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Lectures on topos theory

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We summarize the purpose of topos theory by quoting from Lawvere's Introduction to Lecture Notes in Mathematics #274 (Springer 1972):

"This is the development on the basis of elementary (first-order) axioms ... just good enough to be applicable not only to sheaf theory, algebraic spaces, global spectrum, etc. as originally envisaged by Grothendieck, Giraud, Verdier, and Hakim but also to Kripke semantics, abstract proof theory, and the Cohen-Scott-Solovay method for obtaining independence results in set theory."

"Briefly we may say that the notion of topos summarizes in objective categorical form the essence of 'higher-order logic' (we will explain below how the logical operators become morphisms in a topos) with no axiom of extensionality. This amounts to a natural and useful generalization of set theory to the consideration of 'sets which internally develop'. In a basic example of algebraic geometry, the development may be viewed as taking place along a parameter which varies over 'rings of definition'; in a basic example from intuitionistic logic, the parameter is interpreted as varying over 'stages of knowledge'."

§1. The comprehension scheme

Observe that the definition of a category can be expressed in first order predicate calculus. To give a category $\mathcal{E}$ is to give objects $A, B, \ldots, X, \ldots$; arrows $A \xrightarrow{f} B$, $K \xrightarrow{t} X, \ldots$; for each pair of arrows $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, a composite arrow $A \xrightarrow{gf} C$; and, an identity arrow $1_A : A \to A$ for each object A; satisfying

$$(hg)f = h(gf), \qquad 1_B f = f = f 1_A .$$

The conditions on $\mathcal{E}$ below can also be expressed in elementary language.

An object 1 (not to be confused with an identity arrow) is said to be terminal when, given any object A, there is precisely one arrow $A \to 1$. Terminal objects are unique up to isomorphism.

A diagram

$$\begin{array}{ccc} P & \xrightarrow{v} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

is said to be a pullback when it commutes (that is, $fu = gv$) and, given any $A \xleftarrow{x} X \xrightarrow{y} B$

such that $fx = gy$, there exists a unique arrow $X \xrightarrow{h} P$ (often called $\binom{x}{y}$) such that $x = uh$ and $y = vh$. We sometimes say P is a pullback of f, g (unique up to isomorphism) and sometimes that u is the pullback of g along f.

Say $\mathcal{E}$ is finitely complete when it has a terminal object and every pair of arrows $A \xrightarrow{f} C \xleftarrow{g} B$ has a pullback. Then we construct:

— for each pair of objects A, B, the product $A \times B$ of A and B given by the pullback

$$\begin{tikzcd} A \times B \arrow[r, "proj_2"] \arrow[d, "proj_1"'] & B \arrow[d] \\ A \arrow[r] & 1 \end{tikzcd};$$

— for each pair of arrows $A \underset{g}{\overset{f}{\rightrightarrows}} B$, their equalizer $E \xrightarrow{k} A$ given by the pullback

$$\begin{tikzcd} E \arrow[r] \arrow[d, "k"'] & B \arrow[d, "\binom{1_B}{1_B}"] \\ A \arrow[r, "\binom{f}{g}"'] & B \times B \end{tikzcd}.$$

A relation from A to B is a diagram $A \xleftarrow{d_0} R \xrightarrow{d_1} B$ such that, given $X \underset{v}{\overset{u}{\rightrightarrows}} R$, if $d_0 u = d_0 v$ and $d_1 u = d_1 v$ then $u = v$. An arrow $X \xrightarrow{m} A$ is called a monomorphism when the square

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "1_X"'] & X \arrow[d, "m"] \\ X \arrow[r, "m"'] & A \end{tikzcd}$$

is a pullback. So $A \xleftarrow{d_0} R \xrightarrow{d_1} B$ is a relation if and only if $R \xrightarrow{\binom{d_0}{d_1}} A \times B$ is a monomorphism. Sometimes we just write R for $A \xleftarrow{d_0} R \xrightarrow{d_1} B$, and then we write R^* for $B \xleftarrow{d_1} R \xrightarrow{d_0} A$ which is a relation from B to A. For relations R, S from A to B write $R \leq S$ when there is a commutative diagram

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] \arrow[dd] & \\ A & & B \\ & S \arrow[ul] \arrow[ur] & \end{tikzcd}.$$

Properties of this order are:

$R \leq R$; $R \leq S, S \leq T$ implies $R \leq T$; $R \leq S, S \leq R$ implies $R \cong S$.

An arrow $A \xrightarrow{f} B$ can be regarded as a relation $A \xleftarrow{1} A \xrightarrow{f} B$ from A to B (its "graph") and we sometimes denote this relation also by f.

An <u>order relation</u> on A is a relation R from A to A for which we have commutative diagrams

$$\begin{tikzcd} & A \arrow[dl, "1"'] \arrow[dr, "1"] \arrow[dd] & \\ A & & A \\ & R \arrow[ul, "d_0"] \arrow[ur, "d_1"'] & \end{tikzcd} \qquad \begin{tikzcd} & P \arrow[dl, "d_0 q"'] \arrow[dr, "d_1 p"] \arrow[dd] & \\ A & & A \\ & R \arrow[ul, "d_0"] \arrow[ur, "d_1"'] & \end{tikzcd}$$

where
$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] & R \arrow[d, "d_1"] \\ R \arrow[r, "d_0"'] & A \end{tikzcd}$$
is a pullback. An <u>ordered object</u> is an object together with an order relation on it. Write
$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\leq"] & A \end{tikzcd}$$
when we have a commutative diagram
$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd] & \\ A & & A \\ & R \arrow[ul, "d_0"] \arrow[ur, "d_1"'] & \end{tikzcd}$$
This orders the arrows from X to A.

