Lax monoids, pseudo-operads, and convolution

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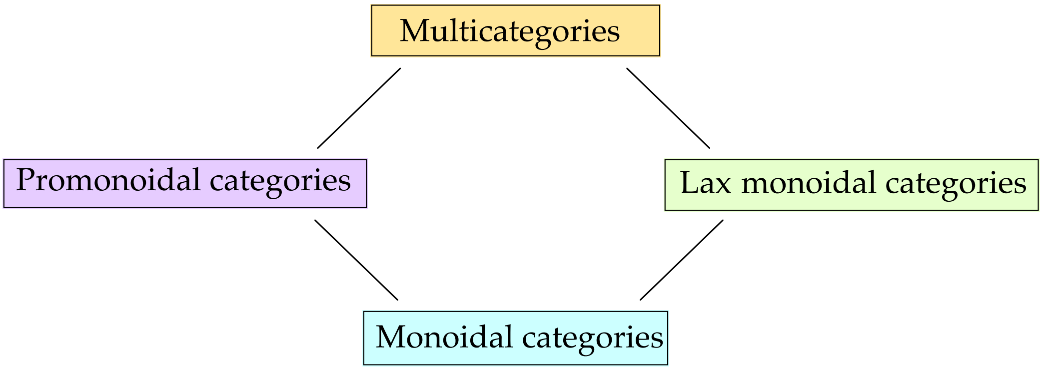
Various weakenings of monoidal category have been in existence almost as long as the notion itself. There are the multicategories of Lambek [Lk], the promonoidal categories of [D1], and the lax monoidal categories involving n-fold tensor products[2] with not-necessarily-invertible associativity and unit constraints. There is a diamond



in which moving down along a side of gradient 1 imposes invertibility on constraints, while moving down along a side of gradient –1 imposes representability on the multihoms. A strong form of representability (see Hermida [H]) leads us from the top of the diamond to the bottom in one step.

Promonoidal categories were introduced to explain a large variety of convolution monoidal structures on functor categories. What we want to point out in this paper is that convolution formulas are available in weaker settings, but, of course, the resultant functor categories bear weaker monoidal structures too.

The central general concept on which our work is based is that of *lax monoid* in a monoidal bicategory. While it is true that lax monoids can be construed as monoids in a suitably modified setting, this does not detract at all from the concept. In making that modification we move away from the familiar. Lax monoids themselves are very closely related to *operads* in that they abstractly express substitution.

We are particularly interested in lax monoids and comonoids in the monoidal bicategory $\mathcal{V}$-**Mod**. The extra freedom allowed by laxness means that convolution structures on functor categories proliferate: we give six such constructions in Section 7.

Monoidal bicategory is precisely the categorical structure in which morphisms can be rigorously depicted as three-dimensional surface diagrams (see [SV], [MT], [BL1] and [BL2]).

[1] The concept of an operad in a **Cat**-operad was suggested to the second author by Michael Batanin in late 1999. A prelude to this work was handwritten in January 2000. The sixth convolution formula in Section 7 was added to our 11 August 2001 preprint when we became aware of the article [BDK].

[2] Actually, the lax monoidal categories in the diamond are "normalized" in the sense that their 1-fold tensor product functor is the identity.

We use the conventions and terminology of [DS], [S5] and [DMS].

§1. Lax monoids

A *lax monoid* M in a Gray monoid $\mathcal{M}$ is a strict-monoidal lax functor

$$M : \mathbf{\Delta} \longrightarrow \mathcal{M}.$$

The objects of the (algebraicists') simplicial category $\mathbf{\Delta}$ are the ordered sets $\mathbf{n} = \{1, 2, \ldots, n\}$ and the arrows $\xi : \mathbf{m} \longrightarrow \mathbf{n}$ are the order-preserving functions. Put $A = M\mathbf{1}$ so that

$$M\mathbf{n} = M(\mathbf{1} + \ldots + \mathbf{1}) = M\mathbf{1} \otimes \ldots \otimes M\mathbf{1} = A^{\otimes n}.$$

Put $s_m = M(\tau_m) : A^{\otimes m} \longrightarrow A$ where $\tau_m : \mathbf{m} \longrightarrow \mathbf{1}$ in $\mathbf{\Delta}$. Since each $\xi : \mathbf{m} \longrightarrow \mathbf{n}$ has the form

$$\tau_{m_1} + \ldots + \tau_{m_n} : \mathbf{m}_1 + \ldots + \mathbf{m}_n \longrightarrow \mathbf{1} + \ldots + \mathbf{1},$$

we see that

$$M(\xi) = s_{m_1} \otimes \ldots \otimes s_{m_n} : A^{\otimes m} \longrightarrow A^{\otimes n}.$$

For each composable pair $\xi : \mathbf{m} \longrightarrow \mathbf{n}$, $\zeta : \mathbf{n} \longrightarrow \mathbf{r}$ in $\mathbf{\Delta}$, we have a constraint

$$\mu_{\xi, \zeta} : M(\zeta)\, M(\xi) \longrightarrow M(\zeta\, \xi).$$

Put

$$\begin{tikzcd}
 & A^{\otimes n} \arrow[dr, "s_n"] \arrow[d, phantom, "\Downarrow \mu_{\xi}"] & \\
A^{\otimes m} \arrow[ur, "s_{m_1} \otimes \ldots \otimes s_{m_n}"] \arrow[rr, "s_m"'] & {} & A
\end{tikzcd}$$

equal to $\mu_{\xi, \tau_n} : M(\tau_n)\, M(\xi) \longrightarrow M(\tau_m)$. Since $\mathbf{m} \xrightarrow{\xi} \mathbf{n} \xrightarrow{\zeta} \mathbf{r}$ can be written as

$$\mathbf{m}_1 + \ldots + \mathbf{m}_r \xrightarrow{\xi_1 + \ldots + \xi_r} \mathbf{n}_1 + \ldots + \mathbf{n}_r \xrightarrow{\tau_{n_1} + \ldots + \tau_{n_r}} \mathbf{1} + \ldots + \mathbf{1},$$

coherence implies that the general $\mu_{\xi, \zeta}$ can be recaptured as

$$\begin{tikzcd}
 & A^{\otimes n} \arrow[dr, "M(\zeta) = s_{n_1} \otimes \ldots \otimes s_{n_r}"] \arrow[d, phantom, "\cong"] & \\
A^{\otimes m} \arrow[ur, "M(\xi) = M(\xi_1) \otimes \ldots \otimes M(\xi_r)"] \arrow[rr, "(s_{n_1} \circ M(\xi_1)) \otimes \ldots \otimes (s_{n_r} \circ M(\xi_r))"] \arrow[rr, bend right=40, "M(\zeta\, \xi) = s_{m_1} \otimes \ldots \otimes s_{m_r}"'] & {} \arrow[d, phantom, "\Downarrow \mu_{\xi_1} \otimes \ldots \otimes \mu_{\xi_r}"] & A^{\otimes r} \\
 & {} &
\end{tikzcd}$$

There is also the constraint $\eta_{\mathbf{n}} : 1_{M(\mathbf{n})} \longrightarrow M(1_{\mathbf{n}})$; we put

$$\eta : 1_A \Rightarrow s_1 \text{ equal to } \eta_{\mathbf{1}}.$$

We recapture $\eta_{\mathbf{n}}$ as $\eta \otimes \ldots \otimes \eta : 1_{A^{\otimes n}} \Rightarrow s_1 \otimes \ldots \otimes s_1$.

A little more work shows that a lax monoid M (in the Gray monoid $\mathcal{M}$) can equally be described as consisting of:

an object A ;

arrows $s_m : A^{\otimes m} \longrightarrow A$ for all $m \in \mathbf{N}$;

2-cells $\mu_{\xi} : s_n \circ (s_{m_1} \otimes \ldots \otimes s_{m_n}) \Rightarrow s_m$ for all partitions

$\xi : m_1 + \ldots + m_n = m$; and,

a 2-cell $\eta : 1_A \Rightarrow s_1$;

subject to the conditions

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
A^{\otimes m} \arrow[rr, "s_{m_{11}} \otimes \ldots \otimes s_{m_{1n_1}} \otimes \ldots \otimes s_{m_{r1}} \otimes \ldots \otimes s_{m_{rn_r}}"] \arrow[drr, bend left=15, "s_{n_1} \circ (s_{m_{11}} \otimes \ldots \otimes s_{m_{1n_1}}) \otimes \ldots \otimes s_{n_r} \circ (s_{m_{r1}} \otimes \ldots \otimes s_{m_{rn_r}})"' {name=U}] \arrow[drr, bend right=15, "s_{m_1} \otimes \ldots \otimes s_{m_r}" {name=L}] \arrow[d, "s_m"'] & & A^{\otimes n} \arrow[d, "s_{n_1} \otimes \ldots \otimes s_{n_r}"] \arrow[dl, phantom, "\cong"] \\
A \arrow[r, phantom, "\overset{\mu_{\zeta\xi}}{\Longleftarrow}"] & {} & A^{\otimes r} \arrow[ll, "s_r"]
\arrow[from=U, to=L, Rightarrow, "\mu_{\xi_1} \otimes \ldots \otimes \mu_{\xi_r}"]
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
A^{\otimes m} \arrow[r, "s_{m_{11}} \otimes \ldots \otimes s_{m_{1n_1}} \otimes \ldots \otimes s_{m_{r1}} \otimes \ldots \otimes s_{m_{rn_r}}"] \arrow[d, "s_m"'] \arrow[dr, phantom, "\overset{\mu_{\xi}}{\Longleftarrow}" near start] & A^{\otimes n} \arrow[d, "s_{n_1} \otimes \ldots \otimes s_{n_r}"] \arrow[dl, "s_n"] \arrow[dl, phantom, "\overset{\mu_{\zeta}}{\Longleftarrow}" near end] \\
A & A^{\otimes r} \arrow[l, "s_r"]
\end{tikzcd}
$$

