrm{Hom}_R(\mathcal{F}(X), L) \cong L^k$. Under these isomorphisms ρ^M_L carries across to the morphism $R^k \otimes_R L \longrightarrow L^k$ with $(r_1, \ldots, r_k) \otimes l \longmapsto (r_1 l, \ldots, r_k l)$ which has inverse $(l_1, \ldots, l_k) \longmapsto \sum_{i=1}^k u_i \otimes l_i$, in which $u_i \in R^k$ projects to 0 in all components except the i-th where it projects to 1. So ρ^M_L is an isomorphism.

Given rings R and S, from any ring morphism $f : S \longrightarrow R$ we obtain two modules ${}_fR : S \nrightarrow R$ and $R_f : R \nrightarrow S$, which have R as underlying abelian group. They have scalar multiplicatons

$$\begin{array}{lcl} S \times {}_fR \longrightarrow {}_fR & , & {}_fR \times R \longrightarrow {}_fR \\ R_f \times S \longrightarrow R_f & , & R \times R_f \longrightarrow R_f \end{array}$$

given by, respectively

$$\begin{array}{lcl} (s, r) \longmapsto f(s)\, r & , & (r, r') \longmapsto r\, r' \\ (r, s) \longmapsto r\, f(s) & , & (r', r) \longmapsto r'\, r\ . \end{array}$$

For any module $L : R \nrightarrow T$ we have canonical isomorphisms

$$\begin{array}{ccccc} {}_fR \otimes_R L & \cong & L & \cong & \mathrm{Hom}_R(R_f, L) \\ r \otimes l & \longmapsto & l & & \\ & & u(1) & \longleftarrow\!\!\!\shortmid & u\ . \end{array}$$

It follows easily from this that

$$(R_f)^* \;\cong\; {}_fR$$

and that R_f is cauchy.

A module $M : R \looparrowright S$ is called *convergent* when there exists a ring morphism $f : S \longrightarrow R$ and a module isomorphism $M \cong R_f$.

The *product* $\prod_{i\in I} M_i : R \looparrowright S$ of a family of modules $M_i : R \looparrowright S$ with $i \in I$, has as elements the families $\mathbf{m} = (m_i)_{i\in I}$ with $m_i \in M_i$; addition and scalar multiplication are given by

$$\mathbf{m} + \mathbf{m}' \;=\; (m_i + m_i')_{i\in I} \quad , \quad r\,\mathbf{m}\,s = (r\,m_i\,s)_{i\in I} \;\; .$$

There are *projections*

$$pr_j : \prod_{i\in I} M_i \longrightarrow M_j \qquad \text{for each } j \in I$$

given by $pr_j(\mathbf{m}) = m_j$. There are also injective module morphisms

$$in_j : M_j \longrightarrow \prod_{i\in I} M_i \qquad \text{for each } j \in I$$

given by $in_j(m) = \mathbf{m}$ where $m_j = m$ and $m_i = 0$ for all $i \neq j$; we can use these to identify each M_j with the submodule of $\prod_{i\in I} M_i$ consisting of those $\mathbf{m}$ with $m_i = 0$ for all $i \neq j$.

The *direct sum* $\sum_{i\in I} M_i : R \looparrowright S$ is the submodule of $\prod_{i\in I} M_i$ which consists of those $\mathbf{m}$ for which $m_i = 0$ for all but finitly many $i \in I$. This is the submodule generated by the union $\cup_{i\in I} M_i$, hence we can write $\sum_{i\in I} m_i$ instead of $\mathbf{m} \in \sum_{i\in I} M_i$. Of course the injections in_j actually land in $\sum_{i\in I} M_i$.

Proposition 5.4 *There are module isomorphisms*

(a)
$$\begin{array}{ccc} \mathrm{Hom}_R\Big(\sum_{i\in I} M_i, L\Big) & \cong & \prod_{i\in I} \mathrm{Hom}_R(M_i, L) \\ f & \longleftrightarrow & (f \circ in_i)_{i\in I} \end{array}$$

(b)
$$\begin{array}{ccc} \Big(\sum_{i\in I} M_i\Big)\otimes_S N & \cong & \sum_{i\in I} M_i \otimes_S N \\ \big(\sum_i m_i\big)\otimes n & \longleftrightarrow & \sum_i (m_i \otimes n)\;. \end{array}$$

Proof. (a) *Injectivity.* If $f \circ in_i = 0$ for all $i \in I$ then f is zero on each M_i and hence on $\sum M_i$.

Surjectivity. Given $f_i : M_i \longrightarrow L$ for all $i \in I$, define $f : \sum M_i \longrightarrow L$ by $f(\sum m_i) = \sum f_i\,(m_i)$.

(b)
$$\begin{aligned}
\mathrm{Hom}_R\Big(\big(\sum_i M_i\big)\otimes_S N\,, L\Big) &\cong \mathrm{Hom}^S\Big(\sum_i M_i, \mathrm{Hom}^R(N, L)\Big) \\
&\cong \prod_i \mathrm{Hom}^S\Big(M_i, \mathrm{Hom}^R(N, L)\Big) \\
&\cong \prod_i \mathrm{Hom}_R(M_i \otimes_S N\,, L) \\
&\cong \mathrm{Hom}_R\Big(\sum_i (M_i \otimes_S N)\,, L\Big)
\end{aligned}$$

and the composite isomorphism is induced by the given map in (b). This proves it. (Why?)

When I is finite, notice that $\sum_{i\in I} M_i = \prod_{i\in I} M_i$. This is also frequently written $\oplus_{i\in I} M_i$. So $M \oplus N = M \times N = M + N$.

Exercises

1. Suppose M is a finitely generated projective module over a commutative ring R. Show that M^* is a finitely generated projective module and that the canonical morphism $M \to M^{**}$ is bijective.

2. Prove directly from the definition of "Cauchy module" that a retract of a Cauchy module is Cauchy.

6
Algebras

Let R be any ring. An *algebra over* R (or R-*algebra*) is a module $A : R \twoheadrightarrow R$ together with module morphisms

$$\mu : A \otimes_R A \longrightarrow A \quad , \quad \eta : R \longrightarrow A$$

such that

(Associativity)
$$\begin{tikzcd} A\otimes_R A\otimes_R A \arrow[r, shift left, "\mu\otimes 1_A"] \arrow[r, shift right, "1_A\otimes\mu"'] & A\otimes_R A \arrow[r, "\mu"] & A \end{tikzcd}$$

(Identity)
$$\begin{tikzcd} A \arrow[r, shift left, "\eta\otimes 1_A"] \arrow[r, shift right, "1_A\otimes\eta"'] \arrow[rr, bend right, "1_A"'] & A\otimes_R A \arrow[r, "\mu"] & A \end{tikzcd} .$$

Notice that A becomes a ring with multiplication $a\,b = \mu(a\otimes b)$ and identity $1 = \eta(1)$.

For R-algebras $A, B : R \twoheadrightarrow R$ an *algebra morphism* $f : A \longrightarrow B$ is a module morphism satisfying

$$\begin{tikzcd} A\otimes_R A \arrow[r, "f\otimes f"] \arrow[d, "\mu"'] & B\otimes_R B \arrow[d, "\mu"] \\ A \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} & R \arrow[dl, "\eta"'] \arrow[dr, "\eta"] & \\ A \arrow[rr, "f"] & & B \end{tikzcd} .$$

We write $\mathbf{Alg}_R(A, B)$ for the set of algebra morphisms from A to B.

Example 6.1 *For any module* $M : R \nrightarrow S$, *the* endomorphism algebras, *over* S *and* R *respectively, are given by*

$$\begin{aligned} \mathrm{End}_R(M) &= \mathrm{Hom}_R(M,M) : S \nrightarrow S \\ \mathrm{End}^S(M) &= \mathrm{Hom}^S(M,M) : R \nrightarrow R \end{aligned}$$

In each case the multiplication is given by composition.

A module morphism

$$\hat{\mu} : A \Rightarrow \mathrm{End}^S(M) : R \nrightarrow R$$

corresponds to a module morphism

$$\mu : A \otimes_R M \Rightarrow M : R \nrightarrow S \quad .$$

To say that $\hat{\mu}$ *is an algebra morphism is precisely to say that* μ *is a scalar multiplication enriching* M *with the structure of left* A*-module.*

Example 6.2 *For any module* $M : R \nrightarrow R$, *write*

$$M^{\otimes n} = M \otimes_R \cdots \otimes_R M \qquad (n \text{ terms}) .$$

The tensor algebra *on* M *is defined by the "geometric series"*

$$T(M) = \sum_{n=0}^{\infty} M^{\otimes n}$$

with multiplication $\mu : T(M) \otimes_R T(M) \longrightarrow T(M)$ *induced by the canonical isomorphisms*

$$M^{\otimes p} \otimes_R M^{\otimes q} \xrightarrow{\cong} M^{\otimes(p+q)}$$

and unit $\eta : R \longrightarrow T(M)$ *equal to the injection*

$$in_0 : R = M^{\otimes 0} \longrightarrow \sum_{n=0}^{\infty} M^{\otimes n} \quad .$$

Composition with the injection $in_1 : M \longrightarrow T(M)$ *gives a bijection*

$$\mathbf{Alg}_R(T(M), A) \cong \mathrm{Hom}_R^R(M, A)$$

for any algebra A. *The inverse takes* $f : M \longrightarrow A$ *to* $g : T(M) \longrightarrow A$ *given by* $g(m_1 \otimes \cdots \otimes m_r) = f(m_1) \cdots f(m_r)$. *In particular, if we take* $M = A$ *and* $f = 1_A$, *we obtain an algebra morphism*

$$\mu : T(A) \longrightarrow S \quad \textit{with} \quad \mu(a_1 \otimes \cdots \otimes a_r) = a_1 \cdots a_r \ .$$

Example 6.3 *Let G be any monoid. There is an R-algebra $R(G)$ which is just the free module $\mathcal{F}^R_R(G)$ on the underlying set of G together with the multiplication μ which extends that of G in the sense that*

$$\begin{tikzcd}
G \times G \arrow[r, hook] \arrow[d, "\mu"'] & \mathcal{F}^R_R(G \times G) \cong \mathcal{F}^R_R(G) \otimes_R \mathcal{F}^R_R(G) \arrow[d, "\mu"] \\
G \arrow[r, hook] & \mathcal{F}^R_R(G) .
\end{tikzcd}$$

This $R(G)$ is called the monoid R-algebra *of G; or when G is a group, the* group R-algebra *of G. Each monoid morphism $G \longrightarrow A$ into the multiplicative monoid of A extends uniquely to an R-algebra morphism $R(G) \longrightarrow A$.*

A representation of G on M *is an $R(G)$-module. Scalar multiplication $R(G) \otimes_R M \longrightarrow M$ can be viewed as a monoid morphism*

$$G \longrightarrow \mathrm{End}_R(M) \ .$$

The subset of M given by $\{gm - m \mid g \in G\,, m \in M\}$ generates a submodule $(gm - m \mid g \in G\,, m \in M)$ and we write M/G for the quotient module $M/(gm - m \mid g \in G\,, m \in M)$.

An *ideal* in an algebra A is a submodule I such that $a\,x\,b \in I$ for all $x \in I$ and $a\,, b \in A$. There is a unique structure of algebra on the quotient module A/I for which the canonical $\rho : A \longrightarrow A/I$ is an algebra morphism. The kernel of any algebra morphism $f : A \longrightarrow B$ is an ideal in A.

If X is a subset of an algebra A, we write (X) for the smallest ideal of A containing X. This should not cause confusion with the module notation; the ideal (X) is precisely the submodule $(A\,X\,A)$ generated by the subset $A\,X\,A = \{a\,x\,b \mid a\,, b \in A\,, x \in X\}$ of A. Given any algebra morphism $g : A \longrightarrow B$ satisfying $g(x) = 0$ for all $x \in X$, then an algebra morphism $f : A/(X) \longrightarrow B$ is uniquely determined via the equation $f \circ \rho = g$.

Now suppose that R is a commutative ring. Then left R-modules are "the same thing" as right R-modules. Moreover, each left R-module M can be naturally regarded as a module $M : R \looparrowright R$ by defining

$$r\,m\,s \;=\; (r\,s)\,m \qquad \text{for all} \quad r\,, s \in R \quad \text{and} \quad m \in M \ .$$

In dealing with modules over a commutative ring, we happily regard left modules as two-sided via this process. Thus for R-modules $M_1, \ldots, M_n$ we have a tensor product R-module

$$M_1 \otimes_R \cdots \otimes_R M_n \ .$$

Furthermore every permutation ξ on the set $\{1, \ldots, n\}$ induces a canonical module isomorphism

$$\begin{array}{rcl} \sigma_\xi \; : \; M_1 \otimes_R \cdots \otimes_R M_n & \xrightarrow{\cong} & M_{\xi(1)} \otimes_R \cdots \otimes_R M_{\xi(n)} \\ m_1 \otimes \cdots \otimes m_n & \longmapsto & m_{\xi(1)} \otimes \cdots \otimes m_{\xi(n)} \; . \end{array}$$

Given an algebra A over R with multiplication μ and unit η, we obtain an *opposite algebra* A^{op} on the same module A, with multiplication

$$\mu^{\mathrm{op}} : A \otimes_R A \xrightarrow{\sigma} A \otimes_R A \xrightarrow{\mu} A$$

and with the same unit $\eta : R \longrightarrow A$. Call A *commutative* when $A^{\mathrm{op}} = A$ as algebras. It follows that the composite

$$A \otimes_R \cdots \otimes_R A \xrightarrow{\sigma_\xi} A \otimes_R \cdots \otimes_R A \xrightarrow{\mu} A$$

is independent of the permutation ξ.

Example 6.4 *For any set X, the set R^X of all functions from X into the commutative ring R becomes a commutative R-algebra after defining addition, scalar multiplication and multiplication as acting pointwise. The unit $\eta : R \longrightarrow R^X$ is given by $\eta(r)(x) = r$ for all $r \in R$ and $x \in X$.*

Example 6.5 *Let M be any module over the commutative ring R. There is a natural representation of the symmetric group $\mathcal{S}_n$ on $M^{\otimes n}$ given by $\sigma_{\_} : \mathcal{S}_n \longrightarrow \mathrm{End}_R(M^{\otimes n})$; that is $\xi \cdot (m_1 \otimes \cdots \otimes m_n) = m_{\xi(1)} \otimes \cdots \otimes m_{\xi(n)}$.*
The symmetric *R-algebra on M is given by the "exponential series"*

$$\mathcal{S}(M) \; = \; \sum_{n=0}^{\infty} M^{\otimes n} / \mathcal{S}_n \; .$$

Another way of constructing this is as follows. For any R-algebra A we can form a commutative R-algebra by taking the quotient of A by the ideal $(a\,b - b\,a \mid a, b \in A)$. Applying this construction to the tensor algebra $T(M)$ gives $\mathcal{S}(M)$.

For every commutative R-algebra A, we have that

$$\mathbf{Alg}_R(\mathcal{S}(M), A) \;\cong\; \mathrm{Hom}_R(M, A) \; .$$

In particular, corresponding to the identity map $1_A : A \longrightarrow A$ there is an R-algebra morphism $\mu : \mathcal{S}(A) \longrightarrow A$.

The following diagram of “forgetful” and “free” constructions summarizes some of the above.

$$
\begin{tikzcd}
 & \mathbf{Comm\,Alg}_R \arrow[d, bend left] \arrow[ddr, bend left=10] & \\
 & \mathbf{Alg}_R \arrow[u, bend left] \arrow[dl, bend left=10, "\text{forget module}"] \arrow[dr, bend left=10, "\text{forget multiplication}"'] & \\
\mathbf{Monoids} \arrow[ur, bend left=10, "R"] \arrow[dr, bend right=10, "\text{forget multiplication}"'] & & \mathbf{Modules}_R \arrow[ul, bend left=10, "T"'] \arrow[uul, bend right=10, "\mathcal{S}"'] \arrow[dl, bend left=10, "\text{forget module structure}"] \\
 & \mathbf{Sets} \arrow[ul, bend right=10, "\text{word monoid}"'] \arrow[ur, bend left=10, "\mathcal{F}_R"] &
\end{tikzcd}
$$

Skew commutativity $a\,b + b\,a = 0$ for an R-algebra is too strong as a requirement for *all* $a\,, b \in A$. For example taking $b = 1$, it would give $(1 + 1)a = 0$. Hence if R is a field of characteristic other than 2 (meaning $1 + 1 \neq 0$ in R), we would get $a = 0$, and so $A = \{0\}$.

An R-algebra A is said to be *skew commutative* when for all $a \in A$ either $a^2 = 0$ or $a \in \eta(R)$. Then, provided none of $a\,, b$ and $a + b$ are in the image of $\eta : R \longrightarrow A$, we have

$$a\,b + b\,a \;=\; (a + b)^2 - a^2 - b^2 \;=\; 0\,.$$

Example 6.6 *For any R-algebra A we can form the quotient by the ideal $(a^2 \mid a \notin \eta(R))$ to obtain a skew commutative algebra. If we do this to the tensor algebra $T(M)$ we obtain the* exterior algebra $\Lambda(M)$. *Alternatively, let $\Lambda_n(M)$ be the quotient module of $M^{\otimes n}$ by the submodule generated by the elements $m_1 \otimes \cdots \otimes m_n$ with $m_i = m_j$ for some $i \neq j$ (this submodule is $\{0\}$ when $n = 0$ or 1); then*

$$\Lambda(M) \;=\; \sum_{n=0}^{\infty} \Lambda_n(M)\,.$$

We write $m_1 \wedge \cdots \wedge m_n$ for the image of $m_1 \otimes \cdots \otimes m_n$ in $\Lambda(M)$. For all $x\,, y\,, z \in M$ we have

$$
\begin{aligned}
x \wedge x \;&=\; 0 \qquad \text{therefore} \qquad x \wedge y \;=\; -\,y \wedge x \\
(rx + sy) \wedge z \;&=\; r(x \wedge z) + s(y \wedge z)\,.
\end{aligned}
$$

If $M = \mathcal{F}_R\{x_1, \ldots, x_k\}$ *is a free module on a* k*-element set then* $\Lambda_n(M)$ *is a free module on a* $\binom{k}{n}$*-element set; so* $\Lambda(M)$ *is a free module on a set with* 2^k*-elements. In particular* $\Lambda_k(M)$ *is free on the singleton set* $\{x_1 \wedge \cdots \wedge x_k\}$*, so if*

$$y_i \; = \; \sum_{j=1}^{k} r_{ij}\, x_j$$

then $y_1 \wedge \cdots \wedge y_k$ *must be a unique scalar multiple*

$$y_1 \wedge \cdots \wedge y_k \; = \; \det(r_{ij})\, x_1 \wedge \cdots \wedge x_k$$

of $x_1 \wedge \cdots \wedge x_k$. *This can be taken as a definition of the determinant of* $(r_{ij}) \in \mathbf{Mat}\,(k\,, R)$.

If A *is a skew-commutative algebra then we have a bijection*

$$\mathbf{Alg}_R(\Lambda(M)\,, A) \;\; \cong \;\; \mathrm{Hom}_R(M, A)\ .$$

An *R-Lie algebra* is an R-module L together with a module morphism $\beta : L \otimes_R L \longrightarrow L$ satisfying the conditions

$$\beta(x\,, x) \; = \; 0$$

$$\text{(Jacobi identity)} \qquad \beta\big(\beta(x\,, y)\,, z\big) + \beta\big(\beta(z\,, x)\,, y\big) + \beta\big(\beta(y\,, z)\,, x\big) \; = \; 0$$

Call such a β a *Lie bracket* on the module L.

Example 6.7 *For any* R*-algebra* A *the* commutator $[\,a\,, b\,] \; = \; a\,b - b\,a$ *defines a Lie bracket on the underlying* R*-module of* A:

$$\begin{aligned}
&[\,[\,a\,,b\,]\,,c\,] + [\,[\,c\,,a\,]\,,b\,] + [\,[\,b\,,c\,]\,,a\,] \\
&\quad = \; [\,a\,,b\,]\,c - c\,[\,a\,,b\,] \; + \; [\,c\,,a\,]\,b - b\,[\,c\,,a\,] \; + \; [\,b\,,c\,]\,a - a\,[\,b\,,c\,] \\
&\quad = \; (a\,b\,c - b\,a\,c) - (c\,a\,b - c\,b\,a) + (c\,a\,b - a\,c\,b) - (b\,c\,a - b\,a\,c) + \\
&\qquad\qquad + (b\,c\,a - c\,b\,a) - (a\,b\,c - a\,c\,b) \; = \; 0\ .
\end{aligned}$$

So A *becomes a Lie algebra, denoted by* A_L. *It turns out (at least when* R *is a field) that every Lie algebra is a submodule, closed under commutator, of such an example.*

Example 6.8 *Let* A *be an* R*-algebra and* $M : A \nrightarrow A$ *a module. Then a* derivation $D : A \longrightarrow M$ *is an* R*-module morphism satisfying*

$$\text{(Leibniz rule)} \qquad D(a\,b) \;\; = \;\; D(a)\,b \; + \; a\,D(b)\ .$$

Notice that $a = b = 1$ *gives* $D(1) = 2\,D(1)$, *so* $D(1) = 0$. *Let* $\mathbf{Der}_R(A\,,M)$ *denote the submodule of* $\mathrm{Hom}_R(A, M)$ *consisting of the derivations. We write* $\mathbf{Der}_R(A)$ *for* $\mathbf{Der}_R(A\,,A)$. *It is easy to check that* $\mathbf{Der}_R(A)$ *is closed under commutator in the algebra* $\mathrm{End}_R(A)$; *that is, if* $D_1\,,D_2 : A \longrightarrow A$ *are derivations then so is* $[\,D_1\,,D_2\,] = D_1 \circ D_2 - D_2 \circ D_1$.

Example 6.9 *The tangent space at the identity of each Lie group is a Lie algebra. The pioneering work of Sophus Lie and Eli Cartan showed how much information about the Lie group is obtainable from the Lie algebra (especially in the compact case).*

The Lie groups $\mathrm{GL}\,(n\,,\mathbb{R})$ *and* $\mathrm{SL}\,(n\,,\mathbb{R})$ *and* $\mathrm{O}(n\,,\mathbb{R})$ *consist of those matrices* $\mathbf{x} \in \mathbf{Mat}(n\,,\mathbb{R})$ *for which respectively* $\mathbf{x}$ *is invertible*, $\det \mathbf{x} = 1$ *and* $\mathbf{x}\,\mathbf{x}^t = 1$. *They have associated Lie algebras*

$$\begin{array}{rcl} gl(n\,,\mathbb{R}) & = & \mathbf{Mat}\,(n\,,\mathbb{R}) \\ sl(n\,,\mathbb{R}) & = & \{\mathbf{x} \in gl(n\,,\mathbb{R}) \mid \mathrm{trace}(\mathbf{x}) = 0\,\} \\ o(n\,,\mathbb{R}) & = & \{\mathbf{x} \in gl(n\,,\mathbb{R}) \mid \mathbf{x}^t = -\mathbf{x}\,\}\,. \end{array}$$

(We shall not stop to prove this here.) The Lie bracket is $[\,\mathbf{x}\,,\mathbf{y}\,] = \mathbf{x}\,\mathbf{y} - \mathbf{y}\,\mathbf{x}$ *in each case. As an exercise the reader should check that* $sl(n\,,\mathbb{R})$ *and* $o(n\,,\mathbb{R})$ *are closed under commutator.*

Suppose L and L' are R-Lie algebras and $f : L \longrightarrow L'$ is an R-module morphism. Then f is a *Lie algebra morphism* when it satisfies

$$f\big(\,\beta(x\,,y)\big) \;=\; \beta\big(f(x)\,,f(y)\big)\,.$$

Write $\mathbf{Lie}_R(L\,,L')$ for the set of Lie algebra morphisms $f : L \longrightarrow L'$.

We saw in Example 6.1above that each R-algebra A gives rise to an R-Lie algebra A_L using the commutator. We shall describe an "adjoint" for this process*:* for each R-Lie algebra L we obtain an R-algebra $\mathcal{U}(L)$, called the *universal enveloping algebra* of L, such that there is a natural bijection

$$\mathbf{Alg}_R\big(\mathcal{U}(L)\,,A\big) \;\cong\; \mathbf{Lie}(L\,,A_L)\,. \qquad (*)$$

For this we use the tensor algebra $T(L)$ on the underlying R-module of L, and take the quotient by the appropriate ideal:

$$\mathcal{U}(L) \;=\; T(L)/\big(x{\otimes}y - y{\otimes}x - \beta(x\,,y) \mid x\,,y \in L\,\big)\,.$$

We have a Lie algebra morphism i as in:

$$\begin{tikzcd} L \arrow[rr, "i"] \arrow[dr, "in_1"'] & & \mathcal{U}(L)_L \\ & T(L) \arrow[ur, "\rho"'] & \end{tikzcd}$$

and it is composition with i that induces the bijection $(*)$.

The *direct sum* $L_1 \oplus L_2$ of Lie algebras L_1, L_2 is their direct sum as modules together with the Lie bracket

$$\beta\big((x_1, x_2), (y_1, y_2)\big) = \big(\beta(x_1, y_1), \beta(x_2, y_2)\big) .$$

Proposition 6.10 *There is an algebra isomorphism*

$$\mathcal{U}(L_1 \oplus L_2) \cong \mathcal{U}(L_1) \otimes_R \mathcal{U}(L_2)$$

whose composite with $i : L_1 \oplus L_2 \longrightarrow \mathcal{U}(L_1 \oplus L_2)$ *takes the pair* (x_1, x_2) *to* $x_1 \otimes 1 + 1 \otimes x_2$.

Proof. It is left to the reader to check that

$$\begin{array}{rlrcl} L_1 \oplus L_2 \longrightarrow & \big(\mathcal{U}(L_1) \otimes_R \mathcal{U}(L_2)\big)_L & (x_1, x_2) & \longmapsto & x_1 \otimes 1 + 1 \otimes x_2 \\ L_1 \longrightarrow & \mathcal{U}(L_1 \oplus L_2)_L & x_1 & \longmapsto & (x_1, 0) \\ L_2 \longrightarrow & \mathcal{U}(L_1 \oplus L_2)_L & x_2 & \longmapsto & (0, x_2) \end{array}$$

are Lie algebra morphisms. These three must therefore be composites with i of algebra morphisms

$$\begin{array}{rlcl} \phi & : \mathcal{U}(L_1 \oplus L_2) & \longrightarrow & \mathcal{U}(L_1) \otimes_R \mathcal{U}(L_2) \\ \psi_1 & : \mathcal{U}(L_1) & \longrightarrow & \mathcal{U}(L_1 \oplus L_2) \\ \psi_2 & : \mathcal{U}(L_2) & \longrightarrow & \mathcal{U}(L_1 \oplus L_2) \quad . \end{array}$$

Define $\psi : \mathcal{U}(L_1) \otimes_R \mathcal{U}(L_2) \longrightarrow \mathcal{U}(L_1 \oplus L_2)$ by $\psi(a \otimes b) = \psi_1(a)\, \psi_2(b)$. Then we have that

$$\begin{array}{rcl} \psi\big(\phi(x_1, x_2)\big) & = & \psi(x_1 \otimes 1 + 1 \otimes x_2) = (x_1, 0) + (0, x_2) = (x_1, x_2) \\ \phi\big(\psi_1(x_1)\big) & = & \phi(x_1, 0) = x_1 \otimes 1 \\ \phi\big(\psi_2(x_2)\big) & = & \phi(0, x_2) = 1 \otimes x_2 \ . \end{array}$$

Hence ϕ and ψ are mutually inverse.

A deeper result which we shall not prove here is:

Proposition 6.11 *(Poincaré–Birkhoff–Witt) If the R-Lie algebra L is free as an R-module then $i : L \longrightarrow \mathcal{U}(L)_L$ is injective.*

A Lie algebra L is called *commutative* when $\beta(x\,,y) = 0$ for all $x\,,y \in L$. So an algebra A is commutative iff A_L is commutative.

Notice that, for any module M, we can make M into a commutative Lie algebra. Then the universal enveloping algebra of M is precisely the same as the symmetric algebra of M, that is $\mathcal{U}(M) = \mathcal{S}(M)$. In particular, we have (Proposition 6.10):

$$\mathcal{S}(M \oplus M') \;\cong\; \mathcal{S}(M) \otimes_R \mathcal{S}(M')\,.$$

Exercises

1. Let R be a commutative ring and G be a group. Consider left modules M, N, L over the group algebra $R(G)$.

   (a) Show that $M \otimes_R N$ becomes an $R(G)$-module on defining $g(m \otimes n) = (gm) \otimes (gn)$ for $g \in G, m \in M, n \in N$.

   (b) Show that $\mathrm{Hom}_R(M, L)$ becomes an $R(G)$-module on defining $(gu)(n) = gu(g^{-1}m)$ for $g \in G, u \in \mathrm{Hom}_R(M, L), m \in M$.

   (c) Show that evaluation $ev_M : M \otimes_R \mathrm{Hom}_R(M, L) \to L$ is an $R(G)$-module morphism.

   (d) Prove that the evaluation induces an isomorphism of R-modules

   $$\mathrm{Hom}_{R(G)}\big(N, \mathrm{Hom}_R(M, L)\big) \cong \mathrm{Hom}_{R(G)}\big(M \otimes_R N, L\big)$$

7
Coalgebras and bialgebras

Let R be any ring. By a *coalgebra* over R (or R-*coalgebra*) we mean a module $C : R \nrightarrow R$ together with module morphisms

$$\delta : C \longrightarrow C \otimes_R C \qquad \text{and} \qquad \varepsilon : C \longrightarrow R$$

such that

$$\begin{tikzcd} C \arrow[r, "\delta"] & C \otimes_R C \arrow[r, shift left, "\delta \otimes 1_X"] \arrow[r, shift right, "1_X \otimes \delta"'] & C \otimes_R C \otimes_R C \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, "\delta"] \arrow[rr, bend right, "1_C"'] & C \otimes_R C \arrow[r, shift left, "\varepsilon \otimes 1_X"] \arrow[r, shift right, "1_X \otimes \varepsilon"'] & C \end{tikzcd} .$$

We call δ the *comultiplication* and ε the *counit.* This structure provides a module with "formal diagonals". There is a uniquely determined

$$\delta : C \longrightarrow C \otimes_R C \otimes_R \cdots \otimes_R C$$

where for each $c \in C$ we have $\delta(c) = \sum_i c_{1i} \otimes \cdots \otimes c_{ni}$. The notation

$$\delta(c) \;=\; \sum_{(c)} c_{(1)} \otimes \cdots \otimes c_{(n)}$$

is sometimes used even though the representation of $\delta(c)$ in the tensor product is not uniquely determined—we act as though a choice of this representation has been made for each $c \in C$. Given a multilinear function $f : C \times \cdots \times C \longrightarrow A$ we also write

$$f\big(\delta(c)\big) \;=\; \sum_{(c)} f(c_{(1)}, \ldots, c_{(n)}) \; .$$

In terms of this notation the axioms can be rewritten as

$$\sum_{(c)} \delta(c_{(1)})\otimes c_{(2)} = \sum_{(c)} c_{(1)}\otimes c_{(2)}\otimes c_{(3)} = \sum_{(c)} c_{(1)}\otimes\delta(c_{(2)})$$
$$c = \sum_{(c)} \varepsilon(c_{(1)})\otimes c_{(2)} = \sum_{(c)} c_{(1)}\otimes\varepsilon(c_{(2)})\ .$$

Suppose C and D are coalgebras. A *coalgebra morphism* $f : C \longrightarrow D$ is a module morphism such that

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\delta"'] & D \arrow[d, "\delta"] \\ C\otimes_R C \arrow[r, "f\otimes f"] & D\otimes_R D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\varepsilon"'] & & D \arrow[dl, "\varepsilon"] \\ & R & \end{tikzcd}$$

that is,

$$\sum_{(c)} f(c_{(1)})\otimes f(c_{(2)}) = \sum_{(f(c))} f(c)_{(1)}\otimes f(c)_{(2)}$$
$$\varepsilon(f(c)) = \varepsilon(c)\ .$$

We write $\mathbf{Cog}_R(C\,,D\,)$ for the set of coalgebra morphisms from C to D.

Suppose R is commutative. A coalgebra C over R is *cocommutative* when

$$\begin{tikzcd} & C \arrow[dl, "\delta"'] \arrow[dr, "\delta"] & \\ C\otimes_R C \arrow[rr, "\sigma"] & & C\otimes_R C\ . \end{tikzcd}$$

Return now to a general ring R. Suppose that A is an R-algebra and C is an R-coalgebra. Then $\mathrm{Hom}_R(C\,,A\,)$ becomes an R-algebra under the following *convolution structure*.

$$\begin{tikzcd} \mathrm{Hom}_R(C\,,A\,)\otimes_R\mathrm{Hom}_R(C\,,A\,) \arrow[d, "-\otimes-"] \\ \mathrm{Hom}_R(C\otimes_R C\,,A\otimes_R A\,) \arrow[d, "\mu\circ-\circ\delta"] \\ \mathrm{Hom}_R(C\,,A\,) \end{tikzcd} \qquad \begin{tikzcd} R \arrow[d, "\cong"] \\ \mathrm{Hom}_R(R\,,R\,) \arrow[d, "\eta\circ-\circ\varepsilon"] \\ \mathrm{Hom}_R(C\,,A\,) \end{tikzcd}$$

In terms of elements, for left R-module morphisms $f\,,\,g\,:\,C \longrightarrow A$ their *convolution product* is given by

$$f * g \;=\; \mu \circ (f \otimes g) \circ \delta \qquad \text{and} \qquad 1 \;=\; \eta \circ \varepsilon\ .$$

Using the notation for comultiplication this becomes the formula

$$(f * g)(c) \;=\; \sum_{(c)} f(c_{(1)})\, g(c_{(2)})\ .$$

In particular, with $A = R$ each R-coalgebra C gives rise to a convolution R-algebra structure on the dual $C^* = \mathrm{Hom}_R(C\,,R)$. However, we prefer to regard C^* as an R-algebra via the multiplication $\mu\,:\,C^* \otimes_R C^* \longrightarrow C^*$ defined by

$$\begin{tikzcd}
C^* \otimes_R C^* \otimes_R C \arrow[rr, "\mu \otimes 1"] \arrow[d, "1 \otimes 1 \otimes \delta"'] & & C^* \otimes_R C \arrow[d, "e"] \\
C^* \otimes_R C^* \otimes_R C \otimes_R C \arrow[dr, "1 \otimes e \otimes 1"'] & & R \\
& C^* \otimes_R C \arrow[ur, "e"'] &
\end{tikzcd}$$

(This works even for non-commutative R.)