A <u>topos</u> is a category $\mathcal{E}$ which is finitely complete and has <u>power objects</u> in the following sense:

— (comprehension scheme) for each object A there is an object $\mathcal{P}A$ and a relation $A \xleftarrow{d_0} \in_A \xrightarrow{d_1} \mathcal{P}A$ from A to $\mathcal{P}A$ such that, given any object B and any relation $A \xleftarrow{d_0} R \xrightarrow{d_1} B$ from A to B, there exists a unique arrow $B \xrightarrow{r} \mathcal{P}A$ for which there is a commutative diagram
$$\begin{tikzcd} & & R \arrow[dl] \arrow[dr, "d_1"] \arrow[ddll, bend right, "d_0"'] & \\ & \in_A \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & & B \arrow[dl, "r"] \\ A & & \mathcal{P}A & \end{tikzcd}$$
in which the diamond is a pullback.

Loosely, arrows $B \xrightarrow{r} \wp A$ are in 1-1 correspondence with relations R from A to B via $\in_A$.

<u>Example</u>. The category Set of sets is a topos. The terminal object 1 is any set with one element. The pullback of f, g is $\{(a,b) \mid fa = gb\}$. For each set A, the set $\wp A$ is the set of subsets of A, the relation $\in_A = \{(a, U) \mid a \in U \subset A\}$; and, given a relation R from A to B, the arrow $r : B \to \wp A$ is given by $r(b) = \{a \in A \mid a R b\}$; or better, $r(b) = \{a \in A \mid$ there exists $x \in R$ such that $d_0 x = a$, $d_1 x = b\}$. //

We now work in a topos $\mathcal{E}$. Corresponding to the relation $A \xleftarrow{1_A} A \xrightarrow{1_A} A$ we obtain an arrow $\mathrm{sing} : A \to \wp A$ called <u>singleton</u>. Form the pullback

$$\begin{tikzcd}
& & P \arrow[dl] \arrow[dr] & & \\
& \in_A \arrow[dl] \arrow[dr, "d_0"] & & \in_A \arrow[dl, "d_0"'] \arrow[dr] & \\
\wp A & & A & & \wp A
\end{tikzcd}$$

of d_0 along d_0 as indicated; this gives a relation P from A to $\wp A \times \wp A$ which in turn gives an arrow $\wedge : \wp A \times \wp A \to \wp A$ called <u>meet</u>. Let $\subset_A \xrightarrow{\binom{d_0}{d_1}} \wp A \times \wp A$ denote the equalizer of the pair of arrows

$$\wp A \times \wp A \underset{\wedge}{\overset{\mathrm{proj}_1}{\rightrightarrows}} \wp A.$$

It is readily checked that $\subset_A$ is an order relation on $\mathcal{P}A$, called the <u>inclusion relation</u>, with the property that

$$\begin{tikzcd} B \arrow[r, bend left, "r"] \arrow[r, bend right, "s"'] \arrow[r, phantom, "\leq"] & \mathcal{P}A \end{tikzcd} \quad \text{if and only if} \quad R \leq S$$

where r, s correspond to R, S. Note that $r \leq s$, $s \leq r$ imply $r = s$.

Suppose R is a relation from A to B and S is a relation from A to C, define a relation R/S from B to C by the pullback

$$\begin{tikzcd} R/S \arrow[r] \arrow[d] & \subset_A \arrow[d, "\binom{d_0}{d_1}"] \\ B \times C \arrow[r, "r \times s"'] & \mathcal{P}A \times \mathcal{P}A \end{tikzcd}$$

where r, s correspond to R, S. The essential property of this construction is:

$$T \leq R/S \quad \text{if and only if} \quad \begin{tikzcd} T \arrow[r] \arrow[d] & C \arrow[d, "s"] \\ B \arrow[r, "r"'] \arrow[ur, phantom, "\leq"] & \mathcal{P}A \end{tikzcd}$$

(the $\leq$ here means that the lower composite is less than the upper composite). In Set,

$$R/S = \{(b, c) \mid aRb \text{ implies } aSc\}.$$

§2. Quantifiers and finite colimits.

Again we work in a topos $\mathcal{E}$. Let R be a relation from A to B. The arrow $\forall_R : \wp A \longrightarrow \wp B$ corresponding to the relation $R/\in_A$ from B to $\wp A$ is called <u>universal quantification along</u> R. (For Set, $\forall_R(U) = \{b \in B \mid \text{forall } a \text{ with } aRb,\ a \in U\}$).

In particular, when R is $A \xleftarrow{1} A \xrightarrow{f} B$ we have $\forall_f$. (For Set, $\forall_f(U) = \{b \in B \mid \text{forall } a \text{ with } fa = b,\ a \in U\}$). When R is $A \xleftarrow{g} B \xrightarrow{1} B$ we write $\wp g$ for $\forall_R$. (For Set, $(\wp g)(U) = \{b \in B \mid g(b) \in U\}$ = inverse image of U under g). This gives a definition of $\wp$ on arrows so that $\wp$ becomes a functor $\wp : \mathcal{E}^{op} \longrightarrow \mathcal{E}$.

Existential quantification along a relation is a little harder to produce. We shall show that $\mathcal{E}^{op}$ is finitely complete first. This was first proved by Chris Mikkelsen from Aarhus, Denmark but the development we give is due to Robert Paré from Montréal.

For a monomorphism $X \xrightarrow{m} A$, the arrow $\exists_m : \wp X \longrightarrow \wp A$ corresponding to the relation $A \xleftarrow{m} X \longleftarrow \in_X \longrightarrow \wp X$ gives a special case of existential quantification.