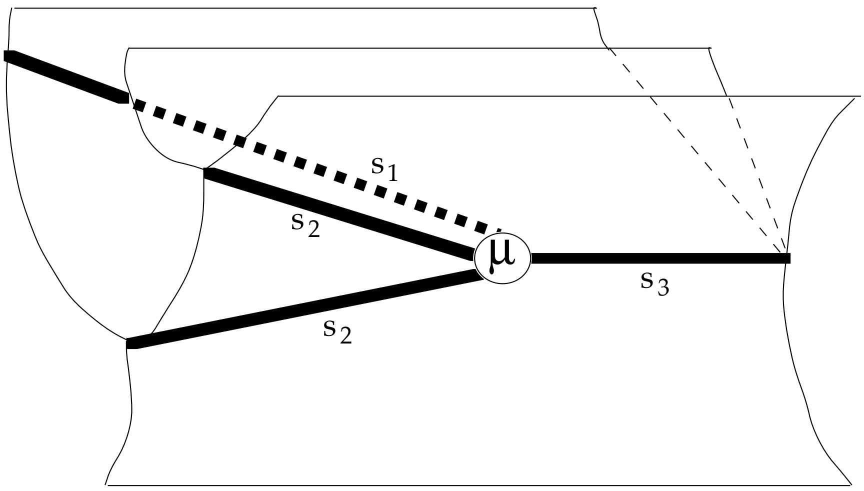
(where $\xi, \zeta, \zeta\xi, \xi_i$ are the partitions

$$m_{11} + \ldots + m_{1n_1} + \ldots + m_{r1} + \ldots + m_{rn_r} = m$$

$$n_1 + \ldots + n_r = n , \qquad m_1 + \ldots + m_r = m , \qquad m_{i1} + \ldots + m_{in_i} = m_i \ ),$$

$$
\begin{tikzcd}
s_n \arrow[r, "1_{s_n} \circ (\eta \otimes \ldots \otimes \eta)"] \arrow[dr, "1_{s_n}"'] & s_n \circ (s_1 \otimes \ldots \otimes s_1) \arrow[d, "\mu_{1_n}"] \\
& s_n
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
s_m \arrow[r, "\eta \circ 1_{s_m}"] \arrow[dr, "1_{s_m}"'] & s_1 \circ s_m \arrow[d, "\mu_{\tau_m}"] \\
& s_m
\end{tikzcd}
.
$$

The lax monoid is called *normal* when $\eta : 1_A \Rightarrow s_1$ is invertible[3].



Surface diagram for μ_ξ where ξ is $2+1=3$

Notice that each lax monoid in $\mathcal{M}$ has an underlying monad in $\mathcal{M}$. To see this, we compose the lax functor $M : \mathbf{\Delta} \longrightarrow \mathcal{M}$ with the functor $\mathbf{1} \longrightarrow \mathbf{\Delta}$ which picks out the singleton ordinal $\mathbf{1}$; the resultant lax functor $\mathbf{1} \longrightarrow \mathcal{M}$ amounts to the required monad s_1 on A.

Suppose M and $N : \mathbf{\Delta} \longrightarrow \mathcal{M}$ are lax monoids. A *lax monoid morphism* is defined to be a strict-monoidal lax natural transformation $\theta : M \longrightarrow N$. Equally, putting $A = M\mathbf{1}$, $B = N\mathbf{1}$, $s_m = M(\tau_m)$, $t_m = N(\tau_m)$, and using μ_ξ and η for both M and N, a lax morphism from A to B consists of a morphism $u : A \longrightarrow B$ together with 2-cells $\rho_n : t_n \circ u^{\otimes n} \Rightarrow u \circ s_n$ such that the following two equations hold.

$$
\begin{tikzcd}
A^{\otimes m} \arrow[rr, "u^{\otimes m}"] \arrow[dr, "s_{m_1} \otimes \cdots \otimes s_{m_n}"'] \arrow[dd, "s_m"'] & & B^{\otimes m} \arrow[dr, "t_{m_1} \otimes \cdots \otimes t_{m_n}"] \arrow[dl, phantom, "\Leftarrow \rho_{m_1} \otimes \cdots \otimes \rho_{m_n}"] & \\
\arrow[r, phantom, "\overset{\mu_\xi}{\Leftarrow}"] & A^{\otimes n} \arrow[rr, "u^{\otimes n}"] \arrow[dl, "s_n"'] \arrow[r, phantom, "\overset{\rho_n}{\Leftarrow}", shift right=6] & & B^{\otimes n} \arrow[dl, "t_n"] \\
A \arrow[rr, "u"'] & & B &
\end{tikzcd}
=
\begin{tikzcd}
A^{\otimes m} \arrow[rr, "u^{\otimes m}"] \arrow[dd, "s_m"'] & & B^{\otimes m} \arrow[dd, "t_m"'] \arrow[dr, "t_{m_1} \otimes \cdots \otimes t_{m_n}"] & \\
& \overset{\rho_m}{\Leftarrow} & & B^{\otimes n} \arrow[dl, "t_n"] \arrow[l, phantom, "\overset{\mu_\xi}{\Leftarrow}"] \\
A \arrow[rr, "u"'] & & B &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "s_1"', ""{name=S}] \arrow[d, bend left=60, "1"] \arrow[d, phantom, "\overset{\eta}{\Leftarrow}", shift left=3] & B \arrow[d, "1"] \\
A \arrow[r, "u"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "s_1"'] \arrow[dr, phantom, "\overset{\rho_1}{\Leftarrow}"] & B \arrow[d, "1"] \arrow[d, bend right=60, "t_1"'] \arrow[d, phantom, "\overset{\eta}{\Leftarrow}", shift left=3] \\
A \arrow[r, "u"'] & B
\end{tikzcd}
.
$$

[3] This term comes from a traditional use; however, normality is not usual.

The concept of weak-monoidal pseudofunctor (or weak-monoidal homomorphism) $T : \mathcal{M} \longrightarrow \mathcal{N}$ between Gray monoids was defined in [DS]; see Definition 2] on page 102. The definition of weak-monoidal lax functor is obtained verbatim by starting with a lax functor T instead of the special case of a pseudofunctor (or homomorphism of bicategories). Suppose $T : \mathbf{\Delta} \longrightarrow \mathcal{M}$ is a weak-monoidal lax functor for which the constraints

$$T\mathbf{m} \otimes T\mathbf{n} \longrightarrow T(\mathbf{m} + \mathbf{n}) \quad \text{and} \quad I \longrightarrow T\mathbf{0}$$

are equivalences. It is possible to construct a strict-monoidal lax functor $M : \mathbf{\Delta} \longrightarrow \mathcal{M}$ (that is, a lax monoid in $\mathcal{M}$) and a monoidal pseudo-natural transformation $\theta : M \longrightarrow T$ (see Definition 3 on page 104 of [DS]) such that each component $\theta_{\mathbf{n}} : M\mathbf{n} \longrightarrow T\mathbf{n}$ is an equivalence and $\theta_{\mathbf{1}}$ is an identity.

Examples of lax monoids

(1) **Monoids** Let $\mathcal{V}$ be any monoidal category regarded as a locally discrete monoidal bicategory $\mathcal{M}$ by taking the only 2-cells to be identities. A lax monoid in this $\mathcal{M}$ is a monoid in $\mathcal{V}$.

(2) **Pseudo-monoids** Each pseudo-monoid $p : A \otimes A \longrightarrow A$, $j : I \longrightarrow A$,

$$\alpha : p \circ (p \otimes 1_A) \cong p \circ (1_A \otimes p), \quad \lambda : p \circ (j \otimes 1_A) \cong 1_A, \quad \rho : p \circ (1_A \otimes j) \cong 1_A,$$

on an object A of the monoidal bicategory $\mathcal{M}$ determines a lax monoid structure on A by taking

$$s_0 = j, \quad s_1 = 1_A, \quad s_m = p \circ (p \otimes 1_A) \circ (p \otimes 1_A \otimes 1_A) \circ \ldots,$$

μ_ξ is the invertible 2-cell uniquely induced by α, λ, ρ,

and η is the identity 2-cell of 1_A.

In particular, each monoidal category becomes a lax monoid in the cartesian monoidal 2-category **Cat**, and each promonoidal category becomes a lax monoid in **Mod**.

(3) **Lax monoidal categories** A lax monoid in the cartesian monoidal 2-category **Cat** is a category $\mathcal{A}$ together with a functor $\underset{n}{\otimes} : \mathcal{A}^n \longrightarrow \mathcal{A}$ (called n-*fold tensor product*), a family of morphisms

$$\alpha_\xi : \underset{n}{\otimes}\big(\underset{m_1}{\otimes}(A_{11}, \ldots, A_{1m_1}), \ldots, \underset{m_n}{\otimes}(A_{n1}, \ldots, A_{n\,m_n})\big) \longrightarrow \underset{m}{\otimes}(A_{11}, \ldots, A_{1m_1}, \ldots, A_{n1}, \ldots, A_{n\,m_n})$$

natural in all A_{ij} where $\xi : m_1 + \ldots + m_n = m$, and a family of morphisms

$$\iota : A \longrightarrow \underset{1}{\otimes} A$$

natural in A, such that

$$\alpha_{\zeta\xi} \circ \underset{r}{\otimes}(\alpha_{\xi_1}, \ldots, \alpha_{\xi_r}) = \alpha_\xi \circ \alpha_\zeta$$

$$\alpha_{1+\ldots+1=n} \circ \underset{n}{\otimes}(\iota, \ldots, \iota) = 1_{\underset{n}{\otimes}} = \alpha_{n=n} \circ \iota \underset{n}{\otimes} \quad .$$

We call this a *lax monoidal category.* Notice that $\underset{1}{\otimes} : \mathcal{A} \longrightarrow \mathcal{A}$ is not necessarily the identity functor (if it is, we have a *normal* lax monoidal category), rather, it is the functor part of the underlying monad of the lax monoidal category. A lax monoid in $\mathbf{Cat}^{co}$ is called an *oplax monoidal category.*

(4) **Operads** Recall from [JS1] that the tensor product $\otimes : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ of a braided monoidal category $\mathcal{V}$ becomes a strong monoidal functor; so $\mathcal{V}$ can be regarded as a one-object bicategory $\Sigma\mathcal{V}$ whose hom-category is $\mathcal{V}$. A (non-permutative) *operad* T in a braided (strict) monoidal category $\mathcal{V}$ is a lax monoid in the "suspension" $\Sigma\mathcal{V}$ of $\mathcal{V}$. We also use the term *$\mathcal{V}$-operad* for such a T.