Example 7.1 *Suppose $\mathcal{X}$ is any category which admits finite products. Suppose $F\,:\,\mathcal{X} \longrightarrow \mathbf{Mod}_R^R$ is a functor into the category of modules from R to R. Suppose there are natural module morphisms*

$$\phi_{X_1,\dots,X_n}\,:\,F(X_1 \times \cdots \times X_n) \longrightarrow FX_1 \otimes_R FX_2 \otimes_R \cdots \otimes_R FX_n$$

compatible with the canonical associativity isomorphisms for product and tensor product. Then for each object X of $\mathcal{X}$ we obtain a coalgebra FX, with comultiplication and counit

$$FX \xrightarrow{F\delta} F(X \times X) \xrightarrow{\phi_{X,X}} FX \otimes_R FX$$

$$FX \xrightarrow{F\varepsilon} F1 \xrightarrow{\phi_0} R\ .$$

If furthermore R is commutative and F is compatible with the twists, then this coalgebra is cocommutative.

sub-example (a). (R commutative) The free R-module construction gives a functor

$$\mathcal{F}_R : \mathbf{Set} \longrightarrow \mathbf{Mod}_R$$

from the category of sets to $\mathbf{Mod}_R$. We have isomorphisms

$$\begin{array}{rccc} \phi \; : & \mathcal{F}_R(X_1 \times \cdots \times X_n) & \xrightarrow{\cong} & \mathcal{F}_R X_1 \otimes_R \cdots \otimes_R \mathcal{F}_R X_n \\ & (x_1 \,, \ldots , x_n) & \longmapsto & x_1 \otimes \cdots \otimes x_n \;. \end{array}$$

Proof. ($n = 2$)

$$\begin{aligned} \mathrm{Hom}_R\big(\mathcal{F}_R(X \times Y)\,, M\big) &\cong M^{X \times Y} \cong (M^X)^Y \\ &\cong \mathrm{Hom}_R(\mathcal{F}_R Y\,, M^X) \\ &\cong \mathrm{Hom}_R\big(\mathcal{F}_R Y\,, \mathrm{Hom}(\mathcal{F}_R X\,, M)\big) \\ &\cong \mathrm{Hom}_R(\mathcal{F}_R X \otimes \mathcal{F}_R Y\,, M)\;. \end{aligned}$$

So *each* $\mathcal{F}_R X$ *becomes an* R*-coalgebra.*

sub-example (b). The universal enveloping algebra provides a functor

$$\mathcal{U} : \mathbf{Lie}_R \longrightarrow \mathbf{Mod}_R$$

and we have already observed

$$\begin{array}{rcl} \mathcal{U}(L \oplus L') & \cong & \mathcal{U}(L) \otimes_R \mathcal{U}(L') \qquad \text{and} \qquad \mathcal{U}(\{0\}) \;\cong\; R \\ (x, x') & \longmapsto & x \otimes 1 + 1 \otimes x' \end{array}$$

Since direct sum is product in $\mathbf{Lie}_R$ we have another standard example. Thus *each universal enveloping algebra* $\mathcal{U}(L)$ *becomes an* R*-coalgebra.* The comultiplication here is determined by

$$\begin{array}{ccccc} L & \xrightarrow{i} & \mathcal{U}(L) & \xrightarrow{\delta} & \mathcal{U}(L) \otimes \mathcal{U}(L) \\ x & \longmapsto & & & x \otimes 1 + 1 \otimes x \end{array}$$

Subexamples (a), (b) suggest two definitions that we can make for any coalgebra C .

- Say that $c \in C$ is *set-like* when $\delta(c) = c \otimes c$ and $\varepsilon(c) = 1$. (In the case of $C = \mathcal{F}_R(X)$ the set-like elements are precisely the elements of X .) Write $\mathcal{D}(C)$ for the set of set-like elements of C .

- Say that $c \in C$ is *primitive* when

$$\delta(c) = c \otimes 1 + 1 \otimes c \ .$$

(In the case of $C = \mathcal{U}(L)$ each element of L is primitive.)
Write $\mathcal{P}(C)$ for the submodule of primitive elements of C.

Proposition 7.2 *(with R a field.) The set-like elements of any coalgebra C form a linearly independent subset $\mathcal{D}(C)$.*

Proof. Suppose $\mathcal{D}(C)$ is linearly dependent. Let $n+1$ be the first natural number for which there is a linearly dependent subset of $\mathcal{D}(C)$ with that many elements. Then any set of n elements of $\mathcal{D}(C)$ must necessarily be linearly independent, while there exist distinct $g, g_1, \ldots, g_n \in \mathcal{D}(C)$ which are linearly dependent. Then we can write

$$g \ = \ \lambda_1 g_1 + \cdots + \lambda_n g_n$$

with the $\lambda_i \in R$ all non-zero. Then

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i \, g_i \otimes g_i \quad &= \quad \sum_{i=1}^{n} \lambda_i \, \delta(g_i) \quad = \quad \delta(g) \quad = \quad g \otimes g \\ &= \quad \sum_{i,j=1}^{n} \lambda_i \lambda_j \, g_i \otimes g_j \ . \end{aligned}$$

Since $\{g_1, \ldots, g_n\}$ is linearly independent in C then $\{g_i \otimes g_j\}$ is linearly independent in $C \otimes C$, so we can equate coefficients: $\lambda_i \lambda_j = 0$ for $i \neq j$ and $\lambda_i = \lambda_i^2$. Since $\lambda_i \neq 0$ this means $n = 1$ and $\lambda_i = 1$. But $g = g_1$ was not allowed.

We shall come back to set-like and primitive elements in the context of bialgebras.

Example 7.3 *(with R commutative.) Let $C = \mathcal{F}_R \mathbb{N}$ be the free R-module on the countable set $\mathbb{N}$. Define*

$$\delta(n) \ = \ \sum_{p+q=n} p \otimes q \qquad \text{and} \qquad \varepsilon(n) \ = \ \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n > 0 \ . \end{cases}$$

This defines a cocommutative coalgebra structure on C.

Take an R-algebra A and look at an example of convolution with this coalgebra C. The convolution structure transports across the R-module isomorphism

$$\mathrm{Hom}_R(C, A) \ \cong \ A^{\mathbb{N}} \qquad (\text{sequences in } A)$$

to give the multiplication

$$a\,b \;=\; \Big(\sum_{p+q=n} a_p\, b_q\Big)$$

for sequences $a = (a_n) = (a_0\,, a_1\,, \ldots)$ *and* $b = (b_n) = (b_0\,, b_1\,, \ldots)$ *in* A. *The unit sequence is* $(1\,,0\,,0\,,0\,,\ldots)$. *A precise definition of* indeterminate *can be taken to mean the sequence*

$$x \;=\; (0\,,1\,,0\,,0\,,0\,,0\,,\ldots) \;\in\; A^{\mathbb{N}}$$

in A. *Each* $u \in A$ *is identified with* $u\,1 = (u\,,0\,,0\,,0\,,\ldots) \in A^{\mathbb{N}}$. *Then each* $a \in A^{\mathbb{N}}$ *can be written as a* formal (*no convergence requirements!*) power series

$$a \;=\; \sum_{n=0}^{\infty} a_n\, x^n\;.$$

Write $A[\![x]\!]$ *for* $A^{\mathbb{N}}$ *with this algebra structure. It is the* R-algebra of formal power series in A. *If* A *is commutative so is* $A[\![x]\!]$. *In particular when* $A = R$ *we obtain the commutative* R*-algebra* $C^* = R[\![x]\!]$.

Example 7.4 *Let* $\mathbf{n} = \{1, 2, \ldots, n\}$ *and put* $C = \mathcal{F}_R(\mathbf{n}\times\mathbf{n})$. *Then* C *becomes an* R*-coalgebra on defining*

$$\delta(i\,,j) \;=\; \sum_{k=1}^{n}(i\,,k)\otimes(k\,,j) \qquad\text{and}\qquad \varepsilon(i\,,j) \;=\; \begin{cases}1 & \text{for } i=j\\ 0 & \text{otherwise}\,.\end{cases}$$

Given any R*-algebra* A, *the convolution structure simply transports across the* R*-module isomorphism*

$$\mathrm{Hom}_R(C\,,A) \;\cong\; A^{\mathbf{n}\times\mathbf{n}} \qquad (\,n\times n \text{ matrices in } A\,)$$

to give the usual matrix multiplication

$$(a_{ij})\,(b_{ij}) \;=\; \Big(\sum_{k=1}^{n} a_{ik}\, b_{kj}\Big)\;.$$

In this way we obtain the R*-algebra* $\mathbf{Mat}(n\,,A) \cong \mathrm{End}_R(A^n)$ *of* $n\times n$ *matrices with entries in* A.

Suppose that R is a commutative ring. An *R-bialgebra* is an R-module B together with algebra and coalgebra structures

$$\mu : B\otimes_R B \longrightarrow B \quad\text{and}\quad \eta : R \longrightarrow B$$
$$\delta : B \longrightarrow B\otimes_R B \quad\text{and}\quad \varepsilon : B \longrightarrow R$$

satisfying the conditions

$$\begin{tikzcd}
B\otimes_R B \arrow[rr, "\mu"] \arrow[d, "\delta\otimes\delta"'] & & B \arrow[d, "\delta"] \\
B\otimes_R B\otimes_R B\otimes_R B \arrow[r, "\cong", "1\otimes\sigma\otimes 1"'] & B\otimes_R B\otimes_R B\otimes_R B \arrow[r, "\mu\otimes\mu"] & B\otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
B\otimes B \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "\mu"'] & R\otimes R \arrow[r, "\eta\otimes\eta"] \arrow[d, "\cong"] & B\otimes B \\
B \arrow[r, "\varepsilon"] & R \arrow[r, "\eta"] & B \arrow[u, "\delta"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & B \arrow[dr, "\varepsilon"] & \\
R \arrow[ur, "\eta"] \arrow[rr, "1_R"] & & R
\end{tikzcd}$$

Notice the complete duality between $(\mu\,,\eta\,)$ and $(\delta\,,\varepsilon\,)$. When expressed in terms of elements the duality is not so apparent:

$$\delta(x\,,y) \;=\; \sum_{(x)}\sum_{(y)} x_{(1)}y_{(1)}\otimes x_{(2)}y_{(2)}$$

$$\varepsilon(x\,y) \;=\; \varepsilon(x)\,\varepsilon(y) \quad \text{and} \quad \delta(1) \;=\; 1\otimes 1 \quad \text{and} \quad \varepsilon(1) \;=\; 1\;.$$

For R commutative, the tensor product $A\otimes_R A'$ of R-algebras A and A' becomes an R-algebra via the multiplication

$$(A\otimes_R A')\otimes_R(A\otimes_R A') \xrightarrow[\substack{1\otimes\sigma\otimes 1\\ \sigma_{1324}}]{\cong} (A\otimes_R A)\otimes_R(A'\otimes_R A') \xrightarrow{\mu\otimes\mu} A\otimes_R A'$$

and unit

$$R \;\cong\; R\otimes_R R \xrightarrow{\eta\otimes\eta} A\otimes_R A' \quad .$$

Also the tensor product $C\otimes_R C'$, of R-coalgebras C and C', becomes an R-coalgebra via the comultiplication

$$C\otimes_R C' \xrightarrow{\delta\otimes\delta} (C\otimes_R C)\otimes_R(C'\otimes_R C') \xrightarrow[1\otimes\sigma\otimes 1]{\cong} (C\otimes_R C')\otimes_R(C\otimes_R C')$$

and counit

$$C\otimes_R C' \xrightarrow{\varepsilon\otimes\varepsilon} R\otimes_R R \;\cong\; R \quad .$$

With this, we can make the observation:

Proposition 7.5 *Suppose* (μ, η) *and* (δ, ε) *are respectively, algebra and coalgebra structures on the* R*-module* B*. Then the following conditions are equivalent:*

(i) B *is a bialgebra*;

(ii) $\mu : B \otimes_R B \longrightarrow B$ *and* $\eta : R \longrightarrow B$ *are coalgebra morphisms*;

(iii) $\delta : B \longrightarrow B \otimes_R B$ *and* $\varepsilon : B \longrightarrow R$ *are algebra morphisms.*

For bialgebras B and B' a *bialgebra morphism* $f : B \longrightarrow B'$ is a function which is both an algebra and coalgebra morphism. Write $\mathbf{Big}_R(B, B')$ for the set of such functions f.

Before giving examples of bialgebras we prove some extra results on the set-like and primitive elements for the bialgebra case.

Proposition 7.6 *If* B *is a bialgebra then the set-like elements are closed under multiplication: so* $\mathcal{D}(B)$ *becomes a monoid.*

Proof. $\delta(bb') = \delta(b)\,\delta(b') = (b \otimes b)(b' \otimes b') = bb' \otimes bb'$ for $b, b' \in \mathcal{D}(B)$; also $\varepsilon(bb') = \varepsilon(b)\,\varepsilon(b') = 1 \cdot 1 = 1$.

Proposition 7.7 *If* B *is a bialgebra then the set of primitive elements is closed under commutator, so* $\mathcal{P}(B)$ *becomes a Lie algebra. Also* $\varepsilon(x) = 0$ *for all* $x \in \mathcal{P}(B)$*.*

Proof. For $x, y \in \mathcal{P}(B)$ we have

$$\begin{aligned}
\delta([x, y]) &= \delta(x)\,\delta(y) - \delta(y)\,\delta(x) \\
&= (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) - (y \otimes 1 + 1 \otimes y)(x \otimes 1 + 1 \otimes x) \\
&= xy \otimes 1 + x \otimes y + y \otimes x + 1 \otimes xy - (yx \otimes 1 + y \otimes x + x \otimes y + 1 \otimes yx) \\
&= [x, y] \otimes 1 + 1 \otimes [x, y]
\end{aligned}$$

so that $[x, y] \in \mathcal{P}(B)$. Also $x = (1 \otimes \varepsilon)\,\delta(x) = (1 \otimes \varepsilon)\,(x \otimes 1 + 1 \otimes x) = x + \varepsilon(x)$; hence $\varepsilon(x) = 0$.

Example 7.8 *Return to the situation of coalgebras in example 7.1. There are two conditions on the functor* $F : \mathcal{X} \longrightarrow \mathbf{Mod}_R$ *which gives rise to bialgebras* FX*.*

(a) When the morphisms ϕ are all invertible.
Then F takes each monoid G in $\mathcal{X}$ to a bialgebra FG. The multiplication and unit for G give an algebra structure

$$FG \otimes_R FG \;\cong\; F(G \times G) \xrightarrow{F\mu} FG \quad \text{and} \quad R \;\cong\; F1 \xrightarrow{F\eta} FG$$

on FG. These are coalgebra morphisms since all arrows in $\mathcal{X}$ "commute with diagonals". By proposition 7.5, each FG becomes a bialgebra. This is the situation for the functor $\mathcal{F}_R : \mathbf{Set} \longrightarrow \mathbf{Mod}_R$, so for each monoid G the monoid algebra $R(G)$ is a cocommutative bialgebra. *Notice here that $G \cong D\big(R(G)\big)$ as monoids (see proposition 7.6).*

(b) When F lifts to $F : \mathcal{X} \longrightarrow \mathbf{Alg}_R$. *In this case each FX is clearly a bialgebra since the comultiplication and counit are algebra morphisms (proposition 7.5). For the functor $\mathcal{U} : \mathbf{Lie}_R \longrightarrow \mathbf{Alg}_R$ this is indeed the situation. Thus we have that* each universal enveloping algebra $\mathcal{U}(L)$ is a cocommutative bialgebra.

Example 7.9 *Return to example 7.3 of a coalgebra. This time, to use the symbol $\mathbb{N}$ to denote our countable set would be confusing. Instead we denote it by $E = \{e_0, e_1, e_2, e_3, \dots\}$. Then the coalgebra structure on $\mathcal{F}_R(E)$ is*

$$\delta(e_n) \;=\; \sum_{p+q=n} e_p \otimes e_q \qquad \text{and} \qquad \varepsilon(e_n) \;=\; \begin{cases} 0 & \text{for } n > 0\,, \\ 1 & \text{for } n = 0\,. \end{cases}$$

We now make $\mathcal{F}_R(E)$ into an algebra via

$$e_p\, e_q \;=\; \frac{(p+q)!}{p!\, q!}\, e_{p+q} \qquad \text{with} \qquad e_0 = 1\,.$$

(The binomial coefficient is an integer and so "lives" in any ring R.) Then $\mathcal{F}_R(E)$ is a bialgebra. If R is a field of characteristic 0 *(i.e. $1 + \dots + 1 \neq 0$ in R for any non-zero number of terms) put $x = e_1$ so that one sees that $e_n = \frac{1}{n!}\, x^n$. Hence, as an algebra, $\mathcal{F}_R(E)$ is isomorphic to the* polynomial algebra $R[x]$ *in one variable. For general R we can think of $\mathcal{F}_R(E)$ as the algebra of* Hurwitz polynomials *in one indeterminate:*

$$\sum_{n=0}^{k} \frac{a_n\, x^n}{n!} \qquad \text{with each} \qquad a_n \in R\,.$$

Example 7.10 *Return to example 7.4 and form the symmetric algebra $\mathcal{S}(C)$ of the coalgebra $C = \mathcal{F}_R(\mathbf{n} \times \mathbf{n})$. Since using $\mathbf{n} \times \mathbf{n}$ can be confusing we replace it by any set $X = \{x_{ij} \mid i, j \in \mathbf{n}\}$ of cardinality n^2. Then we identify $S(\mathcal{F}_R(X))$ with the* polynomial R-algebra $R[(x_{ij})]$ in n^2 *commuting* *indeterminates* x_{ij} *for* $i\,,j \in \mathbf{n}$ *. In example 7.1 we saw that this becomes a bialgebra by virtue of the fact that it is the universal enveloping algebra of a commutative Lie algebra* $\mathcal{F}_R(X)$*, but this is not the structure of interest here. The coalgebra* C *induces the bialgebra structure*

$$\delta(x_{ij}) \;=\; \sum_k x_{ik}{\otimes}x_{kj} \qquad \text{and} \qquad \varepsilon(x_{ij}) \;=\; \begin{cases} 1 & \text{for} \;\; i=j \\ 0 & \text{for} \;\; i\neq j \end{cases}$$

which we call the matrix bialgebra $M(n)$ over R *. This must not be confused with the matrix algebra*

$$\mathbf{Mat}(n\,,R) \;\cong\; \mathbf{Alg}_R(M(n)\,,R\,)$$

(which is the algebra of "points" of $M(n)$ *).*

Exercises

1. For any R-coalgebra C, prove the following identities:

   (a) $\delta(c) = \sum_{(c)} \varepsilon(c_{(2)}){\otimes}\delta(c_{(1)}) = \sum_{(c)} \delta(c_{(2)}){\otimes}\varepsilon(c_{(1)}) \sum_{(c)} c_{(1)}{\otimes}\varepsilon(c_{(3)}){\otimes}c_{(2)}$

   (b) $\sum_{(c)} \varepsilon(c_{(1)}){\otimes}c_{(3)}{\otimes}c_{(2)} = \sum_{(c)} c_{(2)}{\otimes}c_{(1)}$

   (c) $\sum_{(c)} \varepsilon(c_{(1)}){\otimes}\varepsilon(c_{(3)}){\otimes}c_{(2)} = c$

8

Dual coalgebras of algebras

We have seen that the dual C^* of a coalgebra has a natural structure of an algebra. One might expect the dual A^* of an algebra to be a coalgebra in an obvious way, but this is not true because of the failure of the canonical morphism

$$M^* \otimes_R N^* \longrightarrow (M \otimes_R N)^*$$

to be always invertible. If M is cauchy the morphism is invertible since

$$\begin{aligned} (M \otimes_R N)^* &= \mathrm{Hom}_R(M \otimes_R N, R) \\ &\cong \mathrm{Hom}_R(M, N^*) \cong M^* \otimes_R N^* . \end{aligned}$$

So for an algebra A which is cauchy (as a module) we obtain a coalgebra, denoted by A^*, via

$$\begin{aligned} \delta &: A^* \xrightarrow{\mu} (A \otimes_R A)^* \cong A^* \otimes_R A^* \\ \varepsilon &: A^* \xrightarrow{\eta^*} R^* \cong R . \end{aligned}$$

However instead of restricting A, which is unsatisfactory since many of the examples are not cauchy, we modify the definition of the dual A^*.

Let's call an ideal I of an algebra A *cocauchy* when the quotient algebra A/I is cauchy (as a module). Define

$$A^0 = \{ u \in A^* \mid u \text{ is zero on some cocauchy ideal of } A \} .$$

Proposition 8.1 *(with R a field.)*

(a) *A^0 is a submodule of A^*.*

(b) *If $f \in \mathbf{Alg}_R(A, B)$ then $f^* : B^* \longrightarrow A^*$, given by composition with f, takes B^0 into A^0.*

(c) *For any R-algebra B the canonical morphism $A^* \otimes B^* \longrightarrow (A \otimes B)^*$ induces an isomorphism $A^0 \otimes B^0 \xrightarrow{\cong} (A \otimes B)^0$.*

(d) *There exists a unique* $\delta : A^0 \longrightarrow A^0 \otimes A^0$ *satisfying*

$$\begin{tikzcd}
A^0 \arrow[r, "\delta"] \arrow[d, hook] & A^0 \otimes A^0 \arrow[r, "\cong"] & (A\otimes A)^0 \arrow[dl, hook] \\
A^* \arrow[r, "\mu^*"] & (A\otimes A)^* . &
\end{tikzcd}$$

Proof. (a) If $u \in A^0$ and $r \in R$ then $\ker(r\,u) \supseteq \ker u$, so that $r\,u \in A^0$. Take $u\,, v \in A^0$ zero on cocauchy ideals I and J respectively. We can find subspaces U, V and W of A with $A = (I \cap J) \oplus U \oplus V \oplus W$ and $I = (I \cap J) \oplus U$ and $J = (I \cap J) \oplus V$. So $A/I \cong V \oplus W$ and $A/J \cong U \oplus W$ are finite dimensional. Thus $A/I \cap J \cong U \oplus V \oplus W$ is finite dimensional. Hence $I \cap J$ is a cocauchy ideal on which $u + v$ is zero.

(b) Take $v \in B^0$ zero on cocauchy J in B. Then $f^{-1}(J) \subseteq \ker(vf) = \ker f^*(v)$ is an ideal of A; but $f^{-1}(J)$ is the kernel of $A \xrightarrow{f} B \longrightarrow B/J$ so that $A/f^{-1}(J)$ is isomorphic to a subspace of B/J. So $f^{-1}(J)$ is cocauchy.

(c) We shall use the following ...

Exercise. *(with R a field.) When $f : M \longrightarrow N$ is an injective module morphism then $f\otimes_R 1 : M\otimes_R L \longrightarrow N\otimes_R L$ is injective. Furthermore we have that $M^*\otimes_R N^* \longrightarrow (M\otimes_R N)^*$ is injective.*

Before beginning the proof of proposition 8.1(c) notice that for any cocauchy ideal K in $A\otimes B$ we have cocauchy ideals

$$\begin{aligned}
I &= \{a \in A \mid a\otimes 1 \in K\} &&= (\, A \xrightarrow{\ -\otimes 1\ } A\otimes B\,)^*(K) \\
J &= \{b \in B \mid 1\otimes b \in K\} &&= (\, B \xrightarrow{\ 1\otimes -\ } A\otimes B\,)^*(K) \\
A\otimes_R J + I\otimes_R B &&&= \ker(\, A\otimes_R B \longrightarrow (A/I)\otimes_R(B/J)\,)
\end{aligned}$$

of A and B and $A\otimes B$ respectively.

Now take $w \in (A\otimes B)^0$ which is zero on some cocauchy K as above. Then w is zero on $A\otimes J + I\otimes B \subseteq K$. However $A\otimes B/(A\otimes J + I\otimes B) \cong A/I \otimes B/J$ so there exists a unique $\bar{w}$:

$$\begin{tikzcd}
A\otimes B \arrow[rr, "w"] \arrow[dr, "\rho\otimes\rho"'] & & R \\
& (A/I)\otimes(B/J) \arrow[ur, "\bar{w}"'] &
\end{tikzcd}$$

Furthermore since A/I and B/J are finite dimensional, so that we have $(A/I)^*\otimes(B/J)^* \xrightarrow{\ \cong\ } (A/I\otimes B/J)^*$ is invertible, there is some element

$\sum \bar{h}_i \otimes \bar{k}_i \in (A/I)^* \otimes (B/J)^*$ corresponding to $\bar{w}$. In particular, for $x \in A/I$ and $y \in B/J$ we have that

$$\bar{w}(x \otimes y) \quad = \quad \sum_i \bar{h}_i(x)\, \bar{k}_i(y) \quad . \qquad (*)$$

Define the composite $h_i = \bar{h}_i \circ \rho \; : A \longrightarrow A/I \longrightarrow R$ and similarly define $k_i = \bar{k}_i \circ \rho \; : B \longrightarrow B/J \longrightarrow R$. These are in A^0 and B^0 since they are zero on I and J respectively. Hence we have

$$\sum_i h_i \otimes k_i \quad \in \quad A^0 \otimes B^0$$

which is the image of $w \in (A \otimes B)^0$ because of $(*)$.

Conversely, if $h \in A^0$ and $k \in B^0$ vanish on cocauchy I and J (ideals of A and B) then $h \otimes k$ vanishes on $A \otimes J + I \otimes B$ which is a cocauchy ideal of $A \otimes B$.

(d) Suppose $u \in A^0$ vanishes on a cocauchy ideal I. Then $\mu^*(u)(a \otimes b) = (u\mu)(a \otimes b) = u(ab)$, so $\mu^*(u)$ vanishes on $A \otimes I + I \otimes A$ which is a cocauchy ideal of $A \otimes A$. Hence μ^* takes A^0 into $(A \otimes A)^0$ and δ exists as desired.

Corollary 8.2 *(with R a field.) For each algebra A a coalgebra structure on A^0 is given by the δ in 8.1(d) and $\varepsilon = (A^0 \hookrightarrow A^* \xrightarrow{\eta^*} R^* \cong R)$. Also each algebra morphism $f : A \longrightarrow B$ induces a coalgebra morphism $f^0 : B^0 \longrightarrow A^0$ given by restriction of f^* (see proposition 8.1(b)).*

Proof. Draw the diagrams expressing the axioms on μ, η and f. Simply apply $(\_)^*$ then restrict to $(\_)^0$.

Exercise 8.3 *An algebra morphism $f : A \longrightarrow R$ is a set-like element of A^0.*

For each algebra A we obtain a left and right A-module structure on A^* given as follows, for $a \in A$ and $u \in A^*$:

$$\begin{aligned} (au)(x) &= u(xa) \ , \\ (ua)(x) &= u(ax) \ . \end{aligned}$$

In fact $A^* : A \nrightarrow A$. For any $f \in A^*$ write

$$Af \quad \text{and} \quad fA \quad \text{and} \quad AfA$$

for the R-submodules of A^* consisting of those elements of the form af and fa and afb respectively with $a, b \in A$.

Proposition 8.4 *(with R a field.) For $f \in A^*$ these are equivalent:*

(1) $f \in A^0$;
(2) $\mu^*(f)$ *is in the image of* $A^* \otimes A^* \longrightarrow (A \otimes A)^*$;
(3) $\mu^*(f)$ *is in the image of* $(A \otimes A)^0 \hookrightarrow (A \otimes A)^*$;
(4) Af *is cauchy* ;
(5) fA *is cauchy* ;
(6) AfA *is cauchy.*

Proof. (1) $\Rightarrow$ (3) by proposition 8.1(d). Also (3) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (4). Let $\mu^*(f)$ be the image of $\sum_i u_i \otimes v_i \in A^* \otimes A^*$. Then we have that $f(ab) = \sum_i u_i(a)\, v_i(b)$ so $bf = \sum_i v_i(b)\, u_i \in A^*$. Thus bf is in the subspace of A^* spanned by the u_i . Hence Af is finite dimensional.

(4) $\Rightarrow$ (1). Suppose Af is finite dimensional. Then also $\mathrm{End}_R(Af)$ is finite dimensional, so the kernel I of the morphism $A \longrightarrow \mathrm{End}_R(Af)$ given by $a \longmapsto (bf \longmapsto bfa)$, is a cocauchy ideal of A . But $a \in I$ implies $1fa = 0$, so $f(a) = 0$. Hence f is zero on I so that $f \in A^0$.

(5) $\Rightarrow$ (1) is similar to (4) $\Rightarrow$ (1) and (6) $\Rightarrow$ (5) is trivial.

(1) $\Rightarrow$ (6). Take $f \in A^0$ zero on the cofinite ideal I . Then for $c \in I$ we have $(a\,f\,b)(c) = f(b\,c\,a) = 0$. Thus $A\,fA \subseteq I^\perp = \{u \in A^* \mid u(I) = 0\} \cong (A/I)^*$ which is finite dimensional since A/I is so.

Corollary 8.5 *For any coalgebra C the canonical injection $d : C \longrightarrow C^{**}$ given by $d(c)(u) = u(c)$, has image in $(C^*)^0$.*

Proof. Take $c \in C$. Then $C^* d(c) = \{u\, d(c) \mid u \in C^*\} \subseteq C^{**}$. Now $(u\, d(c))(v) = d(c)(v * u) = (v * u)(c) = (v \otimes u)\delta(c) = (v \otimes u) \sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{(c)} v(c_{(1)})\, u(c_{(2)})$ using the definition of multiplication $v * u$ in C . Thus $u\, d(c) = \sum_{(c)} u(c_{(2)})\, d(c_{(1)})$, which is in the subspace of C^{**} spanned by the $d(c_{(1)})$. Hence $C^*\, d(c)$ has finite dimension and by using (4)$\Leftrightarrow$(1) of proposition 8.4 it follows that $d(c) \in (C^*)^0$.

Theorem 8.6 *(with R a field.) For all algebras A and coalgebras C there is a bijection*

$$\begin{array}{ccc} \mathbf{Alg}_R(A, C^*) & \cong & \mathbf{Cog}_R(C, A^0) \\ \left(A \xrightarrow{f} C^*\right) & \mapsto & \left(C \xrightarrow{d} (C^*)^0 \xrightarrow{f^0} A^0\right) . \end{array}$$

Proof. The inclusion $i : A^0 \longrightarrow A^*$ induces $i^* : A^{**} \longrightarrow A^{0*}$, while the inverse to $f \longmapsto f^0 \circ d$ takes $g \in \mathbf{Cog}_R(C, A^0)$ to the composite

$$A \xrightarrow{d} A^{**} \xrightarrow{i^*} A^{0*} \xrightarrow{g^*} C^* .$$

The remaining details are left to the reader.

9

Hopf algebras

Our base ring R will always be assumed commutative, and whenever $(\ )^0$ appears we happily suppose it to be a field.

An *R-Hopf algebra* is an R-bialgebra H together with R-module morphism

$$\nu : H \longrightarrow H$$

called the *antipode*, which satisfies the following diagram.

$$\begin{tikzcd} H \arrow[r, "\delta"] \arrow[drr, "\varepsilon"'] & H{\otimes}H \arrow[r, shift left, "\nu\otimes 1"] \arrow[r, shift right, "1\otimes\nu"'] & H{\otimes}H \arrow[r, "\mu"] & H \\ & & R \arrow[ur, "\eta"'] & \end{tikzcd}$$

For any Hopf algebra H let H^{op} denote the Hopf algebra obtained by replacing μ with $\mu \circ \sigma : H{\otimes}H \xrightarrow{\sigma} H{\otimes}H \xrightarrow{\mu} H$ and replacing δ with $\delta \circ \sigma : H \xrightarrow{\delta} H{\otimes}H \xrightarrow{\sigma} H{\otimes}H$ while keeping the same η, ε and ν.

There is also a bialgebra H' obtained more simply by just replacing δ with $\delta \circ \sigma : H \xrightarrow{\delta} H{\otimes}H \xrightarrow{\sigma} H{\otimes}H$ while keeping the same μ, η, ε and ν. In general however, this H' is not a Hopf algebra.

Proposition 9.1 *Let H be a Hopf algebra. Then*

(a) the antipode ν is uniquely determined;

(b) $\nu : H^{\mathrm{op}} \longrightarrow H$ is a bialgebra morphism;

(c) H' is a Hopf algebra if and only if ν is bijective (moreover the antipode for H' is the inverse for ν);

(d) if H is commutative or cocommutative then $\nu \circ \nu = 1_H$ (that is, ν is an involution).

Proof. (a) Since H is a coalgebra and an algebra, we have the convolution algebra structure on $\mathrm{Hom}_R(H, H)$. An antipode is precisely an inverse for $1_H \in \mathrm{Hom}_R(H, H)$ under convolution. For any monoid, inverses are unique.

