

PARENTHETIC REMARKS

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June 1997

The purpose of this note is to provide a gentle¹ introduction to operads and other related areas of contemporary mathematics by concentrating on an important example: the process of parenthesising.

The bracketing² of a string $abcde$ of entities a, b, c, d, e should produce a new entity $(abcde)$. This example shows bracketing as an operation of arity 5. We do not allow $()$ or (a) . This leads us to the following notion of universal algebra. A *parengebra* is a set A together with, for all $n \geq 2$, exactly one n -ary operation

$$\beta_n : A^n \longrightarrow A$$

subject to no axioms. Let \mathcal{K} denote the category of parengbras and their homomorphisms. The general associativity law implies that every semigroup is canonically a parengebra. In particular, we have the parengebra \mathbb{N} of natural numbers under addition

$$\beta_n(m_1, \dots, m_n) = m_1 + \dots + m_n.$$

Write $F(X)$ for the free parengebra on the set X . The elements of $F(X)$ are built up iteratively as follows:

- (i) each $x \in X$ is in $F(X)$;
- (ii) if $a_1, \dots, a_n \in F(X)$ for $n \geq 2$ then $\beta_n(a_1, \dots, a_n) \in F(X)$.

Writing $(a_1 \dots a_n)$ for $\beta_n(a_1, \dots, a_n)$, we can imagine elements of $F(X)$ as words in X (that is, elements of the free monoid X^* on X) with brackets meaningfully inserted. This provides the left adjoint functor $F : \text{Set} \longrightarrow \mathcal{K}$ to the forgetful functor $U : \mathcal{K} \longrightarrow \text{Set}$ into the category Set of small sets. Let $K = U \circ F : \text{Set} \longrightarrow \text{Set}$ be the monad on Set generated by the adjunction $F \dashv U$. As with any forgetful functor from a category of universal algebras, the functor $U : \mathcal{K} \longrightarrow \text{Set}$ is monadic; we have an isomorphism of categories

$$\mathcal{K} \cong \text{Set}^K,$$

where Set^K is the category of Eilenberg-Moore K -algebras. The concepts needed here can be found in [ML].

Consider the functor $R : \text{Set} \longrightarrow \text{Set}$ given by the power series

¹ A first course in category theory should be a sufficient prerequisite.

² We use the word "bracket", interchangeably with "parenthesis", for the symbols $()$ and $()$; the former word conjugates better. For us, the symbols $[\]$ and $[\]$ are "square brackets".

$$R(X) = \sum_{n \geq 2} X^n.$$

As with any endofunctor, an R -algebra is an object A of the category Set in this case together with an arrow $\beta : R(A) \rightarrow A$, which in this case is the same as a parengebra. Write Set^R for the category of R -algebras; we have the trivial identification:

$$\mathcal{K} = \text{Set}^R.$$

It follows that K is the free (pointwise) monad on the endofunctor R . As such, K is expressible as a power series; that is, there is a sequence k_0, k_1, k_2, \dots of sets and a natural bijection

$$K(X) \cong k_0 + k_1 \times X + k_2 \times X^2 + k_3 \times X^3 + \dots.$$

We can take the set k_n to consist of meaningfully bracketed words of length n . More precisely, we can find the sets k_n using the philosophy of clubs [K1]. This leads to the expectation that the functor K can be recaptured from the free parengebra $k = F(1)$ on the singleton set 1 augmented by some “grounding” homomorphism. Let $\gamma : k \rightarrow \mathbb{N}$ be the unique parengebra homomorphism taking the element of the generating set 1 to the natural number 1 . This gives a grading of k according to the number of occurrences of the generator in the bracketed word; that is,

$$k_n = \gamma^{-1}(n).$$

For any set X , the iterative construction of $F(X)$ given by (i), (ii) can be interpreted as a bijection

$$K(X) \cong X + K(X)^2 + K(X)^3 + K(X)^4 + K(X)^5 + \dots.$$

These are the components of a natural isomorphism

$$K \cong 1_{\text{Set}} + R \circ K$$

between endofunctors of Set . Moreover, we can apply a virtual sets argument to the bijection of the last paragraph to obtain a “formula” for the k_n as mere sets. Put $Y = K(X)$ so that the bijection becomes:

$$Y \cong X + Y^2 + Y^3 + Y^4 + Y^5 + \dots \cong X + Y^2 \times (1 - Y)^{-1}.$$

Multiplying on the right by $(1 - Y)^{-1}$ and rearranging, we obtain the quadratic isomorphism

$$2Y^2 - (1 + X) \times Y + X \cong 0.$$

Solving by radicals and choosing the meaningful minus sign, we obtain

$$Y \cong \frac{1}{4}(1 + X - (1 - 6X + X^2)^{\frac{1}{2}}).$$

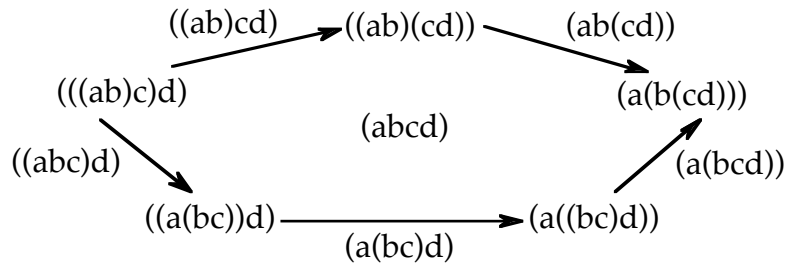
Applying the binomial theorem, we deduce a bijection

$$K(X) \cong X + X^2 + 3X^3 + 11X^4 + 45X^5 + \dots.$$

If the reader is suspicious of virtual sets, at least this argument is valid in terms of generating functions and so gives the cardinalities of $k_0, k_1, k_2, k_3, k_4, k_5, \dots$ as

$$0, 1, 1, 3, 11, 45, \dots.$$

As an example, we see that for k_4 there are the eleven bracketings occurring as in the Stasheff pentagon K_4 :