(5) **Multicategories** Lax monoids in the monoidal bicategory **Span** are precisely multicategories in the sense of Lambek [Lk; p. 103]; also see Linton [Ln]. Here **Span** denotes the bicategory [B] whose objects are sets and whose arrows are spans; the monoidal structure is provided by cartesian product of sets. Recall from [B] that a monad in **Span** is a category; so the underlying monad of the lax monoid in this case is called the *underlying category* of the multicategory.

(6) **Tensor products of lax monoids** Suppose $\mathcal{M}$ is a braided Gray monoid [DS]. If A and B are lax monoids in $\mathcal{M}$ then so is $A \otimes B$. For, let M and N : $\mathbf{\Delta} \longrightarrow \mathcal{M}$ be the strict-monoidal lax functors corresponding to A and B. Using the braiding, we obtain a monoidal structure on the pointwise tensor product $M \otimes N$ of M and N. Then we can replace the monoidal lax functor, up to equivalence, by a strict-monoidal lax functor whose value at $\mathbf{1}$ is $A \otimes B$.

§2. Lax monoids as monads

Suppose our Gray monoid $\mathcal{M}$ has local coproducts (that is, each homcategory has coproducts preserved by composing with arrows on either side). There is a bicategory $\mathcal{M}'$ defined as follows. The objects are those of $\mathcal{M}$, while

$$\mathcal{M}'(A, B) = \prod_{n \geq 0} \mathcal{M}(A^{\otimes n}, B).$$

The composite $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{M}'$ is defined by

$$(g \circ f)_m = \sum_{m_1 + \ldots + m_n = m} g_n \circ (f_{m_1} \otimes \ldots \otimes f_{m_n}).$$

The identity morphism $1_A : A \longrightarrow A$ in $\mathcal{M}'$ is the initial object $0 = 0_A \in \mathcal{M}(A^{\otimes n}, A)$ for $n \neq 1$ and $1_A \in \mathcal{M}(A^{\otimes n}, A)$ for $n = 1$; it is clear that this is an identity up to canonical isomorphisms. Associativity of composition is proved by the following rather familiar argument:

$$(h \circ (g \circ f))_m$$

$$= \sum_{q_1 + \ldots + q_k = m} h_k \circ ((g \circ f)_{q_1} \otimes \ldots \otimes (g \circ f)_{q_k})$$

$$= \sum_{\substack{q_1+\ldots+q_k=m \\ q_i = r_{i1}+\ldots+r_{in_i}}} h_k \circ (g_{n_1} \circ (f_{r_{11}} \otimes \ldots \otimes f_{r_{1n_1}}) \otimes \ldots \otimes g_{n_k} \circ (f_{r_{k1}} \otimes \ldots \otimes f_{r_{kn_k}}))$$

$$\cong \sum_{\substack{q_1+\ldots+q_k=m \\ q_i = r_{i1}+\ldots+r_{in_i}}} h_k \circ (g_{n_1} \otimes \ldots \otimes g_{n_k}) \circ (f_{r_{11}} \otimes \ldots \otimes f_{r_{1n_1}} \otimes \ldots \otimes f_{r_{k1}} \otimes \ldots \otimes f_{r_{kn_k}})$$

Making a change of variables in the summation, we put

$$n = n_1 + \ldots + n_k\,, \quad m_i = r_{1i} \text{ for } 1 \le i \le n_1\,, \quad m_i = n_1 + r_{2i} \text{ for } n_1 \le i \le n_1 + n_2\,, \ldots\,.$$

Then

$$\begin{aligned}
&(h \circ (g \circ f))_m \\
&\qquad \cong \sum_{\substack{m_1+\ldots+m_n=m \\ n_1+\ldots+n_k=n}} h_k \circ (g_{n_1} \otimes \ldots \otimes g_{n_k}) \circ (f_{m_1} \otimes \ldots \otimes f_{m_n}) \\
&\qquad = \sum_{m_1+\ldots+m_n=m} (h \circ g)_n \circ (f_{m_1} \otimes \ldots \otimes f_{m_n}) \\
&\qquad = ((h \circ g) \circ f)_m\,.
\end{aligned}$$

Coherence for these associativity and identity constraints also holds.

There is an obvious inclusion $\mathcal{M} \longrightarrow \mathcal{M}'$ which is the identity on objects and identifies a morphism $u : A \longrightarrow B$ in $\mathcal{M}$ with the sequence (u_n) defined by taking u_n to be the initial object of $\mathcal{M}(A^{\otimes n}, B)$ for $n \neq 1$ and taking $u_1 = u$.

Proposition 2.1 *A lax monoid in $\mathcal{M}$ is the same as a monad in $\mathcal{M}'$.*

In other words, strict-monoidal lax functors $\mathbf{\Delta} \longrightarrow \mathcal{M}$ are the same as lax functors $\mathbf{1} \longrightarrow \mathcal{M}'$.

There is another viewpoint on $\mathcal{M}'$ when $\mathcal{M}$ has global coproducts. Since $\mathcal{M}$ already has local coproducts, the global coproducts are (bicategorical) direct sums and the coprojections have right adjoints [S2]. Define a pseudofunctor

$$D : \mathcal{M} \longrightarrow \mathcal{M}$$

by the geometric series

$$DA = \sum_{n \ge 0} A^{\otimes n}.$$

This gives a pseudomonad using j_A, m_A defined by

$$\begin{tikzcd}
A \arrow[r, "j_A"] \arrow[dr, "\mathrm{copr}_1"'] & DA \arrow[d, equal] \\
& \sum_{n \ge 0} A^{\otimes n}
\end{tikzcd}
\qquad
\begin{tikzcd}
D^2A \arrow[rr, "m_A"] \arrow[d, phantom, "\wr\!\shortparallel"] & & DA \arrow[d, equal] \\
\sum_{m_1,\ldots,m_r} A^{\otimes m_1} \otimes \ldots \otimes A^{\otimes m_r} & & \sum_{n \ge 0} A^{\otimes n} \\
& A^{\otimes (m_1+\ldots+m_r)} \arrow[ul, "\mathrm{copr}_{m_1,\ldots,m_r}"] \arrow[ur, "\mathrm{copr}_{m_1+\ldots+m_r}"'] &
\end{tikzcd}$$

.

A lax algebra for the pseudomonad (D, m, j) is precisely a lax monoid in $\mathcal{M}$. Again from [S2] we know that j_A, m_A have right adjoints $j_A^*: DA \longrightarrow A$, $m_A^*: DA \longrightarrow D^2A$. It follows that we have a pseudocomonad D, j^*, m^* on $\mathcal{M}$. Then

$$\mathcal{M}' \cong \mathcal{M}_{(D, j^*, m^*)},$$

the Kleisli bicategory for this pseudocomonad. The particular case where $\mathcal{M}$ is the bicategory **Span** of sets and spans was considered by Burroni [Bu], Hermida [H] and Leinster [Lr] to show that multicategories could be regarded as monads in an appropriate bicategory (see Section 1, Example 5).

Remark There are general principles involved here. Suppose (T, m, j) is a pseudomonad on any bicategory $\mathcal{K}$. If $m: TT \longrightarrow T$ and $j: 1 \longrightarrow T$ have right adjoints m^*, j^* then we obtain a pseudocomonad (T, j^*, m^*) on $\mathcal{K}$; moreover, to give a lax algebra for (T, m, j) is to give a monad in the Kleisli bicategory for (T, j^*, m^*).

§3. Pseudo-operads

Regarding **Cat** as a cartesian monoidal category, we know what is meant by a **Cat**-operad, or operad in **Cat**; it is a $\mathcal{V}$-operad (Section 1 Example 4) with $\mathcal{V} = \mathbf{Cat}$. However, because of the 2-category structure on **Cat**, there is a more general notion which we call a "pseudo-operad" in **Cat**. Consider the 2-category **Cat/N** of sequences $T = (T_n)$ of categories T_n; it is the countable product of copies of the 2-category **Cat**. There is a "substitution" monoidal structure on the 2-category defined by

$$(T \circ S)_n = \sum_{n_1 + \ldots + n_m = n} T_m \times S_{n_1} \times \ldots \times S_{n_m}.$$

An operad in **Cat** is a monoid in this monoidal 2-category **Cat/N**, whereas a *pseudo-operad* in **Cat** is a pseudomonoid in **Cat/N**. For the substitution operation

$$T_m \times T_{n_1} \times \ldots \times T_{n_m} \longrightarrow T_{n_1 + \ldots + n_m}$$

of a pseudo-operad T, we maintain the notation $(x, y_1, \ldots, y_m) \longmapsto x[y_1, \ldots, y_m]$.

We can define the notion of *operad in a pseudo-operad* T in the sense that we have $s_n \in T_n$ for all $n \in \mathbf{N}$ and arrows

$$\mu_\xi : s_n[s_{m_1}, \ldots, s_{m_n}] \longrightarrow s_m \quad \text{in } T_m,$$

$$\eta : 1 \longrightarrow s_1 \quad \text{in } T_1,$$

satisfying the obvious three conditions.

For example, each object A in a Gray monoid $\mathcal{M}$ gives a pseudo-operad T in **Cat** by defining

$$T_n = \mathcal{M}(A^{\otimes n}, A),$$

and defining substitution

$$T_n \times T_{m_1} \times \ldots \times T_{m_n} \longrightarrow T_m\,, \quad \text{for} \quad m_1 + \ldots + m_n = m\,,$$

by

$$g[f_1, \ldots, f_n] = g \circ (f_1 \otimes \ldots \otimes f_n)\,;$$

also, $1 \in T_1$ is the identity arrow of A. We shall denote this pseudo-operad T by $\mathcal{M}(A)$. Clearly each lax monoid M in $\mathcal{M}$ defines an operad in the pseudo-operad $\mathcal{M}(A)$ where $A = M\mathbf{1}$.