(b) To show $\nu : H^{\mathrm{op}} \longrightarrow H$ preserves multiplication we must show that

$$\left(H\otimes H \xrightarrow{\mu} H \xrightarrow{\nu} H \right) \;=\; \left(H\otimes H \xrightarrow{\sigma} H\otimes H \xrightarrow{\nu\otimes\nu} H\otimes H \xrightarrow{\mu} H \right) \;.$$

We do this by showing that, under convolution, the left-hand side is a left inverse for $\mu \in \mathrm{Hom}_R(H\otimes H, H)$ while the right-hand side is a right inverse.

$$\begin{tikzcd}
H\otimes H \arrow[dr, "\delta\otimes\delta"] \arrow[ddr, bend right, "\mu"] \arrow[ddd, "\varepsilon\otimes\varepsilon"'] \arrow[ddrrrr, bend left, "(\nu\circ\mu)*\mu"] & & & & \\
& H^{\otimes 4} \arrow[r, "1\otimes\sigma\otimes 1"] \arrow[d, phantom, "\text{bialgebra axiom}"] & H^{\otimes 4} \arrow[d, "\mu\otimes\mu"] \arrow[drr, phantom, "\text{definition of convolution}"] & & \\
& H \arrow[r, "\delta"] \arrow[dl, "\varepsilon"] \arrow[l, phantom, "\text{bialgebra axiom}"] \arrow[d, phantom, "\text{antipode axiom}"] & H^{\otimes 2} \arrow[r, "\nu\otimes 1"] & H^{\otimes 2} \arrow[r, "\mu"] & H \\
R \arrow[urrrr, bend right, "\eta"'] & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
H^{\otimes 2} \arrow[r, "\delta\otimes\delta"] \arrow[dr, "\delta\otimes 1"] \arrow[ddr, "1\otimes\varepsilon"'] \arrow[ddddd, "\varepsilon\otimes\varepsilon"'] & H^{\otimes 4} \arrow[r, "1\otimes\sigma\otimes 1"] & H^{\otimes 4} \arrow[r, "\mu\otimes 1\otimes 1"] & H^{\otimes 3} \arrow[dr, "1\otimes\sigma"] & \\
& H^{\otimes 3} \arrow[r, "1\otimes\sigma"] \arrow[dr, "1\otimes 1\otimes\varepsilon"] & H^{\otimes 3} \arrow[r, "1\otimes\delta\otimes 1"] \arrow[d, "1\otimes\varepsilon\otimes 1"] & H^{\otimes 4} \arrow[r, "\mu\otimes 1\otimes 1"] \arrow[d, "1\otimes 1\otimes\nu\otimes 1"] & H^{\otimes 3} \arrow[d, "1\otimes\nu\otimes 1"] \\
& H \arrow[r, "\delta"] \arrow[ddddl, "\varepsilon"] & H^{\otimes 2} \arrow[dr, "1\otimes\eta\otimes 1"] \arrow[ddr, "1\otimes 1"'] & H^{\otimes 4} \arrow[r, "\mu\otimes 1\otimes 1"] \arrow[d, "1\otimes\mu\otimes 1"] & H^{\otimes 3} \arrow[d, "1\otimes 1\otimes\nu"] \arrow[ddl, "\mu\otimes 1"'] \\
& & & H^{\otimes 3} \arrow[d, "\mu\otimes 1"'] & H^{\otimes 3} \arrow[d, shift right, "\mu\otimes 1"'] \arrow[d, shift left, "1\otimes\mu"] \\
& & & H^{\otimes 2} \arrow[r, "1\otimes\nu"] & H^{\otimes 2} \arrow[d, "\mu"] \\
R \arrow[rrrr, "\eta"] & & & & H
\end{tikzcd}$$

While the second commutativity is perhaps more easily seen by looking at elements, the bonus we get on using diagrams is that, formally reversing all the arrows and replacing μ and η by δ and ε, we have the proof that

$\nu : H^{\mathrm{op}} \longrightarrow H$ preserves comultiplication. The following diagram proves ν preserves unit, while the dual diagram proves ν preserves counit.

$$\begin{tikzcd}
R \arrow[rr, "\eta"] \arrow[dr, "\eta"'] \arrow[drr, "\eta\otimes\eta"] \arrow[dd, "1"'] & & H \arrow[rr, "\nu"] & & H \arrow[dl, "1\otimes\eta"'] \arrow[dd, "1"] \\
& H \arrow[r, "\delta"] \arrow[dl, "\varepsilon"] & H^{\otimes 2} \arrow[r, "\nu\otimes 1"] & H^{\otimes 2} \arrow[dr, "\mu"] & \\
R \arrow[rrrr, "\eta"] & & & & H
\end{tikzcd}$$

(c) ν' is a (composition) right inverse for ν $\Leftrightarrow$ $\nu \circ \nu' = 1_H$ $\Leftrightarrow$ $\nu \circ \nu'$ is a convolution left inverse for ν (since 1_H is the convolution inverse for ν)

$$\Leftrightarrow \quad \begin{tikzcd}
H \arrow[r, "\delta"] \arrow[drr, "\varepsilon"'] & H\otimes H \arrow[r, "1\otimes\nu'"] & H\otimes H \arrow[r, "\nu\otimes\nu"] & H\otimes H \arrow[r, "\mu"] & H \\
& & R \arrow[urr, "\eta"'] & &
\end{tikzcd} \qquad \text{(using (b))}$$

$$\Leftrightarrow \quad \begin{tikzcd}
H \arrow[r, "\delta"] \arrow[drr, "\varepsilon"'] & H\otimes H \arrow[r, "1\otimes\nu'"] & H\otimes H \arrow[r, "\sigma"] & H\otimes H \arrow[r, "\mu"] & H \arrow[r, "\nu"] & H \\
& & R \arrow[urrr, "\eta"'] & & &
\end{tikzcd}$$

$$\Leftrightarrow \quad \begin{tikzcd}
H \arrow[r, "\delta"] \arrow[drr, "\varepsilon"'] & H\otimes H \arrow[r, "\sigma"] & H\otimes H \arrow[r, "\nu'\otimes 1"] & H\otimes H \arrow[r, "\mu"] & H \arrow[r, "\nu"] & H \\
& & R \arrow[urr, "\eta"'] & & &
\end{tikzcd}$$

The last condition is the condition that ν' should be a left convolution inverse for $1_{H'}$ in $\mathrm{Hom}(H', H')$, except that ν is applied to the condition. Similarly, we get that ν' is a left (composition) inverse for ν if and only if ν' satisfies the condition to be a right convolution inverse for $1_{H'}$ in $\mathrm{Hom}(H', H')$ with ν applied to the condition. It follows then that ν and ν' are mutually (composition) inverse precisely when ν' and $1_{H'}$ are mutually convolution inverse; that is, if and only if ν' is an anitpode for H'.

(d) If H is cocommutative then $H' = H$ so that H' is a Hopf algebra with antipode $\nu' = \nu$. So ν is its own (composition) inverse; that is, $\nu \circ \nu = 1_H$. For the commutative case replace H by H^{op}.

Remark. Proposition 9.1(d) can also be seen from the observation that *commutative Hopf algebras* are groups in the opposite of the category of commutative algebras, while cocommutative Hopf algebras are groups in the category of cocommutative coalgebras; the antipode is inversion so is clearly involutory.

Proposition 9.2 *Let H and K be any Hopf algebras. Then each bialgebra morphism $f : H \longrightarrow K$ preserves antipode.*

$$\begin{array}{ccc} H & \xrightarrow{f} & K \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\nu} \\ H & \xrightarrow{f} & K \end{array}$$

Proof. Clearly if $f : D \longrightarrow C$ and $g : A \longrightarrow B$ are coalgebra and algebra morphisms respectively, then

$$\mathrm{Hom}(C, A) \longrightarrow \mathrm{Hom}(D, B) \quad \text{whereby} \quad u \longmapsto g \circ u \circ f$$

is a monoid morphism for the convolution structures. In particular, here we have two monoid morphisms

$$\_ \circ f \quad \text{and} \quad f \circ \_ \;: \mathrm{Hom}(H, H) \longrightarrow \mathrm{Hom}(H, K)$$

that both take 1_H to f. Monoid morphisms take inverses to inverses. So

$$\begin{aligned} \nu \circ f &= \big(\text{ convolution inverse of } f \text{ in } \mathrm{Hom}(H, K)\big) \\ &= f \circ \nu \,. \end{aligned}$$

Using other fancier words, the category $\mathbf{Hopf}_R$ of Hopf algebras is a full subcategory of the category $\mathbf{Big}_R$ of bialgebras.

For any algebra H we have seen that H^0 becomes a coalgebra. If H is a bialgebra then H^0 becomes a bialgebra using the multiplication

$$H^0 \otimes H^0 \;\cong\; (H \otimes H)^0 \xrightarrow{\delta^0} H^0$$

and unit

$$R \;\cong\; R^0 \xrightarrow{\varepsilon^0} H^0$$

(recall). Proposition 8.1). Furthermore, if H is a Hopf algebra then so is H^0 with antipode

$$\nu^0 : \; H^0 \longrightarrow H^0 \;.$$

What we have here is a contravariant "self-adjoint" functor

$$(\_)^0 \;:\; \mathbf{Hopf}_R^{\mathrm{op}} \longrightarrow \mathbf{Hopf}_R \;.$$

What "self-adjoint" means in this context is that

$$\mathbf{Big}_R(H, K^0) \;\cong\; \mathbf{Big}_R(K, H^0) \;.$$

Proposition 9.3 *If H is any Hopf algebra then the monoid $\mathcal{D}(H)$ of set-like elements is a group.*

Proof. For $g \in \mathcal{D}(H)$ we have

$$\begin{aligned} \nu(g)\, g &= \big(\mu \circ (\nu \otimes 1)\big)\,(g \otimes g) \\ &= \big(\mu \circ (\nu \otimes 1)\big)\,\delta(g) \\ &= \big(\mu \circ (\nu \otimes 1) \circ \delta\big)(g) \;=\; \eta\big(\varepsilon(g)\big) \;=\; \eta(1) \;=\; 1 \;. \end{aligned}$$

An *A-point* of a Hopf algebra H is an algebra morphism $f : H \longrightarrow A$.

Proposition 9.4 *(a) If $f\,, g : H \longrightarrow A$ are commuting A-points of H (meaning that $[\,f(h)\,, g(k)\,] = 0$ for all $h\,, k \in H$) then $f * g : H \longrightarrow A$ is an A-point of H.*

(b) If $f : H \longrightarrow A$ is an A-point of H then f has a convolution inverse $f \circ \nu : H^{\mathrm{op}} \longrightarrow A$ which is an A-point of H^{op}.

Proof. (a) The commuting property yields that

$$H \otimes H \xrightarrow{\;f \otimes g\;} A \otimes A \xrightarrow{\;\mu\;} A$$

is an algebra morphism. But $\delta : H \longrightarrow H \otimes H$ is an algebra morphism since H is a bialgebra. So $f * g \in \mathbf{Alg}(H, A)$.

(b) Clear from Proposition 9.1(b).

Example 9.5 *For a monoid G, we have seen that the monoid algebra $R(G)$ is a bialgebra. If G is a group then the group algebra $R(G)$ becomes a Hopf algebra with antipode $\nu : R(G) \longrightarrow R(G)$ given by $\nu(g) = g^{-1}$. (The axioms for $(\_)^{-1} : G \longrightarrow G$ expressed diagramatically in* **Set** *are taken by the functor $\mathcal{F}_R : \mathbf{Set} \longrightarrow \mathbf{Mod}_R$ into the axioms which define the antipode.)*

Example 9.6 *For a Lie algebra L, write L^{op} for the Lie algebra with the same module L but with Lie bracket β^{op} given by $\beta^{\mathrm{op}}(x\,, y\,) = \beta(\,y\,, x\,)$. For any algebra A we have $(A^{\mathrm{op}})_L = (A_L)^{\mathrm{op}}$. It follows (why?) that we have a canonical algebra isomorphism*

$$\mathcal{U}(L^{\mathrm{op}}) \;\cong\; \mathcal{U}(L)^{\mathrm{op}} \;.$$

We have a Lie algebra isomorphism $L \longrightarrow L^{\mathrm{op}}$ taking x to $-x$ (note that $[\,-x\,, -y\,] = [\,x\,, y\,] = -[\,y, x\,]$). So we define $\nu : \mathcal{U}(L) \longrightarrow \mathcal{U}(L)^{\mathrm{op}}$ by

$$\begin{tikzcd} L \arrow[r, "-"] \arrow[d, "i"'] & L^{\mathrm{op}} \arrow[d, "i"] \\ \mathcal{U}(L) \arrow[r, "\nu"] & \mathcal{U}(L)^{\mathrm{op}} \cong \mathcal{U}(L^{\mathrm{op}}) . \end{tikzcd}$$

One easily checks that for $x_1 , \dots , x_n \in L$

$$\nu(i(x_1) \cdots i(x_n)) = (-1)^n i(x_1) \cdots i(x_n) .$$

With this antipode $\mathcal{U}(L)$ *becomes a Hopf algebra.*

Example 9.7 *The matrix bialgebra* $M(n)$ (*Example 7.8 of a bialgebra*) *is* not *a Hopf algebra. We need to "adjoin an inverse for the determinant". Recall that* $M(n) = R[X] = \mathcal{S}(\mathcal{F}_R(X))$ *where* $X = \{x_{ij} \mid i , j = 1, \dots , n\}$ *has cardinality* n^2 . *Define*

$$\det(X) = \sum_{\xi \in \mathcal{S}_n} (-1)^{|\xi|} x_{1\,\xi(1)} x_{2\,\xi(2)} \cdots x_{n\,\xi(n)}$$

where $|\xi|$ *is the least number of simple transpositions required to obtain the permutation* ξ . *Form the following commutative polynomial* R*-algebra*: $R[X \cup \{t\}] = R[(x_{ij}), t] = \mathcal{S}(\mathcal{F}_R(X \cup \{t\}))$, *in* $n^2 + 1$ (*commuting*) *indeterminates* t *and* x_{ij} *with* $(1 \le i , j \le n)$. *Put*

$$\mathrm{GL}\,(n) = R[X \cup \{t\}]/(t \det(X) - 1)$$

as a commutative R*-algebra. We make* $\mathrm{GL}\,(n)$ *into a bialgebra by defining*

$$\delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj} \qquad \delta(t) = t \otimes t$$

$$\varepsilon(x_{ij}) = \delta_{ij} \qquad \varepsilon(t) = 1$$

modulo $(t \det(X) - 1)$. *Put* $X_{ij} = \{x_{rs} \mid r \neq i , s \neq j\}$. *Now define the morphism* $\nu : \mathrm{GL}\,(n) \longrightarrow \mathrm{GL}\,(n)$ *by*

$$\nu(x_{ij}) = t \det(X_{ji})$$
$$\nu(t) = \det(X)$$

modulo $(t \det(X) - 1)$. *Then* $\mathrm{GL}\,(n)$ *becomes a Hopf algebra.*

For any commutative R*-algebra* A *we have a canonical isomorphism of groups*

$$\mathbf{Alg}_R(\mathrm{GL}\,(n), A) \cong \mathrm{GL}\,(n, A) .$$

Examples 9.5 and 9.6 above exhibit cocommutative Hopf algebras $R(G)$ and $\mathcal{U}(L)$, while Example 9.7 is a commutative Hopf algebra $\mathrm{GL}\,(n)$. It is only recently that the importance of Hopf algebras which are neither commutative nor cocommutative has been properly understood.

Example 9.8 *We now describe a "quantum deformation" of Example 9.7. This is a generalization to $n \times n$, from the 2×2 case discussed in . Section 3.*

Take $X = \{x_{ij} \mid i, j = 1, \ldots, n\}$ as in Example 9.7. First we form the free algebra $R\langle X \rangle = T(\mathcal{F}_R(X))$ on the (non-commuting) indeterminates x_{ij}. Let $M_q(n)$ denote the quotient of $R\langle X \rangle$ by the ideal generated by the following elements:

$$\begin{array}{cl} x_{ir}\, x_{jk} - x_{jk}\, x_{ir} & \text{for } \ i < j \ \text{ and } \ k < r \\ x_{ir}\, x_{jk} - x_{jk}\, x_{ir} - (q - q^{-1})\, x_{ik}\, x_{jr} & \text{for } \ i < j \ \text{ and } \ r < k \\ x_{ik}\, x_{jk} - q\, x_{jk}\, x_{ik} & \text{for } \ i < j \\ x_{ik}\, x_{ir} - q\, x_{ir}\, x_{ik} & \text{for } \ k < r \ . \end{array}$$

This becomes a coalgebra with comultiplication

$$\delta(x_{ij}) \;=\; \sum_{r=1}^{n} x_{ir} \otimes x_{rj} \quad (\text{ modulo the ideal })$$

and counit

$$\varepsilon(x_{ij}) \;=\; \delta_{ij} \qquad (\text{ Kronecker delta }) \ .$$

Define the "quantum determinant" by

$$\det_q(X) \;=\; \sum_{\xi \in \mathcal{S}_n} (-q)^{|\xi|}\, x_{1\,\xi(1)}\, x_{2\,\xi(2)} \cdots x_{n\,\xi(n)}$$

which is a central element of $M_q(n)$ (that is, it commutes with all other elements). The quantum general linear group *is defined by*

$$\mathrm{GL}_q(n) \;=\; M_q(n)[\,t\,]/(\,t \det_q(X) - 1\,) \ .$$

We adjust the comultiplication and counit of $M_q(n)$ by defining $\delta(t) = t \otimes t$ and $\varepsilon(t) = 1$. Then we have a bialgebra epimorphism

$$\rho : M_q(n) \longrightarrow \mathrm{GL}_q(n) \ .$$

Define $\nu : \mathrm{GL}_q(n) \longrightarrow \mathrm{GL}_q(n)$ by

$$\nu(x_{ij}) \;=\; t \det_q(X_{ji}) \qquad \textit{and} \qquad \nu(t) \;=\; \det_q(X) \ .$$

Then $\mathrm{GL}_q(n)$ becomes a Hopf algebra. Notice that $\mathrm{GL}_q(n)^{\mathrm{op}} = \mathrm{GL}_{q^{-1}}(n)$.

Many claims have been made in this section. For $n = 2$ the calculations in Theorem 3.1 prove them all. (This should be compared with Proposition 9.4 in the present section.) The general case can be verified similarly, but will follow from later work.

Exercises

1. Assume our base ring R is a field and write $\otimes$ for $\otimes_R$. An *ideal* in an algebra A is a submodule I such that $\mu(I\otimes A + A\otimes I) \subseteq I$. We know that A/I becomes an algebra. A *coideal* is a coalgebra C is a submodule I such that $\delta(I) \subseteq I\otimes C + C\otimes I$ and $\varepsilon(I) \subseteq 0$.

   (a) If I is a coideal of a coalgebra C, describe a coalgebra structure on C/I for which $\rho : C \to C/I$ becomes a coalgebra morphism. If C is a bialgebra and I is also an ideal, show that C/I is a bialgebra. What condition on I ensures C/I has an antipode if C has? I is called a ***Hopf ideal*** when this holds.

   (b) Verfy that the polynomial R-algebra $B = R\langle x, y, z\rangle$ on three non-commuting indeterminates becomes a bialgebra with

   $$\delta(x) = x{\otimes}x, \quad \delta(y) = y{\otimes}y, \quad \delta(z) = 1{\otimes}z + z{\otimes}x,$$

   $$\varepsilon(x) = \varepsilon(y) = 1, \quad \varepsilon(z) = 0$$

   (c) Verify that the ideal $(xy - 1, yx - 1)$ is a coideal in B. Let H denote the quotient bialgebra.

   (d) Show that H is a Hopf algebra with antipode v given by

   $$v(x) = y, \quad v(y) = x, \quad v(z) = -zy \quad \text{(modulo the ideal of Q.3)}$$

   Show further that $v^{2n}(z) = x^n z y^n$, $v^{2n+1}(z) = -x^n z y^{n+1}$. Hence, this antipode has an infinite order.

   (e) i. Show that the ideals $I_n = (x^n z y^n - z)$, $J_n = (x^n - 1)$ are Hopf ideals in H.

   ii. Show that the antipodes of both H/I_n and H/J_n have order $2n$.

10

Representations of quantum groups

We mentioned in Example 6.3 that a representation of a group G was an $R(G)$-module. One kind of representation for a Hopf algebra H therefore suggests itself: a module over H. We begin by discussing modules over bialgebras.

First note that if $f : E \longrightarrow A$ is a ring morphism then each (left) A-module M becomes a (left) E-module via the action

$$e\,m = f(e)m \qquad \text{for } e \in E\ ,\ m \in M\ .$$

This is called *restriction of scalars along f.*

Let A be an R-algebra. Then each module is automatically an R-module via restriction of scalars along the unit $\eta : R \longrightarrow A$. Alternatively, we can view an A-module as an R-module M with a ring morphism $\hat{\mu} : A \longrightarrow \mathrm{End}_R(M)$. Later, we want to look at "comodules", and so we want a definition of A-module which dualizes. The good version is: an R-module M with a module morphism $\mu : A \otimes_R M \longrightarrow M$, called the *action* of A on M, satisfying

$$\begin{tikzcd} A\otimes_R A\otimes_R M \arrow[r, shift left, "\mu\otimes 1"] \arrow[r, shift right, "1\otimes\mu"'] & A\otimes_R M \arrow[r, "\mu"] & M \end{tikzcd}$$

$$\begin{tikzcd} & A\otimes_R M \arrow[dr, "\mu"] & \\ M \arrow[ur, "\eta\otimes 1"] \arrow[rr, "1"] & & M\ . \end{tikzcd}$$

We write $\mathbf{Mod}_R(A)$ for $\mathbf{Mod}_A$ just to emphasize that we build it up from $\mathbf{Mod}_R$.

Suppose M and N are (left) modules over the R-algebra A. Regard $M : A \nrightarrow R$ and $N : R \nrightarrow A^{\mathrm{op}}$. We see that $M \otimes_R N : A \nrightarrow A^{\mathrm{op}}$, which means $M \otimes_R N$ becomes an $A \otimes A$-module. If A is a bialgebra then we can restrict scalars along $\delta : A \longrightarrow A \otimes_R A$ to obtain an A-module structure on $M \otimes_R N$. Explicitly, the action is the composite

$$A \otimes_R M \otimes_R N \xrightarrow{\delta \otimes 1 \otimes 1} A \otimes_R A \otimes_R M \otimes_R N \xrightarrow{1 \otimes \sigma \otimes 1} A \otimes_R M \otimes_R A \otimes_R N \xrightarrow{\mu \otimes \mu} M \otimes_R N \,.$$

This generalizes to multiple tensor products (over R) of A-modules. In particular, the empty tensor product R becomes an A-module by restricting scalars along the counit $\varepsilon : A \longrightarrow R$.

With M and N left A-modules as before, we can regard $M : R \nrightarrow A^{\mathrm{op}}$ and $N : R \nrightarrow A^{\mathrm{op}}$, so that $\mathrm{Hom}_R(M, N) : A^{\mathrm{op}} \nrightarrow A^{\mathrm{op}}$; or in other words $\mathrm{Hom}_R(M, N)$ becomes an $A^{\mathrm{op}} \otimes A$-module. Thus if $A = H$ is a Hopf algebra, we can restrict scalars along the R-algebra morphism

$$H \xrightarrow{\delta} H \otimes_R H \xrightarrow{\nu \otimes 1} H^{\mathrm{op}} \otimes_R H$$

to make $\mathrm{Hom}_R(M, N)$ into an H-module. Explicitly, the action of H on $\mathrm{Hom}_R(M, N)$ is the composite

$$\begin{aligned} H \otimes_R \mathrm{Hom}_R(M,N) &\xrightarrow{\delta \otimes 1} H \otimes_R H \otimes_R \mathrm{Hom}_R(M,N) \xrightarrow{\quad 1 \otimes \sigma \quad} \\ H \otimes_R \mathrm{Hom}_R(M,N) \otimes_R H &\xrightarrow{\mu_1 \otimes \nu} \mathrm{Hom}_R(M,N) \otimes_R H \xrightarrow{\mu_2} \mathrm{Hom}_R(M,N) \end{aligned}$$

where μ_1 and μ_2 are the left and right actions ...

$$\begin{array}{l} H \otimes_R \mathrm{Hom}_R(M,N) \xrightarrow{\hat{\mu} \otimes 1} \mathrm{Hom}_R(N,N) \otimes_R \mathrm{Hom}_R(M,N) \xrightarrow{\circ} \mathrm{Hom}_R(M,N) \\ \quad \mu_1 \; : \; h \otimes f \longmapsto (m \longmapsto h(f m)) \\ \mathrm{Hom}_R(M,N) \otimes_R H \xrightarrow{1 \otimes \hat{\mu}} \mathrm{Hom}_R(M,N) \otimes_R \mathrm{Hom}_R(M,M) \xrightarrow{\circ} \mathrm{Hom}_R(M,N) \\ \quad \mu_2 \; : \; f \otimes h \longmapsto (m \longmapsto f(h m)) \,. \end{array}$$

Proposition 10.1 *For left modules M and N over the Hopf algebra H, the canonical R-module morphisms*

$$\begin{array}{lll} e : \mathrm{Hom}_R(M, N) \otimes_R M \longrightarrow N & \text{where} & f \otimes m \longmapsto f(m) \\ d : M \longrightarrow \mathrm{Hom}_R(N, M \otimes_R N) & \text{where} & m \longmapsto (n \longmapsto m \otimes n) \end{array}$$

are left H-module morphisms.

Proof. Omitting $\otimes_R$ and Hom_R from the notation, we obtain the first of these from the following diagram. The second we leave to the reader.

$$
\begin{tikzcd}
H(MN)M \arrow[r, "\delta 11"] \arrow[dd, "1"'] & HH(MN)M \arrow[r, "1\sigma 1"] \arrow[d, shift right, "1\delta 11"'] \arrow[d, shift left, "\delta 111"] \arrow[ddl, bend right, "1\varepsilon 11"'] & H(MN)HM \arrow[dr, "\delta 111"] & \\
& HHH(MN)M \arrow[rr, "11\sigma 1"] \arrow[d, "1\nu 111"] & & HH(MN)HM \arrow[d, "1\sigma 11"] \\
H(MN)M \arrow[dr, "1\eta 11"] \arrow[ddr, "11\eta 1"'] \arrow[dddr, bend right, "1"'] & HHH(MN)M \arrow[r, "11\sigma 1"] \arrow[d, "1\mu 11"] & HH(MN)HM \arrow[dr, "1\sigma 11"] & H(MN)HHM \arrow[d, "11\nu 11"] \\
& HH(MN)M \arrow[d, "1\sigma 1"] & & H(MN)HHM \arrow[dll, "11\mu 1"'] \arrow[d, "1\mu_2 11"] \\
& H(MN)HM \arrow[d, "1\mu_2 1"] & & H(MN)HM \arrow[dll, "1\mu_2 1"'] \arrow[d, "\mu_1 11"] \\
& H(MN)M \arrow[r, "\mu_1 1"'] \arrow[d, "1e"] & (MN)M \arrow[d, "e"] & (MN)HM \arrow[l, "1\mu"'] \\
& HN \arrow[r, "\mu"] & N &
\end{tikzcd}
$$

Corollary 10.2 *For modules M, N and L over a Hopf algebra H, the canonical isomorphism*

$$\mathrm{Hom}_R(M \otimes_R N\,, L) \;\cong\; \mathrm{Hom}_R(M\,, \mathrm{Hom}_R(N, L))$$

restricts to an isomorphism

$$\mathrm{Hom}_H(M \otimes_R N\,, L) \;\cong\; \mathrm{Hom}_H(M\,, \mathrm{Hom}_R(N, L))\,.$$

Proof. The canonical isomorphism is obtained from the evaluation e and the canonical d of Proposition 10.1.

In other words, we have a nice *tensor–hom* situation for the category $\mathbf{Mod}_R(H)$ of (left) H-modules. Both the tensor and the *hom* are preserved by the functor

$$\mathbf{Mod}_R(H) \longrightarrow \mathbf{Mod}_R$$

given by ignoring the H-action.

Although modules over the group algebra are representations of the group, so that the study of modules over a Hopf algebra does suggest itself, the point of view of Section 2 (i.e. space–algebra duality) leads more naturally to "comodules". For here, it is the comultiplication $\delta : H \longrightarrow H \otimes H$ of the Hopf algebra which corresponds to the spatial multiplication.

Suppose C is an R-coalgebra. A (left) C-*comodule* is an R-module M with a module morphism $\delta : M \longrightarrow C \otimes_R M$, called the *coaction* of C on M, satisfying

$$\begin{tikzcd} M \arrow[r, "\delta"] & C\otimes_R M \arrow[r, shift left, "\delta\otimes 1"] \arrow[r, shift right, "1\otimes\delta"'] & C\otimes_R C\otimes_R M \end{tikzcd}$$

$$\begin{tikzcd} & C\otimes_R M \arrow[dr, "\varepsilon\otimes 1"] & \\ M \arrow[ur, "\delta"] \arrow[rr, "1"] & & M \end{tikzcd}$$

We write $\mathbf{Com}_R(C)$ for the category whose objects are C-comodules and whose arrows are C-*comodule morphisms*; that is, R-module morphisms $f : M \longrightarrow N$ such that

$$\begin{tikzcd} M \arrow[r, "\delta"] \arrow[d, "f"'] & C\otimes_R M \arrow[d, "1\otimes f"] \\ N \arrow[r, "\delta"] & C\otimes_R N \end{tikzcd}$$

Each C-comodule M becomes a C^*-module with the action

$$C^*\otimes_R M \xrightarrow{1\otimes\delta} C^*\otimes_R C\otimes_R M \xrightarrow{e\otimes 1} M \ .$$

See Section 7 for the algebra structure on C^*.

By the fundamental theorem of Morita theory (Theorem 5.3), if C is cauchy (as an R-module) then this gives a bijection between C-coactions δ and C^*-actions μ on each R-module M: recover δ as the composite

$$M \xrightarrow{d\otimes 1} C\otimes_R C^*\otimes_R M \xrightarrow{1\otimes\mu} C\otimes_R M \ .$$

So for C cauchy, we have an isomorphism of categories

$$\mathbf{Com}_R(C) \ \cong \ \mathbf{Mod}_R(C^*) \ .$$

If C is an R-bialgebra not necessarily cauchy we obtain, in a manner dual to that for modules, a coaction on the tensor product (over R) of C-comodules. Explicitly for C-modules M and N, the coaction for $M\otimes_R N$ is given by the composite

$$M\otimes_R N \xrightarrow{\delta\otimes\delta} C\otimes_R M\otimes_R C\otimes_R N \xrightarrow{\sigma_{1324}} C\otimes_R C\otimes_R M\otimes_R N \xrightarrow{\mu\otimes 1\otimes 1} C\otimes_R M\otimes_R N \ .$$

The empty tensor product R has the coaction $\eta : R \longrightarrow C\otimes_R R$.

When it comes to Hom our formal duality fails: in reversing arrows we have maintained $\otimes$, yet Hom does not maintain its universal property. However if M is cauchy, $\mathrm{Hom}_R(M,N)$ does have the *reverse-arrow universal property*: there is a bijection between R-module morphisms

$$L \longleftarrow \mathrm{Hom}_R(M,N)$$

and R-module morphisms

$$M\otimes L \longleftarrow N$$

since $\mathrm{Hom}_R(M,N) \cong M^*\otimes_R N$ and $M\otimes_R L \cong \mathrm{Hom}_R(M^*,L)$.

Proposition 10.3 *Each cauchy R-module M gives rise to an R-coalgebra $M\otimes_R M^*$ with counit $e : M\otimes_R M^* \longrightarrow R$ and comultiplication*

$$1\otimes d\otimes 1 : M\otimes_R M^* \longrightarrow M\otimes_R M^*\otimes_R M\otimes_R M^*$$

(see Theorem 5.3). For any R-coalgebra C, the assignment

$$\hat{\delta} \;=\; (M\otimes_R M^* \xrightarrow{\delta\otimes 1} C\otimes_R M\otimes_R M^* \xrightarrow{1\otimes e} C)$$

determines a bijection between coactions

$$\delta : M \longrightarrow C\otimes_R M$$

of C on M and coalgebra morphisms

$$\hat{\delta} : M\otimes_R M^* \longrightarrow C\ .$$

Proof. $M\otimes_R M^* \underset{\sigma}{\cong} M^*\otimes_R M \underset{\rho}{\cong} \mathrm{Hom}_R(M,M)$ has the universal property of Hom under arrow reversal; so the diagrammatic proof that $\mathrm{End}_R(M)$ is an algebra and that an action is an algebra morphism $A \longrightarrow \mathrm{End}_R(M)$, dualizes.

Take $M = R^n$ in the above proposition and let $e_1, \ldots, e_n$ be the standard basis. Now let $e_1^*, \ldots, e_n^*$ be the dual basis for R^{n*} so $e_i^*(e_j) = \delta_{ij}$ (Kronecker-δ). A coaction of C on R^n thus amounts to a coalgebra morphism $\hat{\delta} : R^n\otimes_R R^{n*} \longrightarrow C$, and this is determined by its values on the basis elements $e_i\otimes e_j^*$ of $R^n\otimes_R R^{n*}$:

$$\hat{\delta}(e_i\otimes e_j^*) \;=\; x_{ij} \;\in C\ .$$

So C-comodule structures on R^n are in bijection with *multiplicative matrices* in C; that is, matrices $\vec{\mathbf{x}} = (x_{ij})$ in C satisfying

$$\delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj} \quad , \quad \varepsilon(x_{ij}) = \delta_{ij} \ .$$

Following Manin[Man88], we write the last two equations as

$$\delta(\vec{\mathbf{x}}) = \vec{\mathbf{x}} \otimes \vec{\mathbf{x}} \quad , \quad \varepsilon(\vec{\mathbf{x}}) = \vec{\mathbf{i}}$$

where $\vec{\mathbf{i}}$ is the identity matrix and $\vec{\mathbf{x}} \otimes \vec{\mathbf{y}} = (\sum_k x_{ik} \otimes x_{kj})$ is *not* the usual tensor product of matrices.

Example 10.4 *In the situation of Example 9.8* $\vec{\mathbf{x}} = (x_{ij})$ *and* $\begin{pmatrix} \vec{\mathbf{x}} & 0 \\ 0 & t \end{pmatrix}$ *are multiplicative matrices for* $\mathbf{Mat}_q(n)$ *and* $\mathrm{GL}_q(n)$*, respectively.*

Now suppose $C = H$ is a Hopf algebra and M is a cauchy R-module. By applying Proposition 10.1 to M^* (and using the canonical $M^{**} \cong M$), we see that $M^* \otimes_R M$ becomes a coalgebra with counit

$$M^* \otimes_R M \xrightarrow{\sigma} M \otimes_R M^* \xrightarrow{e} R$$

and comultiplication

$$M^* \otimes_R M \xrightarrow{1 \otimes d \otimes 1} M^* \otimes_R M^* \otimes_R M \otimes_R M \xrightarrow{1 \otimes \sigma \otimes 1} M^* \otimes_R M \otimes_R M^* \otimes_R M \ .$$

Proposition 10.5 (a) $\hat{\delta} : M \otimes_R M^* \longrightarrow H$ *is a coalgebra morphism if and only if the composite*

$$M^* \otimes_R M \xrightarrow{\sigma} M \otimes_R M^* \xrightarrow{\hat{\delta}} H^{\mathrm{op}}$$

is a coalgebra morphism.

(b) *Suppose* M *is an* H*-comodule and put*

$$\hat{\delta}_k = \left(M \otimes_R M^* \xrightarrow{\hat{\delta}} H \xrightarrow{\nu^k} H \right) .$$

For k *even,* $\hat{\delta}_k$ *is a coalgebra morphism and has convolution inverse* $\hat{\delta}_{k+1}$ *in* $\mathrm{Hom}_R(M \otimes_R M^*, H)$ *.*

Proof. (a)

$$
\begin{tikzcd}
M^*M \arrow[r, "\sigma"] \arrow[d, "1\otimes d\otimes 1"'] & MM^* \arrow[r, "\hat{\delta}"] \arrow[d, "1\otimes d\otimes 1"] & H \arrow[d, "\delta"] \\
M^*M^*MM \arrow[r, "\sigma_{4231}"] \arrow[d, "1\otimes\sigma\otimes 1"'] & MM^*MM^* \arrow[r, "\hat{\delta}\otimes\hat{\delta}"] \arrow[d, "\sigma_{3412}"] & H\otimes H \arrow[d, "\sigma"] \\
M^*MM^*M \arrow[r, "\sigma\otimes\sigma"] & MM^*MM^* \arrow[r, "\hat{\delta}\otimes\hat{\delta}"] & H\otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
M^*M \arrow[r, "\sigma"] \arrow[d, "\sigma"'] & MM^* \arrow[r, "\hat{\delta}"] \arrow[dl, equal] \arrow[dd, "e"] & H \arrow[ddl, "\varepsilon"] \\
MM^* \arrow[dr, "e"'] & & \\
& R &
\end{tikzcd}
$$

(b) Since $\nu : H \longrightarrow H^{\mathrm{op}}$ is a coalgebra morphism, by 9.1(b); then $\nu^k : H \longrightarrow H$ is a coalgebra morphism for k even; so $\hat{\delta}_k = \nu^k \circ \hat{\delta}$ is a coalgebra morphism, as required. This also means that right composition with $\hat{\delta}_k$ preserves convolution. Since 1_H and ν are convolution inverses, so are $1_H \circ \hat{\delta}_k$ and $\nu \circ \hat{\delta}_k$; that is, so are $\hat{\delta}_k$ and $\hat{\delta}_{k+1}$.