From this cell complex representation, we see that it is natural to define the *dimension* of an element $a \in F(X)$ to be $n-2$ where n is the largest natural number for which β_n occurs in the iterative construction of a from elements of X . It is also natural to define the *boundary* $\partial(a)$ of $a \in F(X)$ to be the subset of $F(X)$ consisting of those elements obtained from a by meaningfully inserting precisely one further pair of brackets. There is a partial order on $F(X)$ generated by $a \leq b$ if $a \in \partial(b)$. The dimension n of $a \in F(X)$ is the length of a maximal descending chain $a > a_1 > a_2 > \dots > a_n$.

The Stasheff pentagon arose in homotopy theory, specifically, in studying the structure borne by loop spaces. This study was further advanced by May [M] who introduced the concept of operad. We shall see that our sequence $k = (k_n)$ of sets is indeed a planar (or non-permutative) operad.

Given two sequences p, q of sets, we define their substitution product $p \circ q$ to be the sequence of sets

$$(p \circ q)_n = \sum_{m_1 + \dots + m_i = n} p_i \times q_{m_1} \times \dots \times q_{m_i} .$$

We regard a sequence p of sets as a functor from the discrete category \mathbb{N} , whose objects are natural numbers, to the category Set . The substitution product defines a monoidal structure on the category $\text{Set}^{\mathbb{N}}$ of sequences of sets; the unit for substitution is the sequence u with u_1 a singleton set and $u_n = \emptyset$ for $n \neq 1$.

A (*planar*) *operad* t is a monoid for the substitution product on $\text{Set}^{\mathbb{N}}$. Notice that the unit for such a monoid is an element $1 \in t_1$ and the multiplication amounts to a collection of functions

$$t_i \times t_{m_1} \times t_{m_2} \times \dots \times t_{m_i} \longrightarrow t_{m_1 + m_2 + \dots + m_i} ,$$

whose value at $(\tau, \tau_1, \dots, \tau_i)$ is denoted by $\tau[\tau_1, \dots, \tau_i]$, called the result of *substituting* τ_1, \dots, τ_i in τ , such that

$$1[\tau_1] = \tau_1, \quad \tau[1, \dots, 1] = \tau, \\ \tau[\tau_1[\tau_{11}, \dots, \tau_{1j_1}], \dots, \tau_i[\tau_{i1}, \dots, \tau_{ij_i}]] = \tau[\tau_1, \dots, \tau_i][\tau_{11}, \dots, \tau_{1j_1}, \dots, \tau_{i1}, \dots, \tau_{ij_i}].$$

An operad morphism is just a monoid morphism in $\text{Set}^{\mathbb{N}}$ with substitution tensor product.

Example 1 Let \mathcal{A} be a monoidal category. Each object A of \mathcal{A} gives rise to an operad $t = \mathcal{A}(A)$, where $t_n = \mathcal{A}(A^{\otimes n}, A)$, and substitution

$$\mathcal{A}(A^{\otimes i}, A) \times \mathcal{A}(A^{\otimes m_1}, A) \times \dots \times \mathcal{A}(A^{\otimes m_i}, A) \longrightarrow \mathcal{A}(A^{\otimes n}, A)$$

is given by $(f, g_1, \dots, g_i) \longmapsto f \circ (g_1 \otimes \dots \otimes g_i)$. The distinguished element of t_1 is of course the identity arrow of A .

Example 2 Every operad t determines a strict monoidal category \mathcal{V}_t as follows. The objects are the natural numbers and the homsets are given by

$$\mathcal{V}_t(m, n) = \sum_{m_1 + \dots + m_n = m} t_{m_1} \times \dots \times t_{m_n}.$$

Composition

$$\mathcal{V}_t(n, j) \times \mathcal{V}_t(m, n) \longrightarrow \mathcal{V}_t(m, j)$$

takes $(\tau_{n_1}, \dots, \tau_{n_j}, \tau_{m_1}, \dots, \tau_{m_n}) \in t_{n_1} \times \dots \times t_{n_j} \times t_{m_1} \times \dots \times t_{m_n}$ to

$$(\tau_{n_1}[\tau_{m_1}, \dots, \tau_{m_{n_1}}], \tau_{n_2}[\tau_{m_{n_1+1}}, \dots, \tau_{m_{n_1+n_2}}], \dots, \tau_{n_j}[\tau_{m_{n-n_j+1}}, \dots, \tau_{m_n}])$$

where $m_1 + \dots + m_n = m$ and $n_1 + \dots + n_j = n$. The tensor product of \mathcal{V}_t is given on objects by addition and on homsets

$$\mathcal{V}_t(m, n) \times \mathcal{V}_t(i, j) \longrightarrow \mathcal{V}_t(m+i, n+j)$$

by $((\tau_{m_1}, \dots, \tau_{m_n}), (\tau_{i_1}, \dots, \tau_{i_j})) \longmapsto (\tau_{m_1}, \dots, \tau_{m_n}, \tau_{i_1}, \dots, \tau_{i_j})$. Clearly we recapture the original operad t from the object $1 \in \mathcal{V}_t$ by the construction of Example 1. Furthermore, for any strict monoidal category \mathcal{A} and $A \in \mathcal{A}$, the operad morphisms $\phi : t \longrightarrow \mathcal{A}(A)$ are in bijection with the strict monoidal functors $M : \mathcal{V}_t \longrightarrow \mathcal{A}$ with $M(1) = A$; the bijection is determined by putting $M(n) = A^{\otimes n}$ and taking the effect of M on the homsets $\mathcal{V}_t(n, 1) \longrightarrow \mathcal{A}(M(n), M(1))$ to be ϕ_n .