Conversely, suppose T is any pseudo-operad in **Cat**. There is a monoidal bicategory $\mathcal{M}_T$ (the "2-prop" of the pseudo-operad) defined as follows. The objects are the natural numbers. The homcategories are defined by

$$\mathcal{M}_T(m\,,n) = \sum_{m_1+\ldots+m_n=m} T_{m_1} \times \ldots \times T_{m_n}\,.$$

Composition

$$\mathcal{M}_T(n\,,r) \times \mathcal{M}_T(m\,,n) \longrightarrow \mathcal{M}_T(m\,,r)$$

takes $(t_{n_1}, \ldots, t_{n_r}, t_{m_1}, \ldots, t_{m_n}) \in T_{n_1} \times \ldots \times T_{n_r} \times T_{m_1} \times \ldots \times T_{m_n}$ to

$$(\,t_{n_1}[t_{m_1}, \ldots, t_{m_{n_1}}], t_{n_2}[t_{m_{n_1+1}}, \ldots, t_{m_{n_1+n_2}}], \ldots, t_{n_r}[t_{m_{n-n_r+1}}, \ldots, t_{m_n}]\,)$$

where $m_1 + \ldots + m_n = m$ and $n_1 + \ldots + n_r = n$. The tensor product for $\mathcal{M}_T$ is given on objects by addition of natural numbers and on homcategories

$$\mathcal{M}_T(m\,,n) \times \mathcal{M}_T(i\,,j) \longrightarrow \mathcal{M}_T(m+i\,,n+j)$$

by $((t_{m_1}, \ldots, t_{m_n}), (t_{i_1}, \ldots, t_{i_j})) \longmapsto (t_{m_1}, \ldots, t_{m_n}, t_{i_1}, \ldots, t_{i_j})$.

Clearly $\mathcal{M}_T(1) = T$. Moreover, each operad (s_n) inside T gives a lax monoid structure on $1 \in \mathcal{M}_T$.

Batanin [Ba1; page 88] constructed an operad **h** in **Cat** whose algebras are normal lax monoidal categories. In conversation Batanin has also given a description of **h** in terms of the plane trees of [Ba2] (also see [S4]). A (rooted plane) tree T of height m is a diagram

$$T_m \xrightarrow{\xi_m} T_{m-1} \xrightarrow{\xi_{m-1}} \ldots \xrightarrow{\xi_2} T_1 \xrightarrow{\xi_1} T_0$$

in $\mathbf{\Delta}$ with T_0 a singleton whose element is called the *root*. The elements of T_k are called *nodes* of height k. We obtain a directed graph by constructing an edge from each node of positive height k to its image under the function ξ_k. The *arity* of a node is the number of edges into it. A *leaf* is a nullary node (that is, of arity 0); in particular, all nodes of maximum height m are leaves — we call these the *top leaves* and all other leaves are called *lower*. The objects of the category $\mathbf{h}_n$ are those trees T with no unary nodes except perhaps the root, and with precisely n top leaves. In fact, $\mathbf{h}_n$ is a partially ordered set: the reflexive transitive relation is the smallest such that there is an arrow $T \longrightarrow T'$ if T' is obtained from T by contracting an edge (identifying the nodes that the edge joins and moving down all the nodes above to one less height) or by deleting a lower leaf, where, in the case where the deleted leaf has an edge to a binary node, the other edge must be contracted (to maintain no unary nodes). Let $s_n \in \mathbf{h}_n$ denote the tree $\xi_k : \mathbf{n} \longrightarrow \mathbf{1}$ which

Todd Trimble calls the *n-sprout*. There is a unique structure of normal operad in **h** on the sequence of sprouts; in fact, **h** together with (s_n) is the free **Cat** operad containing a normal operad. Moreover, for any object A of a Gray monoid $\mathcal{M}$, an operad morphism $\mathbf{h} \longrightarrow \mathcal{M}(A)$ is a normal lax monoid structure on A.

Batanin has also described to us an explicit construction (in terms of structured trees) of a **Cat** operad $\hat{\mathbf{h}}$ whose algebras are lax monoidal categories (not merely the normal ones); operad morphisms $\hat{\mathbf{h}} \longrightarrow \mathcal{M}(A)$ amount to lax monoid structures on A. Indeed, $\hat{\mathbf{h}}$ together with a particular operad s in it, is the free **Cat** operad containing an operad. We can also consider the free Gray monoid $\mathcal{F}$ containing a lax monoid L; the pseudo-**Cat** operad $\mathcal{F}(L)$ is equivalent to $\hat{\mathbf{h}}$.

§4. Extension and lifting of structure

In ordinary universal algebra, a familiar process is the transport of structure supported by an object A across to an object B by means of an isomorphism $A \longrightarrow B$. The term is also used in homotopy theory to cover the case where $A \longrightarrow B$ is a homotopy equivalence. In 2-dimensional categorical universal algebra, it is used when $A \longrightarrow B$ is an equivalence in a 2-category: the types of structures that so transport exhibit an aspect of flexibility (in the sense of [BKP]). We can also contemplate transport of structure across an adjunction $A \longrightarrow B$; the lax functor generated by an adjunction, as described in [S0], is an example.

Extension of structure is a generalization of transport of structure. The basic idea appeared in [S1] where the extension of a monad along a morphism was described. In any bicategory, given a monad s on an object A and a morphism $u : A \longrightarrow B$, the right extension $t : B \longrightarrow B$ of $u \circ s$ along u (provided it exists) becomes a monad in such a way that u, together with the 2-cell $\rho : t \circ u \Rightarrow u \circ s$ which exhibits the right extension, is a monad morphism. If $f \dashv u : A \longrightarrow B$ is an adjunction with counit $\alpha : f u \Rightarrow 1_A$ then the right extension exists as $t = u \circ s \circ f$ with $\rho = u \circ s \circ \alpha$.

Proposition 4.1 *In any monoidal bicategory $\mathcal{M}$, suppose* A *is a lax monoid and* $u : A \longrightarrow B$ *is a morphism. Assume that, for all* $n \geq 0$, *the right extension* $t_n : B^{\otimes n} \longrightarrow B$ *of* $u \circ s_n : A^{\otimes n} \longrightarrow B$ *along* $u^{\otimes n} : A^{\otimes n} \longrightarrow B^{\otimes n}$ *exists, exhibited by the 2-cell* $\rho_n : t_n \circ u^{\otimes n} \Rightarrow u \circ s_n$. *Then there exists a structure of lax monoid on* B *consisting of the morphisms* $t_n : B^{\otimes n} \longrightarrow B$ *and the unique 2-cells* μ_ξ *and* η *such that* u, *together with the 2-cells* ρ_n, *is a lax monoid morphism.*

Proof In the case where $\mathcal{M}$ has local coproducts, the result follows from the result of [S1], on extension of monads, applied in the bicategory $\mathcal{M}'$. For, it is easy to see that the sequence ρ of 2-cells ρ_n exhibits t as a right extension of $u \circ s$ along u in $\mathcal{M}'$. So t becomes a monad in $\mathcal{M}'$ and hence a lax monoid in $\mathcal{M}$ as required.

In writing out the proof for the general case, the authors found it convenient to write the data for a lax monoid and the 2-cells ρ_n as rewrite rules:

$$\begin{array}{c} s_n \circ \bigotimes_i s_{m_i} \\ -\,-\;\mu_\xi\;-\,- \\ s_m \end{array}\;, \qquad \begin{array}{c} 1_A \\ -\;\eta\;- \\ s_1 \end{array}\;, \qquad \begin{array}{c} t_n \circ u^{\otimes n} \\ -\,-\;\rho_n\;-\,- \\ u \circ s_n \end{array}\;,$$

and the axioms as equalities between derivations

$$\begin{array}{c} s_r \circ \bigotimes_i s_{n_i} \circ \bigotimes_{i,j} s_{m_{ij}} \\ -\,-\,-\;\cong\;-\,-\,- \\ s_r \circ \bigotimes_i (s_{n_i} \circ \bigotimes_j s_{m_{ij}}) \\ -\,-\;\bigotimes_i \mu_{\xi_i}\;-\,- \\ s_r \circ \bigotimes_i s_{m_i} \\ -\,-\;\mu_{\zeta\xi}\;-\,- \\ s_m \end{array} \quad = \quad \begin{array}{c} s_r \circ \bigotimes_i s_{n_i} \circ \bigotimes_{i,j} s_{m_{ij}} \\ -\,-\;\mu_\xi\;-\,- \\ s_n \circ \bigotimes_{i,j} s_{m_{ij}} \\ -\,-\;\mu_\xi\;-\,- \\ s_m \end{array}\;,$$

$$\begin{array}{c} s_n \\ -\;\eta^{\otimes n}\;- \\ s_n \circ s_1^{\otimes n} \\ -\;\mu_{1_n}\;- \\ s_n \end{array} \; = \; \begin{array}{c} s_n \\ -\;=\;- \\ s_n \end{array}\;, \qquad \begin{array}{c} s_m \\ -\;\eta\;- \\ s_1 \circ s_m \\ -\;\mu_{\tau_m}\;- \\ s_m \end{array} \; = \; \begin{array}{c} s_m \\ -\;=\;- \\ s_m \end{array}\;.$$

The data for the lax monoid B consists of t together with μ_ξ and η defined, using the universal property of right extension, by the following equations.