<u>Lemma</u>. <u>If</u> $\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "h"'] & X \arrow[d, "m"] \\ C \arrow[r, "u"'] & A \end{tikzcd}$ <u>is a pullback and</u> m <u>is a monomorphism then the square</u>

$$\begin{tikzcd} \wp X \arrow[r, "\wp k"] \arrow[d, "\exists_h"'] & \wp P \arrow[d, "\exists_h"] \\ \wp A \arrow[r, "\wp u"'] & \wp C \end{tikzcd}$$

<u>commutes</u>.

<u>Proof</u>. One readily sees that both legs of the square correspond to the relation

$$C \xleftarrow{h} P \longleftarrow k^*/\in_X \longrightarrow \wp X$$

from C to $\wp X$. //

<u>Corollary</u>. <u>For any monomorphism</u> $X \xrightarrow{m} A$,

$$\begin{tikzcd} \wp X \arrow[r, "\exists_m"] \arrow[dr, "1"'] & \wp A \arrow[d, "\wp m"] \\ & \wp X \end{tikzcd}$$

<u>commutes</u>.

<u>Proof</u>. Take $u = m$, $h = k = 1_X$ in ~~the~~ lemma. //

<u>Lemma</u>. <u>Every monomorphism</u> $X \xrightarrow{m} A$ <u>is the equalizer of some pair of arrows.</u>

<u>Proof</u>. Let $f : A \to \wp 1$ correspond to the relation $X \xleftarrow{1} X \xrightarrow{m} A$. Then $X \xrightarrow{m} A$ is the equalizer of

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & \wp 1 \\ & 1 \arrow[ur, "\mathrm{sing}"'] & \end{tikzcd}$$

//

Corollary. *An arrow which is both a monomorphism and an epimorphism is an isomorphism.* (By epimorphism we mean monomorphism in $\mathcal{E}^{op}$.) //

We now recall a theorem of Jon Beck on Eilenberg-Moore algebras.

Given a functor $T : \mathcal{A} \to \mathcal{E}$, a *left adjunction* for T is a functor $S : \mathcal{E} \to \mathcal{A}$ and a natural transformation $\eta : 1 \to TS$ such that a bijection between arrows

$$(SX \xrightarrow{f} A) \longleftrightarrow (X \xrightarrow{g} TA)$$

is set up by commutativity of the diagram

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "\eta_X"'] & & TA \\
& TSX \arrow[ur, "Tf"'] &
\end{tikzcd}$$

Let $\varepsilon_A : STA \to A$ correspond to $TA \xrightarrow{1} TA$ under this bijection. Put $M = TS$, $\mu_X = T\varepsilon_{SX}$. This gives a functor $M : \mathcal{E} \to \mathcal{E}$ and natural transformations $\eta : 1 \to M$, $\mu : MM \to M$ satisfying the commutative diagrams

$$\begin{tikzcd}
MX \arrow[r, "M\eta_X"] \arrow[dr, "1"'] & MMX \arrow[d, "\mu_X"] & MX \arrow[l, "\eta_{MX}"'] \arrow[dl] \\
& MX &
\end{tikzcd}
\qquad
\begin{tikzcd}
MMMX \arrow[r, "M\mu_X"] \arrow[d, "\mu_{MX}"'] & MMX \arrow[d, "\mu_X"] \\
MMX \arrow[r, "\mu_X"'] & MX.
\end{tikzcd}$$

We call $\mathcal{E}, M, \mu, \eta$ a *monad*. An *algebra* for the monad is a pair (X, x) where X is an object of $\mathcal{E}$ and $MX \xrightarrow{x} X$ is an arrow satisfying commutative diagrams

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "1"'] & MX \arrow[d, "x"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} MMX \arrow[r, "\mu_X"] \arrow[d, "Mx"'] & MX \arrow[d, "x"] \\ MX \arrow[r, "x"'] & X. \end{tikzcd}$$

A <u>homomorphism</u> $f : (X, x) \longrightarrow (Y, y)$ is an arrow $f : X \longrightarrow Y$ of $\mathcal{E}$ such that $f.x = y.Mf$. The category of algebras and homomorphisms is denoted by $\mathcal{E}^M$. There is a functor $N : \mathcal{A} \to \mathcal{E}^M$ given by $NA = (TA, T\varepsilon_A)$, $Nk = Tk$ for $A \xrightarrow{k} B$ in $\mathcal{A}$. The functor $T : \mathcal{A} \to \mathcal{E}$ is said to be <u>algebraic</u> when it has a left adjunction and $N : \mathcal{A} \to \mathcal{E}^M$ is an equivalence of categories (that is, $LN \cong 1$, $NL \cong 1$ for some $L : \mathcal{E}^M \to \mathcal{A}$).

<u>Theorem</u>. <u>A functor $T : \mathcal{A} \to \mathcal{E}$ is algebraic if it satisfies the following conditions</u>:

(i) T <u>has a left adjunction</u>;

(ii) <u>$\mathcal{A}$ has coequalizers of reflexive pairs of arrows</u>;

(iii) T <u>preserves coequalizers of reflexive pairs of arrows</u>;

(iv) T <u>reflects isomorphisms</u>.

A pair of arrows $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in $\mathcal{A}$ is called <u>reflexive</u> when there is an arrow $B \xrightarrow{t} A$ such that $ft = gt = 1$. A <u>coequalizer</u> of f, g is $B \xrightarrow{h} C$ such that $C \xrightarrow{h} B$ is the equalizer of f, g in $\mathcal{A}^{op}$. By (iii) we mean that, if h is the coequalizer of f, g in $\mathcal{A}$ then Th is the coequalizer of Tf, Tg in $\mathcal{E}$. By (iv) we mean that, if $A \xrightarrow{f} B$ is an arrow of $\mathcal{A}$ such that $TA \xrightarrow{Tf} TB$ is an isomorphism in $\mathcal{E}$, then f is an isomorphism in $\mathcal{A}$.

