Monads and monoidal structures

Ross Street

Macquarie University

Fourth Morgan-Phoa Mathematics Workshop, ANU Canberra

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- In that language, the constructions we have in mind take a monoidal bicategory $\mathscr{M}$ to the 2-category $\mathrm{Mod}(\mathbf{T}, \mathscr{M})$ of models of $\mathbf{T}$ in $\mathscr{M}$.
- However, the twist is that, rather than $\mathrm{Mod}(\mathbf{T}, \mathscr{M})$ itself, we are interested in the two types of lax morphisms between models.

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- A new insight was provided recently by Marek Zawadowski [*The formal theory of monoidal monads*, JPAA 2012].
- He raises a question about a more general context for his work. This talk gives one answer.

Monads

- The remarkable fact about algebraically constructing a category $\mathscr{A}$ from a nice category $\mathscr{X}$ is that the forgetful functor $\mathrm{U} : \mathscr{A} \longrightarrow \mathscr{X}$ has a left adjoint $\mathrm{F} \dashv \mathrm{U}$ and the category $\mathscr{A}$ can be reconstructed as the category $\mathscr{X}^{\mathrm{T}}$ of Eilenberg-Moore algebras for the monad $\mathrm{T} = \mathrm{UF}$ on $\mathscr{X}$.

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- The Kleisli category $\mathscr{X}_{\mathrm{T}}$ is equivalent to the full subcategory of $\mathscr{X}^{\mathrm{T}}$ consisting of the free algebras $\mathrm{T}X$; the adjunction $\mathrm{F} \dashv \mathrm{U}$ restricts to an adjunction $\mathrm{F}_{\mathrm{T}} \dashv \mathrm{U}_{\mathrm{T}} : \mathscr{X}_{\mathrm{T}} \longrightarrow \mathscr{X}$ which generates the same monad $\mathrm{T} = \mathrm{U}_{\mathrm{T}}\mathrm{F}_{\mathrm{T}}$.

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- From the point of view of the 2-category (or bicategory) Cat of categories, the constructions taking $(\mathscr{X}, \mathrm{T})$ to $\mathscr{X}^{\mathrm{T}}$ and $\mathscr{X}_{\mathrm{T}}$ are dual: the first is a limit and the second a colimit.

First motivating example

- Write Abgp for the category of abelian groups. There is an adjunction $\mathrm{F} \dashv \mathrm{U} : \mathrm{Abgp} \longrightarrow \mathrm{Set}$ generating a monad $\mathrm{T} = \mathrm{UF}$ on Set.

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- The tensor product of abelian groups determines a monoidal structure on $\mathrm{Set}^{\mathrm{T}}$. This construction involves coequalizers of morphisms between free T-algebras. Not so easy!

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- The coalgebra structure of H allows the tensor product of k-modules to be lifted to H-modules giving a monoidal structure on Mod^H. An **easy** construction! The right adjoint $\mathrm{Mod}^H \longrightarrow \mathrm{Mod}_k$ preserves the monoidal structure.

Comparison of examples

- The monad T in the first example is **monoidal**: we have a coherent natural family of morphisms $\mathrm{T}X \times \mathrm{T}Y \longrightarrow \mathrm{T}(X \times Y)$ and a compatible distinguished morphism $1 \longrightarrow \mathrm{T}1$. Also the unit and multiplication are monoidal: that is, they respect the monoidal structure.

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- The monad $\mathrm{T} = H \otimes -$ on Mod_k in the second example is **opmonoidal**: we have a coherent natural family of morphisms $\mathrm{T}(M \otimes N) \longrightarrow \mathrm{T}M \otimes \mathrm{T}N$ and a compatible distinguished morphism $\mathrm{T}k \longrightarrow k$. Also the unit and multiplication are opmonoidal: that is, they respect the opmonoidal structure.