$$\begin{array}{c} t_n \circ \bigotimes_i (t_{m_i} \circ u^{\otimes m_i}) \\ -\,-\;\bigotimes_i \rho_{m_i}\;-\,- \\ t_n \circ \bigotimes_i (u \circ s_{m_i}) \\ -\,-\,-\;\cong\;-\,-\,- \\ t_n \circ u^{\otimes n} \circ \bigotimes_i s_{m_i} \\ -\,-\,-\;\rho_n\;-\,-\,- \\ u \circ s_n \circ \bigotimes_i s_{m_i} \\ -\,-\,-\;\mu_\xi\;-\,-\,- \\ u \circ s_m \end{array} \; = \; \begin{array}{c} t_n \circ \bigotimes_i (t_{m_i} \circ u^{\otimes m_i}) \\ -\,-\,-\;\cong\;-\,-\,- \\ t_n \circ \bigotimes_i t_{m_i} \circ u^{\otimes m} \\ -\,-\,-\;\mu_\xi\;-\,-\,- \\ t_m \circ u^{\otimes m} \\ -\,-\,-\;\rho_m\;-\,-\,- \\ u \circ s_m \end{array}\;, \qquad \begin{array}{c} u \\ -\;\eta\;- \\ t_1 \circ u \\ -\;\rho_1\;- \\ u \circ s_1 \end{array} \; = \; \begin{array}{c} u \\ -\;\eta\;- \\ u \circ s_1 \end{array}\;.$$

The proof that the lax monoid axioms hold for B now consists of three sequences of equations between derivations using the above equations and the properties of a monoidal bicategory. We leave this to the reader to reconfirm. What we have not done, but would be nice, is to draw the surface diagrams for these calculations. **q. e. d.**

§5. Multi-lax-functors

Some notation will be helpful. We write $X_\bullet$ for the list $X_1, \ldots, X_n$ and $X_{\bullet\bullet}$ for the list $X_{1\bullet}, \ldots, X_{n\bullet}$ of lists $X_{i1}, \ldots, X_{i\,m_i}$ $(i = 1, \ldots, n)$. We now also write $\otimes_n X_\bullet$ for $X_1 \otimes \ldots \otimes X_n$. Write $\otimes_\xi X_{\bullet\bullet}$ for the list $\otimes_{m_1} X_{1\bullet}, \ldots, \otimes_{m_n} X_{n\bullet}$ where ξ is the partition $m_1 + \ldots + m_n = m$. Write $\otimes_{n,\xi} X_{\bullet\bullet}$ for $\otimes_n \otimes_\xi X_{\bullet\bullet}$. We use the same kind of notation for arrows f in place of objects X.

Let $\mathcal{M}$ and $\mathcal{N}$ be Gray monoids. A *multi-lax-functor* from $\mathcal{M}$ to $\mathcal{N}$ is a lax functor $L : \mathcal{M} \longrightarrow \mathcal{N}$ equipped with an oplax natural transformation $s_n : \otimes_n L \longrightarrow L \otimes_n$ for each natural number n, and with modifications $\eta : 1_L \longrightarrow s_1 : L \longrightarrow L$ and

$$\mu_\xi \ : \ s_{n,\otimes_\xi} \circ \otimes_n s_{m_\bullet} \longrightarrow s_{m,\otimes_n \otimes_\xi} : \ \otimes_{n,\xi} L \longrightarrow L \otimes_{n,\xi} \quad \text{for } \ \xi : m_1 + \ldots + m_n = m ,$$

subject to the obvious three axioms as in the definition of lax monoid. The data for s_n is displayed in the diagram

$$\begin{tikzcd}
\otimes_n LX_\bullet \arrow[r, "s_{n,X_\bullet}"] \arrow[d, "Lf_\bullet"'] \arrow[dr, phantom, "\overset{s_{n,f_\bullet}}{\Longrightarrow}"] & L(\otimes_n X_\bullet) \arrow[d, "L(\otimes_n f_\bullet)"] \\
\otimes_n LY_\bullet \arrow[r, "s_{n,Y_\bullet}"'] & L(\otimes_n Y_\bullet)
\end{tikzcd} .$$

When the lax functor $L : \mathcal{M} \longrightarrow \mathcal{N}$ is a 2-functor, we use the term *multi-2-functor*, and these are all that we require in the present paper. In particular, a multi-2-functor $\mathbf{1} \longrightarrow \mathcal{M}$ is precisely a lax monoid in $\mathcal{M}$. A general multi-lax-functor $\mathbf{1} \longrightarrow \mathcal{M}$ gives rise to a lax monoid A in $\mathcal{M}$ together with an extra monad on A and a distributive law with the monad s_1.

Multi-lax-functors do not compose in general; however, if either one is a multi-2-functor, there is a natural choice of multi-structure on the composite lax functor. The reader will easily see the general problem and provide the structure in the special cases.

Multi-lax-functors take lax monoids to lax monoids; in fact, the image of a lax monoid under a multi-lax-functor includes not only a lax monoid structure but a distributive law of the kind alluded to above.

Suppose $\mathcal{M}$ and $\mathcal{N}$ are Gray monoids with local coproducts. For each multi-lax-functor $L : \mathcal{M} \longrightarrow \mathcal{N}$, we shall define a lax functor $L' : \mathcal{M}' \longrightarrow \mathcal{N}'$. On objects L' agrees with L. The functor $L' : \mathcal{M}'(A, B) \longrightarrow \mathcal{N}'(L'A, L'B)$ on hom-categories is defined to be the composite functor

$$\prod_{n \geq 0} \mathcal{M}(A^{\otimes n}, B) \xrightarrow{\prod_n L} \prod_{n \geq 0} \mathcal{N}(L(A^{\otimes n}), LB) \xrightarrow{\prod_n \mathcal{M}(s_{n,A}, LB)} \prod_{n \geq 0} \mathcal{N}((LA)^{\otimes n}, LB).$$

The composition constraint $\omega_2 : L'(g) \circ L'(f) \longrightarrow L'(g \circ f)$ is the composite

$$
\begin{aligned}
(L'(g) \circ L'(f))_m &= \sum_{m_1+\ldots+m_n = m} L'(g)_n \circ (L'(f)_{m_1} \otimes \ldots \otimes L'(f)_{m_n}) \\
&= \sum_{m_1+\ldots+m_n = m} L(g_n) \circ s_{n,B} \circ ((L(f_{m_1}) \circ s_{m_1,A}) \otimes \ldots \otimes (L(f_{m_n}) \circ s_{m_n,A})) \\
&= \sum_{m_1+\ldots+m_n = m} L(g_n) \circ s_{n,B} \circ (L(f_{m_1}) \otimes \ldots \otimes L(f_{m_n})) \circ (s_{m_1,A} \otimes \ldots \otimes s_{m_n,A}) \\
&\xrightarrow{\text{"}s_{n,f_\bullet}\text{"}} \sum_{m_1+\ldots+m_n = m} L(g_n) \circ L(f_{m_1} \otimes \ldots \otimes f_{m_n}) \circ s_{n,A} \circ (s_{m_1,A} \otimes \ldots \otimes s_{m_n,A}) \\
&\xrightarrow{\text{"}\omega_2 \circ \mu_\xi\text{"}} \sum_{m_1+\ldots+m_n = m} L(g_n \circ (f_{m_1} \otimes \ldots \otimes f_{m_n})) \circ s_{m,A} \\
&\xrightarrow{\text{canon.}} L\Big(\sum_{m_1+\ldots+m_n = m} g_n \circ (f_{m_1} \otimes \ldots \otimes f_{m_n}) \circ s_{m,A} \Big),
\end{aligned}
$$

while the identity constraint $\omega_0 : 1_{L'A} \longrightarrow L'(1_A)$ is the unique 2-cell in all components $n \neq 1$ and is $\omega_0 \circ \eta : 1_A \longrightarrow L(1_A) \circ s_{1,A}$ in component $n = 1$. The coherence conditions for a lax functor do hold.

As you would expect from the last paragraph, if $\mathcal{M}$ and $\mathcal{N}$ also have global coproducts then a multi-structure on a lax functor $L : \mathcal{M} \longrightarrow \mathcal{N}$ equips the lax functor with the structure of the appropriate kind of morphism from the pseudocomonad D on $\mathcal{M}$ to the pseudo-comonad D on $\mathcal{N}$.

Recall that lax functors take monads to monads and that monads in $\mathcal{M}'$ are lax monoids in $\mathcal{M}$. It follows that, if $L : \mathcal{M} \longrightarrow \mathcal{N}$ is a multi-lax-functor, then each lax monoid A in $\mathcal{M}$ determines a lax monoid $L'A$ in $\mathcal{N}$.

A monoidal pseudofunctor $F : \mathcal{M} \longrightarrow \mathcal{N}$ between Gray monoids amounts to a multi-lax-functor for which the underlying lax functor F is a pseudofunctor, each s_n is pseudo-natural, and the modifications η and μ_ξ are all invertible. Of course, it is usual to take the basic data to be s_0 and s_2, to define s_1 to be the identity, and to inductively define s_n for $n > 2$.

§6. Enriched lax promonoidal categories

Let $\mathcal{M}$ be a monoidal bicategory which admits right liftings through morphisms $a : I \longrightarrow A$. Put $\mathcal{V} = \mathcal{M}(I, I)$ as a monoidal category whose tensor product is horizontal composition (or equally, up to unit constraints, tensor product of morphisms) in $\mathcal{M}$. Then $\mathcal{V}$ is braided and closed. Indeed, for all objects A of $\mathcal{M}$, the category $\mathcal{M}(I, A)$ is a $\mathcal{V}$-category, which we denote by $\mathcal{M}[I, A]$, with the $\mathcal{V}$-valued hom $A[a, b] : I \longrightarrow I$ of a and b $: I \longrightarrow A$ defined to be the right lifting of b through a; the right lifting comes equipped with a canonical 2-cell $\rho : a \circ A[a, b] \longrightarrow b$ in $\mathcal{M}$.

