# The natural transformation in mathematics

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# Themes

1. **Relationships to others, not molecules, are what define one. (Anonymous, 2016)**

2. *"One important lesson we have learned from topological quantum field theory is that describing dynamics using group representations is only a special case of describing it using category representations."* (Baez-Dolan, 1995)

# Generalized spaces

Riemann laid foundations for topology, differential geometry, relativity, and analytic number theory. For example, he foresaw that continuity was not dependent on a metric.

In 1914, to define

CONTINUOUS FUNCTIONS

in general, Felix Hausdorff invented

TOPOLOGICAL SPACES

(although the modern meaning is a little more general).

Bernhard Riemann 1826 – 1866

# Groups define geometries

Klein observed (Erlangen Program 1872) that each branch of geometry is defined by its group of symmetries and that that geometry is the study of invariants under the group's action.
For example, the group for congruence geometry consists of invertible metric-preserving functions;
the group for similarity geometry consists of invertible angle-preserving functions;
the group for differential geometry consists of smoothly invertible smooth functions.

Felix Klein 1849 – 1925

# Algebraic structures as invariants

- In Phys, Nöther's First Theorem deduces a conservation law from certain symmetries of a classical physical system. A decade and a half before coming to Macquarie, John Clive Ward (1924 – 2000) obtained a quantum version.
  {*I shall return to this at the end.*}
- In Math, Nöther emphasised algebraic structures as the fundamental invariants of spaces: the numerical invariants can be obtained as dimensions or ranks of those structures.

Emmy Nöther 1882 – 1935

# Founders of Category Theory 1945

What precisely is

NATURALITY

in Mathematics?

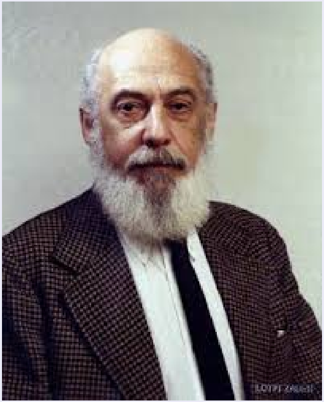
For that they needed

FUNCTORS.

For that they defined

CATEGORIES.

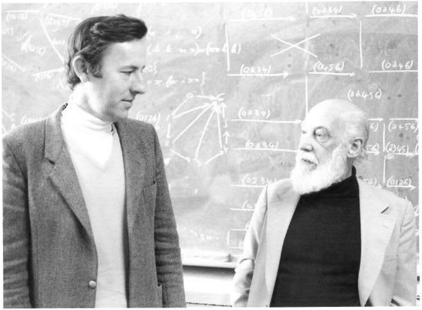
Samuel Eilenberg
1913 – 1998



Saunders Mac Lane
1909 – 2005

# Just for fun

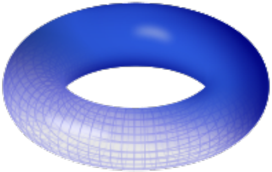
## Eilenberg at 72 at Macquarie



## Mac Lane at 90 in Portugal

## Set Theory

- By the 20th Century, Mathematics was expressed in terms of SETS.
- The Euclidean plane is $\mathbb{R}^2 = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$.
- Many sets of interest are subsets of Euclidean space.
- A torus:



can be seen as a subset of $\mathbb{R}^3$:

$$\mathbb{T}^2 = \{((3 + 2\cos u_1)\cos u_2), (3 + 2\cos u_1)\sin u_2, 2\sin u_1) : u_1, u_2 \in \mathbb{R}\}$$

or as a subset of $\mathbb{R}^4$:

$$\mathbb{S}^1 \times \mathbb{S}^1 = \{x = (x_1, x_2, x_3, x_4) : x_1^2 + x_2^2 = 4,\ x_3^2 + x_4^2 = 9\}\ .$$

# Mathematical Structures and Functions

- Continuous functions lead us to topological spaces. Any subset of Euclidean space of any dimension is a topological space in an obvious way.
- Smooth functions lead us to manifolds. The torus is locally like $\mathbb{R}^2$ and so is a 2-dimensional manifold (surface).
- Linear functions lead us to vector spaces. Any subset of $\mathbb{R}^n$ which contains the plane determined by any two of its points and the origin is a vector space. Linear functions preserve linear combinations.
- The lesson is that each kind of mathematical structure brings with it particular functions which relate to that structure. Klein emphasised the invertible such functions so that groups were central. In category theory, we look at all the specified functions.