Each operad gives rise to a monad. To see this, we identify each natural number $n \in \mathbb{N}$ with the set $\{0, 1, \dots, n-1\}$ and so obtain the natural sequence $\text{nat} : \mathbb{N} \longrightarrow \text{Set}$. Left Kan extension along the functor nat gives a functor

$$\text{Ser} : \text{Set}^{\mathbb{N}} \longrightarrow \text{Set}^{\text{Set}}$$

given by $\text{Ser}(p) = P$, where

$$P(X) = \sum_{n \geq 0} p_n \times X^n$$

is the power series with coefficient sets p_n , $n \geq 0$. There is a monoidal structure on the category Set^{Set} of endofunctors of Set given by composition of functors. A standard

calculation with power series shows that Ser is a strong monoidal functor; that is, it coherently preserves the monoidal structures:

if $P = \text{Ser}(p), Q = \text{Ser}(q)$ then $P \circ Q \cong \text{Ser}(p \circ q)$; also,
 $\text{Ser}(u)$ is isomorphic to the identity functor.

It follows that monoids t in $\text{Set}^{\mathbb{N}}$ are taken to monads $T = \text{Ser}(t)$ on the category Set . An algebra for the operad t is defined to be an Eilenberg-Moore algebra for the monad $T = \text{Ser}(t)$.

Besides substitution product, there are of course other useful operations on sequences of sets. The *sum* $p + q$ of sequences p, q is given pointwise by $(p + q)_n = p_n + q_n$; this is the coproduct of p, q in the category $\text{Set}^{\mathbb{N}}$ and, likewise, infinite sums can be considered. The *convolution product* $p * q$ of p, q is given by $(p * q)_n = \sum_{i+j=n} p_i \times q_j$. It is easy to see

that the functor $\text{Ser} : \text{Set}^{\mathbb{N}} \longrightarrow \text{Set}^{\text{Set}}$ takes sum to coproduct and convolution product to product. We can identify each set z with the sequence z given by $z_0 = z$ and $z_n = 0$ for all $n > 0$. Then each sequence p of sets can be decomposed as a power series

$$p \cong \sum_{n \geq 0} p_n * u^{*n}$$

where u^{*n} is the n -fold convolution power of the unit u for substitution product.

Example 3 Let u^* denote the terminal sequence of sets; that is, each set in the sequence is a singleton. This clearly has a unique operad structure. The monad $\text{Ser}(u^*)$ on Set is given by the full geometric series

$$X^* = \sum_{n \geq 0} X^n.$$

The elements of X^* are written as *words* in the *alphabet* X . The algebras for the terminal operad are the Eilenberg-Moore algebras for the geometric series monad, and so are monoids (in Set with cartesian product).

Example 4 Let z be any set. Consider the sequence $z + u$ of sets given by

$$(z + u)_n = \begin{cases} z & \text{for } n = 0 \\ 1 & \text{for } n = 1 \\ 0 & \text{for } n > 1. \end{cases}$$

There is a unique operad structure on $z + u$ for which the substitution $(z + u)_1 \times (z + u)_0 \longrightarrow (z + u)_0$ is the second projection. The monad $\text{Ser}(z + u)$ takes X to the monic polynomial $z + X$ of degree 1. The monad structure on $\text{Ser}(z + u)$ is induced by the canonical monoid structure on the arbitrary set z with respect to the coproduct as tensor product on Set . The algebras for the operad $z + u$ are sets A together with a function

$z + A \longrightarrow A$ whose restriction to A is the identity; that is, the algebras amount to functions $z \longrightarrow A$ out of z . The category of algebras is the category z/Set of sets under z .

Example 5 Any monad T on Set admits a distributive law [Bk] with the monad $\text{Ser}(z+u)$ of Example 4; that is, there is a natural transformation $\lambda : \text{Ser}(z+u) \circ T \longrightarrow T \circ \text{Ser}(z+u)$ satisfying axioms ensuring that T lifts to a monad on $\text{Ser}(z+u)$ -algebras, and that $T \circ \text{Ser}(z+u)$ gains a monad structure whose algebras are the same as the algebras for the lifted monad; in this case, the category of these algebras is just the category z/Set^T of T -algebras under z . The component $\lambda_X : z + T(X) \longrightarrow T(z + X)$ of λ at X is constructed from the unit of T at z and T of the inclusions of z, X in $z + X$. Suppose now that $T = \text{Ser}(t)$ for some operad t . Then $T \circ \text{Ser}(z+u) = \text{Ser}(t \circ (z+u))$ for the operad $t \circ (z+u)$ calculated as follows:

$$\begin{aligned} (T \circ \text{Ser}(z+u))(X) &= \sum_{n \geq 0} t_n \times (z + X)^n \\ &= \sum_{n \geq 0} t_n \times \sum_{m \leq n} \binom{n}{m} z^{n-m} \times X^m \\ &= \sum_{n \geq 0} t_n \times \sum_{m \leq n} \binom{n}{m} z^{n-m} \times X^m \\ &= \text{Ser}(t \circ (z+u))(X) \end{aligned}$$

where $(t \circ (z+u))_m = \sum_{0 \leq n} \binom{n+m}{m} t_{n+m} \times z^n$. We shall not bother explicitly describing the substitution operation of $t \circ (z+u)$ except in the special case of the next example.