We write $\mathcal{V}$-**CAT** for the usual monoidal 2-category of $\mathcal{V}$-categories (the sets of objects of these $\mathcal{V}$-categories need not be small). From the last paragraph we see that the usual hom pseudofunctor $\mathcal{M}(\mathrm{I},-):\mathcal{M}\longrightarrow\mathbf{CAT}$ lifts to a pseudofunctor $\mathcal{M}[\mathrm{I},-]:\mathcal{M}\longrightarrow\mathcal{V}\text{-}\mathbf{CAT}$. (These pseudofunctors are actually 2-functors when $\mathcal{M}$ is a Gray monoid.) Moreover, $\mathcal{M}[\mathrm{I},-]$ is a monoidal pseudofunctor. To see this we must supply a "comparison" $\mathcal{V}$-functor

$$\mathcal{M}[\mathrm{I},\mathrm{A}]\otimes\mathcal{M}[\mathrm{I},\mathrm{B}]\longrightarrow\mathcal{M}[\mathrm{I},\mathrm{A}\otimes\mathrm{B}]$$

for all objects A and B of $\mathcal{M}$. The value of the $\mathcal{V}$-functor on the object (a,b) is $a\otimes b$. The effect on homs is the 2-cell $\mathrm{A}[a,a']\otimes\mathrm{B}[b,b']\longrightarrow(\mathrm{A}\otimes\mathrm{B})[a\otimes b,a'\otimes b']$, corresponding under the right lifting property and the canonical isomorphism

$$(a\otimes b)\circ(\mathrm{A}[a,a']\otimes\mathrm{B}[b,b'])\;\cong\;(a\otimes\mathrm{A}[a,a'])\circ(b\otimes\mathrm{B}[b,b']),$$

to the 2-cell $\rho\otimes\rho\;:\;(a\otimes\mathrm{A}[a,a'])\circ(b\otimes\mathrm{B}[b,b'])\longrightarrow a'\otimes b'$. It is easily seen that these comparison $\mathcal{V}$-functors form the components of a pseudonatural transformation

$$\mathcal{M}[\mathrm{I},-]\otimes\mathcal{M}[\mathrm{I},?]\longrightarrow\mathcal{M}[\mathrm{I},-\otimes ?]\;:\;\mathcal{M}\times\mathcal{M}\longrightarrow\mathcal{V}\text{-}\mathbf{CAT}.$$

(This is still only pseudonatural and not generally 2-natural even when $\mathcal{M}$ is a Gray monoid.) We must also provide a $\mathcal{V}$-functor

$$I\longrightarrow\mathcal{M}[\mathrm{I},\mathrm{I}]$$

where I is the one object $\mathcal{V}$-category which acts as unit for the tensor product of $\mathcal{V}$-categories; of course, this is the $\mathcal{V}$-functor whose value at the one object of I is the identity morphism of I. These data are easily seen to satisfy the axioms required to make $\mathcal{M}[\mathrm{I},-]:\mathcal{M}\longrightarrow\mathcal{V}\text{-}\mathbf{CAT}$ monoidal.

Corollary 6.1 *If* A *is a lax monoid in* $\mathcal{M}$ *then* $\mathcal{M}[\mathrm{I},\mathrm{A}]$ *is a lax monoid in* $\mathcal{V}$-**CAT** *with the structure induced by the monoidal structure of* $\mathcal{M}[\mathrm{I},-]$.

We are going to examine applications of our results to enriched category theory. Suppose $\mathcal{V}$ is a complete, cocomplete, symmetric, closed, monoidal category. We remind the reader that our notation from [DMS] and [KLSS] is that a $\mathcal{V}$-module $p:\mathrm{A}\longrightarrow\mathrm{B}$ is identified with a $\mathcal{V}$-functor $p:\mathrm{B}^{\mathrm{op}}\otimes\mathrm{A}\longrightarrow\mathcal{V}$ and that the composite of $p:\mathrm{A}\longrightarrow\mathrm{B}$ and $q:\mathrm{B}\longrightarrow\mathrm{C}$ is defined by the coend formula

$$(q\circ p)(c,a)\;=\;\int^{b}p(b,a)\otimes q(c,b)$$

(which certainly exists when B is small). We write $\mathcal{V}$-**Mod** for the monoidal bicategory whose objects are small $\mathcal{V}$-categories, whose morphisms are $\mathcal{V}$-modules (composed according to the formula above), whose 2-cells are module morphisms, and whose tensor product is the usual tensor product of $\mathcal{V}$-categories [Ky]. Write $\mathcal{V}$-**Mat** for the monoidal full subbicategory of $\mathcal{V}$-**Mod** consisting of the discrete $\mathcal{V}$-categories (which we can identify with small sets). Notice that there is a monoidal biequivalence $\mathcal{V}\text{-}\mathbf{Mat}^{\mathrm{op}} \sim \mathcal{V}\text{-}\mathbf{Mat}$. In the case of $\mathcal{V} = \mathbf{Set}$ we just write **Mat** for $\mathcal{V}$-**Mat**; there is a monoidal biequivalence $\mathbf{Mat} \sim \mathbf{Span}$.

In view of example (5) of Section 1, we define a *$\mathcal{V}$-multicategory* to be a lax monoid in $\mathcal{V}$-**Mat**. We do not really need to restrict this definition to the case where the supporting set of the $\mathcal{V}$-multicategory is small: there are other cases where the particular coproducts (required for the matrix composites involved in the definition) exist for other reasons. Every $\mathcal{V}$-multicategory A has an underlying $\mathcal{V}$-category (A, s_1) since monads in $\mathcal{V}$-**Mat** are $\mathcal{V}$-categories. Any full sub-$\mathcal{V}$-category of a $\mathcal{V}$-multicategory becomes a $\mathcal{V}$-multicategory by restriction.

In Proposition 6.3, we shall identify lax monoids in $\mathcal{V}$-**Mod** but first we obtain a useful fact about them.

Lemma 6.2 *In $\mathcal{V}$-**Mod**, each lax monoid structure on X "restricts" along any $\mathcal{V}$-functor $j : A \longrightarrow X$ to give a lax monoid structure on A with j becoming a lax monoid morphism.*

Proof The structure on A is defined by $s_n(b;a_1,\ldots,a_n) = s_n(jb;ja_1,\ldots,ja_n)$ and taking μ_ξ to be the composite

$$\int^{a_1,\ldots,a_n} \bigotimes_i s_{m_i}(ja_i;ja_{i\bullet}) \otimes s_n(ja;ja_\bullet) \xrightarrow{\text{canon.}} \int^{x_1,\ldots,x_n} \bigotimes_i s_{m_i}(x_i;ja_{i\bullet}) \otimes s_n(ja;x_\bullet) \xrightarrow{\mu_\xi} s_m(ja;ja_{\bullet\bullet})$$

where the arrow labelled "canon." is induced by the coprojections for $x_i = ja_i$. The lax monoid axioms can directly be seen to hold; however, we can also apply Proposition 4.1 with u equal to the $\mathcal{V}$-module $j^* : X \longrightarrow A$ defined by $j^*(a,x) = X(ja,x)$. **q. e. d.**

A *lax promonoidal $\mathcal{V}$-category* is a $\mathcal{V}$-functor $\eta : A \longrightarrow M$, which is the identity on objects, together with a $\mathcal{V}$-multicategory structure on M.

Proposition 6.3 *The lax promonoidal $\mathcal{V}$-categories $\eta : A \longrightarrow M$ are in bijection with lax monoid structures on A in $\mathcal{V}$-**Mod**.*

Proof Given a lax monoid A in $\mathcal{V}$-**Mod**, we can apply the monoidal pseudofunctor $\mathrm{ob} : \mathcal{V}\text{-}\mathbf{Mod} \longrightarrow \mathcal{V}\text{-}\mathbf{Mat}$ to obtain a $\mathcal{V}$-multicategory M with the same objects as A. The $\mathcal{V}$-functor $\eta : A \longrightarrow M$ is defined to be the identity on objects and to have its effect on homs equal to the component $\eta : A(a,b) \longrightarrow s_1(a,b) = M(a,b)$ of $\eta : 1_A \Rightarrow s_1$ at a, b. This defines a lax promonoidal $\mathcal{V}$-category. On the other hand, given a lax promonoidal $\mathcal{V}$-category $\eta : A \longrightarrow M$, we can apply Lemma 6.2 to obtain a lax monoid structure on A in $\mathcal{V}$-**Mod**.

These processes are easily seen to be mutually inverse. **q. e. d.**

We therefore see, for example, that each lax monoidal $\mathcal{V}$-category (that is, lax monoid in $\mathcal{V}$-**CAT**) becomes a lax promonoidal $\mathcal{V}$-category by regarding the $\mathcal{V}$-functors s_n as $\mathcal{V}$-modules in the usual way.

We merely write A for a lax promonoidal $\mathcal{V}$-category, identifying it with the corresponding lax monoid in $\mathcal{V}$-**Mod**. Explicitly, the structure consists of $\mathcal{V}$-modules $s_n : A^{\otimes n} \longrightarrow A$, which we identify with $\mathcal{V}$-functors $s_n : A^{op} \otimes A^{\otimes n} \longrightarrow \mathcal{V}$, and $\mathcal{V}$-natural families

$$\eta \ : \ A(a, a') \longrightarrow s_1(a, a') \qquad \text{and}$$

$$\mu_{\xi} \ : \ s_n(a;a_1, \dots, a_n) \otimes s_{m_1}(a_1;a_{11}, \dots, a_{1m_1}) \otimes \dots \otimes s_{m_n}(a_n;a_{n1}, \dots, a_{nm_1}) \longrightarrow s_m(a;a_{11}, \dots, a_{1m_1}, \dots, a_{n1}, \dots, a_{nm_n})$$

satisfying the three axioms. We will call a $\mathcal{V}$-category B *lax procomonoidal* when B^{op} is lax promonoidal. In some ways, lax procomonoidal is the more familiar concept since it is natural to think of $s_n(b_1, \dots, b_n;b)$ as the $\mathcal{V}$-object of "multimorphisms" from $b_1, \dots, b_n$ to b.