Monads in 2-categories

- A *monad* (A, s) in a 2-category $\mathscr{K}$ consists of an object A, a morphism $s : A \longrightarrow A$, and 2-cells $\eta : 1_A \Longrightarrow s$ and $\mu : ss \Longrightarrow s$ making s a monoid in the strict monoidal category $\mathscr{K}(A, A)$ whose tensor product is horizontal composition in $\mathscr{K}$.

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- In particular, $(B, 1_B)$ is a monad for any object B. A monad morphism $(a, \alpha) : (B, 1) \longrightarrow (A, s)$ is called a *(generalized) s-algebra*. (When $\mathscr{K} = \mathrm{Cat}$ and $B = 1$, (a, α) is an Eilenberg-Moore s-algebra in the category A.)

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- Write $\mathrm{Mnd}\mathscr{K}$ for the 2-category of monads, monad morphisms, and obvious 2-cells, all in $\mathscr{K}$.

Eilenberg-Moore objects

- The *Eilenberg-Moore object* A^s for a monad (A, s) in $\mathscr{K}$ is a universal s-algebra $(u, \upsilon) : (A^s, 1) \longrightarrow (A, s)$.
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- Put $\mathrm{Mnd}^{\mathrm{op}}\mathscr{K} = (\mathrm{Mnd}\mathscr{K}^{\mathrm{op}})^{\mathrm{op}}$.
  Then the Kleisli objects are the values of a left adjoint to the 2-functor

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Local cosimplicial objects

- Each monad (A, s) determines a coaugmented cosimplicial object

$$1_A \xrightarrow{\eta} s \underset{\eta s}{\overset{s\eta}{\underset{\xleftarrow{\mu}}{\longrightarrow}}} s^2 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; s^3 \;\ldots$$

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- Then monads (A, s) are in bijection with 2-functors

$$\Sigma\Delta \longrightarrow \mathscr{K} \; .$$

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- For example, a monoidale in Cat (where tensor is cartesian product) is precisely a monoidal category.
- A *monoidal morphism* $f : A \longrightarrow B$ between monoidales A and B is a morphism $f : A \longrightarrow B$ in $\mathscr{M}$ equipped with 2-cells $\phi : p(f \otimes f) \Longrightarrow fp$ and $\phi_0 : j \Longrightarrow fj$ satisfying three axioms. We call f *strong* when ϕ and ϕ_0 are invertible.

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- There is also the 2-category $\mathrm{Mon}^{\mathrm{co}}\mathscr{M} = (\mathrm{Mon}\mathscr{M}^{\mathrm{co}})^{\mathrm{co}}$ of monoidales and opmonoidal morphisms in $\mathscr{M}$.

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- The constructions in each case are of the **easy** type.
- Marek Zawadowski [JPAA (2012)] found an underlying principle for the easy construction to work:

Theorem

If $\mathscr{M}$ is a 2-category with finite products then there are natural isomorphisms

$$\mathrm{Mnd}^{\mathrm{op}}\mathrm{Mon}\mathscr{M} \cong \mathrm{Mon}\mathrm{Mnd}^{\mathrm{op}}\mathscr{M} \ ,$$

$$\mathrm{Mnd}\mathrm{Mon}^{\mathrm{co}}\mathscr{M} \cong \mathrm{Mon}^{\mathrm{co}}\mathrm{Mnd}\mathscr{M} \ ,$$

compatible with the canonical 2-functors.

The lax-Gray closed monoidal structure

- 2-Cat denotes the category of 2-categories and 2-functors as a cartesian closed category; the internal hom is the 2-category $[\mathscr{A}, \mathscr{B}]$ of 2-functors $\mathscr{A} \longrightarrow \mathscr{B}$, 2-natural transformations, and modifications.

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- Gray_{lax} is the same category $2\text{-}\mathrm{Cat}$ but now equipped with the *lax Gray monoidal structure*; see John Gray's book [SLNM **391** (1974)] and paper [*Coherence for the tensor product of 2-categories, and braid groups* (1976)].