Proposition 10.6 *Suppose that M is a comodule over the Hopf algebra H and that M is cauchy as an R-module. Then M^* becomes an H-comodule via*

$$\hat{\delta} = (M^*\otimes_R M \xrightarrow{\ \sigma\ } M\otimes_R M^* \xrightarrow{\ \hat{\delta}\ } H \xrightarrow{\ \nu\ } H)\ .$$

Moreover, the R-module morphisms

$$e : M\otimes_R M^* \longrightarrow R$$
$$d : R \longrightarrow M^*\otimes_R M$$

become H-comodule morphisms. One might therefore say that M becomes a cauchy H-comodule*.*

Proof. By Propositions 10.5(a) and 9.1(b), the stated $\hat{\delta}$ is a coalgebra morphism. So by Proposition 10.5, it determines a coaction of H on M^*. Tracing through, one sees that this is dual to the situation for $\mathrm{Hom}_R(M, R)$ as in Proposition 10.1; so the proof dualizes, but it can also be shown directly that e and d are comodule morphisms.

Remark. To obtain results as in Proposition 10.5(b) for k odd, apply Proposition 10.5(b) to the M^* of Proposition 10.6; compare with [Man88], p.14.

Exercises

1. Give a direct proof of Proposition 10.3 concerning the coalgebra structure on $M\otimes_R M^*$ where M is a Cauchy R-module.

11
Tensor categories

It is clear that specific categories have entered explicitly into the above discussion, but we have made little use of them as categories apart from diagrams and duality. For what follows it is hard to imagine how to express the results without categories.

A *tensor category* is a category $\mathcal{V}$ together with functor $\otimes : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ called *tensor product*, an object I of $\mathcal{V}$ called the *unit object*, and natural families of isomorphisms

$$a_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$$
$$r_A : A \otimes I \longrightarrow A \quad , \quad l_A : I \otimes A \longrightarrow A$$

called respectively the *associativity* constraint, the *right unit* constraint and the *left unit* constraint, subject to the two conditions:

$$\begin{tikzcd}
 & (A\otimes B)\otimes(C\otimes D) \arrow[dr, "a_{A,B,C\otimes D}"] & \\
((A\otimes B)\otimes C)\otimes D \arrow[ur, "a_{A\otimes B,C,D}"] \arrow[d, "a_{A,B,C}\otimes 1"'] & & A\otimes(B\otimes(C\otimes D)) \\
(A\otimes(B\otimes C))\otimes D \arrow[rr, "a_{A,B\otimes C,D}"'] & & A\otimes((B\otimes C)\otimes D) \arrow[u, "1\otimes a_{B,C,D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A\otimes I)\otimes C \arrow[rr, "a_{A,I,C}"] \arrow[dr, "r_A\otimes 1"'] & & A\otimes(I\otimes C) \arrow[dl, "1\otimes l_C"] \\
 & A\otimes C &
\end{tikzcd}$$

Define $A_1 \otimes \ldots \otimes A_n$ to be the object obtained by inserting brackets in some chosen preassigned way, such as from the left $((\ldots(A_1 \otimes A_2) \otimes \ldots) \otimes A_n$. It is an important fact (MacLane's coherence theorem)that the only automorphisms of the form $1\otimes(x\otimes 1)$ or $(1\otimes x)\otimes 1$, where x is a component of a, r, l or their inverses, is the identity arrow of $A_1\otimes\ldots\otimes A_n$. This essentially allows one to work as if the a, r, l are all identities. If all the a, r, l are indeed identities, then the tensor category is called *strict*. The *opposite* $\mathcal{V}^{\mathrm{op}}$ of a tensor category $\mathcal{V}$ consists of the opposite category of $\mathcal{V}$ (obtained by reversing the direction of arrows of $\mathcal{V}$) and the reverse tensor product, so that $A\otimes B$ in $\mathcal{V}^{\mathrm{op}}$ is just $B\otimes A$ in $\mathcal{V}$.

A *braiding* for a tensor category $\mathcal{V}$ is a natural family of isomorphisms

$$c_{A,B} : A\otimes B \longrightarrow B\otimes A$$

subject to the conditions

$$\begin{tikzcd}
 & A\otimes(B\otimes C) \arrow[r, "c_{A,B\otimes C}"] & (B\otimes C)\otimes A \arrow[dr, "a_{B,C,A}"] & \\
A\otimes(B\otimes C) \arrow[ur, "a_{A,B,C}"] \arrow[dr, "c_{A,B}\otimes 1"'] & & & B\otimes(C\otimes A) \\
 & (B\otimes A)\otimes C \arrow[r, "a_{B,A,C}"] & B\otimes(A\otimes C) \arrow[ur, "1\otimes c_{A,C}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (A\otimes B)\otimes C \arrow[r, "c_{A\otimes B,C}"] & C\otimes(A\otimes B) \arrow[dr, "a^{-1}_{C,A,B}"] & \\
A\otimes(B\otimes C) \arrow[ur, "a^{-1}_{A,B,C}"] \arrow[dr, "1\otimes c_{B,C}"'] & & & (C\otimes A)\otimes B \\
 & A\otimes(C\otimes B) \arrow[r, "a^{-1}_{A,C,B}"] & (A\otimes C)\otimes B \arrow[ur, "c_{A,C}\otimes 1"'] &
\end{tikzcd}$$

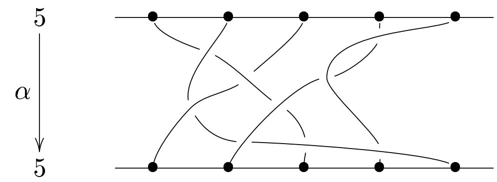
A *braided tensor category* is a tensor category with a chosen braiding.

A *symmetry* for a tensor category is a braiding which satisfies the following extra condition:

$$\begin{tikzcd}
A\otimes B \arrow[rr, "1"] \arrow[dr, "c_{A,B}"'] & & A\otimes B \\
 & B\otimes A \arrow[ur, "c_{B,A}"'] &
\end{tikzcd}$$

A *symmetric tensor category*is a tensor category with a chosen symmetry.

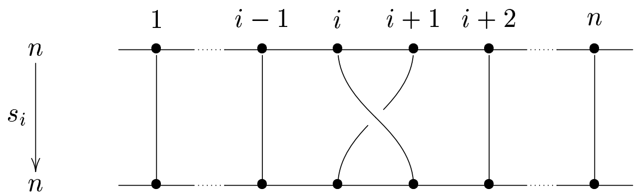
Example 11.1 *The* braid category $\mathcal{B}$ *has as objects the natural numbers* $0, 1, 2, \ldots$ *and as arrows* $\alpha : n \longrightarrow n$ *the braids on* n *strings; there are no arrows* $m \longrightarrow n$ *for* $m \neq n$ *. A braid* α



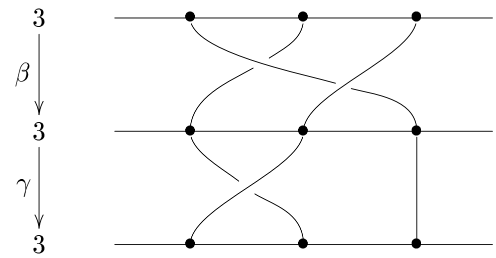
on n *strings can be regarded as an element of the Artin braid group* $\mathcal{B}_n$ *with generators* $s_1, \ldots, s_{n-1}$ *subject to the relations*

$$\begin{aligned} s_i s_j &= s_j s_i \qquad \textit{for } j < i-1 \\ s_{i+1} s_i s_{i+1} &= s_i s_{i+1} s_i \end{aligned}$$

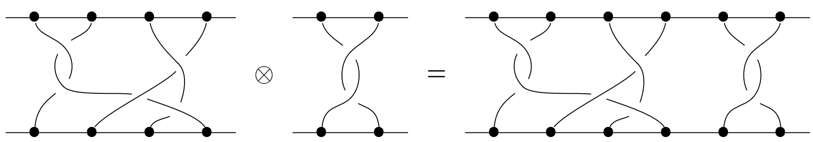
where s_i *is the braid depicted as:*



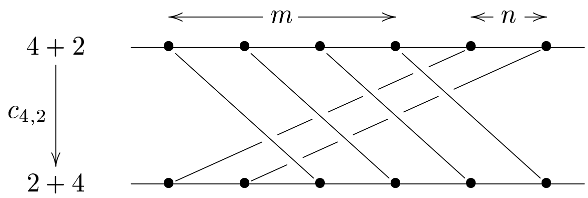
Composition of braids is just multiplication in this group, represented diagramatically by vertical stacking of braids with the same number of strings.



Tensor product of braids adds the number of strings by placing one braid next to the other longitudinally.



This makes $\mathcal{B}$ a strict tensor category. A braiding $c_{m,n} : m+n \longrightarrow n+m$ is given by crossing the first m strings over the remaining n.



The axioms that show $\mathcal{B}$ is braided are easily checked diagramatically.

Example 11.2 *The category $\mathbf{Mod}_R$ of modules over a commutative ring R is a symmetric tensor category with tensor product $\otimes_R$, with the canonical constraints, and with symmetry $\sigma : A\otimes_R B \longrightarrow B\otimes_R A$.*

Example 11.3 *Let A be an R-bialgebra. If M and N are A-modules, we have an A-module structure on $M\otimes_R N$ given by*

$$a\cdot(m\otimes n) \;=\; \sum_{(a)} a_{(1)}m\otimes a_{(2)}n$$

as seen in the last section. So $\mathbf{Mod}_R(A)$ becomes a tensor category with tensor product $\otimes_R$.

If A is cocommutative, the switch morphism $\sigma : M\otimes_R N \longrightarrow N\otimes_R M$ is a symmetry for $\mathbf{Mod}_R(A)$. However, as in the rest of this book, we are more interested in non-cocommutative A.

We ask: *what are the possible braidings on the tensor category $\mathbf{Mod}_R(A)$?*

A braiding $c_{M,N} : M\otimes_R N \longrightarrow N\otimes_R M$ gives, for each A, a morphism $c_{A,A} : A\otimes_R A \longrightarrow A\otimes_R A$ which gives an element $\gamma = c_{A,A}(1\otimes 1) \in A\otimes A$.

Conversely, each element $\gamma = \sum_i u_i\otimes v_i \in A\otimes A$ determines a natural morphism $c_{M,N} : M\otimes_R N \longrightarrow N\otimes_R M$ via the formula

$$c_{M,N}(m\otimes n) = \sum_i (u_i n)\otimes(v_i m)\,.$$

This is a bijection, as can be seen from the following diagram in which $\hat{m} : A \longrightarrow M$ is the unique module morphism with $\hat{m}(1) = m$.

$$\begin{tikzcd}
R\otimes_R R \arrow[r, "\eta\otimes\eta"] & A\otimes_R A \arrow[r, "c_{A,A}"] \arrow[d, "\hat{m}\otimes\hat{n}"'] & A\otimes_R A \arrow[d, "\hat{n}\otimes\hat{m}"] \\
 & M\otimes_R N \arrow[r, "c_{M,N}"] & N\otimes_R M
\end{tikzcd}$$

In order for each $c_{M,N}$ to be an isomorphism it is necessary for $\gamma \in A \otimes_R A$ to be invertible. In order for each $c_{M,N}$ to be a module morphism we need

$$\begin{aligned} c\big(a \cdot (m \otimes n)\big) &= c\Big(\sum_{(a)} (a_{(1)}m) \otimes (a_{(2)}n)\Big) \\ &= \sum_{(a)} \sum_i (u_i n) \otimes (v_i m) \end{aligned}$$

to be equal to

$$\begin{aligned} a \cdot c(m \otimes n) &= a \cdot \sum_i (u_i n) \otimes (v_i m) \\ &= \sum_{(a)} \sum_i (a_{(1)} u_i n) \otimes (a_{(2)} v_i m) \,. \end{aligned}$$

This is equivalent to the requirement

$$\sum_{i,(a)} (u_i a_{(2)}) \otimes (v_i a_{(1)}) = \sum_{i,(a)} (a_{(1)} u_i) \otimes (a_{(2)} v_i) \,.$$

Regarding $\gamma \in A \otimes_R A$ as a morphism $\gamma : R \longrightarrow A \otimes_R A$ whose value at $1 \in R$ is the given γ. We can express this condition diagramatically as

(B0) $$A \xrightarrow{\gamma \otimes \delta} A^{\otimes 4} \underset{\sigma_{3142}}{\overset{\sigma_{1432}}{\rightrightarrows}} A^{\otimes 4} \xrightarrow{\mu \otimes \mu} A^{\otimes 2}$$

For a braiding, we require two more conditions:

$$\begin{aligned} c_{M, N \otimes L}(m \otimes n \otimes l) &= (1_N \otimes c_{M,L})(c_{M,N} \otimes 1_L)(m \otimes n \otimes l) \\ c_{M \otimes N, L}(m \otimes n \otimes l) &= (c_{M,L} \otimes 1_N)(1_M \otimes c_{N,L})(m \otimes n \otimes l) \end{aligned}$$

that is,

$$\begin{aligned} \sum_{i,(u_i)} u_{i(1)} n \otimes u_{i(2)} l \otimes v_i m &= \sum_{i,j} u_i n \otimes u_j l \otimes v_j v_i m \\ \sum_{i,(v_i)} u_i l \otimes v_{i(1)} m \otimes v_{i(2)} n &= \sum_{i,j} u_j u_i l \otimes v_j m \otimes v_i n. \end{aligned}$$

These are equivalent to the two conditions

$$\begin{aligned} \sum_{i,(u_i)} u_{i(1)} \otimes u_{i(2)} \otimes v_i &= \sum_{i,j} u_i \otimes u_j \otimes v_j v_i \\ \sum_{i,(v_i)} u_i \otimes v_{i(1)} \otimes v_{i(2)} &= \sum_{i,j} u_j u_i \otimes v_j \otimes v_i \,. \end{aligned}$$

Diagramatically, these conditions become:

(B1)
$$\begin{tikzcd} R \arrow[r, "\gamma\otimes\gamma"] \arrow[d, "\gamma"'] & A^{\otimes 4} \arrow[r, "\sigma_{1432}"] & A^{\otimes 4} \arrow[d, "1\otimes 1\otimes\mu"] \\ A^{\otimes 4} \arrow[rr, "\delta\otimes 1"] & & A^{\otimes 3} \end{tikzcd}$$

(B2)
$$\begin{tikzcd} R \arrow[r, "\gamma\otimes\gamma"] \arrow[d, "\gamma"'] & A^{\otimes 4} \arrow[r, "\sigma_{3142}"] & A^{\otimes 4} \arrow[d, "\mu\otimes 1\otimes 1"] \\ A^{\otimes 4} \arrow[rr, "1\otimes\delta"] & & A^{\otimes 3} \end{tikzcd}$$

Hence, we define a braiding element *for a bialgebra A to be an invertible element $\gamma \in A\otimes_R A$ which satisfies* (B0), (B1), (B2). *We have proved above that* braiding elements for A are in bijection with braidings *on the tensor category* $\mathbf{Mod}_R(A)$.

A braided bialgebra *(also called "quasitriangular bialgebra") is a bialgebra equipped with a braiding element $\gamma \in A\otimes_R A$. A braiding element γ is called a symmetry element when $\gamma^2 = 1 \in A\otimes_R A$; these are in bijection with symmetries on* $\mathbf{Mod}_R(A)$. *A* symmetric bialgebra *(also sometimes called "triangular algebra") is a bialgebra equipped with a symmetry element.*

Before leaving this example, we point out that conditions (B1), (B2) *can be put in a more familiar form in the case where A is cauchy as an R-module. For in this case, elements $\gamma = \sum_i u_i\otimes v_i \in A\otimes_R A$ are in bijection with R-module morphisms $g : A^* \longrightarrow A$ via the formula*

$$\gamma \;=\; \left(R \xrightarrow{d} A^*\otimes_R A \xrightarrow{g\otimes 1_A} A\otimes_R A \right)$$

Condition (B1) *precisely says that g preserves comultiplication, while condition* (B2) *says that g reverses multiplication. In fact, if γ is a braiding element, $g : A^* \longrightarrow A'^{\mathrm{op}}$ is a bialgebra morphism; preservation of unit and counit follows from $c_{M,I} = c_{I,M} = \mathbf{1}_M$.*

We shall just look at the translation of (B2) *to g. Begin with the defining diagram*

$$\begin{tikzcd} R \arrow[r, "d"] \arrow[d, "d"'] & A^*\otimes_R A \arrow[r, "1\otimes d\otimes 1"] & A^*\otimes_R A^*\otimes_R A\otimes_R A \arrow[d, "\delta^*\otimes 1\otimes 1"] \\ A^*\otimes_R A \arrow[rr, "1\otimes\delta"] & & A^*\otimes_R A\otimes_R A \end{tikzcd}$$

for δ^*, which is the multiplication for A^*. To prove g reverses multiplication is to prove

$$\begin{tikzcd}
A^*\otimes_R A^* \arrow[r, "g\otimes g"] \arrow[d, "d^*"'] & A\otimes_R A \arrow[r, "\sigma"] & A\otimes_R A \arrow[d, "\mu"] \\
A^* \arrow[rr, "g"] & & A
\end{tikzcd}$$

This is equivalent to proving the legs are equal after applying $\_\otimes_R A\otimes_R A$ and composing with

$$R \xrightarrow{d} A^*\otimes_R A^* \xrightarrow{1\otimes d\otimes 1} A^*\otimes_R A^*\otimes_R A\otimes_R A$$

From the defining diagram for δ^*, this amounts to

$$\begin{tikzcd}
R \arrow[r, "d"] & A^*\otimes_R A^* \arrow[r, "1\otimes d\otimes 1"] \arrow[d, "1\otimes\delta"'] & A^*\otimes_R A^*\otimes_R A\otimes_R A \arrow[r, "g\otimes g\otimes 1\otimes 1"] & A^{\otimes 4} \arrow[r, "\sigma"] & A^{\otimes 4} \arrow[d, "\mu\otimes 1\otimes 1"] \\
 & A^*\otimes_R A\otimes_R A \arrow[rrr, "g\otimes 1\otimes 1"] & & & A^{\otimes 3}
\end{tikzcd}$$

Using $\gamma = (g\otimes 1_A)\circ d$, we easily see that this is equivalent to (B2).

Although a braiding is as useful as a symmetry for most purposes, there is sometimes further structure on a braiding which makes it even more like a symmetry without actually forcing it to be one.

Suppose $\mathcal{V}$ is a braided tensor category. A *twist* for $\mathcal{V}$ is a natural family of isomorphisms

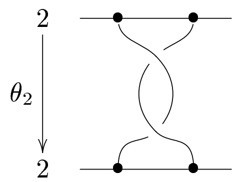
$$\theta_A : A \longrightarrow A$$

such that $\theta_I = 1_I$ and

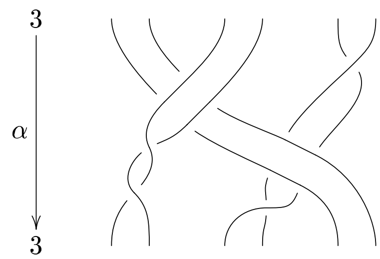
$$\begin{tikzcd}
A\otimes B \arrow[r, "c_{A,B}"] \arrow[d, "\theta_{A\otimes B}"'] & B\otimes A \arrow[d, "\theta_B\otimes\theta_A"] \\
A\otimes B & B\otimes A \arrow[l, "c_{B,A}"]
\end{tikzcd}$$

A *balanced tensor category* is a braided tensor category with a chosen twist. (A braiding is a symmetry if and only if the identity arrows provide a twist.)

Example 11.4 *The braid category $\mathcal{B}$ is canonically balanced. The twist $\theta_n : n \longrightarrow n$ is obtained by taking n vertical parallel strings with ends tied to two horizontal parallel rods, and rotating the bottom rod through a full 2π twist in the right-hand screw direction with thumb vertical. Then θ_0, θ_1 are identities, while θ_2 (which can be written as $(s_1)^2$ using the notation from Example 11.1) is:*



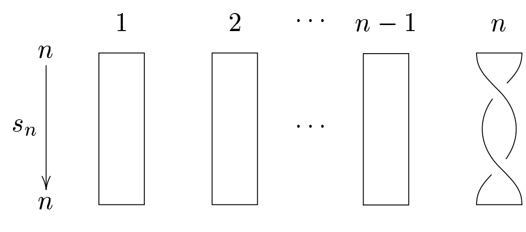
Example 11.5 *There is a tensor category $\tilde{\mathcal{B}}$ which is defined similarly to $\mathcal{B}$, except that the arrows are braids on ribbons (instead of on strings) and it is permissible to twist the ribbons through full 2π turns (as in the following diagram).*



The homsets $\tilde{\mathcal{B}}(n, n) = \tilde{\mathcal{B}}_n$ are groups under composition. A presentation of this group $\tilde{\mathcal{B}}_n$ is given by generators $s_1, \ldots, s_n$ where $s_1, \ldots, s_{n-1}$ satisfy the relations as for $\mathcal{B}_n$. These are depicted by thickened versions of the diagrams in Example 11.1, along with the extra relation

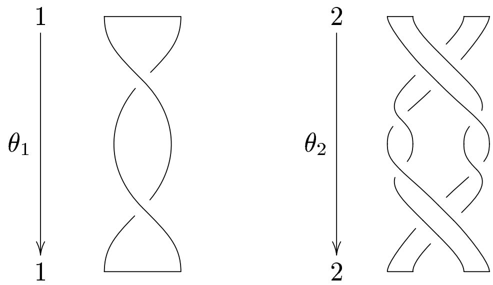
$$s_{n-1} s_n s_{n-1} s_n = s_n s_{n-1} s_n s_{n-1}$$

where s_n is depicted as follows



Composition in $\tilde{\mathcal{B}}$ is vertical stacking of digrams, and tensor product for $\tilde{\mathcal{B}}$ is horizontal placement of diagrams, much as for $\mathcal{B}$. The braiding

$c_{m,n} : m + n \longrightarrow n + m$ *for* $\tilde{\mathcal{B}}$ *is obtaining the first* m *ribbons over the remaining* n *without introducing any twists. The twist* $\theta_n : n \longrightarrow n$ *for* $\tilde{\mathcal{B}}$ *is obtained by regarding the two boundary edges of the ribbons as extra strings and taking* $\theta_{2n} : 2n \longrightarrow 2n$ *in* $\mathcal{B}$*. Then in* $\tilde{\mathcal{B}}$ *we have*



Example 11.6 *Let* A *and* B *be abelian groups and* $f : A \times A \longrightarrow B$ *be a bilinear function. There is a balanced strict tensor category* $\mathcal{C}_f$ *constructed as follows. The objects are the elements of* A*. The homset* $\mathcal{C}_f(x, y)$ *is empty unless* $x = y$ *, in which case* $\mathcal{C}_f(x, x) = B$ *. The tensor product is given by*

$$(x \xrightarrow{\alpha} x) \otimes (y \xrightarrow{\beta} y) \quad = \quad (x + y \xrightarrow{\alpha+\beta} x + y)$$

The braiding is $c_{x,y} = f(x, y) : x + y \longrightarrow y + x$ *and the twist is* $\theta_X = f(x, x) : x \longrightarrow x$ *.*

Example 11.7 *Let* A *be a braided* R*-bialgebra with braiding element* $\gamma = \sum_i u_i \otimes v_i \in A \otimes_R A$ *.* A twist element *for* A *is an invertible central element* $\tau \in A$ *such that* $\varepsilon(\tau) = 1$ *and*

$$\delta(\tau) = \sum_{i,j} (u_i \tau v_j) \otimes (v_i \tau u_j)\,.$$

Diagrammatically the last equation becomes:

$$\begin{tikzcd}
R \arrow[r, "\gamma\otimes\tau\otimes\tau\otimes\gamma"] \arrow[ddd, "\tau"'] & A^{\otimes 6} \arrow[r, "\sigma_{136245}"] & A^{\otimes 6} \arrow[d, "\mu\otimes 1\mu\otimes 1"] \\
 & & A^{\otimes 4} \arrow[dd, "\mu\otimes\mu"] \\
 & & \\
A \arrow[rr, "\delta"] & & A^{\otimes 2}
\end{tikzcd}$$

Twist elements τ for A are in bijection with twists θ *for the braided tensor category* $\mathbf{Mod}_R(A)$*. Naturality of* $\theta_M : M \longrightarrow M$ *means it has the form*

$\theta_M(m) = \tau m$ for some $\tau \in A$; for θ_M to be an A-module morphism, τ needs to be central (meaning $\tau \cdot a = a \cdot \tau$ for all $a \in A$); for θ_M to be an isomorphism, τ needs to be invertible; for $\theta_M = 1_R$, the condition $\varepsilon(\tau) = 1$ is needed; and of course the remaining twist conditions correspond.

A *balanced bialgebra* is a braided bialgebra with a twist.

Exercises

1. (a) In a braided tensor category $\mathcal{V}$ show that (ignoring the constraints a, l, r) $c_{A,I} = c_{I,A} = 1_A$ and

$$\begin{tikzcd} & A\otimes C\otimes B \arrow[r, "c\otimes 1"] & C\otimes A\otimes B \arrow[dr, "1\otimes c"] & \\ A\otimes B\otimes C \arrow[ur, "1\otimes c"] \arrow[dr, "c\otimes 1"'] & & & C\otimes B\otimes A \\ & B\otimes A\otimes C \arrow[r, "1\otimes c"'] & B\otimes C\otimes A \arrow[ur, "c\otimes 1"'] & \end{tikzcd}$$

   (b) For a braided bialgebra A and $\mathcal{V} = \mathbf{Mod}_R(A)$, interpret the properties of c in (a) in terms of the braiding element $\gamma \in A\otimes_R A$.

   (c) Draw diagrams of braids which express the hexagonal diagram of (a) in the braid category B.

2. Define the *centre* $\mathcal{Z}_\mathcal{V}$ of a tensor category $\mathcal{V}$ to be the category whose objects are pairs (A, a) where $A \in \mathcal{V}$ and $a : A\otimes - \longrightarrow - \otimes A$ is a natural isomorphism such that the following conditions hold:

   - $a_I = 1$ (more precisely, a_I is the composite of the canonical isomorphisms $A\otimes I \cong A \cong I\otimes A$).
   - $a_{X\otimes Y} = (1\otimes a_Y) \circ (a_X \otimes 1)$ for all $X, Y \in \mathcal{V}$

   An arrow $f : (A, a) \longrightarrow (B, b)$ in $\mathcal{Z}_\mathcal{V}$ is an arrow $f : A \longrightarrow B$ such that, for all $X \in \mathcal{V}$, we have $b_X \circ (f\otimes 1) = (1\otimes f) \circ a_X$.

   (a) Show that $\mathcal{Z}_\mathcal{V}$ becomes a tensor category with:

$$(A, a)\otimes(B, b) = \big(A\otimes B, (a\otimes 1) \circ (1\otimes b)\big)$$

   (b) Show that the tensor category $\mathcal{Z}_\mathcal{V}$ is braided via

$$c_{(A,a),(B,b)} = a_B : (A, a)\otimes(B, b) \longrightarrow (B, b)\otimes(A, a)$$

12

Internal homs and duals

Suppose $\mathcal{V}$ is a tensor category with A and B being objects of $\mathcal{V}$. A (left) *internal hom* for A and B consists of an object $[A, B]$ of $\mathcal{V}$ together with an arrow

$$e_A : [A, B] \otimes A \longrightarrow B \qquad \text{(called } \textit{evaluation}\text{)}$$

such that, for all arrows $f : C \otimes A \longrightarrow B$, there exists a unique arrow $\hat{f} : C \longrightarrow [A, B]$ with

$$f \quad = \quad \left(C \otimes A \xrightarrow{\hat{f} \otimes 1_A} [A, B] \otimes A \xrightarrow{e_A} B \right) \ .$$

Thus we have a natural bijection

$$\begin{array}{ccc} \mathcal{V}(C, [A, B]) & \cong & \mathcal{V}(C \otimes A, B) \\ g & \longleftrightarrow & e_A \circ (g \otimes 1_A) \ . \end{array}$$

A tensor category is called *left-closed* when each pair of objects has a left internal hom. If $f : C \longrightarrow A$ and $g : B \longrightarrow D$ are arrows of $\mathcal{V}$, then provided the internal homs exist, there is a unique arrow

$$[f, g] : [A, B] \longrightarrow [C, D]$$

such that

$$\begin{tikzcd}[A, B] \otimes C \arrow[rr, "{[f,g] \otimes 1_C}"] \arrow[d, "1_{[A,B]} \otimes f"'] & & {[C, D] \otimes C} \arrow[d, "e_C"] \\ {[A, B] \otimes A} \arrow[r, "e_A"] & B \arrow[r, "g"] & D \ . \end{tikzcd}$$

In this way, when $\mathcal{V}$ is left-closed the internal hom becomes a functor

$$[\_, \_] : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \longrightarrow \mathcal{V} \ .$$

From the universal property, the internal hom for A, B is unique up to isomorphism. An internal hom for I, A always exists; namely

$$[I, A] = A \qquad \text{with} \qquad e_A \;=\; r_A : A \otimes I \longrightarrow A \ .$$

If B, C and $A \otimes B, C$ have internal homs then so do $A, [B, C]$; namely,

$$[A, [B, C]] = [A \otimes B, C] \quad \text{with} \quad e_A = \hat{e}_{A \otimes B} : [A \otimes B, C] \otimes A \longrightarrow [B, C] \ .$$

Usually the internal hom functor $[\_, \_]$ is given *a priori*; in which case all we have are canonical isomorphisms

$$[I, A] \cong A \qquad \text{and} \qquad [A, [B, C]] \cong [A \otimes B, C] \ .$$

There is a *composition arrow*

$$[B, C] \otimes [A, B] \longrightarrow [A, C]$$

in $\mathcal{V}$ (whenever the internal homs exist) which corresponds, by using the universal property of $[A, C]$, to the composite

$$[B, C] \otimes [A, B] \otimes A \xrightarrow{1 \otimes e_A} [B, C] \otimes B \xrightarrow{e_B} C \ .$$

A *right internal hom* $[A, B]'$ for A, B comes equipped with an arrow

$$e'_A : A \otimes [A, B]' \longrightarrow B$$

which induces a bijection

$$\mathcal{V}(C, [A, B]') \;\cong\; \mathcal{V}(A \otimes C, B)$$

for all objects C. If $\mathcal{V}$ is braided, each left internal hom $[A, B]$ gives a right internal hom via $[A, B]' = [A, B]$ and

$$e'_A \;=\; \left(A \otimes [A, B] \xrightarrow{c_{A,[A,B]}} [A, B] \otimes A \xrightarrow{e_A} B \right) .$$

A tensor category is called *closed* when all left- and right-internal homs exist. (In the literature, "closed" is sometimes used for our left closed, while "biclosed" is used for our closed.) When the internal homs exist, we have an arrow

$$\omega'_A : A \longrightarrow [[A, B], B]'$$

which corresponds to $e_A : [A, B] \otimes A \longrightarrow B$ via $e'_{[A,B]}$. Similarly, we have

$$\omega_A : A \longrightarrow [[A, B]', B]$$

corresponding to e'_A.

Example 12.1 *The symmetric tensor category* $\mathbf{Mod}_R$ *of modules over a commutative ring* R *is closed with internal homs given by*

$$[M,N] \;=\; [M,N]' \;=\; \mathrm{Hom}_R(M,N)$$

Example 12.2 *Suppose* H *to be an* R*-Hopf algebra. Then the tensor category* $\mathbf{Mod}_R(H)$ *of* H*-modules (with tensor product* $\otimes_R$*) is left-closed with internal hom given by*

$$[M,N] \;=\; \mathrm{Hom}_R(M,N)$$

see Proposition 10.1.

Suppose the antipode ν *for* H *is invertible. Using Proposition 8.1(c),* H' *becomes a Hopf algebra having antipode* ν^{-1}*. Write* $\mathrm{Hom}'_R(M,N)$ *for* $\mathrm{Hom}_R(M,N)$ *as a left* H'*-module. Clearly,* $\mathbf{Mod}_R(H)$ *is right closed with* $[M,N]' = \mathrm{Hom}'_R(M,N)$*. Therefore* $\mathbf{Mod}_R(H)$ *becomes closed when* ν *is invertible. The forgetful functor*

$$\mathbf{Mod}_R(H) \longrightarrow \mathbf{Mod}_R$$

preserves tensor product, and both left and right internal homs.

Example 12.3 *Let* **Ban** *denote the category of Banach spaces (over the complex numbers), where the arrows* $f : A \longrightarrow B$ *are linear functions for which*

$$\|f(a)\| \leq \|a\| \,.$$

(The analysts in the audience will think these fairly uninteresting functions.) We make **Ban** *into a symmetric tensor category by taking tensor products as vector spaces, completing in the obvious way. The internal hom* $[A\,,B\,]$ *exists for all banach spaces* A, B*; it is the banach space of bounded linear functions from* A *to* B *with the usual norm. (These functions are of more interest to the analyst.) Thus* **Ban** *is a closed symmetric tensor category.*

medskipSuppose $\mathcal{V}$ is a tensor category. An object J is called *left dualizing* when, for all objects A, internal homs $[A,J]$, $[A,J]'$ exist, and the arrow

$$\omega_A : A \longrightarrow [\,[A\,,J\,]', J\,]$$

is invertible. It follows that $\mathcal{V}$ is left closed with

$$[A\,,B\,] \;=\; [A \otimes [B\,,J\,]', J\,]\,,$$

since

$$\begin{aligned}\mathcal{V}(C\,,[A\otimes[B,J\,]',J\,]\,) &\cong \mathcal{V}(C\otimes A\otimes[B,J\,]',J)\\ &\cong \mathcal{V}(C\otimes A\,,[\,[B,J\,]',J\,])\\ &\cong \mathcal{V}(C\otimes A\,,B)\ .\end{aligned}$$

The concept of *right dualizing object* is defined in the same way, with ω replaced by ω'. A *dualizing object* is one which is both left and right dualizing.

