Notes on the tentative abstract of Fahd Al-Agl & Richard Steiner: "Nerves of multiple categories".

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Let D be any set equipped with a sequence of preorders $\leq_n$ such that

$$x \leq_{n+1} y \quad \Longrightarrow \quad x \leq_n y .$$

Write $x \equiv_n y$ when $x \leq_n y$ and $y \leq_n x$.

Suppose $X \subseteq D$. For each $x \in X$, put

$$X_n(x) = \{ y \in X \mid y \equiv_n x \}.$$

Now define

$$\partial_n^- X = \{ x \in X \mid y \in X_n(x) \Rightarrow x \leq_{n+1} y \}$$

$$\partial_n^+ X = \{ x \in X \mid y \in X_n(x) \Rightarrow y \leq_{n+1} x \}.$$

[Letters ε, ζ, η will denote $+, -$ signs.]

Example. $D = \{-1,0,1\}^d = \{x = (x_1, \dots, x_d) \mid x_i = -1, 0 \text{ or } 1 \text{ for } 1 \leq i \leq d\}$ and

$x \leq_n y$ means $x_i \leq y_i$ for $i \leq n$.

Let $e_n^\varepsilon \in D$ be the element whose entries are all 0 except for $\varepsilon 1$ in the nth position. Consider the subset

$$X = \{e_n^\varepsilon \mid \varepsilon = + \text{ or } -,\ n \text{ arb}\}$$
$$= \{x \in D \mid x_i = 0 \text{ for all but one } i\}.$$

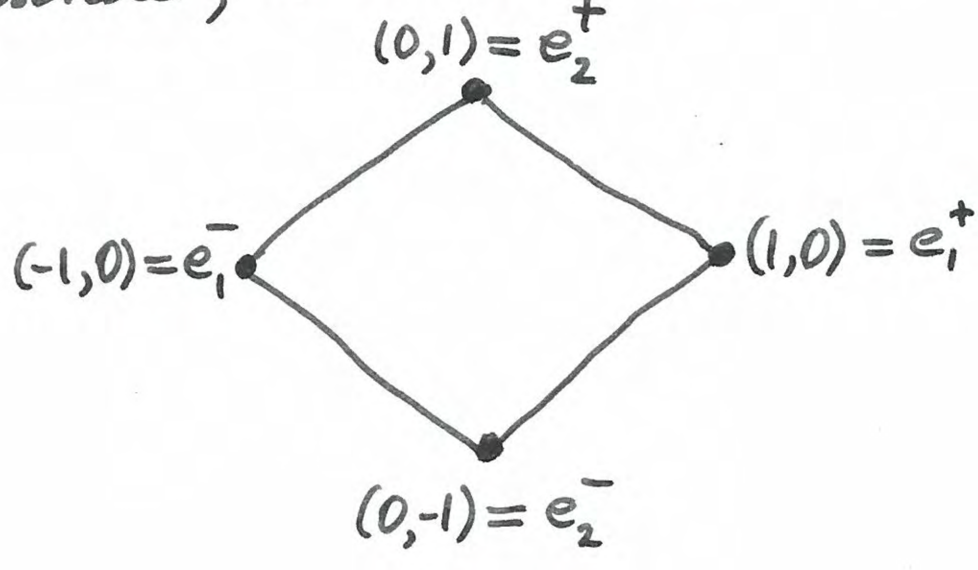
Then

$$X_n(e_m^\varepsilon) = \begin{cases} \{e_m^\varepsilon\} & \text{for } m \leq n \\ \{e_r^\eta \mid r > n\} & \text{for } m > n, \end{cases}$$

$$\partial_n^\varepsilon X = \{e_m^\eta \mid m \leq n\} \cup \{e_{n+1}^\varepsilon\}.$$

To illustrate, take $d = 2$.