# Arrows

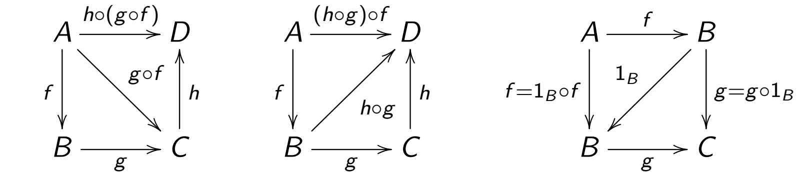
- Since the early 20th century, functions between sets were often denoted by arrows. Thus

$$f\colon X \longrightarrow A$$

  is a name for a specific rule which assigns to each element $x \in X$ an element $f(x) \in A$.
- If $X$ and $A$ were topological spaces, we would be interested in when $f$ was continuous.
- If $X$ and $A$ were vector spaces, we would be interested in when $f$ was linear.
- Arrows $f\colon X \longrightarrow A$, $g\colon A \longrightarrow K$ of these given types compose to give an arrow $g \circ f\colon X \longrightarrow K$ of the same type; first apply $f$ then apply $g$.
- Each set $X$ has an identity function $1_X\colon X \longrightarrow X$ defined by $1_X(x) = x$.

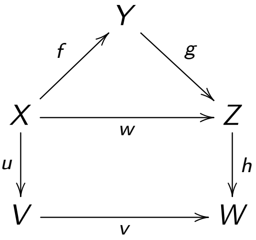
# Definition of Category

A *category* $\mathscr{C}$ consists of a collection of *objects* and, for each pair of objects $A, B$, a set $\mathscr{C}(A, B)$ of morphisms $f : A \longrightarrow B$, together with an associative composition rule $\circ$ with identities $1_A : A \longrightarrow A$.



# Commutative Diagrams

In a category, we can speak of *commutative diagrams*:




$$h \circ w = v \circ u \quad \text{and} \quad g \circ f = w, \quad \text{so } v \circ u = h \circ g \circ f\,.$$

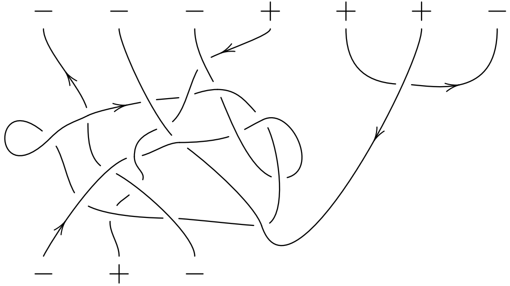
# Examples of categories

- The category **Set** has objects sets and morphisms functions.
- The category **Vect** has objects vector spaces and morphisms linear functions.
- The category **Top** has objects topological spaces and morphisms continuous functions.
- The category $\Delta$ has objects the ordinals $\mathbf{n} = \{0, 1, \ldots, n-1\}$ and morphisms order-preserving functions.
- The category **Mat** has objects the natural numbers $0, 1, 2, \ldots$ and morphisms $a \colon n \longrightarrow m$ the $m \times n$ matrices; composition is matrix multiplication.

# An example from knot theory

The category **Tang** of tangles was defined by David Yetter c.1985.