Example 6 As a particular case of Example 5, take $t = u^*$ to be the terminal sequence as discussed in Example 3. Then $T \circ \text{Ser}(z+u)$ is the monad given by

$$(T \circ \text{Ser}(z+u))(X) = (z + X)^*.$$

The operad $t \circ (z+u) = (z+u)^*$ is given by taking $(z+u)^*_m$ to consist of those elements of the free monoid $(z+1)^*$ which are words in elements of z and the symbol 0 with exactly m occurrences of 0 . For we have the natural bijection

$$(z + X)^* \cong \sum_{m \geq 0} (z+u)^*_m \times X^m$$

$$a_0 x_1 a_1 x_2 a_2 \dots a_{m-1} x_m a_m \longleftarrow \text{---} \longrightarrow ((a_0 0 a_1 0 a_2 \dots a_{m-1} 0 a_m), (x_1, x_2, \dots, x_m))$$

where the $a_0, a_1, \dots, a_m \in z^*$ and $x_1, x_2, \dots, x_m \in X$. We have the substitution functions

$$(z+u)^*_i \times (z+u)^*_{m_1} \times (z+u)^*_{m_2} \times \dots \times (z+u)^*_{m_i} \longrightarrow (z+u)^*_{m_1+m_2+\dots+m_i}$$

$$(a, b_1, \dots, b_i) \longmapsto a_0 b_1 a_1 b_2 a_2 \dots a_{i-1} b_i a_i$$

where $a = a_0 0 a_1 0 a_2 \dots a_{i-1} 0 a_i$. The distinguished element of $(z+u)^*_1$ is of course $0 \in 1$.

Proposition 1 *Free monads on power series endofunctors of Set are all of the form Ser(t) for an operad t.*

This will follow from Proposition 2 which we lead in to by recalling the result of Michael Barr (see [K3]) that the free monad on any endofunctor P on Set is the monad generated by the underlying functor $\text{Set}^P \rightarrow \text{Set}$ from the category of P -algebras and its left adjoint. Let P be a power series endofunctor of Set with coefficients given by the sequence p of sets. We regard the elements of p_n as n -ary operations. A P -algebra is a set A together with, for each $n \in \mathbb{N}$ and each $\omega \in p_n$, a function $A^n \rightarrow A$ which is also denoted by ω . The left adjoint of underlying functor $\text{Set}^P \rightarrow \text{Set}$ can be described iteratively as we did earlier for the case $P = R$. Then the adjunction generates the free monad on P . We shall give another construction³ related to Example 6.

Consider the free monoid $(P(1) + X)^*$ on the set $P(1) + X$ and regard the set \mathbb{Z} of integers as a monoid under addition. Let $(\bar{\quad}) : (P(1) + X)^* \rightarrow \mathbb{Z}$ be the monoid homomorphism given on generators by:

$$\bar{x} = -1 \text{ for all } x \in X \quad \text{and} \quad \bar{\omega} = n-1 \text{ for all } \omega \in p_n.$$

Call $a \in (P(1) + X)^*$ *well formed* when

- (i) $\bar{a} = -1$, and
- (ii) $a = b c$ implies $\bar{c} < 0$.

Let $W_p(X)$ denote the subset of $(P(1) + X)^*$ consisting of the well-formed words. We can equip $W_p(X)$ with the structure of P -algebra by defining

$$\omega(a_1, \dots, a_n) = \omega a_1 \dots a_n$$

for all $\omega \in p_n$ and $a_1, \dots, a_n \in W_p(X)$. To see that $\omega a_1 \dots a_n$ is well formed notice that

- (i) $\overline{\omega a_1 \dots a_n} = \bar{\omega} + \bar{a}_1 + \dots + \bar{a}_n = (n-1) + (-1) + \dots + (-1) = -1$ and
- (ii) if $a_i = b c$, then $\overline{c a_{i+1} \dots a_n} = \bar{c} + \bar{a}_{i+1} + \dots + \bar{a}_n = \bar{c} + (-1) + \dots + (-1) = \bar{c} + (i-n) < 0$.

Proposition 2 $W_p(X)$ is the free P -algebra on the set X .

Proof Given a function $f : X \rightarrow A$ into a P -algebra A , we must show that there exists a unique extension of f to a P -algebra homomorphism $g : W_p(X) \rightarrow A$. That is, we must show that the equations

$$\begin{aligned} g(x) &= f(x) \text{ for } x \in X, \\ g(\omega a_1 \dots a_n) &= \omega(g(a_1), \dots, g(a_n)) \text{ for } \omega \in p_n \text{ and } a_1, \dots, a_n \in W_p(X), \end{aligned}$$

³The criterion for when an expression in a universal algebra, written in Polish notation, is well formed I learned from Samuel Eilenberg. I believe it is an exercise in Bourbaki.

uniquely determine g satisfying the equations. To define $g(a)$ for $a \in W_p(X)$, we use induction on the number r of occurrences of elements of $P(1)$ in the word a . By well formedness, if $r = 0$ then $a \in X$ and the definition $g(a) = f(a)$ is forced. Suppose $r > 0$. By well formedness, a must have length > 1 ; so put $a = t b$ where $t \in X + P(1)$. But $\bar{b} < 0$ and $\bar{t} + \bar{b} = -1$; so $\bar{t} > -1$; so $t \in p_n$ for some $n = -\bar{b}$ and we can write $b = c_1 \dots c_n$ where $\bar{c}_1 = \dots = \bar{c}_n = -1$. Since a is well formed, it follows that $c_1, \dots, c_n \in W_p(X)$. Define

$$g(a) = \omega(g(c_1), \dots, g(c_n))$$

as we are forced to, and can do since each c_i has fewer than r occurrences of elements of $P(1)$. It remains to prove that g is a P -algebra homomorphism, but that is a direct inductive argument. **Q. E. D.**

To construct the operad from W_p , notice that \mathbb{N} becomes a P -algebra by taking each n -ary operation to be n -fold addition. Then we obtain a P -algebra homomorphism

$$\gamma : W_p(1) \longrightarrow \mathbb{N}$$

which restricts to the function $1 \longrightarrow \mathbb{N}$ whose value at the one element 0 of 1 is $1 \in \mathbb{N}$. So $\gamma(a)$ is the number of occurrences of the element 0 of 1 in the word a . Let $t_n = \gamma^{-1}(n)$ be the fibre of γ over n . If we put $z = P(1) = p_0 + p_1 + p_2 + \dots$ we see that W_p is a submonad of the monad $\text{Ser}(z + u)^*$ described in Example 6. Furthermore, the gradings are respected so that t is a suboperad of $(z + u)^*$.