Example 12.4 *The category of finite dimensional vector spaces over a field has a dualizing object, namely the field itself. In this case, the dualizing object is the unit for tensor product.*

Example 12.5 *Fix a field* $\mathbf{k}$ *. A* quadratic algebra *is a pair* $(V\,,R)$ *where* V *is a finite dimensional vector space and* R *is a subspace of* $V\otimes V$*. A quadratic algebra morphism* $f : (V\,,R)\longrightarrow(W\,,S)$ *is a linear function* $f : V\longrightarrow W$ *for which*

$$(f\otimes f)(R) \subseteq S\ .$$

Write $\mathcal{QA}$ *for the category of quadratic algebras. Each quadratic algebra* $(V\,,R)$ *determines an actual algebra*

$$T(V)/R$$

where $T(V)$ *is the tensor algebra on* V *(see Example 6.2). The category* $\mathcal{QA}$ *has a symmetric tensor product given by*

$$(V\,,R)\otimes(W\,,S) \;=\; (V\otimes W\,,\sigma_{1324}(R\otimes S))\ .$$

The unit object is $I = (\mathbf{k}\,,\mathbf{k}\otimes\mathbf{k})$*.*

We claim that $J = (\mathbf{k}\,,0)$ is a dualizing object. *It is easy to see that*

$$[(V\,,R),J\,] \;=\; (V^*\,,R^{\perp})$$

where $V^* = \mathrm{Hom}_{\mathbf{k}}(V,\mathbf{k})$ *and* $R^{\perp}$ *is the kernel of the composite surjection*

$$V^*\otimes V^* \xrightarrow{\;\cong\;} (V\otimes V)^* \xrightarrow{\;i^*\;} R^*$$

with $i : R\longrightarrow V\otimes V$ *being inclusion. It follows that* R *is the kernel of*

$$V\otimes V \xrightarrow{\;\omega\otimes\omega\;} V^{**}\otimes V^{**} \xrightarrow{\;\cong\;} (V^*\otimes V^*)^* \xrightarrow{\;i^*\;} R^{\perp\,*}$$

so we have an isomorphism

$$\omega : (V\,,R)\longrightarrow(V^{**}\,,R^{\perp\perp})\ .$$

The quadratic algebra which we identify with the quantum plane $\mathbb{A}_q^{2|0}$ *(recall Section 3) is*

$$\mathbb{A}_q^{2|0} \;=\; \big(\mathbf{k}^2\,,(y\otimes x - q\,x\otimes y)\big)$$

where $x = (1,0)$, $y = (0,1) \in \mathbf{k}^2$. *Let* $\xi\,,\eta \in \mathbf{k}^{2*}$ *be the dual basis given by* $\xi(x) = \eta(y) = 1$, $\xi(y) = \eta(x) = 0$. *Then*

$$(y\otimes x - q\,x\otimes y)^{\perp} \;=\; (\,\xi\otimes\xi\,,\eta\otimes\eta\,,\xi\otimes\eta + q\,\eta\otimes\xi\,)$$

as a subspace of $\mathbf{k}^{2*}\otimes\mathbf{k}^{2*}$. *Hence the quantum superplane arises as the quadratic algebra*

$$\begin{aligned}\mathbb{A}_q^{0|2} &= \big(\mathbf{k}^{2*}\,,(\,\xi\otimes\xi\,,\eta\otimes\eta\,,\xi\otimes\eta + q\,\eta\otimes\xi\,)\big)\\ &= [\,\mathbb{A}_q^{2|0}\,,J\,]\,.\end{aligned}$$

Notice also that

$$\begin{aligned}\mathbb{A}_q^{0|2}\otimes\mathbb{A}_q^{2|0} &= \big(\mathbf{k}^{2*}\otimes\mathbf{k}^{2},(\,b\otimes a - q\,a\otimes b\,,d\otimes c - q\,c\otimes d\,,\\ &\qquad q^{-1}b\otimes c - q\,c\otimes b + d\otimes a - a\otimes d\,)\big)\end{aligned}$$

(which should be compared with equations $(**)$ *in the proof of Theorem 3.2) where*

$$a = \xi\otimes x \quad,\quad b = \xi\otimes y \quad,\quad c = \eta\otimes x \quad,\quad d = \eta\otimes y$$

gives half the relations required for $M_q(2)$.

Let $\mathcal{V}$ be a tensor category. Write $d\,,e \;:\; B \dashv A$, or briefly $B \dashv A$, for objects A, B of $\mathcal{V}$ and arrows

$$e\,:\,B\otimes A \longrightarrow I \quad, \qquad d\,:\,I \longrightarrow A\otimes B$$

when the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "d\otimes 1_A"] \arrow[dr, "1_A"'] & A\otimes B\otimes A \arrow[d, "1_A\otimes e"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "1_B\otimes d"] \arrow[dr, "1_B"'] & B\otimes A\otimes B \arrow[d, "e\otimes 1_B"] \\ & B \end{tikzcd} \;.$$

We call B a *left dual* for A, and we call A a *right dual* for B. We call e the *counit* and d the *unit*. Duals are uniquely determined up to isomorphism.

The tensor category is called *left autonomous* when each object A has a left dual A^*. Each arrow $f\,:\,A \longrightarrow B$ determines a unique arrow $f^*\,:\,B^* \longrightarrow A^*$ given by the composite

$$B^* \xrightarrow{1\otimes d} B^*\otimes A\otimes A^* \xrightarrow{1\otimes f\otimes 1} B^*\otimes B\otimes A^* \xrightarrow{e\otimes 1} A^* \;.$$

This makes *left dual* into a functor

$$(\_)^* : \mathcal{V}^{\mathrm{op}} \longrightarrow \mathcal{V} .$$

We also have $(A \otimes B)^* \cong B^* \otimes A^*$ and $I^* \cong I$. The tensor category is called *autonomous* when each object A has both a left dual A^* and right dual $A^\vee$.

If $\mathcal{V}$ is a braided tensor category then each left dual A^* is also a right dual with counit and unit, respectively

$$A \otimes A^* \xrightarrow{c_{A,A^*}} A^* \otimes A \xrightarrow{\quad e \quad} I$$

$$I \xrightarrow{\quad d \quad} A \otimes A^* \xrightarrow{c^{-1}_{A,A^*}} A^* \otimes A \quad .$$

This implies $A^{**} \cong A$.

A *tortile tensor category* is an autonomous balanced tensor category in which the twist is related to the dual via the condition

$$\theta_{A^*} \;=\; \theta_A{}^* : A^* \longrightarrow A^* .$$

A left dual A^* for A gives a left internal hom $[A, B] = B \otimes A^*$ with "evaluation":

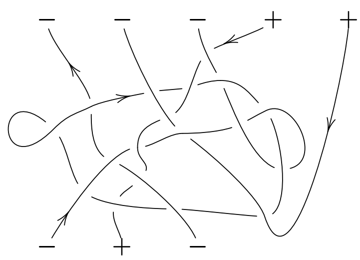
$$e_A \;=\; 1_B \otimes e : B \otimes A^* \otimes A \longrightarrow B$$

for all objects B of $\mathcal{V}$. A right dual $A^\vee$ for A gives a right internal hom $[A, B]' = A^\vee \otimes B$. Hence a *left/right autonomous* tensor category is left-/right-closed.

Example 12.6 *For each commutative ring R, an object M of $\mathbf{Mod}_R$ has a (left) dual if and only if it is cauchy (Theorem 5.3). Write $\mathbf{Prf}_R$ for the full subcategory of $\mathbf{Mod}_R$ consisting of cauchy R-modules. Since $\mathbf{Prf}_R$ is closed under tensor, it is an autonomous (symmetric) tensor category.*

Example 12.7 *Let H be an R-Hopf algebra. An object M of the tensor category $\mathbf{Mod}_R(H)$ has a left dual precisely when it is cauchy as an R-module; in this case, $M^* = \mathrm{Hom}_R(M, R)$ (Proposition 10.1). If H has an invertible antipode then each such M has a right dual $M^\vee = \mathrm{Hom}'_R(M, R)$. Write $\mathbf{Prf}_R(H)$ for the full subcategory of $\mathbf{Mod}_R(H)$ consisting of those H-modules M which are cauchy when viewed as R-modules. For H with invertible antipode, $\mathbf{Prf}_R(H)$ is an autonomous tensor category.*

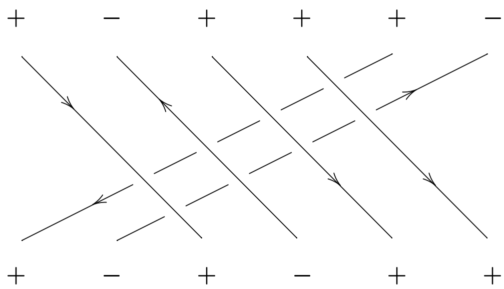
Example 12.8 Tangles on strings. *(This example was discovered by Freyd–Yetter[FY89].) Let P be an Euclidean plane. A* geometric tangle *T is a compact 1-dimensional oriented submanifold of $[0, 1] \times P$ which is* *tamely embedded and whose boundary ∂T is equal to $T \cap \partial([0,1] \times P)$. Thus a geometric tangle T is a disjoint union of directed (topological) circles contained in $(0,1) \times P$ and of directed paths connecting two points on the boundary $\partial([0,1] \times P)$. The* target *of T is the subset $\partial T \cap (\{1\} \times P)$ as an oriented 0-dimensional manifold. The* source *of T is the subset $\partial T \cap (\{0\} \times P)$ but with orientation reversed. A geometric tangle can be pictured as follows:*



A tangle *is an isotopy class of geometric tangles where the isotopies keep the boundaries fixed. The source and target of a tangle are regarded as signed subsets of P. Let $1, 2, 3, \dots$ denote equally-spaced collinear points on P.*

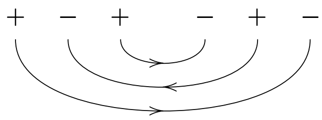
Now we can define the autonomous braided tensor category $\mathcal{T}$ of tangles. *The objects are functions $A : \{1, 2, \dots, n\} \longrightarrow \{+, -\}$ for $n \geq 0$, called* signed sets. *The arrows of $\mathcal{T}$ are the tangles which have these signed sets as sources and targets. Composition and tensor-product are as for braids.*

The braiding is illustrated below.

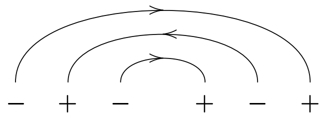


The left dual A^* of a signed set A is given by reversing the order and the sign of the points; that is, $A^*(i)$ and $A(n-i+1)$ are opposite signs for $1 \leq i \leq n$. The counit and unit are illustrated below for $A = \{- + -\}$.

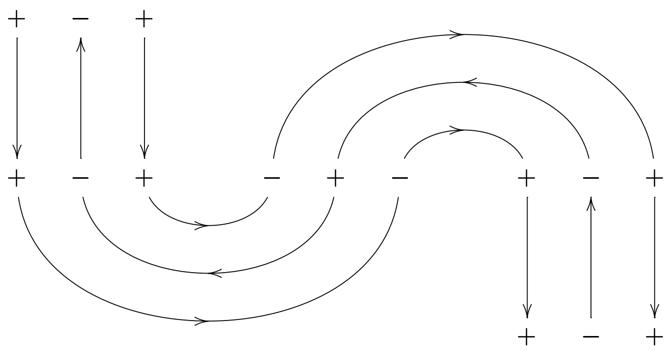
$e : A^{*}\otimes A \longrightarrow I$



$d : I \longrightarrow A\otimes A^{*}$

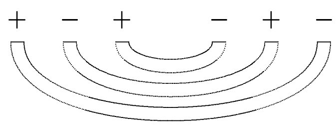


The next diagram proves one triangle for e *and* d*; the other is similar.*

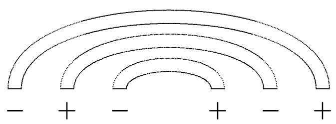


Example 12.9 Tangles on ribbons. *(The full details of this example appear in the thesis of Shum [Shu89]) The category* $\widetilde{\mathcal{T}}$ *of tangles on ribbons is obtained from* $\mathcal{T}$ *just as we obtained* $\widetilde{\mathcal{B}}$ *from* $\mathcal{B}$ *in Example 11.5. The directed strings of tangles are thickened into (directed) ribbons. Ribbons obtained from strings with boundary may be twisted through complete turns. Those which are thickenings of closed strings may have twists, as long as they remain 2-sided 2-manifolds; the Möbius ribbon is not allowed.*

Again we obtain an autonomous tensor category. The counits and units look like this:

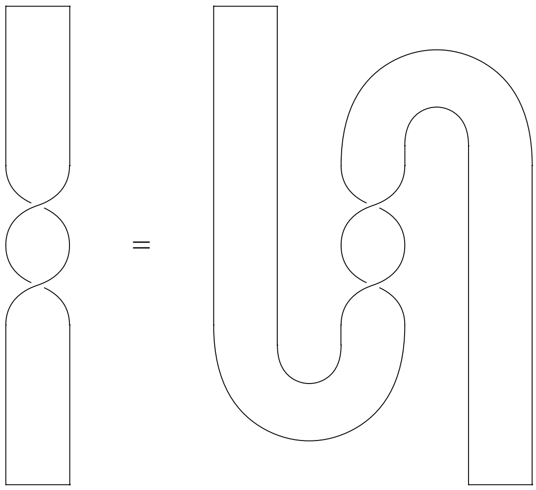


e



d

In fact, $\widetilde{\mathcal{T}}$ is a tortile tensor category. *The twist is as for* $\widetilde{\mathcal{B}}$ *and the identity* $\theta_{A^*} = \theta_A^*$ *can be seen from the following diagram.*



13
Tensor functors and Yang–Baxter operators

Suppose $\mathcal{C}$ and $\mathcal{V}$ are tensor categories. A *tensor functor* $F : \mathcal{C} \longrightarrow \mathcal{V}$ consists of a functor $F : \mathcal{C} \longrightarrow \mathcal{V}$ (denoted by the same symbol) together with a natural isomorphism $\phi_{A,B} : FA \otimes FB \xrightarrow{\cong} F(A \otimes B)$ and another isomorphism $\phi_0 : I \xrightarrow{\cong} FI$, such that

$$\begin{tikzcd}
FA \otimes FB \otimes FC \arrow[r, "\phi_{A,B} \otimes 1"] \arrow[d, "1 \otimes \phi_{B,C}"'] & F(A \otimes B) \otimes FC \arrow[d, "\phi_{A \otimes B \otimes C}"] \\
FA \otimes F(B \otimes C) \arrow[r, "\phi_{A,B \otimes C}"] & F(A \otimes B \otimes C)
\end{tikzcd}$$

and

$$\begin{tikzcd}
FA \arrow[r, "1 \otimes \phi_0"] \arrow[d, "\phi_0 \otimes 1"'] \arrow[dr, "1"] & FA \otimes FI \arrow[d, "\phi_{A,I}"] \\
FI \otimes FA \arrow[r, "\phi_{I,A}"] & FA
\end{tikzcd}$$

(where we have suppressed the constraints a, r, ℓ as usual). If the condition that ϕ, ϕ_0 be invertible is dropped, we have a *weak tensor functor*. If in fact ϕ and ϕ_0 are identities, then F is called a *strict tensor functor*.

If $\mathcal{C}$ and $\mathcal{V}$ are braided, we describe a tensor functor as *braided* when

$$\begin{tikzcd}
FA \otimes FB \arrow[r, "\phi_{A,B} \otimes 1"] \arrow[d, "c_{FA,FB}"'] & F(A \otimes B) \arrow[d, "F(c_{A,B})"] \\
FB \otimes FA \arrow[r, "\phi_{B,A}"] & F(B \otimes A)
\end{tikzcd} \quad .$$

If $\mathcal{C}$ and $\mathcal{V}$ are symmetric, we say F is *symmetric* instead of "braided". If $\mathcal{C}$ and $\mathcal{V}$ are balanced, we say F is *balanced*, when it is braided and

$$F(\theta_A) \;=\; \theta_{FA} : FA \longrightarrow FA \ .$$

Suppose $F : \mathcal{C} \longrightarrow \mathcal{V}$ is a weak tensor functor and $\mathcal{C}$ and $\mathcal{V}$ are left-closed tensor categories. Then the composite

$$F[A\,,B\,]{\otimes}FA \xrightarrow{\ \phi_{[A,B],A}\ } F([A\,,B\,]{\otimes}A) \xrightarrow{\ F\,e_A\ } FB$$

corresponds, using the defining property of $[FA\,,FB\,]$, to an arrow

$$\tilde{\phi}_{A,B} : F[A\,,B\,] \longrightarrow [FA\,,FB\,] \ .$$

We call F a *left-closed tensor functor* when each $\tilde{\phi}_{A,B}$ is invertible; *right closed* and *closed* are now defined in the obvious way. [This differs from the notion of "closed functor" in the literature].

When it comes to duals, the situation is better: *tensor functors preserve duals.* More precisely, if $F : \mathcal{C} \longrightarrow \mathcal{V}$ is a tensor functor and $d\,,e\ :\ B \dashv A$ in $\mathcal{C}$, then $FB \dashv FA$ in $\mathcal{V}$ with unit

$$I \xrightarrow{\ \phi_0\ } FI \xrightarrow{\ Fd\ } F(B{\otimes}A) \xrightarrow{\ \phi^{-1}_{[A,B],A}\ } FA{\otimes}FB$$

and counit

$$FB{\otimes}FA \xrightarrow{\ \phi_{[B,A]}\ } F(B{\otimes}A) \xrightarrow{\ Fe\ } FI \xrightarrow{\ \phi_0^{-1}\ } I \ .$$

Hence, if $\mathcal{C}$ is (left-) autonomous and $\mathcal{V}$ is (left-) closed, then each tensor functor $\mathcal{C} \longrightarrow \mathcal{V}$ is (left-) closed (since $[A\,,B\,] = B{\otimes}A^*$ in $\mathcal{C}$).

Example 13.1 *The category* **Set** *of (small) sets is a tensor category using cartesian product as tensor product. For each commutative ring R, the "free module functor" (see Section 4)*

$$\mathcal{F}_R : \mathbf{Set} \longrightarrow \mathbf{Mod}_R$$

is a tensor functor. It is certainly not *closed since we have that*

$$\mathrm{Hom}_R(\mathcal{F}_R X, \mathcal{F}_R Y) \;\cong\; (\mathcal{F}_R Y)^X \;\ncong\; Y^X \ .$$

The functor $|\_| : \mathbf{Mod}_R \longrightarrow \mathbf{Set}$ *which takes each module to its underlying set $|M|$ is a good example of a weak tensor functor: we have functions*

$$\begin{aligned} \phi_{M,N} \ : \ |M| \times |N| &\longrightarrow |M{\otimes}N| \\ (m\,,n) &\longmapsto m{\otimes}n \\ \phi_0 \ = \ \eta \ : \quad 1 \quad &\longrightarrow |R| \end{aligned}$$

which are not *invertible.*

Example 13.2 Universal Algebra. *The "universal enveloping algebra" functor* $\mathcal{U} : \mathbf{Lie}_R \longrightarrow \mathbf{Alg}_R$ *is a tensor functor (see Proposition 6.10).*

Example 13.3 Yang–Baxter operators. *We want to examine what is involved in giving a tensor functor* $F : \mathcal{B} \longrightarrow \mathcal{V}$ *from the braid category into an arbitrary tensor category* $\mathcal{V}$.

A Yang–Baxter (YB) operator on an object A *of* $\mathcal{V}$ *is an invertible arrow* $y : A{\otimes}A \longrightarrow A{\otimes}A$ *such that the following hexagon commutes.*

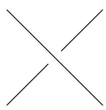
$$\begin{tikzcd} & A{\otimes}A{\otimes}A \arrow[r, "1{\otimes}y"] & A{\otimes}A{\otimes}A \arrow[dr, "y{\otimes}1"] & \\ A{\otimes}A{\otimes}A \arrow[ur, "y{\otimes}1"] \arrow[dr, "1{\otimes}y"'] & & & A{\otimes}A{\otimes}A \\ & A{\otimes}A{\otimes}A \arrow[r, "y{\otimes}1"] & A{\otimes}A{\otimes}A \arrow[ur, "1{\otimes}y"'] & \end{tikzcd}$$

For example, the object 1 *of* $\mathcal{B}$ *admits the following YB-operator:*

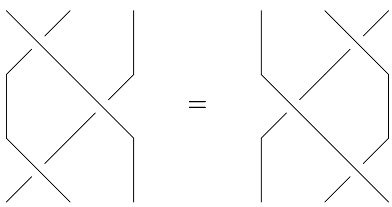
$$c_{1,1} = \ s_1 \ : 1 + 1 \longrightarrow 1 + 1$$

which is the element of the braid group $\mathcal{B}_\mathbf{2}$ *depicted by the diagram:*

s_1 :



The YB-hexagon becomes the following simple identity.



Since any tensor functor $F : \mathcal{B} \longrightarrow \mathcal{V}$ *preserves tensor products "up to coherent isomorphism", we obtain a YB-operator* y *on* $F(1) = A$ *; namely,*

$$y \; : \; A \otimes A \xrightarrow{\phi_{1,1}} F(1+1) \xrightarrow{Fs_1} F(1+1) \xrightarrow{\phi_{1,1}^{-1}} A \otimes A \; .$$

Conversely, given a YB-operator y *on an object* A *of* $\mathcal{V}$*, we can determine a tensor functor* $F : \mathcal{B} \longrightarrow \mathcal{V}$ *such that* $F(1) = A$ *and* y *is the above composite. In fact,* F *is unique up to isomorphism, arising from the different possible choices for the* n*-fold tensor product* $A^{\otimes n}$*). Since* F *is to be a tensor functor, we are forced to have*

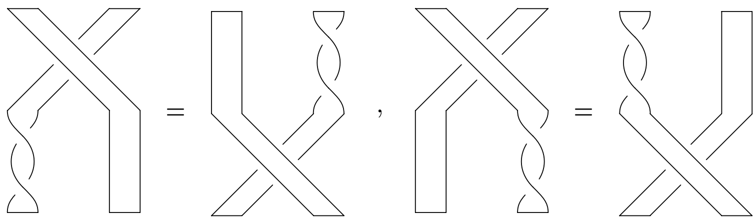
$$F(n) \; = \; F(1+1+\ldots+1) \; \cong \; A^{\otimes n} \; .$$

Each generator s_i *of* $\mathcal{B}_{\mathbf{n}}$ *can be written as* $s_i = \mathbf{1}_{i-1} \otimes s_1 \otimes \mathbf{1}_{n-i-1}$ *in* $\mathcal{B}$*, so the definition of* $Fs_i : A^{\otimes n} \longrightarrow A^{\otimes n}$ *is forced. We just need to check that this is compatible with the braid relations (Example 11.1); but this follows from the YB-hexagon and the functoriality of tensor product. Details are left as an exercise (which is worth doing).*

Hence, up to the appropriate notion of isomorphism, tensor functors $F : \mathcal{B} \longrightarrow \mathcal{V}$ *correspond to pairs* (A, y) *consisting of an object* A *of* $\mathcal{V}$ *and a YB-operator* y *on* A*. We can express this by saying:*

> $(\mathcal{B}, 1, s_1)$ is the free tensor category having an object equipped with a *YB*-operator.

Example 13.4 *A tensor functor from ribbons* $F : \widetilde{\mathcal{B}} \longrightarrow \mathcal{V}$ *also determines a YB-operator* y *on* $F1 = A$ *as in Example* **??***ex133 This time the twist* $\theta_1 : 1 \longrightarrow 1$ *in* $\widetilde{\mathcal{B}}$ *gives an isomorphism* $z \; = \; F\theta_1 : A \longrightarrow A$ *. Due to the equalities in* $\widetilde{\mathcal{B}}$ *...*



this gives an example of the following concept.

A YB-operator y *on an object* A *of* $\mathcal{V}$ *is said to be* balanced *when it is equipped with an isomorphism* $z : A \longrightarrow A$ *such that*

$$\begin{tikzcd} A \otimes A \arrow[r, "y"] \arrow[d, "1 \otimes z"'] & A \otimes A \arrow[d, "z \otimes 1"] \\ A \otimes A \arrow[r, "y"] & A \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[r, "y"] \arrow[d, "z \otimes 1"'] & A \otimes A \arrow[d, "1 \otimes z"] \\ A \otimes A \arrow[r, "y"] & A \end{tikzcd}$$

Each balanced YB-operator determines a unique (once the n-fold tensors $A^{\otimes n}$ are chosen) tensor functor $F : \widetilde{\mathcal{B}} \longrightarrow \mathcal{V}$ from which A, y, z are recovered as above. So ...

> $(\widetilde{\mathcal{B}}, 1, c_{1,1}, \theta_1)$ is the free tensor category containing an object equipped with a balanced YB-operator.

The easier part of Example **??**ex133 can be obtained from two observations.

- tensor functors take YB-operators into YB-operators.
  More precisely, if $F : \mathcal{C} \longrightarrow \mathcal{V}$ is a tensor functor and if y is a YB-operator on X in $\mathcal{C}$, then $Fy : F(X \otimes X) \longrightarrow F(X \otimes X)$ carries across the isomorphism $\phi_{X,X}$ to a YB-operator y on FX in $\mathcal{V}$.
- a braiding on a tensor category gives, on each object X, a YB-operator. $c_{X,X} : X \otimes X \longrightarrow X \otimes X$;

Moreover, tensor functors take balanced YB-operators into balanced YB-operators. Furthermore, in a balanced tensor category there is a balanced YB-operator $(c_{X,X}, \theta_X)$ on each object X. (See Example **??**ex134)

We now look at compatibility of YB-operators with duals .

A YB-operator y on A is called *(left-) dualizable* when A has a left dual A^* and both the arrows $u, v : A^* \otimes A \longrightarrow A^*$ given by the composites:

$$A^* \otimes A \xrightarrow{1 \otimes 1 \otimes d} A^* \otimes A \otimes A \otimes A^* \underset{1 \otimes y^{-1} \otimes 1}{\overset{1 \otimes y \otimes 1}{\rightrightarrows}} A^* \otimes A \otimes A \otimes A^* \xrightarrow{e \otimes 1 \otimes 1} A \otimes A^*$$

are invertible. It follows that the composite w given by ...

$$A^* \otimes A^* \xrightarrow{1 \otimes 1 \otimes d} A^* \otimes A^* \otimes A \otimes A^* \xrightarrow{1 \otimes u \otimes 1} A^* \otimes A \otimes A^* \otimes A^* \xrightarrow{e \otimes 1 \otimes 1} A^* \otimes A^*$$

has inverse w^{-1} given by the composite:

$$A^* \otimes A^* \xrightarrow{1 \otimes 1 \otimes d} A^* \otimes A^* \otimes A \otimes A^* \xrightarrow{1 \otimes v \otimes 1} A^* \otimes A \otimes A^* \otimes A^* \xrightarrow{e \otimes 1 \otimes 1} A^* \otimes A^* \cdot$$

Proposition 13.5 *In a braided tensor category, if an object A has a dual, then $Y = c_{A,A}$ is a dualizable YB-operator on A with*

$$u = c^{-1}_{A,A^*} \quad , \quad v = c^{-1}_{A^*,A} \quad , \quad w = c^{-1}_{A^*,A^*} \ .$$

Proof. To prove $c_{A,A^*} \circ u = 1_{A^* \otimes A}$, it suffices (by the property of duals) to show that equality holds after applying $A \otimes \_$ to both sides and composing with $d \otimes 1_A$. Thus the following diagram gives the first equation.

$$
\begin{tikzcd}
 & A\otimes A^*\otimes A\otimes A\otimes A^* \arrow[rrrr, "1\otimes 1\otimes c_{A,A}\otimes 1"] & & & & A\otimes A^*\otimes A\otimes A\otimes A^* \arrow[dr, "1\otimes e\otimes 1\otimes 1"] & \\
A\otimes A^*\otimes A \arrow[ur, "1\otimes 1\otimes 1\otimes d"] & & A\otimes A\otimes A^* \arrow[ul, "d\otimes 1\otimes 1\otimes 1"'] \arrow[rr, "c_{A,A}\otimes 1"] \arrow[drrrr, "c_{A,A\otimes A^*}"] & & A\otimes A\otimes A^* \arrow[ur, "d\otimes 1\otimes 1\otimes 1"] \arrow[rr, "1"] & & A\otimes A\otimes A^* \arrow[d, "1\otimes c_{A,A^*}"] \\
A \arrow[u, "d\otimes 1"] \arrow[urr, "1\otimes d"'] \arrow[rrr, "c_{A,I} = 1"'] & & & A \arrow[rrr, "d\otimes 1"'] & & & A\otimes A^*\otimes A
\end{tikzcd}
$$

The result for v follows by using the braiding $c^{-1}_{B,A}$ in place of $c_{A,B}$. For w, consider A^* in place of A.

Tensor functors $F : \mathcal{C} \longrightarrow \mathcal{V}$ preserve dualizability. So if $\mathcal{C}$ is braided and X has a dual in $\mathcal{V}$, we obtain a dualizable *YB*-operator on FX in $\mathcal{V}$.

A *YB*-operator on A is called *tortile* when it is balanced, dualizable, and

$$
\begin{tikzcd}
 & & & A\otimes A\otimes A^* \arrow[drr, "1\otimes v^{-1}"] & & & \\
 & A\otimes A\otimes A^* \arrow[urr, "y\otimes 1"] & & & & A\otimes A^*\otimes A \arrow[dr, "1\otimes e"] & \\
A \arrow[ur, "1\otimes d"] \arrow[rrr, "z"] & & & A \arrow[rrr, "z"] & & & A
\end{tikzcd}
$$

Proposition 13.6 *In a balanced tensor category, if an object A has a dual then the pair $(c_{A,A}, \theta_A)$ is a tortile YB-operator precisely when $\theta_{A^*} = (\theta_A)^*$.*

Proof. The following diagram proves the equation

$$(\theta_{A^*})^* \theta_A = (1\otimes e) \circ (1\otimes v^{-1}) \circ (y\otimes 1) \circ (1\otimes d) \ .$$

$$\begin{tikzcd}
A \arrow[r, "\theta_A"] \arrow[d, "1\otimes d"'] & A \arrow[r, "1\otimes d"] & A\otimes A\otimes A^* \arrow[r, "1\otimes c^{-1}_{A^*,A}"] & A\otimes A^*\otimes A \arrow[d, "1\otimes\theta_{A^*}\otimes 1"] \\
A\otimes A\otimes A^* \arrow[rr, "1\otimes c^{-1}_{A^*,A}"] \arrow[dr, "c^{-1}_{A^*,A}"] \arrow[ddd, "y\otimes 1"', "= c_{A,A}\otimes 1"] & & A\otimes A^*\otimes A \arrow[r, "\theta_A\otimes\theta_{A^*}\otimes 1"'] \arrow[ur, "\theta_A\otimes 1\otimes 1"] \arrow[dl, "c^{-1}_{A^*,A}\otimes 1"] & A\otimes A^*\otimes A \arrow[d, "c^{-1}_{A,A^*}\otimes 1"] \\
 & A^*\otimes A\otimes A \arrow[rr, "\theta_{A^*A}\otimes 1"'] \arrow[d, "1\otimes c_{A,A}"'] \arrow[dr, "e\otimes 1"] \arrow[ddr, "c_{A^*A,A}"] & & A^*\otimes A\otimes A \arrow[d, "e\otimes 1"] \\
 & A^*\otimes A\otimes A \arrow[dr, "c_{A^*,A}\otimes 1"'] & A \arrow[r, "\theta_I = I"] \arrow[dr, "c_{I,A} = 1"] & A \\
A\otimes A\otimes A^* \arrow[ur, "c^{-1}_{A^*,A\,A}"] \arrow[rr, "1\otimes v^{-1} = 1\otimes c^{-1}_{A^*,A}"'] & & A\otimes A^*\otimes A \arrow[r, "1\otimes e"'] & A \arrow[u, "1"']
\end{tikzcd}$$

So the balanced *YB*-operator (y,z) is tortile if and only if $(\theta_{A^*})^*\theta_A = z^2$; that is, if and only if $(\theta_{A^*})^*\theta_A = (\theta_A)^2 \;\Leftrightarrow\; (\theta_{A^*})^* = \theta_A \;\Leftrightarrow\; \theta_{A^*} = (\theta_A)^*$.

It follows that, in a tortile tensor category, each object A is equipped with a tortile *YB*-operator $(c_{A,A}\,,\theta_A)$.

Example 13.7 *Since $\widetilde{\mathcal{T}}$ is a tortile tensor category, we obtain a tortile YB-operator $(c_{+,+}\,,\theta_+)$ on the object $+$ of $\widetilde{\mathcal{T}}$. Thus, each tensor functor $F : \widetilde{\mathcal{T}} \longrightarrow \mathcal{V}$ yields a tortile YB-operator on $F(+)$ in $\mathcal{V}$.*

In fact, $\left(\widetilde{\mathcal{T}}\,,+\,,c_{+,+}\,,\theta_+\right)$ is the free tensor category equipped with a tortile YB-operator ([Shu89] together with [JS91b]). This means that, given a tortile YB-operator (y,z) on an object A in a tensor category $\mathcal{V}$, there exists a (unique up to isomorphism) tensor functor $F : \widetilde{\mathcal{T}} \longrightarrow \mathcal{V}$ which takes $\left(+\,,c_{+,+}\,,\theta_+\right)$ to $\left(A\,,y\,,z\right)$. We do not intend to prove this here; after all, our geometric description of $\widetilde{\mathcal{T}}$ was incomplete. We hope the result is believable. All we really need is that such a free $\widetilde{\mathcal{T}}$ should exist; but the description of the realistic model is too pretty to omit.