The construction for the functor $L : \mathcal{E}^M \longrightarrow \mathcal{A}$ needed in the proof of the above theorem is given by taking the coequalizer $SX \longrightarrow L(X, x)$ of the pair of arrows $Sx, \varepsilon_{SX} : STSX \longrightarrow SX$ (which are reflexive with common right inverse $S\eta_X$).

Theorem. *If $T : \mathcal{A} \longrightarrow \mathcal{E}$ is algebraic then $\mathcal{A}$ possesses a terminal object and pullbacks provided $\mathcal{E}$ does.*

Proof. It suffices to prove that $\mathcal{E}^M$ has a terminal object and pullbacks which is easy. //

Theorem (Mikkelsen–Paré) *For any topos $\mathcal{E}$, the functor $\mathcal{P} : \mathcal{E}^{op} \longrightarrow \mathcal{E}$ is algebraic.*

<u>Proof.</u> (i) Relations from A to X are the same as relations from X to A. So there is a bijection between arrows $X \longrightarrow \mathcal{P}A$ and arrows $A \longrightarrow \mathcal{P}X$ in $\mathcal{E}$. So there is a bijection between arrows $X \longrightarrow \mathcal{P}A$ in $\mathcal{E}$ and arrows $\mathcal{P}X \longrightarrow A$ in $\mathcal{E}^{op}$ as required for a left adjunction for $\mathcal{P} : \mathcal{E}^{op} \longrightarrow \mathcal{E}$.

(ii) $\mathcal{E}^{op}$ has all coequalizers since $\mathcal{E}$ has equalizers.

(iii) Let $A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{h} C$ denote a coequalizer in $\mathcal{E}^{op}$ where f, g are a reflexive pair. In $\mathcal{E}$, $C \xrightarrow{h} B \overset{f}{\underset{g}{\rightrightarrows}} A$ is an equalizer and f, g have a common left inverse $A \xrightarrow{t} B$. It follows that f, g are monomorphisms and

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

is a pullback. So we have a commutative diagram

$$\begin{tikzcd} \mathcal{P}B \arrow[r, "\mathcal{P}h"] \arrow[d, "\exists_f"'] & \mathcal{P}C \arrow[d, "\exists_h"] \\ \mathcal{P}A \arrow[r, "\mathcal{P}g"'] & \mathcal{P}B. \end{tikzcd}$$

This condition together with $\mathcal{P}f . \exists_f = 1$, $\mathcal{P}h . \exists_h = 1$ readily yield that

$$\mathcal{P}A \overset{\mathcal{P}f}{\underset{\mathcal{P}g}{\rightrightarrows}} \mathcal{P}B \xrightarrow{\mathcal{P}h} \mathcal{P}C$$

exhibits $\mathcal{P}h$ as a coequalizer of $\mathcal{P}f, \mathcal{P}g$. So $\mathcal{P}$ preserves coequalizers of reflexive pairs.

(iv) We first prove that, given $B \underset{g}{\overset{f}{\rightrightarrows}} A$, if $\mathcal{P}f = \mathcal{P}g$ then $f = g$; that is, $\mathcal{P}$ is <u>faithful</u>. For, if $\mathcal{P}f = \mathcal{P}g$ then $(A \xrightarrow{\text{sing}} \mathcal{P}A \xrightarrow{\mathcal{P}f} \mathcal{P}B) = (A \xrightarrow{\text{sing}} \mathcal{P}A \xrightarrow{\mathcal{P}g} \mathcal{P}B)$. So the relations $B \xleftarrow{1} B \xrightarrow{f} A$, $B \xleftarrow{1} B \xrightarrow{g} A$ corresponding to each side are ~~equal~~ isomorphic. So $f = g$.

But faithful functors reflect monomorphisms and epimorphisms and hence isomorphisms in this case (last corollary). //

<u>Corollary</u>. <u>Any topos is finitely cocomplete.</u> //

Note in particular we have an <u>initial object</u> 0 (there is precisely one arrow $0 \longrightarrow A$ for any A). If we trace through the above proof we see that the way we form colimits is as follows. Suppose we wish to form the coequalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$. Apply $\mathcal{P}$ to obtain $\mathcal{P}B \underset{\mathcal{P}g}{\overset{\mathcal{P}f}{\rightrightarrows}} \mathcal{P}A$; take the equalizer $E \longrightarrow \mathcal{P}B$ of $\mathcal{P}f$, $\mathcal{P}g$ and observe that E carries an algebra structure; hence $E \cong \mathcal{P}C$ for some C; and then $B \longrightarrow C$ is the coequalizer of f, g.

Even in the case $\mathcal{E} = \mathcal{S}et$ this theorem is interesting. It is known that $\mathcal{S}et^{op} \xrightarrow{\mathcal{P}} \mathcal{S}et$ is algebraic and the algebras are also equivalent to the complete atomic boolean algebras. The more usual construction of coequalizers in the $\mathcal{S}et$ case is: take the equivalence relation generated by the relations $f(a) \sim g(a)$ for all $a \in A$ and let C be the set of equivalence classes; then $B \to C$ is the coequalizer of f, g. In many ways the former construction is preferable.

By a standard technique we can now construct <u>images</u> in a topos $\mathcal{E}$. Given an arrow $f : A \to B$, form the pushout (= pullback in $\mathcal{E}^{op}$)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "y"] \\ B \arrow[r, "x"'] & C \end{tikzcd}$$

and then the equalizer $f(A) \longrightarrow B \overset{x}{\underset{y}{\rightrightarrows}} C$. Of course $f(A) \to B$ is a monomorphism, but also the unique arrow $A \to f(A)$ such that

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \\ & f(A) \arrow[ur] & \end{tikzcd}$$

commutes is an epimorphism. Furthermore, $f(A) \to B$ is the smallest monomorphism into B through which f factors.