Corollary 1 *The operad t is free on the sequence p of sets. The monad $T = \text{Ser}(t)$ is free on the endofunctor P .*

Proof Suppose o is an operad and $\theta_n : p_n \longrightarrow o_n$ are functions for all $n \geq 0$. We can make $\sum_{m \geq 0} o_m$ into a P -algebra by defining, for each $\omega \in p_n$ a function

$$\omega : \left(\sum_{m \geq 0} o_m \right)^n = \sum_{m_1, \dots, m_n} o_{m_1} \times \dots \times o_{m_n} \longrightarrow \sum_{m \geq 0} o_m$$

by $\omega(v_1, \dots, v_n) = \theta_n(\omega)[v_1, \dots, v_n]$. By Proposition 2, there is a unique P -algebra homomorphism $\phi : W_p(1) \longrightarrow \sum_{m \geq 0} o_m$ determined by $\phi(0) = 1 \in u_1$. This clearly respects the

gradings and so gives functions $\phi_n : t_n \longrightarrow o_n$. Since ϕ is a P -algebra homomorphism, the functions $\phi_n, n \in \mathbb{N}$, commute with the substitution operations and ϕ_1 preserves the distinguished object. So (ϕ_n) is the unique operad morphism extending (θ_n) .

The free monad on the endofunctor P on Set is the monad generated by the underlying functor $\text{Set}^P \longrightarrow \text{Set}$ and its left adjoint. By Proposition 2, this left adjoint is

provided by W_p . It is easily seen that the monad structure transports across our isomorphism $W_p \cong \text{Ser}(t)$ to the monad structure induced by the operad t . **Q. E. D.**

In particular, taking the sequence r of sets given by $r_n = 0$ for $n = 0, 1$ and $r_n = 1$ for $n \geq 2$, so that the power series endofunctor $\text{Ser}(r)$ is R , we obtain a parengebra isomorphism

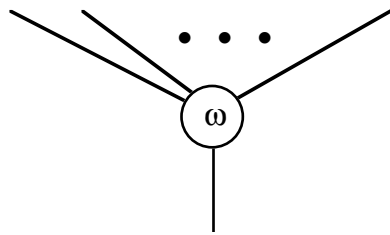
$$W_r(X) \cong F(X),$$

an operad structure on k producing the monad structure on $K = \text{Ser}(k)$. (A topological version of k is the Stasheff A_∞ -operad.)

We shall make a connection with free monoidal categories on certain⁴ tensor schemes [JS]. Take $p \in \text{Set}^{\mathbb{N}}$. Consider the tensor scheme \mathcal{D}_p with one object D and one arrow

$$\omega : D^{\otimes n} \longrightarrow D$$

for each natural number n and each $\omega \in p_n$. We represent the arrow ω by a planar string diagram where there are n input strings above the node ω and 1 output string below⁵.



Let \mathcal{D}_p^* denote the free strict monoidal category on the tensor scheme \mathcal{D}_p : the objects are of the form $D^{\otimes n}$ for $n \in \mathbb{N}$ and the arrows can be identified with the deformation classes of planar string diagrams generated by those representing arrows of \mathcal{D} . Write $\text{Mon}(\mathcal{D}_p^*, \text{Set})$ for the category of (strong) monoidal functors from the monoidal category \mathcal{D}_p^* to Set with its cartesian monoidal structure. Clearly there is an equivalence of categories

$$\text{Set}^P \xrightarrow{\sim} \text{Mon}(\mathcal{D}_p^*, \text{Set}).$$

Corollary 2 *If t is the free operad on the sequence p of sets then there is a monoidal isomorphism*

$$\mathcal{D}_p^* \xrightarrow{\cong} \mathcal{V}'_t$$

taking D to 1 and $\omega : D^{\otimes n} \longrightarrow D$ to $\omega 0 \dots 0 \in t_n = \mathcal{V}'_t(n, 1)$ for $\omega \in p_n$.

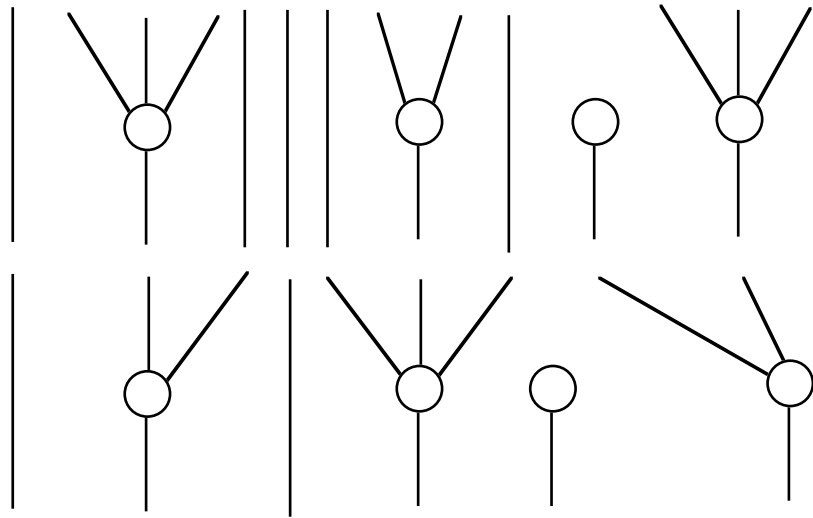
Proof We must prove that, given a strict monoidal category \mathcal{A} , an object A of \mathcal{A} , and

⁴ One might call them opetopic [BD] tensor schemes or multigraphs.