- The objects are words $- + + - - +$ in symbols $+$ and $-$
- Morphisms are tangles




- Composition is vertical gluing of tangles

# New categories from old

- The opposite or dual of a category $\mathscr{C}$ is the category $\mathscr{C}^{\mathrm{op}}$ whose objects are the same as for $\mathscr{C}$ however a morphism $f\colon A \to B$ in $\mathscr{C}^{\mathrm{op}}$ is a morphism $f\colon B \to A$ in $\mathscr{C}$. The composite $g \circ f$ in $\mathscr{C}^{\mathrm{op}}$ is $f \circ g$ in $\mathscr{C}$.
- The product of two categories $\mathscr{A}$ and $\mathscr{X}$ is the category $\mathscr{A} \times \mathscr{X}$ whose objects are pairs $(A, X)$ where $A$ is an object of $\mathscr{A}$ and $X$ is an object of $\mathscr{X}$. The morphisms $(f, u)\colon (A, X) \to (B, Y)$ consist of a morphism $f\colon A \to X$ in $\mathscr{A}$ and a morphism $u\colon B \to Y$ in $\mathscr{X}$.

# The release from curly brackets

- No longer do we define an object in terms of its elements.
- Each object is determined by how it is observed, via morphisms, by other objects of the category.
- The one-element set $\mathbf{1}$ as an object of **Set** has exactly one morphism $X \to \mathbf{1}$ from any other object. Such an object is called terminal in the category. The zero vector space $\{0\}$ is terminal in **Vect**.
- The zero vector space is also initial in **Vect**: there is exactly one linear function $\{0\} \to V$ into any other vector space $V$.
- However, the one-element set $\mathbf{1}$ is not initial in **Set**. In fact, functions $\mathbf{1} \to X$ can be identified with elements of $X$.
- Morphisms $U \to A$ in a category $\mathscr{C}$ can be thought of generalized elements or $U$-elements of $A$. Categories naturally permit us to think also of $U$-coelements $A \to U$ of $A$.
- Objects in quite different categories can exhibit similar categorical properties.

# Isomorphisms

A morphism $f\colon A \to B$ in a category $\mathscr{C}$ is invertible or an isomorphism when there exists a morphism $g\colon B \to A$ in $\mathscr{C}$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. The morphism $g$ can be proved to be unique and so is denoted by $f^{-1}$.

When an invertible morphism exists $A \to B$, we write $A \cong B$. This is an equivalence relation.

An invertible morphism $f\colon A \to A$ is called an automorphism of $A$. Every group arises as a group $\mathrm{Aut}_{\mathscr{C}}(A)$ of automorphisms in some category $\mathscr{C}$; we can regard the group as a subcategory of $\mathscr{C}$ with one object $A$ and automorphisms as morphisms.

# Functors

Categories are themselves mathematical structures: so we should look at morphisms between them.

A *functor* $T : \mathscr{C} \longrightarrow \mathscr{H}$ assigns

- an object $TA$ of $\mathscr{H}$ to each object $A$ of $\mathscr{C}$,
- a morphism $Tf : TA \longrightarrow TB$ in $\mathscr{H}$ to each $f : A \longrightarrow B$ in $\mathscr{C}$,

such that

$$T1_A = 1_{TA} \quad \text{and} \quad T(g \circ f) = Tg \circ Tf.$$

So categories (with some restriction on size) form a category **Cat** with functors as the morphisms.

# Examples of functors

A functor is a construction of objects of a category from the objects of another category which allows induced morphisms.

- There is a functor $\mathbb{R}^{-} \colon \textbf{Mat} \to \textbf{Vect}$ taking each $n$ to $\mathbb{R}^{n}$ and each matrix to the linear function represented by that matrix.
- There is a functor $\mathrm{C} \colon \textbf{Top}^{\mathrm{op}} \to \textbf{Vect}$ taking each space $X$ to the vector space $\mathrm{C}(X)$ of continuous functions from $X$ into $\mathbb{R}$.
- For each object $U$ of any category $\mathscr{C}$, there is a functor $\mathrm{E}_U \colon \mathscr{C} \to \textbf{Set}$ taking each object $A$ to the set of $U$-elements of $A$.
- Cartesian product is a functor $- \times - \colon \textbf{Set} \times \textbf{Set} \to \textbf{Set}$.
- A functor $R \colon \mathrm{Aut}_{\mathscr{C}}(A) \longrightarrow \textbf{Vect}$ is precisely a linear representation of the group $\mathrm{Aut}_{\mathscr{C}}(A)$.