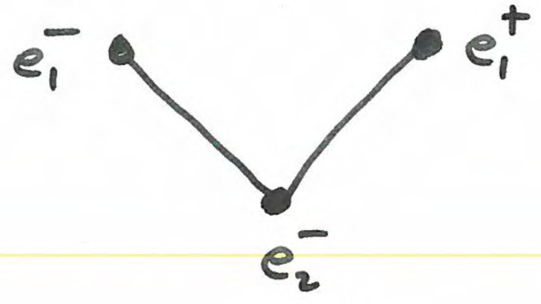
X



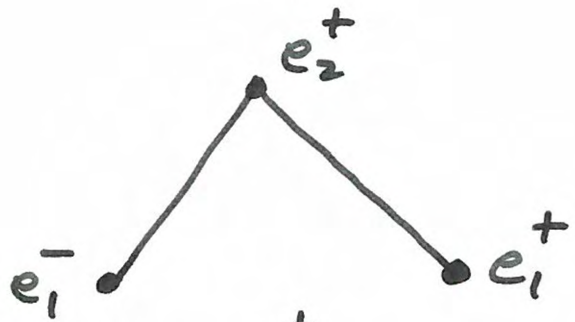
$\partial_0^- X$

$\partial_0^+ X$

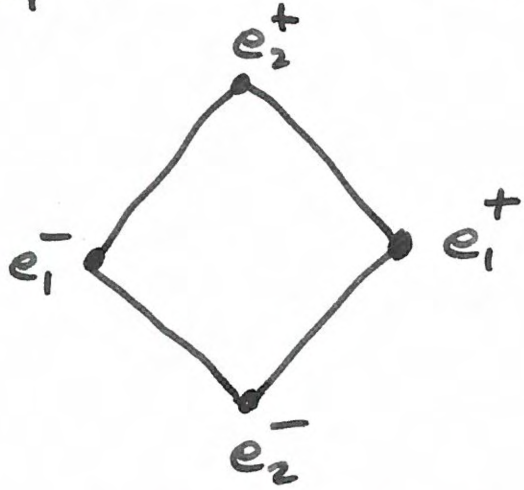
$\partial_1^- X$



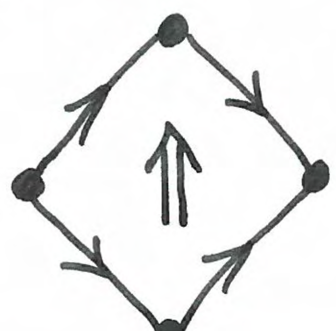
$\partial_1^+ X$



$\partial_2^\varepsilon X$



This fits well with our picture of the oriented square (= 2-cube) as below.



<u>Definition.</u> Call $X \subseteq D$ <u>ruled</u> when each $X_n(x) \cap \partial_n^{\varepsilon} X$ is a singleton. Denote the unique element by x_n^{ε} when X is understood.

<u>Proposition 1.</u> <u>If $X \subseteq D$ is ruled then</u>

(a) $\partial_m^{\eta} X \subseteq \partial_n^{\varepsilon} X$ <u>for</u> $m < n$,

(b) $\partial_m^{\eta} \partial_n^{\varepsilon} X = \partial_n^{\varepsilon} X$ <u>for</u> $m \geqslant n$,

(c) $\partial_m^{\eta} \partial_n^{\varepsilon} X = \partial_m^{\eta} X$ <u>for</u> $m < n$,

(d) <u>each</u> $\partial_n^{\varepsilon} X$ <u>is ruled</u>.

<u>Proof.</u> (a) If $x \in \partial_m^{\eta} X$ and $m < n$ then $x = x_m^{\eta} = x_n^{\varepsilon} \in \partial_n^{\varepsilon} X$.

(b) Take $\eta = -$ as the other case is dual. Now $\partial_m^{-} \partial_n^{\varepsilon} X = \{ x \in \partial_n^{\varepsilon} X \mid y \in \partial_n^{\varepsilon} X \cap X_m(x) \Rightarrow x \leqslant_{n+1} y \} \subseteq \partial_n^{\varepsilon} X$. But, if $x \in \partial_n^{\varepsilon} X$ and $m \geqslant n$, then $\partial_n^{\varepsilon} X \cap X_m(x) \subseteq \partial_n^{\varepsilon} X \cap X_n(x) = \{x\}$. Certainly $y \in \{x\} \Rightarrow x \leqslant_{n+1} y$. So $x \in \partial_m^{-} \partial_n^{\varepsilon} X$.

(c) It follows from (a) that $\partial_m^- X \subseteq \partial_m^- \partial_n^\varepsilon X$. Take $x \in \partial_m^- \partial_n^\varepsilon X$. To test whether $x \in \partial_m^- X$, take $y \in X_m(x)$. Then $y_n^\varepsilon \in \partial_n^\varepsilon X \cap X_m(x)$; so $x \leqslant_{m+1} y_n^\varepsilon \equiv_{m+1} y$.