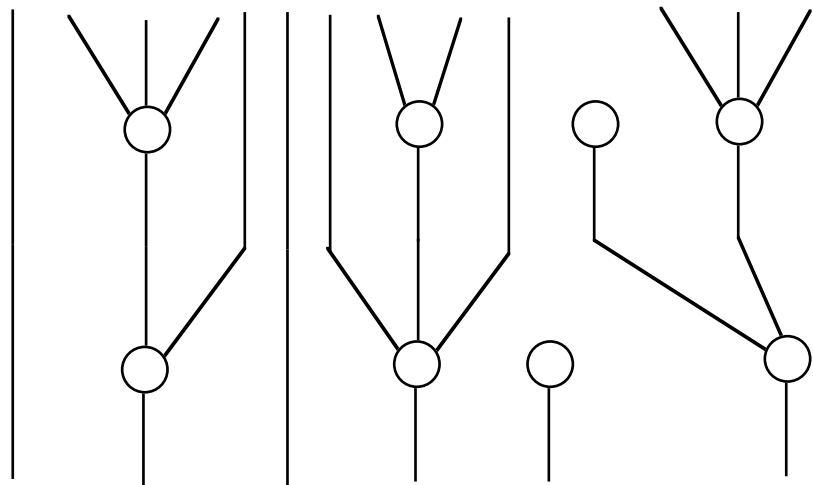
⁵ Here we take the progressive direction to be downward which is opposite to [JS] but the same as [S].

an arrow $f_\omega : A^{\otimes n} \rightarrow A$ for each $\omega \in p_n$, there exists a unique strict monoidal functor $M : \mathcal{V}_t \rightarrow \mathcal{A}$ with $M(1) = A$ and $M(\omega 0 \dots 0) = f_\omega$. Since t is the free operad on p , there is a unique operad morphism $\phi : t \rightarrow \mathcal{A}(A)$ (see Example 1) whose restriction to p is given by $\omega \mapsto f_\omega$. By Example 2, this operad morphism determines a strict monoidal functor M as required. **Q. E. D.**

The arrows of \mathcal{D}_p^* can be identified with p -labelled (planar) forests. To see this, recall from [JS] how the general string diagrams are built from the generators in \mathcal{D}_p . We take some arrows from \mathcal{D}_p and some identity strings and tensor them; this amounts to placing the representing string diagrams next to each other. Here are two examples.

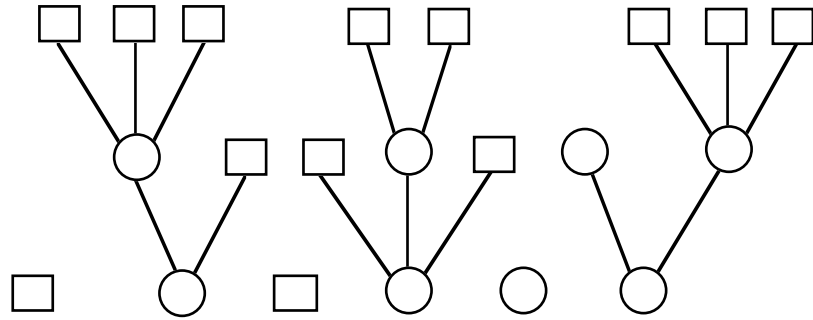


Next we stack such diagrams vertically splicing each lower loose string of one diagram with precisely one upper loose string of the diagram below. The two example diagrams are composable and the composite is represented as follows.



But such a planar diagram can be replaced by a more combinatorial structure. We can

represent this last composite by a diagram



where we have removed the bottom loose strings, put square nodes on the top loose strings, and shortened the strings so that they connect nodes of consecutive height. We must remember that, although we have omitted it from the diagram, each round node with n strings attached at top is labelled by an element of p_n . The structure that arises in this way is precisely a *planar forest*; that is, a functor $f : [k]^{\text{op}} \rightarrow \Delta$ where Δ is the category whose objects are the linearly ordered sets $[k] = \{0, 1, \dots, k\}$ ⁶ and whose arrows are order-preserving functions. Such a functor gives linearly ordered sets $f(i)$ for $0 \leq i \leq k$ (whose elements are called *vertices* of f of *height* i) and order-preserving functions $f_i : f(i+1) \rightarrow f(i)$ for $0 \leq i < k$. In our example, $f(0) = [5]$, $f(1) = [6]$, $f(2) = [7]$, $f_0(0) = f_0(1) = 1$, $f_0(2) = f_0(3) = f_0(4) = 3$, $f_0(5) = f_0(6) = 5$, $f_1(0) = f_1(1) = f_1(2) = 0$, $f_1(3) = f_1(4) = 3$, $f_1(5) = f_1(6) = f_1(7) = 6$. A forest $f : [k]^{\text{op}} \rightarrow \Delta$ is called a *tree* when $f(0) = [0]$. Each forest can be identified with a linearly ordered set of component trees: the number of component trees is n where $f(0) = [n-1]$.