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- Here $\mathrm{Fun}_\ell(\mathscr{B}, \mathscr{C})$ is the 2-category of 2-functors $\mathscr{B} \longrightarrow \mathscr{C}$, oplax natural transformations, and modifications; $\mathrm{Fun}_r(\mathscr{A}, \mathscr{C})$ is the 2-category of 2-functors $\mathscr{A} \longrightarrow \mathscr{C}$, lax natural transformations, and modifications; and $\mathscr{A} \square \mathscr{B}$ is the lax-Gray tensor product.

Generalized symmetries and the main commutativity

- We have natural isomorphisms

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  in $2\text{-}\mathrm{Cat}$. Call this our **main commutativity**.
- Gray_{ps} denotes the category $2\text{-}\mathrm{Cat}$ with the pseudo-Gray tensor product $\mathscr{A} \boxtimes \mathscr{B}$. This structure is symmetric closed monoidal. A *Gray monoid* is a monoid $\mathscr{M}$ in Gray_{ps}.

Models and lax morphisms

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- We would like these last two 2-categories to be Gray monoids so that we might restrict the main commutativity by replacing Fun by Mor throughout. Of course this is absurd since monoid morphisms between two given monoids do not form a monoid unless the codomain is commutative.

Pointwise tensor

- Suppose $\mathscr{N}$ is a braided Gray monoid in the sense of [DaySt1997]. We attempt to define the tensor product in $\mathrm{Mor}_\ell(\mathscr{M},\mathscr{N})$ pointwise by $(S \otimes T)M = SM \otimes TM$, however, this is only a pseudofunctor in M. When it comes to preservation of tensor, we must use the braiding and consequently end up with preservation up to equivalence.

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- This suggests we look at the Gray monoid $\mathrm{SMPs}_\ell(\mathscr{M},\mathscr{N})$ of strong monoidal pseudofunctors $\mathscr{M} \longrightarrow \mathscr{N}$, monoidal opnatural transformations, and modifications; the tensor product is pointwise.
- Similarly, let $\mathrm{SMPs}_r(\mathscr{M},\mathscr{N})$ be the Gray monoid of strong monoidal pseudofunctors $\mathscr{M} \longrightarrow \mathscr{N}$, monoidal natural transformations, and modifications; the tensor product is pointwise.

Embeddings

There are fully faithful inclusion 2-functors

$$\mathrm{Mor}_\ell(\mathscr{M},\mathscr{N}) \longrightarrow \mathrm{SMPs}_\ell(\mathscr{M},\mathscr{N})$$

and

$$\mathrm{Mor}_r(\mathscr{M},\mathscr{N}) \longrightarrow \mathrm{SMPs}_r(\mathscr{M},\mathscr{N})$$

which allow us to define

$$\mathrm{Mor}_r(\mathscr{L},\mathrm{Mor}_\ell(\mathscr{M},\mathscr{N}))$$

and

$$\mathrm{Mor}_\ell(\mathscr{M},\mathrm{Mor}_r(\mathscr{L},\mathscr{N}))$$

as follows.

The way around the problem

Suppose the 2-category $\mathscr{F}$ is supplied with a fully faithful 2-functor K into a Gray monoid $\mathscr{X}$. Define the 2-categories $\mathrm{Mon}_\ell(\mathscr{L}, \mathscr{F})$ to be the following pullback of 2-categories.

$$\begin{array}{ccc}
\mathrm{Mor}_\ell(\mathscr{L}, \mathscr{F}) & \longrightarrow & \mathrm{Fun}_\ell(\mathscr{L}, \mathscr{F}) \\
\big\downarrow & & \big\downarrow {\scriptstyle \mathrm{Fun}_\ell(1_{\mathscr{L}}, K)} \\
\mathrm{Mor}_\ell(\mathscr{L}, \mathscr{X}) & \xrightarrow[\text{forget}]{} & \mathrm{Fun}_\ell(\mathscr{L}, \mathscr{X})
\end{array}$$

Similarly define $\mathrm{Mor}_r(\mathscr{L}, \mathscr{F})$ as a pullback with the ℓ subscripts replaced by r subscripts.