Composition of relations $A \xleftarrow{d_0} R \xrightarrow{d_1} B$, $B \xleftarrow{d_0} S \xrightarrow{d_1} C$ is now defined by first forming the pullback

$$\begin{tikzcd} P \arrow[r, "v"] \arrow[d, "u"'] & S \arrow[d, "d_0"] \\ R \arrow[r, "d_1"'] & B \end{tikzcd}$$

and then taking $\begin{pmatrix} d_0 \\ d_1 \end{pmatrix} : S \circ R \longrightarrow A \times C$ to be the image of $\begin{pmatrix} d_0 u \\ d_1 v \end{pmatrix} : P \longrightarrow A \times C$. This gives a relation $A \longleftarrow S \circ R \longrightarrow C$. In the next section we see that this composition is associative.

<u>Theorem</u>. $S \circ R \leq T$ <u>if and only if</u> $S \leq R/T$.

<u>Proof</u>. $S \leq R/T$ means
$$\begin{tikzcd} S \arrow[r] \arrow[d] & C \arrow[d, "t"] \\ B \arrow[r, "r"'] \arrow[ur, phantom, "\leq"] & 8A \end{tikzcd}$$
, which means

$(\text{reln corres } S \longrightarrow B \xrightarrow{r} 8A) \leq (\text{reln corres } S \longrightarrow C \xrightarrow{t} 8A)$, which means $(A \longleftarrow R \longleftarrow P \longrightarrow S) \leq (A \longleftarrow T \longleftarrow Q \longrightarrow S)$ where
$$\begin{tikzcd} Q \arrow[r] \arrow[d] & S \arrow[d] \\ T \arrow[r] & C \end{tikzcd}$$
is a pullback,

which means there is a comm. diag.
$$\begin{tikzcd} & & P \arrow[dl] \arrow[dd] \arrow[dr] & \\ & R \arrow[dl] & & S \\ A & & & \\ & T \arrow[ul] & Q \arrow[l] \arrow[uur] & \end{tikzcd}$$
,

which means there is a comm. diag.
$$\begin{tikzcd} & P \arrow[dl] \arrow[d] \arrow[dr] & \\ A & T \arrow[l] \arrow[r] & C \end{tikzcd}$$
by the pullback property of Q, which means there is a comm. diag.
$$\begin{tikzcd} S \circ R \arrow[r] \arrow[d] & A \times C \\ T \arrow[ur] & \end{tikzcd}$$
by the smallestness of image, which means $S \circ R \leq T$. //

For each relation R from A to B, the arrow $\exists_R : 8A \longrightarrow 8B$ is defined by the condition that it corresponds to the relation $\in_A \circ R^*$; it is called <u>existential quantification along</u> R. For Sets, $\exists_R(U) = \{ b \in B \mid \text{there exists } a \in U \text{ such that } (a,b) \in R \}$.

<u>Corollary</u>. $\exists_R \dashv \forall_{R^*}$. <u>This means</u>

$$\begin{tikzcd} \mathcal{P}A \arrow[rr, "1"] \arrow[dr, "\exists_R"'] & \arrow[d, phantom, "\leq"] & \mathcal{P}A \\ & \mathcal{P}B \arrow[ur, "\forall_{R^*}"'] & \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{P}A \arrow[dr, "\exists_R"] \arrow[d, phantom, "\leq"] & \\ \mathcal{P}B \arrow[ur, "\forall_{R^*}"] \arrow[rr, "1"'] & {} & \mathcal{P}B. \end{tikzcd}$$ //

There are arrows $\cap : \mathcal{P}\mathcal{P}A \to \mathcal{P}A$, $\cup : \mathcal{P}\mathcal{P}A \to \mathcal{P}A$ called <u>intersection</u> and <u>union</u> corresponding to the relations $(\in_{\mathcal{P}A} / \in_A^*)^*$, $\in_A \circ \in_A^*$. These arrows also have characterizing adjunction properties.

§3. <u>Internal completeness.</u>

From the notion of arrow we obtained a notion of relation. We already have an "object of relations" from A to B, namely, $\mathcal{P}(A\times B)$. We shall construct (following Kock-Mikkelsen) an "object of arrows" $[A,B]$ from A to B.

The relation $\in_{A\times B}$ from $A\times B$ to $\mathcal{P}(A\times B)$ can be regarded as a relation from B to $\mathcal{P}(A\times B)\times A$ and so corresponds to an arrow $\mathcal{P}(A\times B)\times A \to \mathcal{P}B$. Compose this with the arrow $\mathcal{P}B \to \mathcal{P}1$ corresponding to $1 \leftarrow B \xrightarrow{\text{sing}} \mathcal{P}B$ to yield an arrow $\mathcal{P}(A\times B)\times A \to \mathcal{P}1$. This corresponds to a relation from 1 to $\mathcal{P}(A\times B)\times A$, and so a relation from A to $\mathcal{P}(A\times B)$, and so an arrow $\mathcal{P}(A\times B) \to \mathcal{P}A$ (which in **Sets** takes $R \subset A\times B$ to $\{a\in A \mid$ there exists a unique $b\in B$ such that $(a,b)\in R\}$).

Now form the pullback:

$$\begin{tikzcd}
{[A,B]} \arrow[r] \arrow[d] & \wp(A\times B) \arrow[d] \\
1 \arrow[r, "\ulcorner A \urcorner"'] & \wp A
\end{tikzcd}$$

where $\ulcorner A \urcorner$ corresponds to the relation $A \xleftarrow{1} A \longrightarrow 1$.