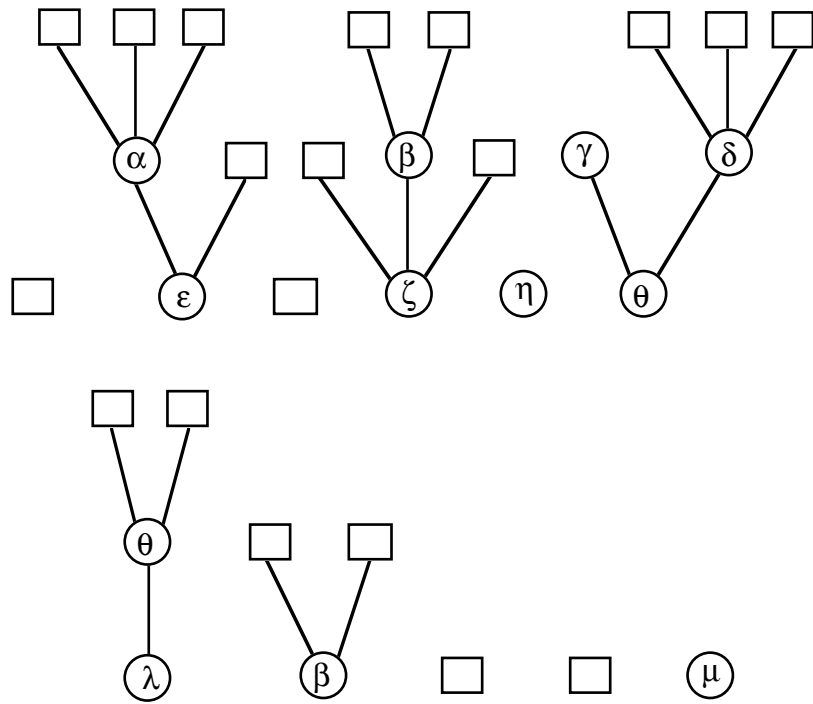
For a forest $f : [k]^{\text{op}} \rightarrow \Delta$ of *height* k , a vertex $v \in f(i)$ is called a *leaf* when the fibre of f_i over v is empty. Notice in our example that the square nodes are all leaves; but there are also two round nodes that are leaves. Let p be a sequence of sets. A *labelling* of a forest f in p assigns to each vertex $v \in f(i)$, which is not a leaf, an element $\omega(v) \in p_n$ where $n > 0$ is the cardinality of the fibre of f_i over v , and assigns to some leaves $v \in f(i)$, which are not of height k , an element $\omega(v) \in p_0$. A leaf which is not labelled will be called *fallen*; so all leaves of height k are fallen. A forest together with a labelling in p will be called a *p-forest*.

This leads to our new view of the strict monoidal category⁷ \mathcal{D}_p^* . We see that the arrows $D^{\otimes m} \rightarrow D^{\otimes n}$ can be identified with p -forests having m fallen leaves and n vertices of height 0. The tensor product of two p -forests is given by placing them next to each other in the plane; we shall not describe it combinatorially. Composition is given by *grafting*: we shall just give an example. Let us take the following example of two

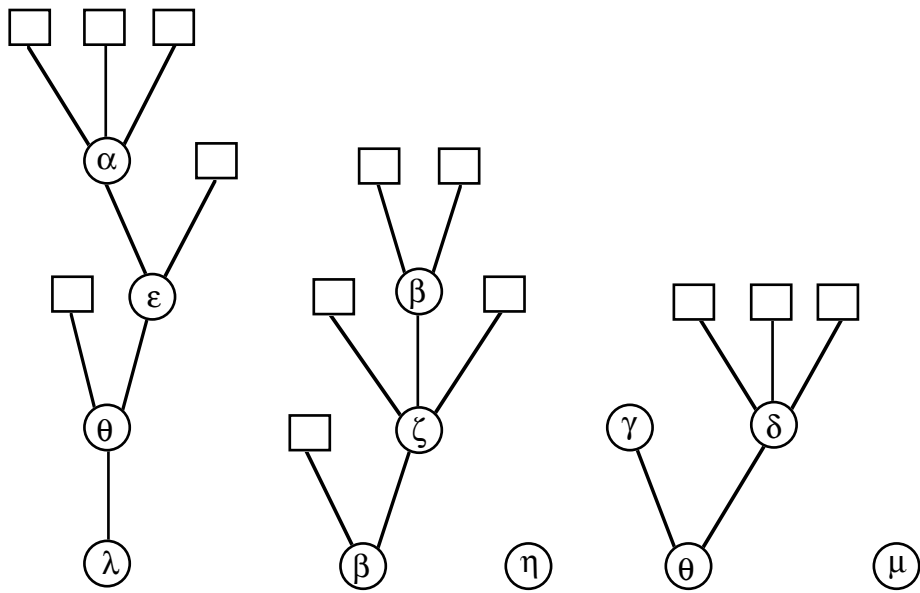
⁶ We include the case $k = -1$ so that $[-1]$ is the empty set.

⁷ I dedicate the identity arrow of the tensor unit to the Harris-Daishowa company.

composable p-forests.