# Natural transformations

- Suppose $S\colon \mathscr{A} \to \mathscr{X}$ and $T\colon \mathscr{A} \to \mathscr{X}$ are functors.
- A natural transformation $\theta\colon S \Longrightarrow T$ consists of a morphism

$$\theta_A\colon SA \longrightarrow TA$$

in $\mathscr{X}$ for each object $A$ in $\mathscr{A}$, subject to commutativity of the following square for every morphism $f\colon A \to B$ in $\mathscr{A}$.

$$\begin{array}{ccc} SA & \xrightarrow{\ \theta_A\ } & TA \\ {\scriptstyle Sf}\downarrow & & \downarrow{\scriptstyle Tf} \\ SB & \xrightarrow[\ \theta_B\ ]{} & TB \end{array}$$

# Yoneda Lemma

- Suppose $T\colon \mathscr{C} \to \mathbf{Set}$ is any functor and $t \in TU$ is any element. There is a natural transformation $\theta\colon \mathrm{E}_U \Longrightarrow T$ defined by

$$\theta_A(a) = (Ta)(t)$$

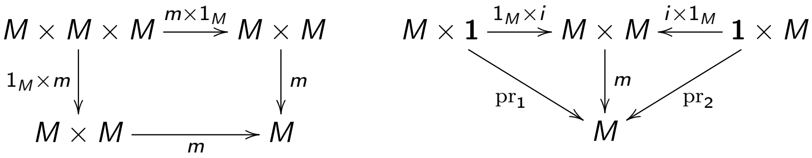
for all $a\colon U \to A$.

- The Yoneda Lemma says that these natural transformations are distinct for distinct $t$ and that they are the only natural transformations $\mathrm{E}_U \Longrightarrow T$.

# Functor categories

- Let $R, S, T\colon \mathscr{A} \to \mathscr{X}$ be functors. If $\phi\colon R \Longrightarrow S$ and $\theta\colon S \Longrightarrow T$ are natural transformations then, by composing components, we obtain a composite natural transformation $\theta \circ \phi\colon R \Longrightarrow T$.
- Therefore we obtain a category $[\mathscr{A}, \mathscr{X}]$ of functors from $\mathscr{A}$ to $\mathscr{X}$ where the morphisms are natural transformations.
- The functor category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ shares many properties with the category **Set** and, as a consequence of Yoneda, faithfully contains $\mathscr{C}$ itself.

# Monoids

- A monoid is a set $M$ with binary operation $m(a,b) = a \cdot b$ and a distinguished element $i$ such that the operation is associative and the element acts as an identity.
- We can write the binary operation and element as functions $m = - \cdot - \colon M \times M \to M$ and $i \colon \mathbf{1} \to M$. We can express the associativity and identity laws as commutative diagrams.
- 

# Monoidal categories

- Many categories $\mathcal{A}$ have a canonical choice of functor
$$- \otimes - : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A} \; ,$$
which is associative up to components
$$\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C) \; ,$$
of a natural isomorphism, and for which there is an object $I$ with natural isomorphisms having components $I \otimes A \cong A \cong A \otimes I$ .
- With a couple of axioms, this defines monoidal category (aka "tensor category").
- Sometimes there is a choice of a natural commutativity isomorphism $\mathrm{c}_{A,B} : A \otimes B \longrightarrow B \otimes A$ .
- With conditions expressing $\mathrm{c}_{A \otimes B, C}$ in terms of $\mathrm{c}_{A,C}$ and $\mathrm{c}_{B,C}$, and expressing $\mathrm{c}_{A, B \otimes C}$ in terms of $\mathrm{c}_{A,B}$ and $\mathrm{c}_{A,C}$, we have a braiding.
- If further $\mathrm{c}_{B,A} \circ \mathrm{c}_{A,B} : A \otimes B \to A \otimes B$ is the identity then we have a symmetry.