(d) We must prove that each $W = X_m(x) \cap \partial_n^\varepsilon X \cap \partial_m^\eta \partial_n^\varepsilon X$ is a singleton for $x \in \partial_n^\varepsilon X$. For $m < n$, (c) and (a) imply $W = X_m(x) \cap \partial_n^\varepsilon X \cap \partial_m^\eta X = X_m(x) \cap \partial_m^\eta X = \{x_m^\eta\}$. So suppose $m \geqslant n$. By (b), $W = X_m(x) \cap \partial_n^\varepsilon X \subseteq X_n(x) \cap \partial_n^\varepsilon X = \{x_n^\varepsilon\}$. But $x \in W$; so $W = \{x\}$. $\square$

<u>Proposition 2.</u> *If* $X, Y \subseteq D$ *are ruled with* $\partial_m^+ X = \partial_m^- Y$, *and if* $Z = X \cup Y$, *then*

(a) $\partial_m^+ X = \partial_m^- Y = X \cap Y$,

(b)

$$Z_n(z) = \begin{cases} X_n(u) \cup Y_n(u) & \underline{\text{for}}\ n \leq m \\ X_n(z) & \underline{\text{for}}\ z \in X,\ n > m \\ Y_n(z) & \underline{\text{for}}\ z \in Y,\ n > m, \end{cases}$$

(c)

$$\partial_n^\varepsilon Z = \begin{cases} \partial_n^- X & \underline{\text{for}}\ \varepsilon = -,\ n \leq m \\ \partial_n^+ Y & \underline{\text{for}}\ \varepsilon = +,\ n \leq m \\ \partial_n^\varepsilon X \cup \partial_n^\varepsilon Y & \underline{\text{for}}\ n > m, \end{cases}$$

(d) Z <u>is ruled</u>.

<u>Proof</u>. (a) Clearly $\partial_m^+ X = \partial_m^- Y \subseteq X \cap Y$, so take $z \in X \cap Y$. Since X, Y are ruled, $X_m(z) \cap \partial_m^+ X = \{x\}$, $Y_m(z) \cap \partial_m^- Y = \{y\}$. Then $x \in \partial_m^+ X = \partial_m^- Y \subseteq Y$. So $x = y$. So $z \leq_{m+1} x = y \leq_{m+1} z$. So $z \equiv_{m+1} x$. To prove $z \in \partial_m^+ X$, take $u \in X_m(z) = X_m(x)$.

But $x \in \partial_m^+ X$ implies $u \leq_{m+1} x \equiv_{m+1} z$.

(b) Take $z = x \in X$ as the other case is dual. Let $X_m(x) \cap \partial_m^+ X = \{u\}$. So $u \in \partial_m^- Y \subseteq Y$. If $n \leq m$ then clearly $Z_n(x) = X_n(u) \cup Y_n(u)$. So assume $n > m$. Clearly $Z_n(x) \cap X = X_n(x)$. So suppose $y \in Y$ with $y \equiv_n x$. Then $u \equiv_{m+1} y$. So $y \in Y_m(u) \cap \partial_m^- Y = \{u\}$. So $y = u$. So $y \in X_n(x)$. Hence $Z_n(x) = X_n(x)$.

(c) For $n > m$, $\partial_n^\varepsilon Z = \partial_n^\varepsilon X \cup \partial_n^\varepsilon Y$ is immediate from (b). So suppose $n \leq m$. Clearly $(\partial_n^- Z) \cap Y \subseteq \partial_n^- Y$ and $(\partial_n^- Z) \cap X \subseteq \partial_n^- X$. By Proposition 1(a), $\partial_n^- Y \subseteq \partial_m^- Y = \partial_m^+ X \subseteq X$; so $\partial_n^- Z \subseteq \partial_n^- X$. Take $x \in \partial_n^- X$. To test whether $x \in \partial_n^- Z$, we take $w \in Z_n(x) = X_n(u) \cup Y_n(u)$ where $X_m(x) \cap \partial_m^+ X = \{u\}$. If $w \in X_n(u) = X_n(x)$ then $x \leq_{n+1} w$ since $x \in \partial_n^- X$. So suppose $w \in Y_n(u)$. Now $u \in X_n(x)$ and $x \in \partial_n^- X$, so $x \leq_{n+1} u$.

Let $Y_n(u) \cap \partial_n^- Y = \{v\}$. Then $v \leqslant_{n+1} w$. But $v \in \partial_n^- Y \subseteq X$. So $v \in X_n(u) = X_n(x)$. But $x \in \partial_n^- X$, so $x \leqslant_{n+1} v$. So $x \leqslant_{n+1} w$. Thus $\partial_n^- Z = \partial_n^- X$.