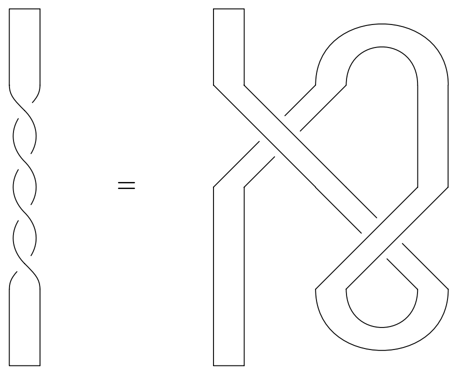
It should be clear how to define F in terms of A, y, z. For example,

$$F(+---+-) \quad = \quad A\otimes A^*\otimes A^*\otimes A^*\otimes A\otimes A^*$$

$$\begin{aligned} F(c_{+,-}) &= u^{-1} \\ F(c_{-,-}) &= w \\ F(\theta_+) &= z \\ F(\theta_-) &= z^* \end{aligned}$$

and so on. Any tangle of ribbons can be decomposed, using composition and tensor product in $\widetilde{\mathcal{T}}$, *into single crossings* $\big(c_{+,+}\,, c_{+,-}\,, c_{-,+}\,, c_{-,-}\big)$ *and their inverses), turnings (e and d) and twists* (θ_+, θ_-, *and their inverses). So the value of F on the tangle is forced. The hard part, which we shall not include in these notes, is to show that this value is independent of the decomposition.*

It is instructive to see in this example what is meant by the equation $z^2 = (1{\otimes}e)\circ(1{\otimes}v^{-1})\circ(y{\otimes}1)\circ(1{\otimes}d)$. *It is expressed by the following diagram (which can be tested be taking off your belt).*



14

A tortile Yang–Baxter operator for each finite-dimensional vector space

Let $\mathbf{k}$ be a field and let $q \in \mathbf{k}$ be a fixed non-zero element. Let V be a vector space over $\mathbf{k}$ with basis $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$. Define a linear function

$$y : V \otimes V \longrightarrow V \otimes V$$

on the basis elements $\varepsilon_i \otimes \varepsilon_j$ of $V \otimes V$ by

$$y(\varepsilon_i \otimes \varepsilon_j) = \begin{cases} \varepsilon_j \otimes \varepsilon_i & \text{for } i > j \\ \varepsilon_j \otimes \varepsilon_i + (q - q^{-1})\, \varepsilon_i \otimes \varepsilon_j & \text{for } i < j \\ q\, \varepsilon_i \otimes \varepsilon_i & \text{for } i = j\ . \end{cases}$$

In order to check the *YB*-hexagon for y, we look at $(y \otimes 1)(1 \otimes y)(y \otimes 1)$, $(1 \otimes y)(y \otimes 1)(1 \otimes y)$ at each $\varepsilon_i \otimes \varepsilon_j \otimes \varepsilon_k$. There are thirteen of these cases to check to account for all possible relative positions of i, j and k. We shall only give three of these cases as an illustration: put $r = q - q^{-1}$ and omit the ε and $\otimes$ symbols from the notation.

$$\begin{aligned} i < j < k \ :\ (ijk) \ &\xmapsto{y \otimes 1}\ (jik) + \rho\,(ijk) \\ &\xmapsto{1 \otimes y}\ (jki) + \rho\,(jik) + \rho\,(ikj) + \rho^2(ijk) \\ &\xmapsto{y \otimes 1}\ (kji) + \rho\,(jki) + \rho\,(ijk) + \rho\,(kij) + \rho^2(ikj) \\ &\qquad\qquad + \rho^2(jik) + \rho^3(ijk) \end{aligned}$$

$$\begin{aligned} (ijk) \ &\xmapsto{1 \otimes y}\ (ikj) + \rho\,(ijk) \\ &\xmapsto{y \otimes 1}\ (kij) + \rho\,(ikj) + \rho\,(jik) + \rho^2(ijk) \\ &\xmapsto{1 \otimes y}\ (kji) + \rho\,(kij) + \rho\,(ijk) + \rho\,(jki) + \rho^2(jik) \\ &\qquad\qquad + \rho^2(ijk) + \rho^3(ijk) \end{aligned}$$

$$
\begin{array}{rlll}
j<i<k\ :\ (ijk) & \xmapsto{y\otimes 1} & (jik) \\
 & \xmapsto{1\otimes y} & (jki)+\rho\,(jik) \\
 & \xmapsto{y\otimes 1} & (kji)+\rho\,(jki)+\rho\,(ijk)+\rho^2(jik) \\
(ijk) & \xmapsto{1\otimes y} & (ikj)+\rho\,(ijk) \\
 & \xmapsto{y\otimes 1} & (kij)+\rho\,(ikj)+\rho\,(jik) \\
 & \xmapsto{1\otimes y} & (kji)+\rho\,(ijk)+\rho\,(jki)+\rho^2(jik)
\end{array}
$$

$$
\begin{array}{rlll}
i=j<k\ :\ (iik) & \xmapsto{y\otimes 1} & q\,(iik) \\
 & \xmapsto{1\otimes y} & q\,(iki)+\rho q\,(iik) \\
 & \xmapsto{y\otimes 1} & q\,(kii)+\rho q\,(iki)+\rho q^2(iik) \\
(iik) & \xmapsto{1\otimes y} & (iki)+\rho\,(iik) \\
 & \xmapsto{y\otimes 1} & (kii)+\rho\,(iki)+\rho q\,(iik) \\
 & \xmapsto{1\otimes y} & q\,(kii)+\rho\,(iik)+\rho q\,(iki)+\rho^2 q\,(iik)
\end{array}
$$

(Note that $q^2 r = r + q r^2$ since $r = q - q^{-1}$.)

Clearly y is invertible with inverse given by:

$$
y^{-1}(\varepsilon_i\otimes\varepsilon_j) \;=\; \begin{cases} \varepsilon_j\otimes\varepsilon_i & \text{for } i<j \\ \varepsilon_j\otimes\varepsilon_i + (q^{-1}-q)\,\varepsilon_i\otimes\varepsilon_j & \text{for } i>j \\ q^{-1}\varepsilon_i\otimes\varepsilon_i & \text{for } i=j. \end{cases}
$$

Hence, *y is a YB-operator on the object V of* $\mathbf{Mod}_{\mathbf{k}}$.

It is now possible to calculate the operators u, v, w and their inverses (see the definition of dualizable *YB*-operator in Section 13). For this, let $\varepsilon_1^*,\ldots,\varepsilon_n^* \in V^*$ be the dual basis for $\varepsilon_1,\ldots,\varepsilon_n \in V$; this means

$$
\varepsilon_i^*(\varepsilon_j) = \delta_{ij}\ .
$$

Recall that $e : V^*\otimes V \longrightarrow \mathbf{k}$ is the evaluation functor and $d = \sum_k \varepsilon_k\otimes\varepsilon_k^*$. Now we obtain ...

$$
\begin{aligned}
u(\varepsilon_i^*\otimes\varepsilon_j) &= (e\otimes 1\otimes 1)(1\otimes y\otimes 1)(1\otimes 1\otimes d)(\varepsilon_i^*\otimes\varepsilon_j) \\
&= \sum_k (e\otimes 1\otimes 1)(1\otimes y\otimes 1)(\varepsilon_i^*\otimes\varepsilon_j\otimes\varepsilon_k\otimes\varepsilon_k^*) \\
&= \sum_{k>j} (e\otimes 1\otimes 1)\left(\varepsilon_i^*\otimes\varepsilon_k\otimes\varepsilon_j\otimes\varepsilon_k^* + (q-q^{-1})\,\varepsilon_i^*\otimes\varepsilon_j\otimes\varepsilon_k\otimes\varepsilon_k^*\right)
\end{aligned}
$$

$$
\begin{aligned}
& + q\,(e\otimes 1\otimes 1)\,(\varepsilon_i^*\otimes\varepsilon_j\otimes\varepsilon_j\otimes\varepsilon_j^*) + \sum_{k<j}(e\otimes 1\otimes 1)\,(\varepsilon_i^*\otimes\varepsilon_k\otimes\varepsilon_j\otimes\varepsilon_k^*) \\
= \ & \sum_{k>j}\big(\delta_{ik}\,\varepsilon_j\otimes\varepsilon_k^* + (q-q^{-1})\,\delta_{ij}\,\varepsilon_k\otimes\varepsilon_k^*\big) + q\,\delta_{ij}\,(\varepsilon_j\otimes\varepsilon_j^*) \\
& + \sum_{k<j}\delta_{ik}\,\varepsilon_j\otimes\varepsilon_k^* \\
= \ & \begin{cases} \varepsilon_j\otimes\varepsilon_i^* & \text{for } j<i \\ \varepsilon_j\otimes\varepsilon_i^* & \text{for } i<j \\ q^{-1}\,\varepsilon_i\otimes\varepsilon_i^* \ + \ \sum_{k>i}(q-q^{-1})\,\varepsilon_k\otimes\varepsilon_k^* & \text{for } i=j\ . \end{cases}
\end{aligned}
$$

The other operators are calculated similarly. We record the results below:

$$
\begin{aligned}
u(\varepsilon_i^*\otimes\varepsilon_j) &= \begin{cases} \varepsilon_j\otimes\varepsilon_i^* & \text{for } i\neq j \\ q\,\varepsilon_i\otimes\varepsilon_i^* \ + \ \sum_{k>i}(q-q^{-1})\,\varepsilon_k\otimes\varepsilon_k^* & \text{for } i=j \end{cases} \\
u^{-1}(\varepsilon_i\otimes\varepsilon_j^*) &= \begin{cases} \varepsilon_j^*\otimes\varepsilon_i & \text{for } i\neq j \\ q^{-1}\varepsilon_i^*\otimes\varepsilon_i \ + \ \sum_{k>i}(q^{-1}-q)q^{-2(k-i)}\,\varepsilon_k^*\otimes\varepsilon_k & \text{for } i=j \end{cases} \\
v(\varepsilon_i^*\otimes\varepsilon_j) &= \begin{cases} \varepsilon_j\otimes\varepsilon_i^* & \text{for } i\neq j \\ q^{-1}\,\varepsilon_i\otimes\varepsilon_i^* \ + \ \sum_{k<i}(q^{-1}-q)\,\varepsilon_k\otimes\varepsilon_k^* & \text{for } i=j \end{cases} \\
v^{-1}(\varepsilon_i\otimes\varepsilon_j^*) &= \begin{cases} \varepsilon_j^*\otimes\varepsilon_i & \text{for } i\neq j \\ q\,\varepsilon_i^*\otimes\varepsilon_i \ + \ \sum_{k<i}(q-q^{-1})q^{2(i-k)}\,\varepsilon_k^*\otimes\varepsilon_k & \text{for } i=j \end{cases} \\
w(\varepsilon_i^*\otimes\varepsilon_j^*) &= \begin{cases} \varepsilon_j^*\otimes\varepsilon_i^* & \text{for } i<j \\ \varepsilon_j^*\otimes\varepsilon_i^* \ + \ (q-q^{-1})\,\varepsilon_i^*\otimes\varepsilon_j^* & \text{for } i>j \\ q\,\varepsilon_i^*\otimes\varepsilon_i^* & \text{for } i=j \end{cases} \\
w^{-1}(\varepsilon_i^*\otimes\varepsilon_j^*) &= \begin{cases} \varepsilon_j^*\otimes\varepsilon_i^* & \text{for } i>j \\ \varepsilon_j^*\otimes\varepsilon_i^* \ + \ (q^{-1}-q)\,\varepsilon_i^*\otimes\varepsilon_j^* & \text{for } i<j \\ q^{-1}\varepsilon_i^*\otimes\varepsilon_i^* & \text{for } i=j\ . \end{cases}
\end{aligned}
$$

Hence y is dualizable. It enriches to a balanced *YB*-operator on defining $z : V \longrightarrow V$ simply to be the homothety

$$z(x) \ = \ q^n\,x \quad .$$

Proposition 14.1 *The YB-operator (y, z) defined above is tortile.*

Proof. First observe that, for $i \neq j$, the value of $e\,v^{-1}$ at $\varepsilon_i \otimes \varepsilon_j^*$ is 0; while for $i = j$ the value is

$$\begin{aligned}
& q + \sum_{k<i} (q - q^{-1}) q^{2(i-k)} \\
= \; & q + (q - q^{-1}) \big(q^{2(i-1)} + q^{2(i-2)} + \ldots + q^{2(i-(i-1))} \big) \\
= \; & q + q(q^2 - 1)\, \frac{(q^2)^{i-1} - 1}{q^2 - 1} \\
= \; & q + q(q^2)^{i-1} - q \;\; = \;\; q^{2i-1} \;.
\end{aligned}$$

Hence, we have the following remarkable calculation:

$$\begin{aligned}
& (1 \otimes e)(1 \otimes v^{-1})(y \otimes 1)(1 \otimes d)(\varepsilon_i) \\
= \; & \sum_j (1 \otimes (e\,v^{-1}))(y \otimes 1)(\varepsilon_i \otimes \varepsilon_j \otimes \varepsilon_j^*) \\
= \; & \sum_{j<i} (1 \otimes (e\,v^{-1}))(\varepsilon_j \otimes \varepsilon_i \otimes \varepsilon_j^*) + (1 \otimes (e\,v^{-1}))\, q\, (\varepsilon_i \otimes \varepsilon_i \otimes \varepsilon_i^*) \\
& \quad + \sum_{i<j} (1 \otimes (e\,v^{-1})) \big((\varepsilon_j \otimes \varepsilon_i \otimes \varepsilon_j^*) + (q - q^{-1})\, (\varepsilon_i \otimes \varepsilon_j \otimes \varepsilon_j^*) \big) \\
= \; & 0 \; + \; q\, q^{2i-1}\, \varepsilon_i \; + \; 0 \; + \; \sum_{i<j} (q - q^{-1}) q^{2j-1}\, \varepsilon_i \\
= \; & \big(q^{2i} + (q - q^{-1})(q^{2i+1} + q^{2i+3} + \ldots + q^{2n-1}) \big)\, \varepsilon_i \\
= \; & \big(q^{2i} + (q^2 - 1)\, q^{2i} (1 + q^2 + \ldots + q^{2(n-i-1)}) \big)\, \varepsilon_i \\
= \; & \left(q^{2i} + (q^2 - 1)\, q^{2i}\, \frac{(q^2)^{n-i} - 1}{q^2 - 1} \right) \varepsilon_i \\
= \; & \big(q^{2i} + q^{2i}(q^{2n-2i} - 1) \big)\, \varepsilon_i \\
= \; & q^{2n}\, \varepsilon_i \\
= \; & z(z(\varepsilon_i)) \;.
\end{aligned}$$

15

Monoids in tensor categories

A *monoid* in a tensor category $\mathcal{V}$ consists of an object A and arrows

$$\mu : A \otimes A \longrightarrow A \qquad \text{and} \qquad \eta : I \longrightarrow A$$

which satisfy the usual identity and associativity conditions (see Sections 1 and 6). A *monoid arrow* $f : A \longrightarrow B$ is an arrow in $\mathcal{V}$ which preserves μ and η, in the diagrammatically expressed sense.

It is also useful to consider "arrows between monoid arrows". Suppose that $f, g : A \longrightarrow B$ are monoid arrows. A *2-cell*

$$\xi : f \Rightarrow g : A \longrightarrow B$$

is defined to be an arrow $\xi : I \longrightarrow B$ in $\mathcal{V}$ such that

$$\begin{tikzcd} A \arrow[r, shift left, "\xi\otimes f"] \arrow[r, shift right, "g\otimes\xi"'] & B\otimes B \arrow[r, "\mu"] & B \end{tikzcd} .$$

The more 2-dimensional notation

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\xi"] & B \end{tikzcd}$$

is also used. [In the case where $\mathcal{V} = \mathbf{Set}$, such a 2-cell amounts to an element $\xi \in B$ for which $\xi f(a) = g(a)\xi$ for all $a \in A$.]

There are two basic *pasting operations* for 2-cells. Given the situation

$$\begin{tikzcd} A' \arrow[r, "h"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\xi"] & B \arrow[r, "k"] & B' \end{tikzcd}$$

where the arrows are monoid arrows and ξ is a 2-cell, there is a 2-cell

$$\begin{tikzcd} A' \arrow[r, bend left, "kfh"] \arrow[r, bend right, "kgh"'] \arrow[r, phantom, "\Downarrow k\xi"] & B' \end{tikzcd}$$

obtained since we have the following factorization:

$$
\begin{tikzcd}
A' \arrow[r, "h"] \arrow[rr, bend left, "\xi\otimes(fh)"] \arrow[rr, bend right, "(gh)\otimes\xi"'] & A \arrow[r, shift left, "\xi\otimes f"] \arrow[r, shift right, "g\otimes\xi"'] & B\otimes B \arrow[r, "\mu"] \arrow[d, "k\otimes k"] & B \arrow[d, "k"] \\
& & B'\otimes B' \arrow[r, "\mu"] & B'
\end{tikzcd}
$$

This is called *whiskering* ξ by h and k.

The other basic pasting operation is *vertical composition*, which takes a pair of 2-cells ξ and ζ, as in the following situation

$$
\begin{tikzcd}
A \arrow[r, bend left=50, "f"{name=F}] \arrow[r, "g"'{name=G}] \arrow[r, bend right=50, "h"'{name=H}] & B
\arrow[from=F, to=G, phantom, "\Downarrow\xi"] \arrow[from=G, to=H, phantom, "\Downarrow\zeta"]
\end{tikzcd}
$$

to a 2-cell

$$
\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\zeta*\xi"] & B
\end{tikzcd}
$$

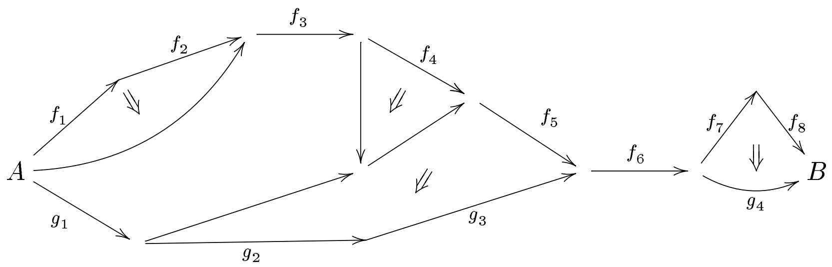
where $\zeta * \xi \;=\; \big(\, I \xrightarrow{\zeta\otimes\xi} B\otimes B \xrightarrow{\mu} B \,\big)$.

The *identity 2-cell* $\eta : f \Rightarrow f : A \longrightarrow B$ is an identity for the operation of vertical composition.

When we write a diagram such as

$$
\begin{tikzcd}
& {} \arrow[rr, "f_2"] & & {} \arrow[dr, "f_3"] & \\
A \arrow[ur, "f_1"] \arrow[dr, "g_1"'] & & \Downarrow\xi & & B \\
& {} \arrow[r, "g_2"'] & {} \arrow[r, "g_3"'] & {} \arrow[ur, "g_4"'] &
\end{tikzcd}
$$

it is intended that $\xi : f_3\, f_2\, f_1 \Rightarrow g_4\, g_3\, g_2\, g_1 : A \longrightarrow B$. This allows us to define a more general *pasting* operation, which assigns to a diagram like



a 2-cell $f_8\, f_7\, f_6\, f_5\, f_4\, f_3\, f_2\, f_1 \Rightarrow g_4\, f_6\, g_3\, g_2\, g_1 : A \longrightarrow B$, obtained (quite possibly in several different ways) by first whiskering the 2-cells in the diagram to be of the form

$$\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

as well as being vertically composable, and then composing vertically.

As an example of this pasting, consider the diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "h"'] \arrow[rr, phantom, shift right=6, "\Downarrow\xi"] & & {} \arrow[dr, "g"] & \\ & {} \arrow[ur, "l"] \arrow[rr, "k"'] \arrow[rr, phantom, shift left=6, "\Downarrow\zeta"] & & B \end{tikzcd}$$

First whisker ξ and ζ appropriately, as in

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "lh"'] \arrow[r, phantom, "\Downarrow\xi"] & {} \arrow[r, "g"] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "h"] & {} \arrow[r, bend left, "gl"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\zeta"] & B \end{tikzcd}$$

to obtain two vertically composable 2-cells

$$\begin{tikzcd} A \arrow[rr, bend left=40, "gf"] \arrow[rr, "glk" description] \arrow[rr, bend right=40, "kh"'] \arrow[rr, phantom, shift left=4, "\Downarrow g\xi"] \arrow[rr, phantom, shift right=4, "\Downarrow g\zeta"] & & B \end{tikzcd} .$$

Then vertically compose to obtain a 2-cell

$$\begin{tikzcd} A \arrow[r, bend left, "gf"] \arrow[r, bend right, "kh"'] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

which is called the *pasted composite* of the original diagram. In this case there was only one way of performing the pasting.

As another example, consider the diagram

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\xi"] & \cdot \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\zeta"] & B \end{tikzcd}$$

On the one hand we can whisker ξ and ζ as in

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\xi"] & \cdot \arrow[r, "h"] & B \end{tikzcd} \quad , \quad \begin{tikzcd} A \arrow[r, "g"] & \cdot \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\zeta"] & B \end{tikzcd}$$

and then vertically compose; while on the other hand we can whisker ξ and ζ as in

$$\begin{tikzcd} A \arrow[r, "f"] & \cdot \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\zeta"] & B \end{tikzcd} \quad , \quad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\xi"] & \cdot \arrow[r, "k"] & B \end{tikzcd}$$

and then vertically compose. The reader should verify that the resultant 2-cells of the form

$$\begin{tikzcd} A \arrow[r, bend left, "hf"] \arrow[r, bend right, "kg"'] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

are actually equal.

It is a general fact that the result of pasting is independent of the way it is broken down into basic pasting operations. In fact, all ambiguities in the method can be traced back to instances of the last example. For the particular diagrams we shall use here, it is easily shown that they have a uniquely determined pasted composite.

Write $\mathbf{Mon}(\mathcal{V})$ to denote the category of monoids in $\mathcal{V}$; the arrows are monoid arrows. With the extra structure of 2-cells, $\mathbf{Mon}(\mathcal{V})$ is an example of a "2-category".

(For a commutative ring R, we have that $\mathbf{Mon}(\mathbf{Mod}_R) = \mathbf{Alg}_R$ and also $\mathbf{Mon}(\mathbf{Mod}_R^{\mathrm{op}}) = \mathbf{Cog}_R^{\mathrm{op}}$ where $\mathbf{Mod}_R$ and $\mathbf{Mod}_R^{\mathrm{op}}$ have the same tensor product $\otimes_R$.)

Weak tensor functors take monoids to monoids.

More precisely, if $F : \mathcal{V} \longrightarrow \mathcal{W}$ is a weak tensor functor, each monoid A in $\mathcal{V}$ gives a monoid $F(A)$ in $\mathcal{W}$ with multiplication

$$F(A)\otimes F(A) \xrightarrow{\phi_{A,A}} F(A\otimes A) \xrightarrow{F(\mu)} F(A)$$

and unit

$$I \xrightarrow{\phi_0} F(I) \xrightarrow{F(\eta)} F(A) \ .$$

In fact, we obtain a functor

$$\mathbf{Mon}(F) : \mathbf{Mon}(\mathcal{V}) \longrightarrow \mathbf{Mon}(\mathcal{W})$$

which preserves the basic pasting operations of 2-cells (and so is an example of a "2-functor").

For each monoid A in $\mathcal{V}$ there is a category called $\mathbf{Mod}_{\mathcal{V}}(A)$, of (left) A-modules. An *A-module* consists of an object M of $\mathcal{V}$ and an arrow $\mu : A\otimes M \longrightarrow M$, called the *action*, which satisfies all the usual defining diagrams for a module (see Section 9). An *A-module arrow* $\mu : M \longrightarrow N$ is an arrow in $\mathcal{V}$ such that

$$\begin{tikzcd} A\otimes M \arrow[r, "1\otimes u"] \arrow[d, "\mu"'] & A\otimes N \arrow[d, "\mu"] \\ M \arrow[r, "u"] & N \end{tikzcd}$$

There is a "forgetful" functor $U_A : \mathbf{Mod}_{\mathcal{V}}(A) \longrightarrow \mathcal{V}$ which forgets the module action.

Each monoid arrow $f : A \longrightarrow B$ determines a functor

$$\mathbf{Mod}(f) : \mathbf{Mod}_{\mathcal{V}}(B) \longrightarrow \mathbf{Mod}_{\mathcal{V}}(A)$$

given by "restriction of scalars" along f. That is, for a B-module M, we take $\mathbf{Mod}(f)(M)$ to be M with A-action

$$A\otimes M \xrightarrow{f\otimes 1} B\otimes M \xrightarrow{\mu} M \ .$$

Each B-module arrow becomes an A-module arrow, thereby giving the following commutative triangle of categories and functors:

$$\begin{tikzcd} \mathbf{Mod}_{\mathcal{V}}(B) \arrow[rr, "\mathbf{Mod}(f)"] \arrow[dr, "U_B"'] & & \mathbf{Mod}_{\mathcal{V}}(A) \arrow[dl, "U_A"] \\ & \mathcal{V} & \end{tikzcd}$$

Furthermore, each 2-cell $\xi : f \Rightarrow g : A \longrightarrow B$ between monoid arrows f and g in $\mathcal{V}$ determines a natural transformation

$$\mathbf{Mod}(\xi) : \mathbf{Mod}(f) \longrightarrow \mathbf{Mod}(g)$$

whose component at the B-module M is the composite

$$M \xrightarrow{\xi \otimes 1} B \otimes M \xrightarrow{\mu} M$$

which is an A-module arrow, as can be seen from the diagram

$$\begin{tikzcd}
A \otimes M \arrow[r, "1 \otimes \xi \otimes 1"] \arrow[d, "f \otimes 1"'] & A \otimes B \otimes M \arrow[r, "1 \otimes \mu"] \arrow[d, "g \otimes 1 \otimes 1"] & A \otimes M \arrow[d, "g \otimes 1"] \\
A \otimes M \arrow[r, "\xi \otimes 1 \otimes 1"] \arrow[d, "\mu"'] & B \otimes B \otimes M \arrow[r, "1 \otimes \mu"] \arrow[d, shift right, "1 \otimes \mu"'] \arrow[d, shift left, "\mu \otimes 1"] & B \otimes M \arrow[d, "\mu"] \\
M \arrow[r, "\xi \otimes 1"] & B \otimes M \arrow[r, "\mu"] & M
\end{tikzcd}$$

Naturality follows from the following diagram involving a B-module arrow $u : M \longrightarrow N$.

$$\begin{tikzcd}
M \arrow[r, "\xi \otimes 1"] \arrow[d, "u"'] & B \otimes M \arrow[r, "\mu"] \arrow[d, "1 \otimes u"] & M \arrow[d, "u"] \\
N \arrow[r, "\xi \otimes 1"] & B \otimes N \arrow[r, "\mu"] & N
\end{tikzcd}$$

The assignment $\xi \longmapsto \mathbf{Mod}(\xi)$ turns the two basic pasting operations into corresponding familiar operations on natural transformations. This gives an example of a "2-functor"

$$\mathbf{Mod}_{\mathcal{V}} : \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \longrightarrow \mathbf{Cat}$$

where **Cat** is some appropriate "2-category" of categories.

For our purposes, it is important to remember the forgetful functors $U_A : \mathbf{Mon}_{\mathcal{V}}(A) \longrightarrow \mathcal{V}$. So, rather than **Cat**, we consider $\mathbf{Cat}/\mathcal{V}$, whose objects are functors $F : \mathcal{C} \longrightarrow \mathcal{V}$, whose arrows

$$T : (\mathcal{C}, F) \longrightarrow (\mathcal{D}, G)$$

are functors $T : \mathcal{C} \longrightarrow \mathcal{D}$ such that $GT = F$, and for which the 2-cells $\alpha : T \longrightarrow T'$ are arbitrary natural transformations from T to T' (that is, no condition relating it to F and G). Observe that $\mathbf{Mod}_{\mathcal{V}}$ really lands in $\mathbf{Cat}/\mathcal{V}$ by taking $A \in \mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$ to $\left(\mathbf{Mod}_{\mathcal{V}}(A)\,, U_A\right)$. So we have that

$$\mathbf{Mod}_{\mathcal{V}} : \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \longrightarrow \mathbf{Cat}/\mathcal{V}$$

is a 2-functor.
There is an obvious candidate for a tensor product on $\mathbf{Cat}/\mathcal{V}$, namely

$$(\mathcal{C}\,,F)\otimes(\mathcal{D}\,,G) \;=\; \left(\mathcal{C}\times\mathcal{D},\ \mathcal{C}\times\mathcal{D}\xrightarrow{F\times G}\mathcal{V}\times\mathcal{V}\xrightarrow{\otimes}\mathcal{V}\right).$$

This tensor product respects all of the pasting operations for natural transformations. Ignoring the 2-cells, $\mathbf{Cat}/\mathcal{V}$ becomes a tensor category with unit given by $\left(1\,,\ I : 1 \longrightarrow \mathcal{V}\right)$.

Suppose now that $\mathcal{V}$ is braided, For monoids A and B in $\mathcal{V}$, we enrich $A\otimes B$ with the multiplication

$$A\otimes B\otimes A\otimes B \xrightarrow{1\otimes c_{B,A}\otimes 1} A\otimes A\otimes B\otimes B \xrightarrow{\mu\otimes\mu} A\otimes B$$

and unit $\eta\otimes\eta : I \longrightarrow A\otimes B$; the braiding properties imply (exercise!) that this makes $A\otimes B$ into a monoid. Thus $\mathbf{Mon}(\mathcal{V})$ becomes a tensor category such that the forgetful functor

$$\mathbf{Mon}(\mathcal{V}) \longrightarrow \mathcal{V}$$

is a strict tensor functor. The tensor product on $\mathbf{Mon}(\mathcal{V})$ respects the basic pasting operations of 2-cells (exercise!).

We shall now see that

$$\mathbf{Mod}_{\mathcal{V}} : \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \longrightarrow \mathbf{Cat}/\mathcal{V}$$

is *essentially* a weak tensor functor For this, observe that $\mathbf{Mod}_{\mathcal{V}}(I) = \mathcal{V}$ and we have arrows

$$\begin{tikzcd} 1 \arrow[r, "I"] \arrow[dr, "I"'] & \mathcal{V} \arrow[d, "1_{\mathcal{V}}"] \\ & \mathcal{V} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Mod}_{\mathcal{V}}(A)\times\mathbf{Mod}_{\mathcal{V}}(B) \arrow[r, "\Phi"] \arrow[d, "U_A\otimes U_B"'] & \mathbf{Mod}_{\mathcal{V}}(A\otimes B) \arrow[d, "U_{A\otimes B}"] \\ \mathcal{V}\times\mathcal{V} \arrow[r, "\otimes"] & \mathcal{V} \end{tikzcd}$$

in $\mathbf{Cat}/\mathcal{V}$, where $\Phi(M\,,N) = M\otimes N$, with action

$$A\otimes B\otimes M\otimes N \xrightarrow{1\otimes c_{B,M}\otimes 1} A\otimes M\otimes B\otimes N \xrightarrow{\mu\otimes\mu} M\otimes N\ .$$

The reason for the word "essentially" is that the axioms for a weak tensor functor (see the beginning of Section 12) hold only up to isomorphism (instead of equality); in fact, the isomorphisms are precisely provided by the associativity and unit constraints a, r, ℓ for the tensor product.

Just as weak tensor functors take monoids to monoids, the 2-functor $\mathbf{Mod}_{\mathcal{V}}$ takes *tensor objects* in $\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$ to tensor objects in $\mathbf{Cat}/\mathcal{V}$.

A tensor object is "essentially" a monoid.

A tensor object in $\mathbf{Cat}$ is precisely a tensor category; whereas a monoid in $\mathbf{Cat}$ is a *strict* tensor category

A tensor object in $\mathbf{Cat}/\mathcal{V}$ is pair $\big(\mathcal{C}\,, F\big)$ consisting of a tensor category $\mathcal{C}$ and a strict tensor functor $F:\ \mathcal{C} \longrightarrow \mathcal{V}$.

A tensor object in $\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$ will be called a *quasi-bimonoid* in $\mathcal{V}$. This consists of a monoid A in $\mathcal{V}$, monoid arrows

$$\delta : A \longrightarrow A{\otimes}A \quad , \quad \epsilon : A \longrightarrow I$$

and 2-cells:

$$\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\swarrow\alpha"] & A^{\otimes 2} \arrow[d, "\delta\otimes 1"] \\
A^{\otimes 2} \arrow[r, "1\otimes\delta"'] & A^{\otimes 3}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[dl, "1_A"'] \arrow[d, "\delta"] \arrow[dr, "1_A"] & \\
A & A^{\otimes 2} \arrow[l, "\epsilon\otimes 1"] \arrow[r, "1\otimes\epsilon"'] \arrow[ul, phantom, "\lambda\Uparrow"] \arrow[ur, phantom, "\rho\Uparrow"'] & A
\end{tikzcd}$$

which are invertible under vertical composition and satisfy the following equalities between parted diagrams.

$$\begin{tikzcd}
A \arrow[rr, "\delta"] \arrow[dd, "\delta"'] \arrow[ddrr, phantom, "\swarrow\alpha"] & & A^{\otimes 2} \arrow[dd, "\delta\otimes 1"'] \arrow[dr, "\delta\otimes 1"] & \\
 & & & A^{\otimes 3} \arrow[dd, "\delta\otimes 1\otimes 1"] \arrow[dl, phantom, "\Leftarrow\alpha\otimes 1"] \\
A^{\otimes 2} \arrow[rr, "1\otimes\delta"] \arrow[dr, "1\otimes\delta"'] & & A^{\otimes 3} \arrow[dr, "1\otimes\delta\otimes 1"'] \arrow[dl, phantom, "1\otimes\alpha\Downarrow"] & \\
 & A^{\otimes 3} \arrow[rr, "1\otimes 1\otimes\delta"'] & & A^{\otimes 4}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[rr, "\delta"] \arrow[dr, "\delta"] \arrow[dd, "\delta"'] & & A^{\otimes 2} \arrow[dr, "\delta\otimes 1"] \arrow[d, phantom, "\Downarrow\alpha"] & \\
 & A^{\otimes 2} \arrow[rr, "1\otimes\delta"'] \arrow[dd, "\delta\otimes 1"] \arrow[dl, phantom, "\Leftarrow\alpha"] & & A^{\otimes 3} \arrow[dd, "\delta\otimes 1\otimes 1"] \\
A^{\otimes 2} \arrow[dr, "1\otimes\delta"'] & & & \\
 & A^{\otimes 3} \arrow[rr, "1\otimes 1\otimes\delta"'] & & A^{\otimes 4}
\end{tikzcd}$$

$$
\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\swarrow\alpha"] & A^{\otimes 2} \arrow[d, "\delta\otimes 1"] \\
A^{\otimes 2} \arrow[r, "1\otimes\delta"] \arrow[dr, "1"'] & A^{\otimes 3} \arrow[d, "1\otimes\epsilon\otimes 1"] \arrow[ld, phantom, "\swarrow 1\otimes\lambda"] \\
 & A^{\otimes 2}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "\delta"] \arrow[d, "\delta"'] & A^{\otimes 2} \arrow[d, "\delta\otimes 1"] \arrow[dl, "1"'] \\
A^{\otimes 2} \arrow[dr, "1"'] \arrow[r, phantom, "\swarrow\rho\otimes 1"] & A^{\otimes 3} \arrow[d, "1\otimes\epsilon\otimes 1"] \\
 & A^{\otimes 2}
\end{tikzcd}
$$

A *bimonoid* in $\mathcal{V}$ is a strict quasi-bimonoid; that is, one for which the 2-cells α, λ, ρ are actually identity 2-cells. (A bimonoid in $\mathbf{Mod}_R$ is an R-bialgebra; see Proposition 7.2.)

Hence we have that a (quasi-) bimonoid A in $\mathcal{V}$ determines the structure of a tensor category on $\mathbf{Mod}_{\mathcal{V}}(A)$ as well as a tensor functor structure on $U_A : \mathbf{Mod}_{\mathcal{V}}(A) \longrightarrow \mathcal{V}$ (see Section 10 for those cases when $\mathcal{V} = \mathbf{Mod}_R$ and when $\mathcal{V} = \mathbf{Mod}_R^{\mathrm{op}}$).