## Examples of monoidal categories

- **Set** becomes symmetric monoidal using cartesian product $A \times B$.
- The cartesian product of two vector spaces is a vector space so **Vect** becomes symmetric monoidal using cartesian product. However, here it is written as direct sum $V \oplus W$ since, for finite dimensional vector spaces,

$$\dim(V \oplus W) = \dim V + \dim W \ .$$

- There is also tensor product $V \otimes W$ of vector spaces defining a different symmetric monoidal structure on **Vect**. For finite $V, W$,

$$\dim(V \otimes W) = \dim V \times \dim W \ .$$

- **Tang** becomes braided monoidal: the tensor product of two words in $+$ and $-$ is juxtaposition and, of tangles, is horizontal placement.

# Abstract monoids

- Looking back at the categorical definition of monoid, we can see that we can define monoids in any monoidal category $\mathscr{C}$ by replacing $\times$ by the abstract $\otimes$ of $\mathscr{C}$.
- Now $M$ is an object of $\mathscr{C}$ and we have morphisms $m\colon M \otimes M \to M$ and $i\colon I \to M$ in $\mathscr{C}$ subject to some commuting diagrams.
- A monoid in **Vect** is called an algebra.
- $\Delta$ becomes monoidal via ordinal sum with unit object $I$ equal to $\mathbf{0}$. The object $\mathbf{1}$ becomes a monoid via the unique morphisms $\mathbf{1} + \mathbf{1} = \mathbf{2} \to \mathbf{1}$ and $\mathbf{0} \to \mathbf{1}$.

**Theorem**

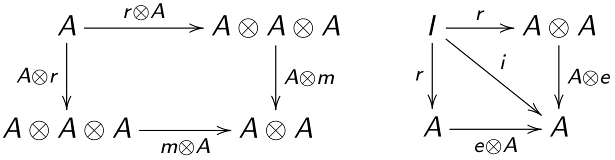
*Let $\mathscr{C}$ be a monoidal category. The category of functors $F\colon \Delta \to \mathscr{C}$, which preserve tensor and unit object up to coherent isomorphism, is equivalent to the category of monoids in $\mathscr{C}$.*

# Other such theorems

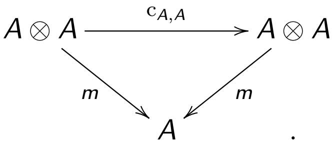
- The previous Theorem means that $\Delta$ is produced when you want to freely generate a monoidal category containing a monoid. Every morphism of $\Delta$ can be obtained from the monoid structure on $\mathbf{1}$ by tensoring and composing. The equality of any two morphisms obtained by this process is a consequence of the monoid axioms.
- There are many structures that can be defined in a monoidal category and even more in a braided monoidal category.
- Commutative monoids make sense in a braided monoidal category. An important viewpoint on this is that monoids in a braided monoidal category $\mathscr{C}$ form a monoidal category $\mathrm{Mon}\mathscr{C}$. Monoids in $\mathrm{Mon}\mathscr{C}$ are precisely the commutative monoids in $\mathscr{C}$.

# Frobenius monoids

A monoid $A$ in a monoidal category $\mathscr{C}$ is called Frobenius when it is equipped with a morphism $e\colon A \to I$ for which there exists a morphism $r\colon I \to A \otimes A$ satisfying the following commutative diagrams.




As mentioned, when the monoidal category is symmetric, we can also speak of commutative monoids: the condition is

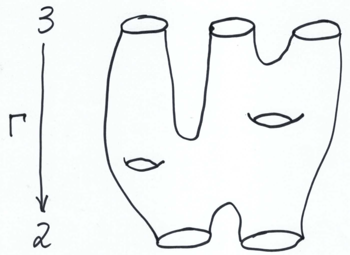


# Cobordisms

A 2D-cobordism $\Gamma$ from natural number $m$ to natural number $n$ is a compact surface with boundary consisting of $m + n$ circles, $m$ of which are allocated as inputs and $n$ of which are allocated as outputs.