<u>Theorem</u>. <u>There is an evaluation arrow</u>

$$\mathrm{eval} : [A,B]\times A \longrightarrow B$$

<u>which establishes a bijection</u>

$$C\times A \xrightarrow{f} B \quad \longleftrightarrow \quad C \xrightarrow{g} [A,B]$$

<u>via commutativity of the diagram</u>

$$\begin{tikzcd}
C\times A \arrow[r, "g\times 1"] \arrow[dr, "f"'] & {[A,B]\times A} \arrow[d, "\mathrm{eval}"] \\
& B
\end{tikzcd}$$

//

A finitely complete category with $[A,B]$ as in the theorem for all pairs of objects A, B is called <u>cartesian closed</u>.

Certain constructions on a topos yield new topoi. For any object B, let $\mathcal{E}_B$ denote the category of objects over B; that is, an object is an arrow $E \xrightarrow{p} B$ in $\mathcal{E}$ (which we sometimes abbreviate to E thinking of p as a structure map) and an arrow $f : (E \xrightarrow{p} B) \longrightarrow (F \xrightarrow{q} B)$ is an arrow $f : E \longrightarrow F$ such that

$$\begin{tikzcd}
E \arrow[rr, "f"] \arrow[dr, "p"'] & & F \arrow[dl, "q"] \\
& B &
\end{tikzcd}$$

commutes.

Fundamental theorem of topoi. If $\mathscr{E}$ is a topos and B is an object of $\mathscr{E}$ then $\mathscr{E}_B$ is a topos.

Proof. The constuction of the power object is the only point that requires thought. The power object $\mathcal{P}_B E$ of $E \xrightarrow{p} B$ is just $\in_E / p$. (In Set, $\mathcal{P}_B E = \{(U, b) \mid e \in U \text{ implies } pe = b\}$.) //

Corollary. $\mathscr{E}_B$ is cartesian closed. //

Given $A \xrightarrow{f} B$, we obtain a functor $f^* : \mathscr{E}_B \to \mathscr{E}_A$ by pulling back along f. Composition with f gives a functor $\Sigma_f : \mathscr{E}_A \to \mathscr{E}_B$. There are bijections

$$\left(\Sigma_f(X \xrightarrow{u} A) \dashrightarrow (E \xrightarrow{p} B)\right) \longleftrightarrow \begin{array}{ccc} X & \dashrightarrow & E \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[f]{} & B \end{array} \longleftrightarrow \begin{array}{ccc} X & \dashrightarrow & f^*E \\ & {\scriptstyle u}\searrow \quad \swarrow & \\ & A & \end{array}$$

so that we have $\Sigma_f \dashv f^*$; no topos theory needed for this. In a topos we further have:

Theorem. f^* has a right adjoint Π_f.

Proof. For $X \xrightarrow{u} A$ in $\mathscr{E}_A$, form the pullback in $\mathscr{E}_B$

$$\begin{array}{ccc} \Pi_f(X \xrightarrow{u} A) & \longrightarrow & [A \xrightarrow{f} B, X \xrightarrow{fu} B] \\ \downarrow & & \downarrow{\scriptstyle [1,u]} \\ (B \xrightarrow{1} B) & \longrightarrow & [A \xrightarrow{f} B, A \xrightarrow{f} B] \end{array}$$

where the bottom arrow corresponds to $\begin{array}{ccc} A & \xrightarrow{1} & A \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f} & \\ & B & \end{array}$.

We then have bijections

$$(E \xrightarrow{p} B) \dashrightarrow \Pi_f(X \xrightarrow{u} A) \quad \longleftrightarrow \quad \begin{tikzcd} (E \xrightarrow{p} B) \arrow[r, dashed] \arrow[d] & {[A \xrightarrow{f} B, X \xrightarrow{fu} B]} \arrow[d] \\ (B \xrightarrow{1} B) \arrow[r] & {[A \xrightarrow{f} B, A \xrightarrow{f} B]} \end{tikzcd}$$

$$\longleftrightarrow \quad \begin{tikzcd} (E \xrightarrow{p} B)\times(A \xrightarrow{f} B) \arrow[rr, dashed] \arrow[dr] & & (X \xrightarrow{fu} B) \arrow[dl, "u"] \\ & (A \xrightarrow{f} B) & \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} f^*E \arrow[rr, dashed] \arrow[dr] & & X \arrow[dl, "u"] \\ & A & \end{tikzcd}$$

since $(E \xrightarrow{p} B)\times(A \xrightarrow{f} B)$ in $\mathcal{E}_B$ is precisely $f^*E \to A$. //

Corollary. A pullback of an epimorphism along any arrow is an epimorphism.

Proof. Functors with right adjoints preserve epimorphisms. Pulling back along f has a right adjoint. If $C \xrightarrow{e} B$ is an epi. in $\mathcal{E}$ then
$$\begin{tikzcd} C \arrow[rr, "e"] \arrow[dr, "e"'] & & B \arrow[dl, "1"] \\ & B & \end{tikzcd}$$
is an epi. in $\mathcal{E}_B$. //

Corollary. Composition of relations is associative. //

Corollary. $\exists_{S\circ R} = \exists_S \exists_R$ and $\forall_{S\circ R} = \forall_S \forall_R$. //

Corollary (Chevalley condition). If
$$\begin{tikzcd} P \arrow[r, "v"] \arrow[d, "u"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$
is a pullback then the following commute

$$\begin{tikzcd} \mathcal{P}P \arrow[r, "\exists v"] & \mathcal{P}B \\ \mathcal{P}A \arrow[u, "\mathcal{P}u"] \arrow[r, "\exists f"'] & \mathcal{P}C \arrow[u, "\mathcal{P}g"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}P \arrow[d, "\forall_u"'] & \mathcal{P}B \arrow[l, "\mathcal{P}v"'] \arrow[d, "\forall_g"] \\ \mathcal{P}A & \mathcal{P}C \arrow[l, "\mathcal{P}f"] \end{tikzcd} .$$