Each monoidal $\mathcal{V}$-category A (including $\mathcal{V}$ itself) becomes a lax promonoidal $\mathcal{V}$-category by defining

$$s_n(a;a_1, \dots, a_n) = A(a,\, a_1 \otimes \dots \otimes a_n);$$

but, since A^{op} is also monoidal, A also becomes a lax procomonoidal $\mathcal{V}$-category by defining

$$s_n(a_1, \dots, a_n;a) = A(\, a_1 \otimes \dots \otimes a_n,\, a).$$

§7. Convolution

We are interested in lax promonoidal structures on the $\mathcal{V}$-category $[A, B]$ of $\mathcal{V}$-functors from A to B. Recall (for example, from [Ky]) that the $\mathcal{V}$-valued hom for $[A, B]$ is "the $\mathcal{V}$-object of $\mathcal{V}$-natural transformations from f to g" defined by the end formula

$$[A, B](f, g) \ = \ \int_a B(fa, ga)$$

(which certainly exists when A is small).

The first convolution structure we wish to distinguish makes $[A^{op}, \mathcal{V}]$ into a lax monoidal $\mathcal{V}$-category for any small lax promonoidal $\mathcal{V}$-category A. The structure is obtained merely by applying Corollary 6.1 to $\mathcal{V}$-**Mod**. For, notice that $\mathcal{M}[I, A]$ is precisely the $\mathcal{V}$-category $[A^{op}, \mathcal{V}]$; the hom called $A[f, g]$ above is precisely $[A^{op}, \mathcal{V}](f, g) = \int_a [fa, ga]$ where $[u,v]$ is the internal hom of $\mathcal{V}$. To be explicit, the $\mathcal{V}$-functor

$$\underset{n}{\otimes} : [A^{op}, \mathcal{V}]^{\otimes n} \longrightarrow [A^{op}, \mathcal{V}]$$

is defined by the coend formula

$$\underset{n}{\otimes}(f_1, \ldots, f_n)(a) = \int^{a_1, \ldots, a_n} f_1 a_1 \otimes \ldots \otimes f_n a_n \otimes s_n(a; a_1, \ldots, a_n) .$$

The $\mathcal{V}$-natural transformation $\mu_\xi : s_n \circ (s_{m_1} \otimes \ldots \otimes s_{m_n}) \Rightarrow s_m$ has components

$$\int^{a_\bullet, a_{\bullet\bullet}} \underset{i}{\otimes} \left\{ \left(\underset{j}{\otimes} f_{ij} a_{ij} \right) \otimes s_{m_i}(a_i; a_{i\bullet}) \right\} \otimes s_n(b; a_\bullet) \longrightarrow \int^{a_{\bullet\bullet}} \underset{ij}{\otimes} (f_{ij} a_{ij}) \otimes s_m(b; a_{\bullet\bullet})$$

isomorphic to

$$\int^{a_{\bullet\bullet}} 1 \otimes \mu_\xi : \int^{a_{\bullet\bullet}} \underset{ij}{\otimes} (f_{ij} a_{ij}) \otimes \int^{a_\bullet} \underset{i}{\otimes} s_{m_i}(a_i; a_{i\bullet}) \otimes s_n(b; a_\bullet) \longrightarrow \int^{a_{\bullet\bullet}} \underset{ij}{\otimes} (f_{ij} a_{ij}) \otimes s_m(b; a_{\bullet\bullet}),$$

while $\eta : 1_{[A^{op}, \mathcal{V}]} \Rightarrow s_1$ is the composite

$$f b \cong \int^a fa \otimes A(b, a) \xrightarrow{\int^a 1 \otimes \eta} \int^a fa \otimes s_1(b, a).$$

What we notice in this construction is that we could actually reverse the μ and η of A and find that the structure on $[A^{op}, \mathcal{V}]$ also has its μ and η reversed; the reversed arrows are called δ and ε as with coalgebras and comonads. In other words, if A is a lax monoid in $\mathcal{V}$-**Mod**^{co} (that is, an *oplax promonoidal $\mathcal{V}$-category*) then $[A^{op}, \mathcal{V}]$ is an oplax monoidal category. In fact, if we go ahead with that reversal we see that we only need the oplax direction for $\mathcal{V}$.

All told, we obtain our second convolution construction. We start with any (small) oplax promonoidal $\mathcal{V}$-category A and any cocomplete oplax monoidal $\mathcal{V}$-category C, and obtain an oplax monoidal $\mathcal{V}$-category $[A^{op}, C]$. Explicitly, the $\mathcal{V}$-functor

$$\underset{n}{\otimes} : [A^{op}, C]^{\otimes n} \longrightarrow [A^{op}, C]$$

is defined by the coend formula

$$\underset{n}{\otimes}(f_1, \ldots, f_n)(a) = \int^{a_1, \ldots, a_n} s_n(a; a_1, \ldots, a_n) \bullet \underset{n}{\otimes}(f_1 a_1, \ldots, f_n a_n)$$

where $v \bullet c$ denotes the tensor of $v \in \mathcal{V}$ with $c \in C$ (which exists since C is cocomplete). The components of δ for $[A^{op}, C]$ are given by composites

$$\underset{m}{\otimes}(f_{\bullet\bullet})(a) = \int^{a_{\bullet\bullet}} s_m(a; a_{\bullet\bullet}) \bullet \underset{m}{\otimes}(f_{\bullet\bullet} a_{\bullet\bullet}) \xrightarrow{\int^{a_{\bullet\bullet}} \delta \bullet \delta}$$

$$\int^{a_{\bullet\bullet}, a_\bullet} s_n(a; a_\bullet) \otimes \underset{n}{\otimes} s_{m_\bullet}(a_\bullet; a_{\bullet\bullet}) \bullet \underset{n}{\otimes}(\underset{m_\bullet}{\otimes}(f_{\bullet\bullet} a_{\bullet\bullet}))(a_\bullet) \xrightarrow{\text{canon.}}$$

$$\int^{a_\bullet} s_n(a; a_\bullet) \bullet \underset{n}{\otimes}\left(\int^{a_{1\bullet}} s_{m_1}(a_1; a_{1\bullet}) \bullet \underset{m_1}{\otimes} f_{1\bullet} a_{1\bullet}, \ldots, \int^{a_{n\bullet}} s_{m_n}(a_n; a_{n\bullet}) \bullet \underset{m_n}{\otimes} f_{n\bullet} a_{n\bullet}\right)$$

$$= \underset{n}{\otimes}\,(\underset{m_1}{\otimes} f_{1\bullet}a_{1\bullet}\,, \ldots , \underset{m_n}{\otimes} f_{n\bullet}a_{n\bullet})(a)\,,$$

where the arrow labelled "canon." is the usual comparison from the colimit of a functor at a diagram to the functor at the colimit of the diagram, and the components of ε are given by

$$\underset{1}{\otimes}(f_1)(a) = \int^{a_1} s_1(a;a_{\bullet\bullet}) \bullet \underset{1}{\otimes}(f_1 a) \xrightarrow{\int^{a_1}\varepsilon\bullet\varepsilon} \int^{a_1} A(a,a_1) \bullet f_1 a_1 \cong f_1 a\,.$$

A dual of this second construction is obtained by noting that $(\ )^{op}$ reverses 2-cells but not morphisms in $\mathcal{V}$-**Cat**. So we replace C by X^{op} where X is any complete lax monoidal $\mathcal{V}$-category, and we obtain a lax monoidal structure on $[A, X] = [A^{op}, C]^{op}$. The formula for the $\mathcal{V}$-functor

$$\underset{n}{\otimes} : [A, X]^{\otimes n} \longrightarrow [A, X]$$

is defined by the end formula

$$\underset{n}{\otimes}(f_1, \ldots, f_n)(a) = \int_{a_1,\ldots,a_n} \{s_n(a;a_1,\ldots,a_n), \underset{n}{\otimes}(f_1 a_1,\ldots,f_n a_n)\}\,,$$

where $\{v, x\}$ denotes the cotensor product of $v \in \mathcal{V}$ with $x \in X$.

Our third convolution construction starts with A a lax procomonoidal $\mathcal{V}$-category and B a lax promonoidal $\mathcal{V}$-category, both small, and produces a lax promonoidal structure on $[A, B]$. This is obtained by applying Lemma 6.2 to the $\mathcal{V}$-functor $j : [A, B] \longrightarrow [B^{op} \otimes A, \mathcal{V}]$ defined by

$$j(f)(b, a) = B(b, f(a)).$$

By Example 6 of Section 1 we know that $B^{op} \otimes A = (A^{op} \otimes B)^{op}$ is a lax promonoidal $\mathcal{V}$-category, so, by our first convolution construction, we have a lax monoidal $\mathcal{V}$-category $[B^{op} \otimes A, \mathcal{V}]$. The formula for the $\mathcal{V}$-module $s_n : [A, B]^{\otimes n} \longrightarrow [A, B]$ can be calculated as follows :

$$s_n(f; f_1, \ldots, f_n) =$$

$$\int_{a,b}\left[B(b, fa), \int^{a_1,\ldots,a_n,b_1,\ldots,b_n} B(b_1, f_1 a_1) \otimes \ldots \otimes B(b_n, f_n a_n) \otimes s_n(a_1,\ldots,a_n;a) \otimes s_n(b;b_1,\ldots,b_n)\right]$$

$$= \int_a \int^{a_1,\ldots,a_n,b_1,\ldots,b_n} B(b_1, f_1 a_1) \otimes \ldots \otimes B(b_n, f_n a_n) \otimes s_n(a_1,\ldots,a_n;a) \otimes s_n(fa;b_1,\ldots,b_n)$$

$$= \int_a \int^{a_1,\ldots,a_n} s_n(a_1,\ldots,a_n;a) \otimes s_n(fa; f_1 a_1,\ldots,f_n a_n).$$

This last expression holds even when B is not small and clearly still yields a lax promonoidal structure on $[A, B]$. In particular, this can be applied when A is replaced by A^{op} and B by $\mathcal{V}$

$$s_n(f\,;f_1,\ldots,f_n) \;=\; \int_a \int^{a_1,\ldots,a_n} s_n(a;a_1,\ldots,a_n)\otimes[fa,f_1a_1\otimes\ldots\otimes f_na_n]$$

which is different from the first convolution construction on $[A^{op}, \mathcal{V}]$ in general; it does agree when each $f(a)$ has a dual in $\mathcal{V}$, however there is much more to be said when duals exist, and we do not wish to pursue that here.