A *(quasi-) Hopf monoid* in $\mathcal{V}$ is a (quasi-) bimonoid H together with an arrow $\nu : H \longrightarrow H$, called the *antipode*, such that

$$
\begin{tikzcd}
H \arrow[r, "\delta"] \arrow[drr, "\epsilon"'] & H\otimes H \arrow[r, shift left, "\nu\otimes 1"] \arrow[r, shift right, "1\otimes\nu"'] & H\otimes H \arrow[r, "\mu"] & H \\
 & & I \arrow[ur, "\eta"'] &
\end{tikzcd}
$$

Drinfeld [Dri89] has recently obtained interesting examples of quasi-Hopf monoids in $\mathbf{Mod}_{\mathbf{k}}$.

Proposition 15.1 *Suppose $\mathcal{V}$ is a braided tensor category and H is a Hopf monoid in $\mathcal{V}$. If M is an H-module which has a left dual M^* as an object of $\mathcal{V}$ then M^* becomes the left dual of M in $\mathbf{Mod}_{\mathcal{V}}(H)$ via the action*

$$
H\otimes M^* \xrightarrow{\nu\otimes 1} H\otimes M^* \xrightarrow{c_{H,M^*}} M^*\otimes H \xrightarrow{1\otimes d} M^*\otimes H\otimes M\otimes M^* \longrightarrow \cdots
$$
$$
\cdots \xrightarrow{1\otimes\mu\otimes 1} M^*\otimes M\otimes M^* \xrightarrow{e\otimes 1} M^* .
$$

Proof. This is a matter of proving that e and d are module arrows. For this, compare with Propositions 10.1 and 10.5.

A *braiding* for a (quasi-) bimonoid A in $\mathcal{V}$ is a 2-cell

$$\begin{tikzcd}
& A \arrow[dl, "\delta"'] \arrow[dr, "\delta"] \arrow[d, phantom, "\Lleftarrow\gamma"] & \\
A\otimes A \arrow[rr, "c_{A,A}"'] & {} & A\otimes A
\end{tikzcd}$$

which is invertible (with respect to vertical composition) and satisfies the two equalities below.

$$\begin{tikzcd}[column sep=small]
& & A \arrow[dll, "\delta"'] \arrow[drr, "\delta"] \arrow[dl, "\delta"'] \arrow[dr, "\delta"] \arrow[dd, phantom, "\Lleftarrow\gamma"] & & \\
A\otimes A \arrow[ddr, "1\otimes\delta"'] \arrow[dr, phantom, "\Lleftarrow\alpha"] & & & & A\otimes A \arrow[ddl, "\delta\otimes 1"] \arrow[dl, phantom, "\Lleftarrow\alpha"] \\
& A\otimes A \arrow[rr, "c_{A,A}"'] \arrow[d, "\delta\otimes 1"] & {} & A\otimes A \arrow[d, "1\otimes\delta"'] & \\
& A\otimes A\otimes A \arrow[rr, "c_{A\otimes A,A}"'] & & A\otimes A\otimes A &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& & A \arrow[dll, "\delta"'] \arrow[drr, "\delta"] \arrow[d, phantom, "\Leftarrow\alpha"] & & \\
A\otimes A \arrow[ddr, "1\otimes\delta"'] \arrow[drr, "1\otimes\delta"] \arrow[dr, phantom, "\Downarrow 1\otimes\gamma"] & & {} & & A\otimes A \arrow[ddl, "\delta\otimes 1"] \arrow[dll, "\delta\otimes 1"'] \arrow[dl, phantom, "\Lleftarrow\gamma\otimes 1"] \\
& {} & A\otimes A\otimes A \arrow[dr, "c_{A,A}\otimes 1"] & {} & \\
& A\otimes A\otimes A \arrow[ur, "1\otimes c_{A,A}"] \arrow[rr, "c_{A\otimes A,A}"'] & & A\otimes A\otimes A &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& & A \arrow[dll, "\delta"'] \arrow[drr, "\delta"] \arrow[dl, "\delta"'] \arrow[dr, "\delta"] \arrow[dd, phantom, "\Lleftarrow\gamma"] & & \\
A\otimes A \arrow[ddr, "\delta\otimes 1"'] \arrow[dr, phantom, "\Lleftarrow\alpha^{-1}"] & & & & A\otimes A \arrow[ddl, "1\otimes\delta"] \arrow[dl, phantom, "\Lleftarrow\alpha^{-1}"] \\
& A\otimes A \arrow[rr, "c_{A,A}"'] \arrow[d, "1\otimes\delta"] & {} & A\otimes A \arrow[d, "\delta\otimes 1"'] & \\
& A\otimes A\otimes A \arrow[rr, "c_{A\otimes A,A}"'] & & A\otimes A\otimes A &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& & A \arrow[dll, "\delta"'] \arrow[drr, "\delta"] \arrow[d, phantom, "\Leftarrow\alpha^{-1}"] & & \\
A\otimes A \arrow[ddr, "\delta\otimes 1"'] \arrow[drr, "\delta\otimes 1"] \arrow[dr, phantom, "\Downarrow \gamma\otimes 1"] & & {} & & A\otimes A \arrow[ddl, "1\otimes\delta"] \arrow[dll, "1\otimes\delta"'] \arrow[dl, phantom, "\Lleftarrow 1\otimes\gamma"] \\
& {} & A\otimes A\otimes A \arrow[dr, "1\otimes c_{A,A}"] & {} & \\
& A\otimes A\otimes A \arrow[ur, "c_{A,A}\otimes 1"] \arrow[rr, "c_{A,A\otimes A}"'] & & A\otimes A\otimes A &
\end{tikzcd}$$

A *twist* for A is a 2-cell

$$\begin{tikzcd}
A \arrow[r, bend left, "1_A"] \arrow[r, bend right, "1_A"'] \arrow[r, phantom, "\Downarrow\tau"] & A
\end{tikzcd}$$

which is invertible, with respect to $*$, and satisfies

$$
\begin{tikzcd}
A \arrow[r, bend left, "1"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\tau"] & A \arrow[r, "\delta"] & A\otimes A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[rrr, "\delta"] \arrow[dr, "\delta"] \arrow[dd, "\delta"'] \arrow[rrrd, phantom, "\Downarrow\gamma"] & & & A\otimes A \\
 \arrow[r, phantom, "\Swarrow\gamma"] & A\otimes A \arrow[r, bend left, "1"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\tau\otimes\tau"] & A\otimes A \arrow[ur, "c_{A,A}"'] & \\
A\otimes A \arrow[ur, "c_{A,A}"'] \arrow[rrruu, bend right, "1"'] & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, bend left, "1"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\tau"] & A \arrow[r, "\epsilon"] & I
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, bend left, "\epsilon"] \arrow[r, bend right, "\epsilon"'] \arrow[r, phantom, "\Downarrow 1"] & I
\end{tikzcd}
\;.
$$

A (quasi-) bimonoid with a braiding and twist is called *balanced.*

The reader should interpret the above pasting diagrams in the special case where $\mathcal{V} = \mathbf{Mod_k}$, to see that these definitions agree with the definitions of braiding element and twist for bialgebras, as in Examples 11.3 and 11.7. We now have a conceptual version of the calculations in those examples.

Proposition 15.2 *Suppose $\mathcal{V}$ is a symmetric tensor category and A is a bimonoid in $\mathcal{V}$. There is a bijection between braidings γ for A and braidings c for $\mathbf{Mod}_{\mathcal{V}}(A)$ determined by $c_{M,N}$ as the composite:*

$$M\otimes N \xrightarrow{\gamma\otimes 1\otimes 1} A\otimes A\otimes M\otimes N \xrightarrow{1\otimes c_{A\otimes M,N}} A\otimes N\otimes A\otimes M \xrightarrow{\mu\otimes\mu} N\otimes M$$

There is a bijection between twists τ for A and twists θ for $\mathbf{Mod}_{\mathcal{V}}(A)$ determined by

$$\theta_M \;=\; \left(M \xrightarrow{\tau\otimes 1} A\otimes M \xrightarrow{\mu} M \right).$$

Proof. Apply the 2-functor $\mathbf{Mod}_{\mathcal{V}}$ to the triangle containing γ, and paste below it a square containing a natural isomorphism whose components are the symmetry of $\mathcal{V}$ (we omit the subscripts $\mathcal{V}$ on maps in the diagram):

$$\begin{tikzcd}
& \mathbf{Mod}_{\mathcal{V}}(A) \arrow[d, phantom, "\Leftarrow \mathbf{Mod}(\gamma)"] & \\
\mathbf{Mod}_{\mathcal{V}}(A\otimes A) \arrow[ur, "\mathbf{Mod}(\delta)"] & \arrow[d, phantom, "\Leftarrow c"] & \mathbf{Mod}_{\mathcal{V}}(A\otimes A) \arrow[ul, "\mathbf{Mod}(\delta)"'] \arrow[ll, "\mathbf{Mod}(c_{A,A})"] \\
\mathbf{Mod}_{\mathcal{V}}(A) \times \mathbf{Mod}_{\mathcal{V}}(A) \arrow[u, "\Phi"] & & \mathbf{Mod}_{\mathcal{V}}(A) \times \mathbf{Mod}_{\mathcal{V}}(A) \arrow[u, "\Phi"'] \arrow[ll, "\sigma"']
\end{tikzcd}$$

The result is a natural isomorphism, whose component at the pair $(M, N) \in \mathbf{Mod}_{\mathcal{V}}(A) \times \mathbf{Mod}_{\mathcal{V}}(A)$ is the $c_{M,N}$ as stated in the proposition. The axioms on γ convert to the braiding axioms for c.

Conversely, to recapture γ from c, take the composite

$$\gamma \;=\; \Big(I \xrightarrow{\eta\otimes\eta} A\otimes A \xrightarrow{c_{A,A}} A\otimes A \Big)$$

where we regard A as an object of $\mathbf{Mod}_{\mathcal{V}}(A)$ with action μ. The proof of the twist bijection is similar.

A *tortile bimonoid* in $\mathcal{V}$ is a balanced Hopf monoid H such that

$$\begin{tikzcd}
& I \arrow[dl, "\tau"'] \arrow[dr, "\tau"] & \\
H \arrow[rr, "\nu"] & & H
\end{tikzcd} .$$

Proposition 15.3 *Suppose that $\mathcal{V}$ is a symmetric autonomous tensor category. If H is a tortile bimonoid in $\mathcal{V}$ then $\mathbf{Mod}_{\mathcal{V}}(H)$ is a tortile tensor category.*

Proof. By Propositions 15.1 and 15.2 $\mathbf{Mod}_{\mathcal{V}}(H)$ becomes an autonomous balanced tensor category. All that remains to see is that $\theta_{M^*} = (\theta_M)^*$, which follows from $\nu \circ \tau = \tau$.

Consider the replacement of the tensor category $\mathcal{V}$ by its opposite tensor category $\mathcal{V}^{\mathrm{op}}$. Monoids become comonoids, but bimonoids and Hopf monoids are unchanged. For a bimonoid H in $\mathcal{V}$, a *cobraiding* $\gamma : H\otimes H \longrightarrow I$

on H in $\mathcal{V}$ is defined to be a braiding on H in $\mathcal{V}^{\mathrm{op}}$; a *cotwist* $\tau : H \longrightarrow I$ on H in $\mathcal{V}$ is defined to be a twist on H in $\mathcal{V}^{\mathrm{op}}$.

Thus we have the corresponding notions of *cobraided*, *cobalanced* and *cotortile* bimonoid in $\mathcal{V}$.

Our proposal for a definition of a *quantum group over* R is that it should be a cotortile bimonoid in $\mathbf{Mod}_R$. In Section 16 we shall see that our main example, the quantum general linear group, does indeed give an instance of this concept.

16

Tannaka duality

Given a compact group G, the set $\mathbf{Rep}\,G$ of isomorphism classes of *appropriate representations* admits various operations; for example direct sum and tensor product. Tannaka's duality theorem (1939) provided a recipe for recovering a compact group $\mathbf{Gp}\,R$ from a structure R such as $\mathbf{Rep}\,G$ whereby $\mathbf{Gp}\,\mathbf{Rep}\,G \cong G$.

For algebraic groups, Saavedra Rivano [Riv72] considered the category of appropriate representations together with the tensor structure and the underlying functor into vector spaces. He gave criteria on a tensor functor into vector spaces under which it should be equivalent to such an underlying functor. A non-commutative generalization of this was given by Ulbrich [Ulb89]. We shall lead into this Hopf algebra version by examining the 2-functor $\mathbf{Mod}_{\mathcal{V}}$ of the previous section.

For simplicity of exposition we suppose our tensor category $\mathcal{V}$ is strict. This loses no generality in fact since every tensor category is equivalent to a strict one (MacLane's coherence theorem). We also suppose that $\mathcal{V}$ is symmetric (but we cannot suppose the tensor product is strictly commutative). A consequence of this simplification is that we really do have a weak tensor functor

$$\mathbf{Mod}_{\mathcal{V}} : \mathbf{Mon}_{\mathcal{V}}^{\mathrm{op}} \longrightarrow \mathbf{Cat}/\mathcal{V}$$

We are now interested in going back from $\mathbf{Cat}/\mathcal{V}$ to $\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$. A possible way to do this is via a left adjoint to $\mathbf{Mod}_{\mathcal{V}}$, if it exists. Under reasonable conditions, a left adjoint $E_F \in \mathbf{Mon}(\mathcal{V}))$ does exist at an object $(\mathcal{C}, F)$ of $\mathbf{Cat}/\mathcal{V}$. It is constructed as follows.

If each internal hom $[FX, FX]$ exists in $\mathcal{V}$ and if $\mathcal{V}$ is suitably complete, we put

$$E_F = \int_{X \in \mathcal{C}} [FX, FX]$$

where the integral sign denotes the *end* (see MacLane [Mac71]) of the functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{V}$ taking (X, Y) to $[FX, FY]$; it is the equalizer of the obvious pair of arrows

$$\prod_X [\,FX\,,FX\,] \rightrightarrows \prod_{f:X\to Y} [\,FX\,,FY\,] \ \ .$$

There are projection arrows

$$\pi_X\,:\,E_F \longrightarrow [\,FX\,,FX\,]$$

for each object $X \in \mathcal{C}$. These correspond, using the definition of *internal hom*, to arrows

$$\mu_X\,:\,E_F \otimes FX \longrightarrow FX \ \ .$$

The univeral properties of *end* and *internal hom* show that there exists a bijection between the arrows $f\,:\,A \longrightarrow E_F$ in $\mathcal{V}$ and natural families of arrows $\theta_X\,:\,A \otimes FX \longrightarrow FX$, given by

$$\theta_X \ = \ \mu_X \circ (f \otimes 1_X)\ .$$

The natural families

$$E_F \otimes E_F \otimes FX \xrightarrow{1\otimes\mu_X} E_F \otimes FX \xrightarrow{\mu_X} FX \quad , \quad FX \xrightarrow{1_{FX}} FX$$

induce, under such bijections, the monoid structure on E_F :

$$\mu\,:\,E_F \otimes E_F \longrightarrow E_F \quad , \quad \eta\,:\,I \longrightarrow E_F \ \ .$$

Example 16.1 *Take* $\mathcal{V} = \mathbf{Mod}_R$ *for some commutative ring* R *. Then we have that* $\mathbf{Mon}(\mathcal{V}) = \mathbf{Alg}_R$ *. Now for any functor* $F\,:\,\mathcal{C} \longrightarrow \mathcal{V}$*, the algebra* E_F *has as elements the natural families* $\theta = (\theta_X)_{X\in G}$ *of* R*-linear morphisms* $\theta_X\,:\,FX \longrightarrow FX$ *; addition and multiplication by scalars are done componentwise, while multiplication is componentwise composition. In particular,* for any R-algebra A , if we have that

$$F \ = \ U_A\,:\,\mathbf{Mod}_R(A) \longrightarrow \mathbf{Mod}_R \quad , \quad \eta\,:\,I \longrightarrow E_F$$

then there is a natural isomorphism of algebras

$$E_F \ \cong \ A\ .$$

To see this, notice that each element m *of an* A*-module* M *determines a unique* $\hat{m}\,:\,A \longrightarrow M$ *in* $\mathbf{Mod}_R(A)$ *with* $\hat{m}(1) = m$ *; so for a natural* $\theta\,:\,U_A \longrightarrow U_A$ *, we have*

$$\begin{tikzcd} A \arrow[r, "\theta_A"] \arrow[d, "\hat{m}"'] & A \arrow[d, "\hat{m}"] \\ M \arrow[r, "\theta_M"] & M \end{tikzcd}$$

which implies $\theta_M(m) = \theta_A(1)\,m$ *; so* θ *is determined by* $\theta_A(1) \in A$.

Suppose A is a monoid in $\mathcal{V}$. The two axioms which are required for an arrow $f : A \longrightarrow E_F$ to be in $\mathbf{Mon}(\mathcal{V})$ translate to the two conditions on the corresponding family of arrows $\theta_X : A \otimes FX \longrightarrow FX$ which say that each θ_X is an *action* of A on FX. This is precisely what is needed to lift F to a functor $T : \mathcal{C} \longrightarrow \mathbf{Mod}_{\mathcal{V}}(A)$ such that $U_A T = F$; just put $TX = (FX, \theta_X)$. This gives a natural bijection

$$\mathbf{Mon}(\mathcal{V})(A, E_F) \cong (\mathbf{Cat}/\mathcal{V})\Big((\mathcal{C}, F), (\mathbf{Mod}_{\mathcal{V}}(A), U_A)\Big)$$

between hom sets, which means that $(\mathcal{C}, F) \longmapsto E_F$ *is left adjoint to* $\mathbf{Mod}_{\mathcal{V}} : \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \longrightarrow \mathbf{Cat}/\mathcal{V}$. In fact, the above bijection becomes an isomorphism of categories, since it extends to 2-cells:

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\xi"] & E_F \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} \mathcal{C} \arrow[rr, bend left, "T"] \arrow[rr, bend right, "T'"'] \arrow[rr, phantom, "\Downarrow"] \arrow[dr, "F"'] & & \mathbf{Mod}_{\mathcal{V}}(A) \\ & \mathcal{V} \arrow[ur, "U_A"'] & \end{tikzcd}$$

This is expressed by saying that $(\mathcal{C}, F) \longmapsto E_F$ is *left 2-adjoint to* $\mathbf{Mod}_{\mathcal{V}}$.

Taking $A = E_F$ in the above bijection and looking at the image of the identity arrow, we obtain

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "N"] \arrow[dr, "F"'] & & \mathbf{Mod}_{\mathcal{V}}(E_F) \arrow[dl, "U_{E_F}"] \\ & \mathcal{V} & \end{tikzcd}$$

where $NX = (FX, \mu_X)$. We obtain a (partial) 2-functor

$$E_{\_} : \mathbf{Cat}/\mathcal{V} \longrightarrow (\mathbf{Mon}(\mathcal{V}))^{\mathrm{op}}$$

by taking the 2-cell $\alpha : T \Rightarrow T' : (\mathcal{C}, F) \longrightarrow (\mathcal{D}, G)$ in $\mathbf{Cat}/\mathcal{V}$ into the 2-cell $E_\alpha : E_T \Rightarrow E_{T'} : E_G \longrightarrow E_F$ in $\mathbf{Mon}(\mathcal{V})$ corresponding (under the 2-adjunction) to the 2-cell in $\mathbf{Cat}/\mathcal{V}$:

$$N\alpha : NT \Rightarrow NT' : (\mathcal{C}, F) \longrightarrow (\mathbf{Mod}_{\mathcal{V}}(E_G), U_{E_G}) .$$

Remark 16.2 Formal Tannaka Duality *criteria on* $F : \mathcal{C} \longrightarrow \mathcal{V}$ *are that* $N : \mathcal{C} \longrightarrow \mathbf{Mod}_{\mathcal{V}}(E_F)$ *should be faithful and also that every "appropriate"* E_F*-module should be isomorphic to some* NX.

We can equally well regard $E_{\_}$ as a 2-functor

$$E_{\_} : (\mathbf{Cat}/\mathcal{V})^{\mathrm{op}} \longrightarrow \mathbf{Mon}(\mathcal{V})$$

whereupon (for general reasons as an adjoint to $\mathbf{Mod}_{\mathcal{V}}$) it is a weak tensor functor. It preserves the unit in the sense that $E_I \cong I$, while we have a canonical arrow ϕ such that

$$\begin{tikzcd}
E_F \otimes F_G \arrow[rr, "\phi_{F,G}"] \arrow[d, "\pi_X \otimes \pi_Y"'] & & E_{F\otimes G} \arrow[d, "\pi_{X\otimes Y}"] \\
{[FX, FX] \otimes [GY, GY]} \arrow[rr, "-\otimes-"] & & {[FX\otimes GY, FX\otimes GY]}
\end{tikzcd}$$

where the bottom arrow corresponds to the composite

$$[FX, FX]\otimes[GY, GY]\otimes FX\otimes GY \xrightarrow{1\otimes c\otimes 1} [FX, FX]\otimes FX\otimes[GY, GY]\otimes GY$$

$$\cdots \xrightarrow{e\otimes e} FX\otimes GY \ .$$

So $E_{\_}$ takes monoids in $(\mathbf{Cat}/\mathcal{V})^{\mathrm{op}}$ to monoids in $\mathbf{Mon}(\mathcal{V})$, the latter being the commutative monoids in $\mathcal{V}$, but this is of no interest to us here.

Our real interest is in to what extent

$$E_{\_} : (\mathbf{Cat}/\mathcal{V}) \longrightarrow \mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$$

takes monoids to monoids. This will be true of those monoids $(\mathcal{C}, F)$ in $\mathbf{Cat}/\mathcal{V}$ for which $\phi_{F,F} : E_F \otimes E_F \longrightarrow E_{F\otimes F}$ is invertible.

Is this a reasonable condition? At first glance, invertibility of

$$\phi_{F,G} : \int_X [FX, FX] \otimes \int_Y [GY, GY] \longrightarrow \int_{X,Y} [FX\otimes GY, FX\otimes GY]$$

looks unlikely. It would be implied by the two conditions:

(a) *each* $A\otimes\_ : \mathcal{V} \longrightarrow \mathcal{V}$ *preserves ends*; and

(b) *each* $[A, B]\otimes[C, D] \xrightarrow{-\otimes-} [A\otimes C, B\otimes D]$ *is invertible, for every* $A = FX$ *and every* $C = GY$.

However these look unlikely too, if we think in terms of Example 16.1.

We shall look at the conditions (a) and (b) more closely. If $\mathcal{V}$ is a closed tensor category then $A\otimes\_$ preserves colimits (since it has a right adjoint

$[A\,,\,\_\,]$; see Mac Lane ([Mac71]; Chapter V §5). But *end* is a limit, not a colimit. So (a) can be ensured by taking $\mathcal{V}$ to be the opposite of a complete closed tensor category. We need to be careful here since we still need the internal homs of the form $[FX, FX\,]$ in $\mathcal{V}$, not in $\mathcal{V}^{\mathrm{op}}$.

Condition (b) is true, for example for finite-dimensional vector spaces. What is needed is that A and C should have duals; then we have canonical isomorphisms

$$\begin{aligned} [\,A\,,B\,]\otimes[\,C\,,D\,] &\cong A^{*}\otimes B\otimes C^{*}\otimes D \\ &\cong (A\otimes C)^{*}\otimes(B\otimes D) \cong [\,A\otimes C\,,B\otimes D\,]\,. \end{aligned}$$

Hence, conditions (a) and (b) are not unreasonable after all. They are satisfied when $\mathcal{V}$ is the opposite of a closed symmetric tensor category which is cocomplete enough for co-ends over $\mathcal{C}$ to exist, and when each FX and GY has a dual.

Suppose then that $\mathcal{V}^{\mathrm{op}}$ is a closed symmetric (strict) tensor category which is (small) cocomplete. Suppose $\mathcal{C}$ is a left autonomous small (strict) tensor category and $F : \mathcal{C} \longrightarrow \mathcal{V}$ is a (strict) tensor functor. Then each FX has a dual FX^{*}. Since $\big(\mathcal{C}\,,F\big)$ is a monoid in $\mathbf{Cat}/\mathcal{V}$, we obtain a monoid E_F in $\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$; that is, a bi-monoid E_F in $\mathcal{V}$. This gives a factorization

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "N"] \arrow[dr, "F"'] & & \mathbf{Mod}_{\mathcal{V}}(E_F) \arrow[dl, "U_{E_F}"] \\ & \mathcal{V} & \end{tikzcd}$$

of our tensor functor F into tensor functors N and U_{E_F}.

In fact, E_F *is a Hopf monoid.* To see this, define $F^{*} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{V}$ by $F^{*}X = FX^{*}$. We obtain a monoid arrow

$$(\,)^{*} : (\mathcal{C}, F) \longrightarrow (\mathcal{C}^{\mathrm{op}}, F^{*})$$

in $\mathbf{Cat}/\mathcal{V}$. This induces a monoid arrow ν with

$$\begin{tikzcd} E_{F^{*}} \arrow[rr, "\nu"] \arrow[dr, "\pi_{X^{*}}"'] & & E_F \arrow[dl, "\pi_{X^{*}}"] \\ & FX\otimes FX^{*} & \end{tikzcd}$$

It is easy to see that $E_{F^{*}} = {E_F}^{\mathrm{op}}$ as bi-monoids in $\mathcal{V}$ (that is, $E_{F^{*}}$ is just E_F with switched multiplication and switched comultiplication), and ν is an antipode for the bi-monoid E_F.

Now suppose $\mathcal{C}$ is braided. The braiding can be regarded as an invertible 2-cell:

$$\begin{tikzcd}
(\mathcal{C},F)\otimes(\mathcal{C},F) \arrow[rr, "\sigma"] \arrow[dr, "\otimes"'] & \arrow[d, phantom, "\overset{c}{\Rrightarrow}"] & (\mathcal{C},F)\otimes(\mathcal{C},F) \arrow[dl, "\otimes"] \\
& (\mathcal{C},F) &
\end{tikzcd}$$

in $\mathbf{Cat}/\mathcal{V}$. Applying $E_{\_}$, we obtain an invertible 2-cell in $\mathbf{Mon}(\mathcal{V})$:

$$\begin{tikzcd}
E_F\otimes E_F & \arrow[d, phantom, "\underset{\gamma}{\Lleftarrow}"] & E_F\otimes E_F \arrow[ll, "c"'] \\
& E_F \arrow[ul, "\delta"] \arrow[ur, "\delta"'] &
\end{tikzcd}$$

The braiding arrows for c on $\mathcal{C}$ carry over precisely to those for γ on E_F. Moreover, $N : \mathcal{C} \longrightarrow \mathbf{Mod}_{\mathcal{V}}(E_F)$ becomes a braided tensor functor.

Next suppose $\mathcal{C}$ is balanced. The twist on $\mathcal{C}$ can be regarded as being an invertible 2-cell:

$$\begin{tikzcd}
(\mathcal{C},F) \arrow[r, bend left, "1_{\mathcal{C}}"] \arrow[r, bend right, "1_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\theta"] & (\mathcal{C},F)
\end{tikzcd}$$

in $\mathbf{Cat}/\mathcal{V}$, and, applying $E_{\_}$, we obtain a twist

$$\begin{tikzcd}
E_F \arrow[r, bend left, "1"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\tau"] & E_F
\end{tikzcd}$$

for the braided bi-monoid E_F . So E_F becomes a balanced Hopf monoid and $N : \mathcal{C} \longrightarrow \mathbf{Mod}_{\mathcal{V}}(E_F)$ becomes a balanced tensor functor.

Finally, if $\mathcal{C}$ is a tortile tensor category, E_F is a tortile bimonoid in $\mathcal{V}$.

To obtain Ulbrich's [Ulb89] setting, we take $\mathcal{V} = \mathbf{Mod}_R^{\mathrm{op}}$ for a commutative ring R. For each R-coalgebra C, we have

$$\mathbf{Mod}_{\mathcal{V}}(\mathcal{C})^{\mathrm{op}} \;=\; \mathbf{Comod}_R(\mathcal{C})\ .$$

We use the notation $\mathbf{Comod}_R(\mathcal{C})_c$ to denote the full subcategory consisting of C-comodules M for which the underlying R-module $U_C M$ is cauchy.

Consider a small category $\mathcal{C}$ and a functor $F : \mathcal{C} \longrightarrow \mathbf{Mod}_R$ whose values FX are cauchy R-modules. The coend

$$E'_F = \int^X FX \otimes_R (FX)^*$$

becomes an R-coalgebra and we have

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "N"] \arrow[dr, "F"'] & & \mathbf{Comod}_R(E'_F) \arrow[dl, "U"] \\ & \mathbf{Mod}_R & \end{tikzcd}$$

(since we can apply our previous theory to F regarded as going from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{V} = \mathbf{Mod}_R^{\mathrm{op}}$). Notice that N actually lands in $\mathbf{Comod}_R(E'_F)_c$.

If $\mathcal{C}$ is a tensor category and F is a tensor functor then E'_F becomes an R-bialgebra and N becomes a tensor functor. If $\mathcal{C}$ is left autonomous then E'_F becomes a Hopf algebra with invertible antipode. If $\mathcal{C}$ is a tortile tensor category then E'_F becomes a cotortile R-bialgebra (quantum group!) and N becomes a balanced tensor functor.

An important case of Tannaka duality is the characterization of those $F : \mathcal{C} \longrightarrow \mathbf{Mod}_R$ equivalent to $U_H : \mathbf{Comod}_R(H)_c \longrightarrow \mathbf{Mod}_R$ for some Hopf algebra H. This can be investigated by looking at when the functor $N : \mathcal{C} \longrightarrow \mathbf{Comod}_R(E'_F)_c$ is an equivalence.

The question arises here as to whether $E'_F \cong C$ when the equality $F = U_C : \mathbf{Comod}(C)_c \longrightarrow \mathbf{Mod}_R$ holds for a coalgebra C. We cannot use the technique of Example 16.1 since, although C is a C-comodule, it is generally not cauchy as an R-module.

Proposition 16.3 *If C is a coalgebra over a field R and U denotes the forgetful functor $U : \mathbf{Comod}(C)_c \longrightarrow \mathbf{Mod}_R$, then there is a coalgebra isomorphism*

$$E'_U \cong C .$$

Proof. We need to show that C has the universal property of E'_U ; that is, the assignment $f \longmapsto (f \otimes 1) \circ \delta$ determines a natural bijection between R-module morphisms $F : C \longrightarrow X$ and families of R-module morphisms $\theta_M : U(M) \longrightarrow X \otimes U(M)$ natural in $M \in \mathbf{Comod}(C)_c$.

We need to apply the *fundamental theorem on coalgebras* (see Sweedler [Swe69], p.46): (when R is a field) "the sub-coalgebra generated by an element of C is cauchy".

Given a family θ_M, we must define $f(c)$ for each $c \in C$. Let M be any sub-coalgebra of C which contains c and is finite dimensional. Such M exist by the above *fundamental theorem*, and can be regarded as C-comodules. Put $f(c) = (1 \otimes \varepsilon)\theta_M(c)$. This is independent of the choice of M since θ_M is natural in M. The proof that this gives the inverse to $f \longmapsto (f \otimes 1) \circ \delta$ is now easy.

17

Adjoining an antipode to a bialgebra

Tannaka duality allows the possibility of taking an R-bialgebra A, applying some categorical construction to $\mathbf{Comod}_R(A)_c$, and asking whether the result again has the form $\mathbf{Comod}_R(B)_c$ for some R-bialgebra B.

An example of an appropriate categorical construction is adjoining left-dual objects to a tensor category. To each tensor category $\mathcal{C}$, there is a left-autonomous tensor category $\mathcal{A}_\ell(\mathcal{C})$ and a tensor functor $\mathcal{C} \longrightarrow \mathcal{A}_\ell(\mathcal{C})$ which induces a natural equivalence between the category of tensor functors $\mathcal{A}_\ell(\mathcal{C}) \longrightarrow \mathcal{D}$ and the category of tensor functors $\mathcal{C} \longrightarrow \mathcal{D}$ for all left autonomous tensor categories $\mathcal{D}$. (See Joyal–Street [JS91a] and paper II in the series.)

Suppose that $F : \mathcal{C} \longrightarrow \mathbf{Mod}_R$ is a tensor functor whose values FX are cauchy R-modules. Then we obtain a corresponding tensor functor $\hat{F} : \mathcal{A}_\ell(\mathcal{C}) \longrightarrow \mathbf{Mod}_R$.

Proposition 17.1 *$E'_{\hat{F}}$ is the reflection of the R-bialgebra E'_F into the category of Hopf R-algebras.*

Proof. Let H be a Hopf R-algebra. Then we have that $\mathbf{Comod}_R(H)_c$ is a left autonomous tensor category. Thus the tensor functors over $\mathbf{Mod}_R$, $\mathcal{A}_\ell(\mathcal{C}) \longrightarrow \mathbf{Comod}_R(H)_c$, correspond to $\mathcal{C} \longrightarrow \mathbf{Comod}_R(H)_c$, also as tensor functors over $\mathbf{Mod}_R$. By the left adjoint property of E'_-, it follows that bialgebra morphisms $E'_{\hat{F}} \longrightarrow H$ correspond to bialgebra morphisms $E'_F \longrightarrow H$, as required.

This gives a construction for adjoining an antipode to a bialgebra over a field R; that is, a construction for a left adjoint to the inclusion of the category $\mathbf{Hopf}_R$ of Hopf algebras in the category $\mathbf{Big}_R$ of bialgebras. Given a bialgebra A, put $F = U_A : \mathbf{Comod}_R(A)_c \longrightarrow \mathbf{Mod}_R$. By Proposition 16.3, we have $A \cong E'_F$. By Proposition 17.1, the Hopf algebra $H = E'_{\hat{F}}$ is the required reflection.

If we require the adjoined antipode to be invertible, we must replace $\mathcal{A}_\ell(\mathcal{C})$ in the above by $\mathcal{A}(\mathcal{C})$ which is the free autonomous tensor category on the tensor category $\mathcal{C}$. And so on.

18

The quantum general linear group again

Let V be an n-dimensional vector space over a field $\mathbf{k}$. Given an invertible $q \in \mathbf{k}$, let (y, z) be the tortile Yang–Baxter-operator on V defined in Section 14. By Examples 13.4 and 13.7, there are strict tensor functors

$$M : \widetilde{\mathcal{B}} \longrightarrow \mathbf{Mod_k} \quad , \quad G : \widetilde{\mathcal{T}} \longrightarrow \mathbf{Mod_k}$$

taking $(+, c_{+,+}, \theta_+)$ to be (V, y, z) (where we are identifying $\widetilde{\mathcal{B}}$ with the subcategory of $\widetilde{\mathcal{T}}$ whose objects are positively signed sets, with arrows being ribbons which do not bend around).

Applying Tannaka duality ideas (Section 16) to M and G, we obtain a co-balanced bi-algebra E'_M and a co-tortile bi-algebra E'_G.