(d) We must prove each $W = Z_n(z) \cap \partial_n^\varepsilon Z$ is a singleton. First consider the case $n \leq m$. Take $\varepsilon = -$ as the other case is dual. Then $W = (X_n(u) \cup Y_n(u)) \cap \partial_n^- X = (X_n(u) \cap \partial_n^- X) \cup (Y_n(u) \cap \partial_n^- X) = X_n(u) \cap \partial_n^- X$, a singleton, since $Y_n(u) \cap \partial_n^- X \subseteq X_n(u) \cap \partial_n^- X$. So the case $n > m$ remains. Take $z = x \in X$ as the other case is similar. Then $W = X_n(x) \cap (\partial_n^\varepsilon X \cup \partial_n^\varepsilon Y) = (X_n(x) \cap \partial_n^\varepsilon X) \cup (X_n(x) \cap \partial_n^\varepsilon Y)$. But $X_n(x) \cap \partial_n^\varepsilon Y \subseteq X \cap Y = \partial_m^+ X \subseteq \partial_n^\varepsilon X$. So $W = X_n(x) \cap \partial_n^\varepsilon X$, a singleton. $\square$

<u>Corollary 3</u>. <u>The ruled subsets of</u> D <u>form an ω-category</u> $R(D)$ <u>with</u> <u>n-source</u> ∂_n^-, <u>n-target</u> ∂_n^+ <u>and</u> <u>n-composition union</u>. $\square$

Fahd Al-Agl & Richard Steiner < 7 Aug 1990

Nerves of multiple categories

Tentative abstract

Define functions $w_0, w_1, w_2, \ldots : \mathbb{R}^N \to \mathbb{R}$ by

$$w_n(t_1,\ldots,t_N) = \sum [t_{i(0)} - t_{i(1)} + t_{i(2)} - \ldots \pm t_{i(n)}],$$

where the sum is over (n+1)-tuples such that $i(0) < i(1) < \ldots < i(n)$. For x a subset of $\mathbb{R}^N$, let $d_n^- x$ $[d_n^+ x]$ be the subset of x consisting of points at which w_n is minimal [maximal] for given $w_0, \ldots, w_{n-1}$. For x a subspace of $\mathbb{R}^N$ and n = 0, 1, ..., an *n-fibre* of x is a non-empty fibre of

$$(w_0,\ldots,w_{n-1}): x \to \mathbb{R}^n.$$

A *complex* in $\mathbb{R}^N$ is a space which is expressed as the union of a finite collection of compact sets closed under union and intersection. A complex x is *globelike* if

(i) x is non-empty,

(ii) each $d_n^{\pm} x$ is a subcomplex of x,

(iii) each n-fibre of x is connected and meets each of $d_n^- x$, $d_n^+ x$ in a unique point.

If C is a complex in $\mathbb{R}^N$, then the globelike subcomplexes of C form an ∞-category G(C) with source and target maps $d_n^{\pm}$ and with all the compositions $\times_n$ given by union. In particular, let $I^N = [-1,1]^N$ be the standard N-cube and let Δ^N be the N-simplex in $\mathbb{R}^N$ with vertices

$$(0,\ldots,0),\ (1,0,\ldots,0),\ (1,1,0,\ldots,0),\ \ldots,\ (1,\ldots,1),$$

where in each case the subcomplexes are the unions of the faces. Then $G(\Delta^N)$ is the Nth oriental and $G(I^N)$ is the cubic analogue.

Let C be a subcomplex of I^N or Δ^N. Then G(C) has a presentation in which the generators are the faces contained in C and the relations are

(i) $d_n^- \sigma = d_n^+ \sigma = \sigma$ for σ an n-face,

(ii) $d_{n-1}^- \sigma$ and $d_{n-1}^+ \sigma$ are the appropriate subcomplexes of σ for σ an n-face, $n \geq 1$.

It follows that

$$G(C) = \operatorname{colim} G(\sigma),$$

where the colimit is taken over faces σ in C and their inclusions.

Let $\mathcal{K}$ [$\mathcal{S}$] be the full subcategory of the category of ∞-categories whose objects are the G(C) with C a globelike subcomplex of some I^N [Δ^N]. Define a *cube ∞-category* [*simplex ∞-category*] to be a contravariant functor S from $\mathcal{K}$ [$\mathcal{S}$] to sets such that

$$S(G(C)) = \lim S(G(\sigma)).$$

Morphisms of cube and simplex ∞-categories are to be natural transformations. Then a cube ∞-category [simplex ∞-category] is a cubical set [simplicial set] with additional structure. There are inverse equivalences

$$\infty\text{-categories} \underset{\gamma}{\overset{\lambda}{\rightleftarrows}} \text{cube } \infty\text{-categories [simplex } \infty\text{-categories]}$$

given by

$$(\lambda X)(G(C)) = \operatorname{Hom}(G(C), X) \quad \text{(nerve)},$$

$$\gamma S = \operatorname{colim}_{s \in S(G(C))} G(C) \quad \text{(realisation - a coend)},$$

where, in the definition of an ∞-category, every element is required to be finite-dimensional.

There is a biclosed (but not symmetric) monoidal structure on cube ∞-categories coming from a bifunctor

$$(G(C),G(D)) \mapsto G(C \times D)\colon \mathcal{K} \times \mathcal{K} \to \mathcal{K},$$

and this transfers to ∞-categories and simplex ∞-categories.