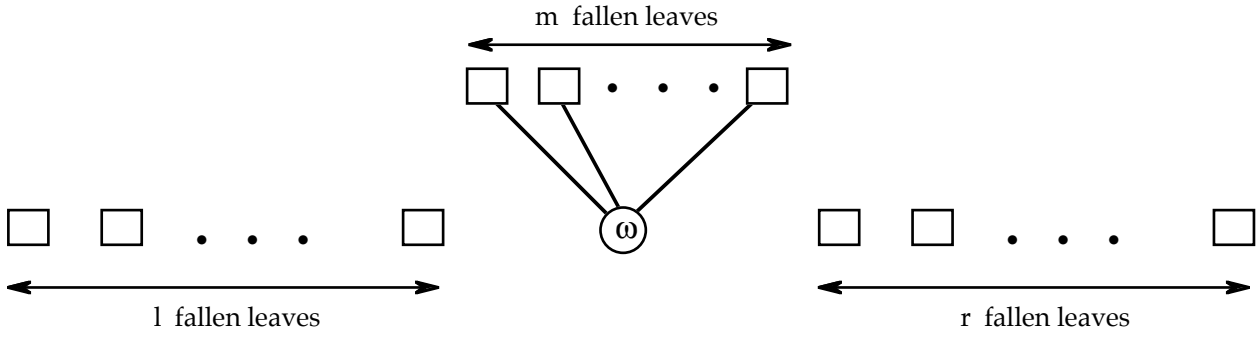


Their composite is the following p-forest in which the component trees of the top forest are grafted on at the fallen leaves of the bottom forest.



From this, we deduce another view of the free operad t on p in which elements of t_n are identified with p-trees with n fallen leaves.

A view of \mathcal{D}_p^* that even a computer can understand can be obtained using *rewrite systems* [S]. The basic p-forest



can be identified with triplets (l, ω, r) where l, m, r are natural numbers and $\omega \in p_m$. Consider the directed graph \mathcal{G}_p whose vertices are natural numbers and whose edges are the expressions

$$(\omega, r) : a \longrightarrow b$$

where a, b, r are natural numbers such that $r < b \leq a + 1$ and $\omega \in p_m$ where $m = a - b + 1$. The idea is that each $\omega \in p_m$ is regarded as the name of a (first-order) rewrite rule and $(\omega, r) : a \longrightarrow b$ represents an allowable application of the rule; $l = b - 1 - r$ and r are the left and right "whiskers", respectively. We introduce some (second-order) rewrite rules on directed paths in the graph \mathcal{G}_p .

$$\frac{a \xrightarrow{(\omega, r)} b \xrightarrow{(\omega \oplus r \oplus)} c}{a \xrightarrow{(\omega \oplus r \oplus a - b)} a - b + c \xrightarrow{(\omega, r)} c} \quad \text{for } r < r \oplus$$

It is easy to see that the p-forests corresponding to the top and bottom of this rewrite rule are equal. Two directed paths in \mathcal{G}_p with same source and target are said to be equivalent when there is a sequence of applications of the (second-order) rewrite rules which takes one path to the other. A directed path

$$a_0 \xrightarrow{(\omega_1, r_1)} a_1 \xrightarrow{(\omega_2, r_2)} \dots \xrightarrow{(\omega_n, r_n)} a_n$$

is said to be in *normal form* when $r_i \geq r_{i+1}$ for $1 \leq i < n$. Each directed path is equivalent to a unique path in normal form; indeed, the normal form can be achieved by directed applications of the (second-order) rewrite rules (the proof of confluence and termination is similar to the case considered in [S]).

In this way, we obtain a strict monoidal category isomorphic to \mathcal{D}_p^* . The objects are natural numbers and the arrows are equivalence classes of directed paths in \mathcal{G}_p . The composition is induced on equivalence classes by concatenation of paths. The tensor product $a \otimes b$ is given on objects a, b by addition $a + b$ of natural numbers. The functor $c \otimes -$ is given on arrows by

$$c \otimes [(\omega_n, r_n) \circ \dots \circ (\omega_1, r_1)] = [(\omega_n, r_n) \circ \dots \circ (\omega_1, r_1)]$$

and the functor $- \otimes d$ is given on arrows by

$$[(\omega_n, r_n) \circ \dots \circ (\omega_1, r_1)] \otimes d = [(\omega_n, r_n + d) \circ \dots \circ (\omega_1, r_1 + d)];$$

the tensor product $\pi \otimes \rho : a \otimes c \longrightarrow b \otimes d$ of two arrows $\pi : a \longrightarrow b$, $\rho : c \longrightarrow d$ is then given by either route around the following square, the commutativity of which is precisely what we have achieved by our equivalence relation on paths.

$$\begin{array}{ccc}
 a \otimes c & \xrightarrow{\pi \otimes c} & b \otimes c \\
 a \otimes \rho \downarrow & & \downarrow b \otimes \rho \\
 a \otimes d & \xrightarrow{\pi \otimes d} & b \otimes d
 \end{array}$$

We now want to define an order inspired by the order on $F(X)$ coming from the Stasheff polytopes. Suppose the sequence p of sets is equipped with a “substitution into the i -th position” operation

$$\text{sub}_i : p_m \times p_n \longrightarrow p_{m+n-1}$$

for $1 \leq i \leq m$, written $\text{sub}_i(\omega, \tau) = \omega \wedge_i \tau$. This allows us to define an order on the sets $W_p(X)$. Let \leq be the smallest reflexive transitive relation such that $a\omega b\tau c \leq a(\omega \wedge_i \tau)bc$ whenever $\omega, \tau \in P(1)$ and b is a product of $i-1$ well-formed elements of the monoid $(P(1) + X)^*$. Notice that, under these conditions, $a\omega b\tau c$ is well formed iff $a(\omega \wedge_i \tau)bc$ is well formed. Also, the order is antisymmetric and satisfies the ascending chain condition (since the length of the word $a(\omega \wedge_i \tau)bc$ is one less than the length of $a\omega b\tau c$). The number of elements of X occurring in the word $a\omega b\tau c$ is the same as in $a(\omega \wedge_i \tau)bc$; so the order respects the grading $\gamma : W_p(1) \longrightarrow \mathbb{N}$, yielding a sequence t of partially ordered sets t_n , $n \geq 0$.

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