Theorem 18.1 *There are $\mathbf{k}$-bialgebra isomorphisms (see example 9.8):*

$$E'_M \cong \mathbf{Mat}_q(n) \quad , \quad E'_G \cong \mathrm{GL}_q(n) .$$

Proof. Let A be the bialgebra E'_M. It comes equipped with a universal linear function $\delta_Z : MZ \longrightarrow A \otimes MZ$ for $Z \in \widetilde{\mathcal{B}}$. In particular, we have $\delta = \delta_+ : V \longrightarrow A \otimes V$, and $\delta_{+,+} : V \otimes V \longrightarrow A \otimes V \otimes V$ is the composite

$$V \otimes V \xrightarrow{\delta \otimes \delta} A \otimes V \otimes A \otimes V \xrightarrow{1 \otimes \sigma \otimes 1} A \otimes A \otimes V \otimes V \xrightarrow{\mu \otimes 1 \otimes 1} A \otimes V \otimes V$$

while $y : V \otimes V \longrightarrow V \otimes V$ becomes a co-module morphism

$$\begin{tikzcd} V \otimes V \arrow[r, "\delta_{+,+}"] \arrow[d, "y"'] & A \otimes V \otimes V \arrow[d, "1 \otimes y"] \\ V \otimes V \arrow[r, "\delta_{+,+}"] & A \otimes V \otimes V \end{tikzcd}$$

Putting $\delta(\varepsilon_i) = \sum_j x_{ij} \otimes \varepsilon_j$, it is a straightforward, but tedious, matter to check that commutativity of the above square is equivalent to the elements

$X = \{x_{ij} \mid i, j = 1, \ldots, n\}$ satisfying the defining relations for the quantum matrix monoid $\mathbf{Mat}_q(n)$ (see Example 9.8). We therefore have a bialgebra morphism $\mathbf{Mat}_q(n) \longrightarrow A$ which can be seen to be invertible.

To introduce an antipode to the bialgebra A and thereby obtain the Hopf algebra H, we must introduce a left dual $\kappa : V^* \longrightarrow A \otimes V^*$ for $\delta : V \longrightarrow A \otimes V$. Since $\widetilde{\mathcal{T}}$ is the free autonomous tensor category on $\widetilde{\mathcal{B}}$, we have $H = E'_G$ (as in Section 17). If we put

$$\kappa(\varepsilon_i^*) \; = \; \sum_j w_{ij} \otimes \varepsilon_j^*$$

and express what it means for $e : V^* \otimes V \longrightarrow \mathbf{k}$ and $d : \mathbf{k} \longrightarrow V^* \otimes V$ to be H-comodule morphisms, we obtain the conditions

$$\sum_m w_{im}\, x_{jm} \; = \; \delta_{ij} \qquad \text{and} \qquad \sum_m x_{mi}\, w_{mj} \; = \; \delta_{ij} \; ,$$

which mean that the matrix (w_{ij}) is the inverse of the transpose of (x_{ij}). The Hopf algebra H is therefore obtained from A by adjoining elements w_{ij} subject to the above two conditions. By means of a quantum Cramer's Rule (checked in Section 3 for the $n = 2$ case), we can take $w_{ij} = t \det_q(X_{ij})$ (see Example 9.8) where t is an adjoined inverse for $\det_q(X)$. In this way we see that $H \cong \mathrm{GL}_q(n)$.

Corollary 18.2 *$M_q(n)$ is a co-balanced bi-algebra and $\mathrm{GL}_q(n)$ is a co-tortile bi-algebra.*

The co-braiding given by $\gamma : \mathrm{GL}_q(n) \otimes \mathrm{GL}_q(n) \longrightarrow \mathbf{k}$, and co-twist given by $\tau : \mathrm{GL}_q(n) \otimes \mathrm{GL}_q(n) \longrightarrow \mathbf{k}$, satisfy the equations

$$\begin{aligned} y(\varepsilon_i \otimes \varepsilon_j) \quad &= \quad \sum_{m,r} \gamma(x_{im}, x_{jr})\, \varepsilon_m \otimes \varepsilon_r \\ q^n \varepsilon_i \quad &= \quad \sum_m \tau(x_{im})\, \varepsilon_m \end{aligned}$$

This means:

$$\begin{aligned} \gamma(x_{im}, x_{jr}) \quad &= \quad \begin{cases} 1 & \text{for } i \neq j\,,\ m = j\,,\ r = i \\ (q - q^{-1}) & \text{for } i < j\,,\ m = i\,,\ r = j \\ q & \text{for } i = j = m = r \\ 0 & \text{otherwise} \end{cases} \\ \tau(x_{ij}) \quad &= \quad q^n\, \delta_{ij} \; . \end{aligned}$$

19

Solutions to Exercises

Chapter 4

1. For $a \in \mathbb{Z}/(2), b \in \mathbb{Z}/(5)$, put $x = a \otimes b \in \mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{Z}/(5)$. So we have:

$$5x = 5(a \otimes b) = a \otimes 5b = a \otimes 0 = a \otimes (0.0) = 0a \otimes 0 = 0 \otimes 0 = 0$$

$$2x = 2(a \otimes b) = 2a \otimes b = 0 \otimes b = 0$$

So $x = (5 - 2.2)x = 0 - 0 = 0$. Elements of the form x generate, so $\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{Z}/(5) = \{0\}$.

2. (a) Let $\otimes$ denote $\otimes_{\mathbb{Z}}$. The multiplication and unit are given by

$$(R \otimes S) \otimes (R \otimes S) \cong^{1 \otimes \sigma \otimes 1} (R \otimes R) \otimes (S \otimes S) \xrightarrow{\mu \otimes \mu} R \otimes S$$

$$\mathbb{Z} \cong \mathbb{Z} \otimes \mathbb{Z} \xrightarrow{\eta \otimes \eta} R \otimes S$$

This makes it clear that μ us an abelian group morphism, so we automatically have distributivity. In terms of generating elements, the multiplication is $(r \otimes s)(r' \otimes s') = (rr') \otimes (ss')$ and the unit is $1 = 1 \otimes 1$. Associativity and unit conditions only need to be checked on generators where they clearly follow from these conditions in R, S.

(b) Yes, $\varphi(r) = r \otimes 1$ does define a ring morphism $\varphi : R \longrightarrow R \otimes S$.

$$\varphi = (R \cong R \otimes \mathbb{Z} \xrightarrow{1_R \otimes \eta} R \otimes S)$$

is clearly an abelian group morphism. It remains to check that multiplication and unit are preserved:

- $\varphi(rr') = (rr') \otimes 1 = (r \otimes 1)(r' \otimes 1) = \varphi(r)\varphi(r')$
- $\varphi(1) = 1 \otimes 1 = 1$ using the definition in (a).

(c) Let $R \xrightarrow{\varphi} R \otimes_{\mathbb{Z}} S \xleftarrow{\psi} S$, with $\varphi(r) = r \otimes 1$ and $\psi(s) = 1 \otimes s$; these are ring morphisms as in (b). These give our "coprojections". Now given $R \xrightarrow{f} T \xleftarrow{g} S$ with f, g ring morphisms and T commutative, we must show there is a unique ring morphism $h : R \otimes S \longrightarrow T$ with $h \circ \varphi = f$, $h \circ \psi = g$. These last equations force us to define $h(r \otimes s) = h\big((r \otimes 1)(1 \otimes s)\big) = h(r \otimes 1)h(1 \otimes s) = f(r)g(s)$. It is easily checked that $R \times S \longrightarrow T$, $(r, s) \longmapsto f(r)g(s)$ is bilinear. So h does give an abelian group morphism. It remains to show h preserves multiplication and unit.

$$\begin{aligned} h\big((r \otimes s)(r' \otimes s')\big) &= h\big(rr' \otimes ss'\big) = f(rr')g(ss') \\ &= f(r)f(r')g(s)g(s') = f()g(s)f(r')g(s') \\ &= h(r \otimes s)h(r' \otimes s') \end{aligned}$$

So h is a ring morphism. Since the definition of h was forced, it is unique.

3. A module $M : R \,{\to\!\!\!\!\!+}\, S$ is an abelian group with an abelian group morphism $\mu : R \otimes M \otimes S \longrightarrow M$, written $\mu(r \otimes m \otimes s) = rms$, satisfying $r'(rms)s' = (r'r)m(ss')$, $1m1 = m$. One can easily see that this agrees with the definition given in lectures (given left R-, right S-scalar multiplications satisfying $(rm)s = r(ms)$ we define $rms = (rm)s$; all the distributive laws precisely summarise to trilinearity (over $\mathbb{Z}$); conversely, given μ, define the two scalar multiplications by $rm = rm1$ and $ms = 1ms$). Now a left $R \otimes S^{\mathrm{op}}$-module is an abelian group M with an abelian group morphism $\overline{\mu} : (R \otimes S) \otimes M \longrightarrow M$, written $\overline{\mu}(r \otimes s \otimes m) = (r \otimes s)m$ satisfying $(1 \otimes 1)m = m$ and $(r'r \otimes ss')m = (r' \otimes s')\big((r \otimes s)m\big)$. Clearly to give the abelian group morphism μ, $\overline{\mu}$ is the "same thing" via the diagram:

$$\begin{tikzcd} R \otimes M \otimes S \arrow[dr, "\mu"] \arrow[dd, phantom, "\cong 1 \otimes \sigma"] & \\ & M \\ R \otimes S \otimes M \arrow[ur, "\overline{\mu}"'] & \end{tikzcd}$$

Moreover the conditions on μ directly translate to those on $\overline{\mu}$.

4. Suppose $R \,{\to\!\!\!\!\!+}^{M}\, S \,{\to\!\!\!\!\!+}^{N}\, T \,{\to\!\!\!\!\!+}^{L}\, U$ As an abelian group we have:

$$M \otimes_S N \otimes_T L = B$$

Where B is a subset of the abelian group $\mathcal{F}_{\mathbb{Z}}(M \times N \times L)$ consisting of all elements of the form:

$$(m + m', n, l) - (m, n, l) - (m', n, l)\ ,$$

$$(m, n+n', l) - (m,n,l) - (m,n',l) \ ,$$
$$(m,n,l+l') - (m,n,l) - (m,n,l') \ ,$$
$$(ms,n,l) - (m,sn,l) \ ,$$
$$(m,nt,l) - (m,n,tl)$$

The equivalence class of (m,n,l) is denoted by $m \otimes n \otimes l$. We now define $r(m\otimes n \otimes l)\mu = (rm)\otimes n \otimes (l\mu)$ yielding $M \otimes_S N \otimes_T L : R \twoheadrightarrow U$. Then

$$\mathrm{Hom}_R^U(M \otimes_S N \otimes_T L, K) \cong \mathbf{Mult}\,(M,N,L;K)$$

Chapter 5

1. We use the Fundamental Theorem of Morita Theory. Suppose M is finitely generated and projective. By Theorem 5.3, we have the morphisms $d : R \longrightarrow M^* \otimes_R M$, $e : M \otimes_R M^* \longrightarrow R$ satisfying

   $$(M^* \xrightarrow{d\otimes 1} M^* \otimes_R M \otimes_R M^* \xrightarrow{1\otimes e} M^*) = 1_{M^*} \quad \text{and} \quad (e\otimes 1)\circ(1\otimes d) = 1_M$$

   Put $d' = (R \xrightarrow{d} M^* \otimes_R M \xrightarrow{\sigma} M \otimes_R M^*)$, $e' = (M^* \otimes_R M \xrightarrow{\sigma} M \otimes_R M^* \xrightarrow{e} R)$. We can now apply Theorem 5.3 (iii) (replacing M, N, e, d with M^*, M, e', d' respectively) and by (iv) M^* is finitely generated and projective.

2. Notice that $\rho_L^M : M^* \otimes_R L \longrightarrow \mathrm{Hom}_R(M,L)$ is "natural" in M (S in L too for that matter), meaning that for any module morphism $f : M \longrightarrow N : R \twoheadrightarrow S$, the following "$f$-square" commutes:

   $$\begin{tikzcd} N^* \otimes_R L \arrow[r, "\rho_L^N"] \arrow[d, "f^* \otimes 1"'] & \mathrm{Hom}_R(N,L) \arrow[d, "- \circ f"] \\ M^* \otimes_R L \arrow[r, "\rho_L^M"'] & \mathrm{Hom}_R(M,L) \end{tikzcd}$$

   Suppose now that M is a retract of a Cauchy module N; so we have $i : M \longrightarrow N$, $r : N \longrightarrow M$, $r \circ i = 1_M$ and ρ_L^N invertible. We can show that the composite

   $$\mathrm{Hom}\,(M,L) \xrightarrow{-\circ r} \mathrm{Hom}\,(n,L) \xrightarrow{(\rho_L^N)^{-1}} N^* \otimes L \xrightarrow{i^* \otimes 1} M^* \otimes L$$

   is an inverse for ρ_L^M. For $\rho_L^M \circ (i^* \otimes 1) \circ (\rho_L^N)^{-1} \circ (-\circ r) = (-\circ i) \circ \rho_L^N \circ (\rho_L^N)^{-1} \circ (-\circ r)$ (by the i-square) $= (-\circ i) \circ 1_{\mathrm{Hom}\,(N,L)} \circ (-\circ r) = (-\circ i)\circ(-\circ r) = -\circ(ri) = -\circ 1_M = 1_{\mathrm{Hom}\,(M,L)}$ and $(i^* \otimes 1) \circ (\rho_L^N)^{-1} \circ (-\circ r) \circ \rho_L^M = (i^* \otimes 1) \circ (\rho_L^N)^{-1} \circ \rho_L^N \circ (r^* \otimes 1)$ (by the r-square) $= (i^* \otimes 1) \circ (r^* \otimes 1) = (ri)^* \otimes 1 = 1_{M^*} \otimes 1_L = 1_{M^* \otimes L}$, as required.

Chapter 6

1. (a) $G \longrightarrow \mathrm{End}_R(M \otimes N)$, $g \longmapsto \big(m{\otimes}n \longmapsto (gm){\otimes}(gn)\big)$ is a monoid morphism since $1 \longmapsto 1_{M\otimes N}$ and $gh \longmapsto \big(m{\otimes}n \longmapsto (hm){\otimes}(hn) \longmapsto (ghm){\otimes}(ghn)\big)$. So this extends to a unique R-algebra morphism $R(G) \longrightarrow \mathrm{End}_R(M \otimes_R N)$. So $M \otimes_R N$ is an $R(G)$-module.

   (b) Let $\hat{\mu}(g) : \mathrm{Hom}_R(M, L) \longrightarrow \mathrm{Hom}_R(M, L)$ be the R-module morphisms given by $\hat{\mu}(g)(\mu)(m) = g\mu(g^{-1}m)$. Then $\hat{\mu}(1)(\mu)(m) = \mu(m)$, $\hat{\mu}(gh)(\mu)(m) = gh\mu(h^{-1}g^{-1}m) = g\hat{\mu}(h)(\mu)(g^{-1}m) = \big(\hat{\mu}(g) \circ \hat{\mu}(h)\big)(\mu)(m)$. So $\hat{\mu} : G \longrightarrow \mathrm{End}_R\big(\mathrm{Hom}_R(M, L)\big)$ is a monoid morphism. So $\mathrm{Hom}_R(M, L)$ becomes an $R(G)$-module.

   (c) $ev_m\big(g \circ (m{\otimes}\mu)\big) = ev_M(gm{\otimes}g\mu) = (g\mu)(gm) = g\mu(g^{-1}gm) = g\mu(m) = gev_M(m{\otimes}\mu)$ so ev_M preserves the $R(G)$-action.

   (d) We also need $d : N \longrightarrow \mathrm{Hom}_R(M, M \otimes_R N)$, $n \longmapsto (m \longmapsto m{\otimes}n)$ to be an $R(G)$-module morphism. We have

   $$d(gn)(m) = m{\otimes}gn = g\big((g^{-1}m){\otimes}n\big) = gd(n)(g^{-1}m) = \big(gd(n)\big)(m)$$

   so $d(gn) = gd(n)$. The required isomorphism is the restriction of

   $$\begin{array}{rcl} \mathrm{Hom}_R\big(N, \mathrm{Hom}_R(M, L)\big) & \cong & \mathrm{Hom}_R(M \otimes_R N, L) \\ f & \longmapsto & e \circ (1_M{\otimes}f) \\ (g \circ -) \circ d & \longleftarrow\!\!\!\!\shortmid & g \end{array}$$

   to $R(G)$-module morphis, f, g; since e, d are such, so are $e \circ (1_M{\otimes}f)$ and $(g \circ -) \circ d$ when f, g are. [This will be generalised from $R(G)$ to an arbitrary Hopf algebra in Section 9].

Chapter 7

1. There is a little abuse of notation here since the four parts to this are elements of $C \otimes_R C$, $R \otimes_R C$, $C \otimes_R C \otimes_R R$, $C \otimes_R R \otimes_R C$ respectively. But these modulesare canonically isomorphic, and so "=" really means "corresponds under the canonical isomorphism to".

(a) To prove the first "equality":

$$\begin{tikzcd}
 & C\otimes C \arrow[r, "\sigma"] \arrow[dr, "1\otimes\varepsilon"] & C\otimes C \arrow[r, "\varepsilon\otimes 1"] & R\otimes C \arrow[dr, "1\otimes\delta"] & \\
C \arrow[ur, "\delta"] \arrow[rr, "\cong"] \arrow[drr, "\delta"'] & & C\otimes R \arrow[r, "\delta\otimes 1"'] \arrow[ur, "\sigma"] & C\otimes C\otimes R \arrow[r, "\cong"] & R\otimes C\otimes C \\
 & & C\otimes C \arrow[urr, "\cong"'] & &
\end{tikzcd}$$

(b) Similarly for the second:

$$\begin{tikzcd}
 & C\otimes C\otimes C \arrow[r, "1\otimes\delta"] \arrow[dr, "\varepsilon\otimes 1\otimes 1"] & C\otimes C\otimes C \arrow[dr, "\varepsilon\otimes 1\otimes 1"] & \\
C \arrow[ur, "\delta"] \arrow[r, "\delta"'] & C\otimes C \arrow[u, "\delta\otimes 1"] \arrow[r, "\cong"'] \arrow[dr, "\sigma"'] & R\otimes C\otimes C \arrow[r, "1\otimes\sigma"'] & R\otimes C\otimes C \\
 & & C\otimes C \arrow[ur, "\cong"'] &
\end{tikzcd}$$

(c) For the third:

$$\begin{tikzcd}
 & & C\otimes C\otimes C \arrow[r, "1\otimes\sigma"] \arrow[d, "\varepsilon\otimes 1\otimes 1"] & C\otimes C\otimes C \arrow[dr, "\varepsilon\otimes\varepsilon\otimes 1"] & \\
C \arrow[urr, "\delta"] \arrow[r, "\delta"'] \arrow[rrrr, bend right=60, "\cong"'] & C\otimes C \arrow[ur, "\delta\otimes 1"] \arrow[r, "\cong"'] \arrow[dr, "1\otimes\varepsilon"'] & R\otimes C\otimes C \arrow[r, "1\otimes\sigma"'] \arrow[dr, "1\otimes 1\otimes\varepsilon"] & R\otimes C\otimes C \arrow[r, "\varepsilon\otimes 1\otimes 1", "1\otimes\varepsilon\otimes 1"'] & R\otimes R\otimes C \\
 & & C\otimes R \arrow[r, "\cong"'] & R\otimes C\otimes R \arrow[ur, "1\otimes\sigma"] &
\end{tikzcd}$$

Chapter 9

1. (a) Suppose I is a coideal of C, that is, a submodule satisfying $\delta(I) \subseteq I\otimes C + C\otimes I$ and $\varepsilon(I) = 0$. The composite morphism

$$C \xrightarrow{\delta} C\otimes C \xrightarrow{\rho\otimes\rho} C/I\otimes C/I$$

maps I to 0 since $(\rho\otimes\rho)\delta(I) \subseteq (\rho\otimes\rho)(I\otimes C + C\otimes I) = \rho(I)\otimes\rho(C) + \rho(C)\otimes\rho(I) = 0\otimes C/I + C/I\otimes 0 = 0$. So there exists a unique module morphism $\delta : C/I \longrightarrow C/I\otimes C/I$ such that $\delta \circ \rho =$

$(\rho\otimes\rho)\circ\delta$. Similarly, $\varepsilon(I) = 0$ implies there exists a unique $\varepsilon : C/I \longrightarrow R$ with $\varepsilon\circ\rho = \varepsilon$. These properties of δ, ε will mean $\rho : C \longrightarrow C/I$ is a coalgebra morphism once we know C/I is a coalgebra. To prove coassociativity of $\delta : C/I \longrightarrow C/I\otimes C/I$, take the coassociativity diagram for C/I and precompose with $\rho : C \longrightarrow C/I$:

$$\begin{tikzcd}
C \arrow[r, "\rho"] \arrow[dr, "\delta"'] & C/I \arrow[r, "\delta"] & C/I\otimes C/I \arrow[r, shift left, "\delta\otimes 1"] \arrow[r, shift right, "1\otimes\delta"'] & C/I\otimes C/I\otimes C/I \\
& C\otimes C \arrow[ur, "\rho\otimes\rho"] \arrow[r, shift left, "\delta\otimes 1"] \arrow[r, shift right, "1\otimes\delta"'] & C\otimes C\otimes C \arrow[ur, "\rho\otimes\rho\otimes\rho"'] &
\end{tikzcd}$$

The result commutes by coassociativity of C. But $\rho$is surjective; so the coassociativity diagram for C/I commutes. Similarly we can prove $\varepsilon : C/I \longrightarrow R$ is a counit. If C is a bialgebra and I is also an ideal, certainly C/I becomes an algebra. All that remains to check are the extra bialgebra conditions (see Proposition 7.2). The main one, showing that δ preserves multiplication, is obtained by precomposing the diagram with $\rho\otimes\rho$ and using the corresponding condition for C; This gives the result since $\rho\otimes\rho$ is surjective.

Since ρ is a bialgebra morphism, the only possible way C/I can become a Hopf algebra is for $\nu\circ\rho = \rho\circ\nu$ this forces us to ask $\nu(I) \subseteq I$ for the antipode of C.

(b) But $B = R\langle x, y, z\rangle$. The given equations define algebra morphisms $\delta : B \longrightarrow B\otimes B$, $\varepsilon : B \longrightarrow R$ since B is free as an algebra. By Proposition 7.2, it remains to see that these morphisms make B a coalgebra. First look at the coassociativity:

$$(\delta\otimes 1)\delta(x) = (\delta\otimes 1)(x\otimes x) = x\otimes x\otimes x = (1\otimes\delta)(x\otimes x) = (1\otimes\delta)\delta(x)$$

Similarly for y:

$$\begin{aligned}
(\delta\otimes 1)\delta(z) &= (\delta\otimes 1)(1\otimes z + z\otimes x) = 1\otimes 1\otimes z + (1\otimes z + z\otimes x)\otimes x \\
&= 1\otimes(1\otimes z + z\otimes x) + z\otimes x\otimes x = (1\otimes\delta)(1\otimes z + z\otimes x) \\
&= (1\otimes\delta)\delta(z)
\end{aligned}$$

Then the counit conditions:

$$(\varepsilon\otimes 1)\delta(x) = (\varepsilon\otimes 1)(x\otimes x) = \varepsilon(x)x = x = (1\otimes\varepsilon)(x\otimes x) = (1\otimes\varepsilon)\delta(x)$$

and similarly for y:

$$\begin{aligned}
(\varepsilon\otimes 1)\delta(z) &= (\varepsilon\otimes 1)(1\otimes z + z\otimes x) = z + 0x = z = 0 + z \\
&= (1\otimes\varepsilon)(1\otimes z + z\otimes x) = (1\otimes\varepsilon)\delta(z)
\end{aligned}$$

(c) We have:

$$\begin{aligned}
\delta(xy-1)\ (x{\otimes}x)(y{\otimes}y)-1{\otimes}1 &= xy{\otimes}xy-1{\otimes}1\\
&= (xy-1){\otimes}xy+1{\otimes}xy-1{\otimes}1\\
&= (xy-1){\otimes}xy+1{\otimes}(xy-1)\\
&\subseteq I{\otimes}B+B{\otimes}I
\end{aligned}$$

(d) We must check:

$$\begin{tikzcd}
H \arrow[r, "\delta"] \arrow[drr, "\varepsilon"'] & H\otimes H \arrow[rr, shift left, "\nu\otimes 1"] \arrow[rr, shift right, "1\otimes\nu"'] & & H\otimes H \arrow[r, "\mu"] & H \\
& & R \arrow[urr, "\eta"'] & &
\end{tikzcd}$$

$$\begin{aligned}
\mu(\nu{\otimes}1)\delta(x) &= \mu(\nu{\otimes}1)(x{\otimes}x)=\mu(y{\otimes}x)=yx\equiv 1=\nu\varepsilon(x)\\
&\equiv xy=\mu(x{\otimes}y)=\mu(1{\otimes}\nu)(x{\otimes}x)=\mu(1{\otimes}\nu)\delta(x)
\end{aligned}$$

Similarly for y:

$$\begin{aligned}
\mu(\nu{\otimes}1)\delta(z) &= \mu(\nu{\otimes}1)(1{\otimes}z+z{\otimes}x)=\mu(1{\otimes}z+(-zy){\otimes}x)=z-zyx\\
&\equiv 0=\eta\varepsilon(z)=0=-zy+zy=\mu(1{\otimes}(-zy)+z{\otimes}y)\\
&= \mu(1{\otimes}\nu)(1{\otimes}z+z{\otimes}x)=\mu(1{\otimes}\nu)\delta(z)
\end{aligned}$$

So H is a Hopf algebra. By Proposition 8.1, ν reverses both multiplication and comultiplication. The formulas for $\nu^r(z)$ are trivial for $r=0,1$. Also $\nu^{2n}(z)=x^nzy^n$ implies:

$$\nu^{2n+1}(z)=\nu(y)^n\nu(z)\nu(x)^n=x^n(-zy)y^n=-x^nzy^{n+1}$$

which gives:

$$\nu^{2n+2}(z)=-\nu(y)^{n+1}\nu(z)\nu(x)^n=-x^{n+1}(-zy)y^n=x^{n+1}zy^{n+1}$$

So formulas follow by induction.

If ν had finite order, we would have either $x^nz=zx^n$ or $x^nz=-zx^{n+1}$ which are false in H.

(e) i.

$$\begin{aligned}
\nu(x^nzy^n-z) &= \nu(y)^n\nu(z)\nu(x)^n-\nu(z)=x^n(-zy)y^n+zy\\
&= -x^nzy^{n+1}+zy=(x^nzy^n-z)(-y)\in I_n\\
\delta(x^nzy^n-z) &= (x^n{\otimes}x^n)(1{\otimes}z+z{\otimes}x)(y^n{\otimes}y^n)-1{\otimes}z-z{\otimes}x\\
&= (x^n{\otimes}x^nz+x^nz{\otimes}x^{n+1})(y^n{\otimes}y^n)-1{\otimes}z-z{\otimes}x
\end{aligned}$$

$$\begin{aligned}
&= x^n y^n {\otimes} x^n z y^n + x^n z y^n {\otimes} x^{n+1} y^n - 1{\otimes} z - z{\otimes} x \\
&= 1{\otimes} x^n z y^n + x^n z y^n {\otimes} x - 1{\otimes} z - z{\otimes} x \\
&= 1{\otimes}(x^n z y^n - z) + (x^n z y^n - z){\otimes} x \\
&\in I_n \otimes H + H \otimes I_n \\
\varepsilon(x^n z y^n - z) &= 0.
\end{aligned}$$

So I_n is a Hopf ideal in H

$$\begin{aligned}
\nu(x^n - 1) &= y^n - 1 \equiv -y^n(x^n - 1) \in J_n \\
\delta(x^n - 1) &= x^n {\otimes} x^n - 1{\otimes} 1 = (x^n - 1){\otimes} x^n + 1{\otimes}(x^n - 1) \\
&\in J_n \otimes H + H \otimes J_n \\
\varepsilon(x^n - 1) &= 1^n - 1 = 0
\end{aligned}$$

So J_n is a Hopf ideal in H.

ii. $\nu^{2n}(x) = x$ and $\nu^{2n}(y) = y$ since ν just switches x and y.

$$\begin{aligned}
\nu^{2n}(z) &= x^n z y^n \equiv z \pmod{I_n} \\
\nu^{2n}(z) &= x^n z y^n \equiv |z| = z \pmod{J_n}
\end{aligned}$$

Chapter 10

1. Let M be a Cauchy R-module. The diagrams (coassociativity and counit) showing E to be a coalgebra are:

$$MM^* \xrightarrow{1\otimes d\otimes 1} MM^*MM^* \underset{1\otimes d\otimes 1\otimes 1\otimes 1}{\overset{1\otimes 1\otimes 1\otimes d\otimes 1}{\rightrightarrows}} MM^*MM^*MM^*$$

$$MM^* \xrightarrow{1\otimes d\otimes 1} MM^*MM^* \underset{1\otimes 1\otimes e}{\overset{e\otimes 1\otimes 1}{\rightrightarrows}} MM^*$$

(they follow from functionality of $\otimes_R$ and Theorem 5.3)

Certainly $\delta \longmapsto \hat{\delta}$ is a bijection between R-linear functions $\delta : M \longrightarrow C\otimes_R M$ and R-linear functions $\omega : M\otimes_R M^* \longrightarrow C$. The inverse assignment $\omega \longmapsto {}^{\vee}\omega$ is given by:

$${}^{\vee}\omega = \left(M \xrightarrow{1\otimes d} MM^*M \xrightarrow{\omega\otimes 1} CM\right)$$

[That these assignments are mutually inverse follows from the properties of e,d in Theorem 5.3] It remains to see that coaction axioms on δ translate precisely to coalgebra morphisms on ω. We'll do the translation for the coaction axiom:

$$M \xrightarrow{\delta} CM \underset{1\otimes\delta}{\overset{\delta\otimes 1}{\rightrightarrows}} CCM$$

(where $\underline{\delta}$ is the comult of C). This is equivalent to:

$$\begin{tikzcd} MM^* \arrow[r, "\delta\otimes 1"] & CMM^* \arrow[r, shift left, "\underline{\delta}\otimes 1\otimes 1"] \arrow[r, shift right, "1\otimes\delta\otimes 1"'] & CCMM^* \arrow[r, "1\otimes 1\otimes e"] & CC \end{tikzcd}$$

Using:

$$\begin{tikzcd} MM^* \arrow[r, "\delta\otimes 1"] \arrow[dr, "\hat{\delta}"'] & CMM^* \arrow[r, "\underline{\delta}\otimes 1\otimes 1"] \arrow[d, "1\otimes e"] & CCMM^* \arrow[d, "1\otimes 1\otimes e"] \\ & C \arrow[r, "\underline{\delta}"'] & CC \end{tikzcd}$$

and

$$\begin{tikzcd} MM^* \arrow[rr, "1\otimes d\otimes 1"] \arrow[dd, "\delta\otimes 1"'] & & MM^*MM^* \arrow[dl, "\delta\otimes 1\otimes 1"'] \arrow[dd, "de\hat{l}ta\otimes 1"] \\ & CMM^*MM^* \arrow[dr, "1\otimes e\otimes 1\otimes 1"'] & \\ CMM^* \arrow[ur, "1\otimes 1\otimes\delta\otimes 1"] \arrow[rr, "1"'] & & CMM^* \end{tikzcd}$$

We see that the axiom becomes:

$$\begin{tikzcd} MM^* \arrow[rr, "\hat{\delta}"] \arrow[d, "1\otimes d\otimes 1"'] & & C \arrow[d, "\underline{\delta}"] \\ MM^*MM^* \arrow[r, "\hat{\delta}\otimes 1"'] & CMM^* \arrow[r, "1\otimes\hat{\delta}"'] & CC \end{tikzcd}$$

which is a coalgebra morphism axiom on $\hat{\delta}$.

Chapter 11

1. (a) $c_{A,I} = c_{A,I} \circ c_{A,I}$ Since $c_{A,I}$ is invertible, $c_{A,I} = 1_A$. Similarly $c_{I,A} = 1_A$.

$$\begin{tikzcd} A \arrow[r, "c_{A,I\otimes I}"] \arrow[d, "c_{A,I}\otimes 1_I"'] & A \\ A \arrow[ur, "1_I\otimes c_{A,I}"'] & \end{tikzcd}$$

$$\begin{tikzcd}
 & ACB \arrow[r, "c\otimes 1"] & CAB \arrow[dr, "1\otimes c"] & \\
ABC \arrow[ur, "1\otimes c"] \arrow[urr, "c_{AB,C}"'] \arrow[dr, "c\otimes 1"'] & & & CBA \\
 & BAC \arrow[r, "1\otimes c"'] \arrow[urr, "c_{BA,C}"] & BCA \arrow[ur, "c\otimes 1"'] &
\end{tikzcd}$$

(b) Put $\gamma = \sum_i \mu_i \otimes v_i \in A\otimes A$ so that $c_{M,N}(m\otimes n) = \sum_i (\mu_i n)\otimes(v_i n)$:

$$\begin{tikzcd}
 & \sum_i m\otimes\mu_i l\otimes v_i n \arrow[r, mapsto, "c\otimes 1"] & \sum_{i,j} \mu_j\mu_i l\otimes v_j m\otimes v_i n \arrow[dr, mapsto, "1\otimes c"] & \\
m\otimes n\otimes l \arrow[ur, mapsto, "1\otimes c"] \arrow[dr, mapsto, "c\otimes 1"'] & & & X \\
 & \sum_i \mu_i n\otimes v_i m\otimes l \arrow[r, mapsto, "1\otimes c"'] & \sum_{i,j} \mu_i n\otimes\mu_j l\otimes v_j v_i m \arrow[ur, mapsto, "c\otimes 1"'] &
\end{tikzcd}$$

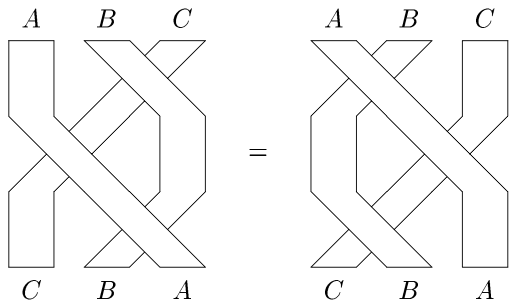
The hexagon gives us the condition:

$$X = \sum_{i,j,k} \mu_j\mu_i l\otimes\mu_k v_i n\otimes v_k v_j m = \sum_{i,j,k} \mu_k\mu_j l\otimes v_k\mu_i n\otimes v_j v_i m$$

in $A\otimes A\otimes A$. Diagramatically this comes:

$$R \xrightarrow{\gamma\otimes\gamma\otimes\gamma} A^{\otimes 6} \underset{\sigma_{536142}}{\overset{\sigma_{315264}}{\rightrightarrows}} A^{\otimes 6} \xrightarrow{\mu\otimes\mu\otimes\mu} A^{\otimes 3}$$

(c)



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Index