Main result

Proposition

For Gray monoids $\mathscr{L}$ and $\mathscr{M}$, and a braided Gray monoid $\mathscr{N}$, restriction of the 'main commutativity' yields a natural isomorphism of 2-categories

$$\Theta : \mathrm{Mor}_r(\mathscr{L}, \mathrm{Mor}_\ell(\mathscr{M}, \mathscr{N})) \cong \mathrm{Mor}_\ell(\mathscr{M}, \mathrm{Mor}_r(\mathscr{L}, \mathscr{N})) \ .$$

Moreover, the following triangle commutes, expressing compatibility with evaluation ev_M at any object M of $\mathscr{M}$.

$$\mathrm{Mon}_r(\mathscr{L}, \mathrm{Mon}_\ell(\mathscr{M}, \mathscr{N})) \xrightarrow{\quad \Theta \quad} \mathrm{Mon}_\ell(\mathscr{M}, \mathrm{Mon}_r(\mathscr{L}, \mathscr{N}$$

$$\mathrm{Mon}_r(\mathscr{L}, \mathrm{Mon}_\ell(\mathscr{M}, \mathscr{N})) \xrightarrow{\mathrm{Mon}_r(1_{\mathscr{L}}, \mathrm{ev}_M)} \mathrm{Mon}_r(\mathscr{L}, \mathscr{N}) \xleftarrow{\mathrm{ev}_M} \mathrm{Mon}_\ell(\mathscr{M}, \mathrm{Mon}_r(\mathscr{L}, \mathscr{N}$$

The monad example

- Let $\mathscr{F}\Sigma\Delta$ denote the free Gray monoid on the 2-category $\Sigma\Delta$ so that there is a natural bijection between Gray monoid morphisms $\mathscr{F}\Sigma\Delta \longrightarrow \mathscr{N}$ into the Gray monoid $\mathscr{N}$ and monads in the underlying 2-category of $\mathscr{N}$. Moreover, we obtain isomorphisms of 2-categories

$$\mathrm{Mor}_r(\mathscr{F}\Sigma\Delta, \mathscr{N}) \cong \mathrm{Fun}_r(\Sigma\Delta, \mathscr{N}) \cong \mathrm{Mnd}(\mathscr{N}),$$

where $\mathrm{Mnd}(\mathscr{N})$ is the 2-category of monads and monad morphisms in the underlying 2-category of $\mathscr{N}$.

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  where $\mathrm{Mnd}(\mathscr{N})$ is the 2-category of monads and monad morphisms in the underlying 2-category of $\mathscr{N}$.
- We also have

$$\mathrm{Mor}_\ell(\mathscr{F}\Sigma\Delta^{\mathrm{op}}, \mathscr{N}) \cong \mathrm{Mnd}^{\mathrm{op}}(\mathscr{N}).$$

The monoidale example

- Steve Lack [*A coherent approach to pseudomonads* (2000)] constructed a Gray monoid Δ' such that monoid morphisms $\Delta' \longrightarrow \mathscr{M}$ are in natural bijection with monoidales in the Gray monoid $\mathscr{M}$. Indeed, we have

$$\mathrm{Mor}_r(\Delta', \mathscr{N}) \cong \mathrm{Mon}(\mathscr{N}),$$

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- As Δ' is locally an equivalence relation (that is, locally posetal and locally groupoidal) we have $\Delta'^{\,\mathrm{co}} = \Delta'$. So

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- Zawadowski's commutation result is an example of our main result with $\mathscr{L}$ and $\mathscr{M}$ taken to be Δ' and $\mathscr{F}\Sigma\Delta$ irrespectively.