Proof. One square is obtained from the other by adjunction. Recall that $\forall_{f^*} = \mathcal{P}f$ and regard P as a relation from B to A. Then

$$\mathcal{P}f \cdot \forall_g = \forall_{f^*}\forall_g = \forall_{f^*\circ g} = \forall_P = \forall_{u\circ v^*} = \forall_u \cdot \forall_{v^*}$$
$$= \forall_u \cdot \mathcal{P}v. //$$

The last theorem we claim demonstrates the internal completeness of a topos. Take the particular case of the arrow $B \longrightarrow 1$. We have then that the functor $\mathcal{E} \longrightarrow \mathcal{E}_B$ which takes X to $X \times B \xrightarrow{proj_2} B$ has both a left and right adjoint. In the case $\mathcal{E} = \mathcal{S}et$, an object $E \xrightarrow{p} B$ of $\mathcal{E}_B$ can be regarded as a B-indexed family of disjoint sets; namely, $(E_b)_{b \in B}$ where $E_b = \{e \in E \mid pe = b\}$. In these terms, the left adjoint $\Sigma : \mathcal{E}_B \longrightarrow \mathcal{E}$ takes (E_b) to $\sum_{b \in B} E_b$ (disjoint union) and the right adjoint takes (E_b) to $\prod_{b \in B} E_b$ (cartesian product).

With a general topos $\mathcal{E}$ we cannot talk about completeness unless we already have an existing set theory; then completeness amounts to the existence of all products indexed over the given sets. In a very useful sense, in a topos we already have, without the set theory, "products indexed over objects of $\mathcal{E}$ itself".

§4. Sheaves.

Let X denote a topological space. A <u>sheaf over</u> X is a topological space E and a local homeomorphism $E \xrightarrow{p} X$ (ie. there is a nhd of each $e \in E$ and a nhd of $pe \in X$ between which p induces a homeomorphism). For $U \subset X$ open, a <u>section</u> of the sheaf over U is a continuous function $s : U \longrightarrow E$ such that $ps : U \longrightarrow X$ is the inclusion.

Let $\Gamma_E U$ denote the set of sections over $U \subset X$. For $U \subset V$ open in X, we have a function

$$\Gamma_E(U \subset V) : \Gamma_E V \longrightarrow \Gamma_E U$$

given by taking each section over V to its restriction to U. Let $\mathbb{T}$ denote the category whose objects are the open sets of X and whose arrows are the inclusions. Then $\Gamma_E : \mathbb{T}^{op} \to \mathcal{S}et$ is a functor.

It is readily proved that the following conditions on a functor $P : \mathbb{T}^{op} \to \mathcal{S}et$ are satisfied when $P = \Gamma_E$:

(a) given any open cover (U_λ) of open $U \subset X$, if $s, t \in P(U)$ are such that $P(U_\lambda \subset U)(s) = P(U_\lambda \subset U)(t)$ for each λ, then $s = t$;

(b) given any open cover (U_λ) of open $U \subset X$ and, for each λ, an element $s_\lambda \in P(U_\lambda)$ such that $P(U_\lambda \cap U_\mu \subset U_\lambda)(s_\lambda) = P(U_\lambda \cap U_\mu \subset U_\mu)(s_\mu)$ for all λ, μ, then there exists $s \in P(U)$ such that $P(U_\lambda \subset U)(s) = s_\lambda$ for all λ.

Furthermore, $E \longmapsto \Gamma_E$ is actually an equivalence of categories between the category of sheaves over X and the full subcategory of $[\mathbb{T}^{op}, \mathcal{S}et]$ consisting of those functors P satisfying (a) and (b).

Conditions (a) and (b) have a nice categorical expression. An open cover (U_λ) of U determines a diagram

$$\begin{tikzcd}[row sep=small] & U_\lambda \\ U_\lambda \cap U_\mu \arrow[ur] \arrow[dr] & \\ & U_\mu \end{tikzcd} \qquad \text{all } \lambda, \mu$$

in $\mathbb{T}$. In fact, U is the colimit of this diagram in $\mathbb{T}$. So U is the limit of the diagram in $\mathbb{T}^{op}$. Conditions (a) and (b) together amount to the requirement that P should preserve this limit: that is, $P(U)$ should be the limit of the diagram as indicated

$$\begin{tikzcd}[row sep=small] & P(U_\lambda) \arrow[dl] & \\ P(U_\lambda \cap U_\mu) & & P(U) \arrow[ul, dashed] \arrow[ll, dashed] \arrow[dl, dashed] \\ & P(U_\mu) \arrow[ul] & \end{tikzcd}$$

Consequently, we do not hesitate to call a functor P with this limit preservation property a sheaf over X.

We have already algebraicized our topological setting to some extent: categories of functors preserving certain limits are generalizations of categories of algebraic theories (Gabriel theories). For example, a group can be regarded as a coproduct preserving functor from a certain category (called "the theory of groups") to Set.

This leads us to consider functor categories $[\mathbb{T}^{op}, Set]$. What I am about to say could also be done replacing Set by any topos $\mathcal{E}$ and taking $\mathbb{T}$ to be a "category object" in $\mathcal{E}$; but for lack of time we stick to the "more concrete" case.