A fourth construction can be obtained by applying Lemma 6.2 to the same $\mathcal{V}$-functor $j : [A, B] \longrightarrow [B^{op}\otimes A, \mathcal{V}]$ making use of the dual second construction on $[B^{op}\otimes A, \mathcal{V}]$ (here $X = \mathcal{V}$). We start this time with A oplax promonoidal and B oplax procomonoidal (both small) to obtain the following lax monoidal structure on $[A, B]$:

$$s_n(f\,;f_1,\ldots,f_n) \;=$$

$$\int_{a;a_1,\ldots,a_n;b_1,\ldots,b_n}[s_n(a;a_1,\ldots,a_n)\otimes s_n(b_1,\ldots,b_n;fa)\,,\,B(b_1,f_1a_1)\otimes\ldots\otimes B(b_n,f_na_n)].$$

The fifth convolution construction again starts, as in case three, with A a lax procomonoidal $\mathcal{V}$-category and B a lax promonoidal $\mathcal{V}$-category, both small, and produces a lax promonoidal structure on $[A, B]$. This time we begin with the $\mathcal{V}$-module

$$u : A^{op}\otimes B \longrightarrow [A, B]$$

defined by $u(f, a, b) = B(fa, b)$. We apply Proposition 4.1 to extend the lax monoid structure from $A^{op}\otimes B$ to $[A, B]$. Using the formula for right extension along $\mathcal{V}$-modules and using the Yoneda lemma, we obtain

$$s_n(f\,;f_1,\ldots,f_n) \;=$$

$$\int_{a_1,\ldots,a_n,\,b_1,\ldots,b_n}\left[B(f_1a_1,b_1)\otimes\ldots\otimes B(f_na_n,b_n),\int^{a,\,b} s_n(a_1,\ldots,a_n;a)\otimes s_n(b;b_1,\ldots,b_n)\otimes B(fa,b)\right]$$

$$= \int_{a_1,\ldots,a_n}\int^{a} s_n(a_1,\ldots,a_n;a)\otimes s_n(fa;f_1a_1,\ldots,f_na_n).$$

Our sixth[4] and final construction is added in the light of the article [BDK] which came to our notice after completion of the rest of the paper. For this we take A to be a small lax procomonoidal $\mathcal{V}$-category and obtain a lax procomonoidal $\mathcal{V}$-category structure on $[A^{op}, \mathcal{V}]$ defined by

$$s_n(\,f_1,\ldots,f_n;f\,) \;=\; \int_{a_1,\ldots,a_n}\left[f_1a_1\otimes\ldots\otimes f_na_n\,,\int^{a} s_n(a_1,\ldots,a_n;a)\otimes f\,a\right].$$

There are several ways to understand that this formula works. One way is to directly calculate the substitution structure on the $s_n(\,f_1,\ldots,f_n;f\,)$ induced by that on the $s_n(a_1,\ldots,a_n;a)$. Another way is to apply Lemma 6.2 to the $\mathcal{V}$-functor

$$\Phi = \exists_Y : [A^{op}, \mathcal{V}] \longrightarrow [[A, \mathcal{V}], \mathcal{V}]$$

[4] In the presence of appropriate duals, this construction is a special case of the fifth construction.

defined by left Kan extension along the Yoneda embedding $\mathbf{Y} : A^{op} \longrightarrow [A, \mathcal{V}]$; this is given explicitly by the formulas

$$(\Phi f)h = \int^{a}[A,\mathcal{V}](\mathbf{Y}a,h)\otimes fa = \int^{a} ha\otimes fa .$$

In order to apply Lemma 6.2 we require a lax $\mathcal{V}$-procomonoidal structure on $[[A, \mathcal{V}], \mathcal{V}]$; indeed, we obtain a representable such structure: we actually make $[[A, \mathcal{V}], \mathcal{V}]$ into an oplax monoidal $\mathcal{V}$-category. To obtain this we first note that the first convolution construction gives $[A, \mathcal{V}]$ as a lax monoidal $\mathcal{V}$-category:

$$\underset{n}{\otimes}(h_1, \ldots, h_n)(a) = \int^{a_1, \ldots, a_n} h_1a_1 \otimes \ldots \otimes h_n a_n \otimes s_n(a_1, \ldots, a_n; a) .$$

Then we can apply the first convolution construction again to obtain an oplax monoidal $\mathcal{V}$-category $[[A, \mathcal{V}], \mathcal{V}]$ defined by

$$\underset{n}{\otimes}(r_1, \ldots, r_n)(h) = \int^{h_1, \ldots, h_n} r_1h_1 \otimes \ldots \otimes r_n h_n \otimes [A, \mathcal{V}](\underset{n}{\otimes}(h_1, \ldots, h_n), h).$$

There is a small word of warning here: the coend over $h_1, \ldots, h_n \in [A, \mathcal{V}]$ may not exist in general because $[A, \mathcal{V}]$ is not small. However, as you will see, the coends we need do exist[5]. Let us calculate the lax procomonoidal $\mathcal{V}$-category structure on $[A^{op}, \mathcal{V}]$ obtained by restriction along Φ according to Lemma 6.2:

$$s_n(f_1, \ldots, f_n; f) = [[A,\mathcal{V}],\mathcal{V}](\underset{n}{\otimes}(\Phi f_1, \ldots, \Phi f_n), \Phi f) \qquad \cong$$

$$\int_{h, h_1, \ldots, h_n}\left[\int^{a_1}[A,\mathcal{V}](\mathbf{Y}a_1,h_1)\otimes f_1 a_1 \otimes \ldots \otimes \int^{a_n}[A,\mathcal{V}](\mathbf{Y}a_n,h_n)\otimes f_n a_n \otimes [A,\mathcal{V}](\underset{n}{\otimes}(h_1, \ldots, h_n),h), (\Phi f)h\right]$$

$$\cong \int_{h, a_1, \ldots, a_n}\left[f_1a_1 \otimes \ldots \otimes f_na_n \otimes [A,\mathcal{V}](\underset{n}{\otimes}(\mathbf{Y}a_1, \ldots, \mathbf{Y}a_n), h), (\Phi f)h\right]$$

$$\cong \int_{a_1, \ldots, a_n}\left[f_1a_1 \otimes \ldots \otimes f_na_n, (\Phi f)\, s_n(a_1, \ldots, a_n; -)\right]$$

$$\cong \int_{a_1, \ldots, a_n}\left[f_1a_1 \otimes \ldots \otimes f_na_n, \int^{a} s_n(a_1, \ldots, a_n; a)\otimes fa\right].$$

The advantage of this viewpoint is that we have a fully faithful lax $\mathcal{V}$-comonoidal embedding Φ of $[A^{op}, \mathcal{V}]$ into $[[A, \mathcal{V}], \mathcal{V}]$. In the case where A is a "premonoidal" $\mathcal{V}$-category in the sense of [D1], we have that $[A, \mathcal{V}]$ becomes a closed monoidal $\mathcal{V}$-category under convolution; so, by the same result again, the structure on $[[A, \mathcal{V}], \mathcal{V}]$ is monoidal, and we have a lax comonoidal embedding Φ of our lax comonoidal $[A^{op}, \mathcal{V}]$ into a genuinely monoidal category.

A third (perhaps more conceptual or abstract) way of obtaining this sixth construction is to revisit the material at the beginning of Section 6. We can exploit the fact that the

[5] There are other ways around this problem involving a "change of $\mathcal{V}$-universe" [D2].

comparison $\mathcal{M}[\mathrm{I},\mathrm{A}] \otimes \mathcal{M}[\mathrm{I},\mathrm{B}] \longrightarrow \mathcal{M}[\mathrm{I},\mathrm{A}\otimes\mathrm{B}]$ is actually a $\mathcal{V}$-functor to replace it by its right adjoint $\mathcal{V}$-module $\mathcal{M}[\mathrm{I},\mathrm{A}\otimes\mathrm{B}] \longrightarrow \mathcal{M}[\mathrm{I},\mathrm{A}] \otimes \mathcal{M}[\mathrm{I},\mathrm{B}]$ and so obtain a comonoidal pseudo-functor $\mathcal{M}[\mathrm{I},-] : \mathcal{M} \longrightarrow \mathcal{V}$-**MOD** [6]. So this $\mathcal{M}[\mathrm{I},-]$ takes lax comonoids to lax comonoids. Applying this in the case $\mathcal{M} = \mathcal{V}$-**Mod** gives the desired lax comonoidal structure on $[\mathrm{A}^{\mathrm{op}}, \mathcal{V}]$.

The special case of this sixth construction of prime importance in [BDK] (see their page 24) has $\mathcal{V}$ the monoidal category of vector spaces and A^{op} a Hopf algebra H; this implies that $[\mathrm{A}^{\mathrm{op}}, \mathcal{V}]$ is the $\mathcal{V}$-category of left H-modules with the $\mathcal{V}$-multicategory structure given by

$$s_n(m_1,\ldots,m_n;m) = [H^{\otimes n},\mathcal{V}](m_1 \otimes \ldots \otimes m_n, H^{\otimes n} \underset{H}{\otimes} m).$$

Actually [BDK] makes use of the action of the symmetric groups on $s_n(m_1,\ldots,m_n;m)$ but we will leave it to a later paper to discuss symmetric lax monoids in symmetric monoidal bicategories and the like.

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[6] By $\mathcal{V}$-**MOD** we mean the bicategory whose objects are not-necessarily-small $\mathcal{V}$-categories and whose morphisms are $\mathcal{V}$-modules $\mathrm{M} : \mathcal{A} \longrightarrow \mathcal{B}$ which are such that, for each object A of $\mathcal{A}$, the $\mathcal{V}$-functor M(–,A) : $\mathcal{B}^{\mathrm{op}} \longrightarrow \mathcal{V}$ is accessible (that is, a small colimit of representables).

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