Let **2Cob** be the category whose objects are natural numbers and whose morphisms are input/output-preserving topological isomorphism classes of cobordisms. Composition is done vertically by sewing output circles of the first to input circles of the second.

## Cobordism example



# 2D-Topological Quantum Field Theories

The category **2Cob** is symmetric monoidal. The tensor product on objects is addition of natural numbers and on morphisms is induced by horizontal placement of surfaces.

### Definition

A 2D-topological quantum field theory (2DTQFT) is a symmetry-and-tensor-preserving functor

$$T\colon \mathbf{2Cob} \longrightarrow \mathbf{Vect}\ .$$

# Birth, death and marriage

The object 1 is a commutative Frobenius monoid in **2Cob**.

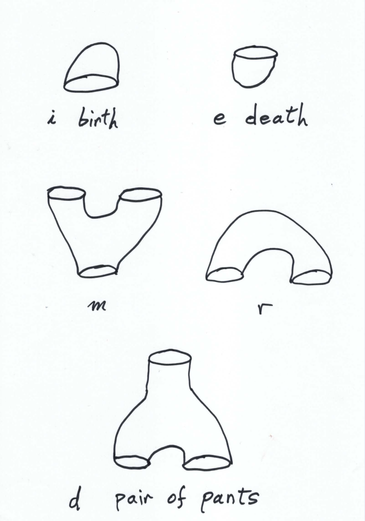
The monoid and Frobenius structural morphisms

$$i\colon 0 \to 1\ ,\ \ e\colon 1 \to 0\ ,$$
$$m\colon 2 \to 1\ ,\ \ r\colon 0 \to 2$$

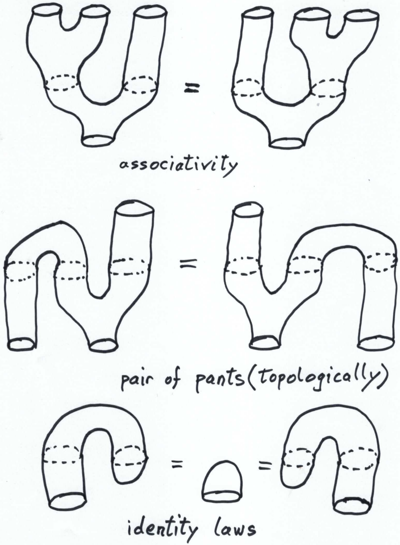
are as shown on the right.

A comultiplication $d\colon 1 \to 2$ can be constructed, as we will now see.

## Cobordism example



# How some axioms look



# Classification of 2DTQFTs

**Theorem**

*The category of 2DTQFTs is equivalent to the category of commutative Frobenius algebras. The equivalence takes $T\colon \mathbf{2Cob} \to \mathbf{Vect}$ to $T1$.*

- The physicist would see this as a construction for 2DTQFTs from a commutative Frobenius algebra.
- The mathematician obtains from it invariants for compact closed surfaces $\Sigma$ (since $\Sigma\colon 0 \to 0$ in **2Cob**, so $T\Sigma$ is a scalar).

# 3DTQFTs, 2DCFTs and modular tensor categories

- Ed Witten showed that the space of conformal blocks of a 2D Conformal Field Theory can be identified with the space of states of a 3DTQFT.
- Modular tensor categories define 3DTQFTs via a Reshetikhin-Turaev surgery construction.
- The category of representations of the chiral vertex algebra of a rational CFT is a modular tensor category.
- The part of the CFT beyond its chiral aspect involves a consistent system of correlators for the fields.
- Symmetries strongly constrain the possible correlators owing to the Ward identities.

John Ward 1924 – 2000; MqU days 1967 – 1984

# Thank you!