<u>Theorem</u>. *For any small category $\mathbb{T}$, the functor category $[\mathbb{T}^{op}, Set]$ is a topos.*

<u>Proof outline</u>. For $P, Q : \mathbb{T}^{op} \to Set$, the functor $[P,Q] : \mathbb{T}^{op} \to Set$ is given by

$$[P,Q](U) = \text{the set of natural transfns } \mathbb{T}(-,U) \times P- \to Q-.$$

For U in $\mathbb{T}$, a U-<u>crible</u> is a set C of arrows in $\mathbb{T}$ with target U such that:

$$W \xrightarrow{w} U \text{ in } C \text{ implies } V \xrightarrow{wu} U \text{ in } C \text{ for all } V \xrightarrow{v} W \text{ in } \mathbb{T}.$$

Let $\Omega : \mathbb{T}^{op} \to Set$ be the functor given by

$$\Omega(U) = \text{set of } U\text{-cribles}.$$

Then $\wp P = [P, \Omega]$ is the power object of P. //

<u>Corollary</u>. *The category of permutation representations of a group is a topos.* //

In any topos $\mathcal{E}$ we call the object $\Omega = \wp 1$ the <u>subobject classifier</u>. A monomorphism $X \xrightarrow{m} A$ is the same as a relation $1 \leftarrow X \xrightarrow{m} A$ and so corresponds to an arrow $\text{char}\, m : A \to \Omega$ related by the pullback

$$\begin{array}{ccc} X & \xrightarrow{m} & A \\ \downarrow & & \downarrow{\scriptstyle \text{char}\, m} \\ 1 & \xrightarrow[\text{sing}]{} & \Omega \end{array}.$$

A <u>topology</u> on $\mathcal{E}$ is an arrow $j : \Omega \to \Omega$ such that the following diagrams commute

$$
\begin{tikzcd}
\Omega \arrow[r, "j"] \arrow[dr, "j"'] & \Omega \arrow[d, "j"] \\
& \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "\mathrm{sing}"] \arrow[dr, "\mathrm{sing}"'] & \Omega \arrow[d, "j"] \\
& \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega\times\Omega \arrow[r, "\wedge"] \arrow[d, "j"'] & \Omega \arrow[d, "j"] \\
\Omega\times\Omega \arrow[r, "\wedge"'] & \Omega
\end{tikzcd}
.
$$

Then we define the closure $\overline{(X \xrightarrow{m} A)}$ of the monomorphism $X \xrightarrow{m} A$ is defined to be the monomorphism into A corresponding to the arrow $A \xrightarrow{\mathrm{char}\,m} \Omega \xrightarrow{j} \Omega$ from A to Ω. This gives ~~an~~ obvious definitions of closed and dense. Say $X \xrightarrow{m} A$ is <u>closed</u> when $\overline{(X \xrightarrow{m} A)} \cong (X \xrightarrow{m} A)$ and <u>dense</u> when $\overline{(X \xrightarrow{m} A)} \cong (A \xrightarrow{1} A)$. An object A of $\mathcal{E}$ is said to be a j-<u>sheaf</u> when, given any dense monomorphism $Y \xrightarrow{n} B$ and any arrow $Y \xrightarrow{f} A$, there exists a unique arrow $B \xrightarrow{g} A$ such that $f = gn$.

<u>Theorem (Lawvere-Tierney)</u> <u>Suppose $j : \Omega \to \Omega$ is a topology on a topos $\mathcal{E}$. The full subcategory Sh_j of $\mathcal{E}$ consisting of the j-sheaves is a topos. The inclusion $Sh_j \longrightarrow \mathcal{E}$ has a left adjoint which preserves finite limits.</u> //

Let us return to the topos $\mathcal{E} = [\mathbb{T}^{op}, Set]$ where $\mathbb{T}$ is the category of open sets of a topological space X. In this case, a U-crible $\underline{C}$ is a subset of the open subsets of U such that $W \subset V$, $V \in \underline{C}$ implies $W \in \underline{C}$. It can be shown that $j : \Omega \to \Omega$ given by $j_U : \Omega(U) \to \Omega(U)$ takes $\underline{C}$ to $\{V \in \mathbb{T} \mid V \subset \bigcup_{W \in \underline{C}} W\}$, is a topology on $\mathcal{E}$.

Theorem. If $\mathbb{T}$ is the category of open sets of a space X, $\mathcal{E} = [\mathbb{T}^{op}, Set]$ and $j : \Omega \to \Omega$ is as given above, then $P : \mathbb{T}^{op} \to Set$ is a j-sheaf if and only if it satisfies (a) and (b). //

Corollary. The category $Sh(X)$ of sheaves on a topological space X is a topos and the section functor

$$Sh(X) \xrightarrow{\Gamma} [\mathbb{T}^{op}, Set]$$

has a finite limit preserving left adjoint. //

Given topoi $\mathcal{E}, \mathcal{E}'$, a functor $G : \mathcal{E} \to \mathcal{E}'$ is called geometric when it has a finite limit preserving left adjoint. A continuous function $f : X \to Y$ between topological spaces induces a geometric functor $Sh(f) : Sh(X) \to Sh(Y)$.

$$\begin{tikzcd} Sh(X) \arrow[r, "Sh(f)"] \arrow[d, phantom, "\cap"] & Sh(Y) \arrow[d, phantom, "\cap"] \\ {[\mathbb{T}_X^{op}, Set]} \arrow[r, "{[f^{-1}, 1]}"'] & {[\mathbb{T}_Y^{op}, Set]} \end{tikzcd}.$$

Thus we have a functor $Top\,Spaces \xrightarrow{Sh} Topoi$ via which topoi can be regarded as generalized topological spaces. A "point" is a geometric functor $Set \to Sh(X)$.

§5. Relation to logical theories.

This will be discussed in the lecture by Bob Walters at La Trobe University.

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