LOW–DIMENSIONAL TOPOLOGY
AND
HIGHER–ORDER CATEGORIES

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Authors of some contributing works:

Roger Penrose (diagrammatic tensor calculus)

G. Max Kelly and M.L. Laplaza (combinatorial string diagrams)
Peter Freyd and David Yetter (category of tangles versus duality)
N.Yu. Reshetikhin and V.G. Turaev (invariants via categories)
Mei Chee Shum (Macquarie PhD thesis on tangles of ribbons)

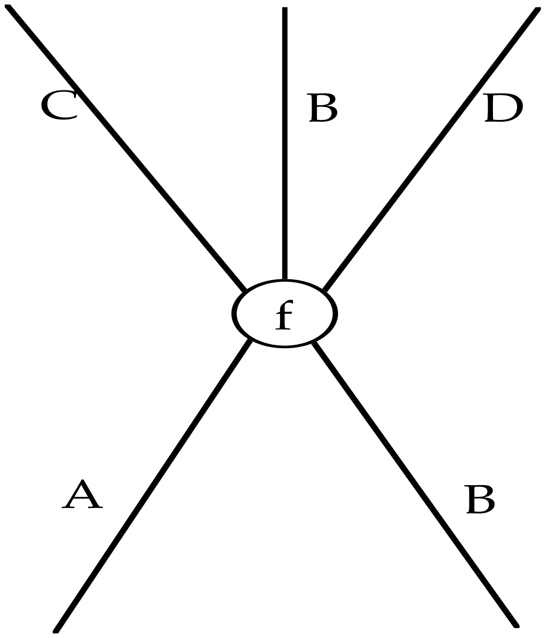
J.S. Carter and M. Saito (movies)

J. E. Fischer Jr (the monoidal bicategory of 2-tangles)

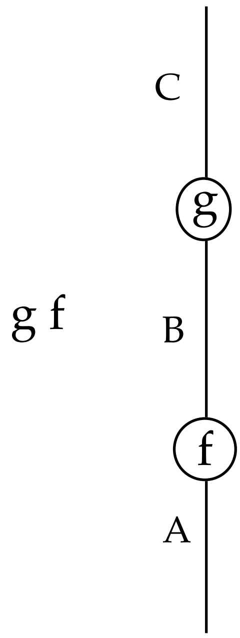
Good textbook for background applications:

Christian Kassel *Quantum Groups* **Grad Texts in Math 155 (Springer-Verlag, 1995)**

An arrow $f : A \otimes B \longrightarrow C \otimes B \otimes D$ in a monoidal category is depicted as follows.



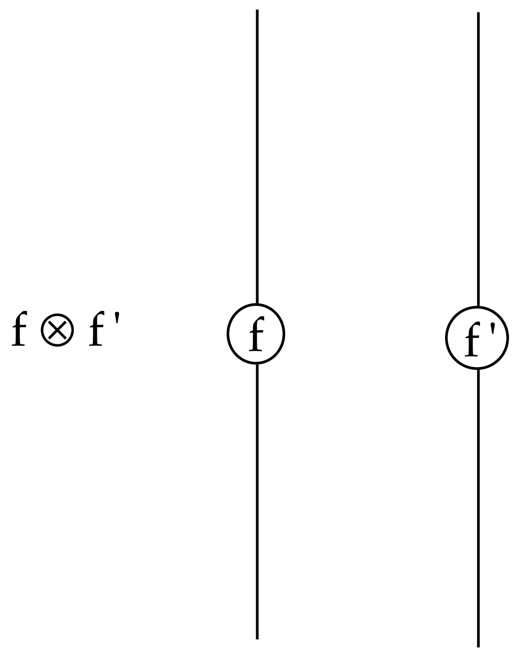
Composition $g\,f : A \longrightarrow C$ of arrows $f : A \longrightarrow B$, $g : B \longrightarrow C$ is performed vertically up the plane (electronics term: *in series*):



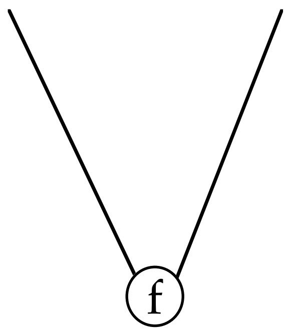
Tensoring $f : A \longrightarrow B$, $f' : A' \longrightarrow B'$ to get

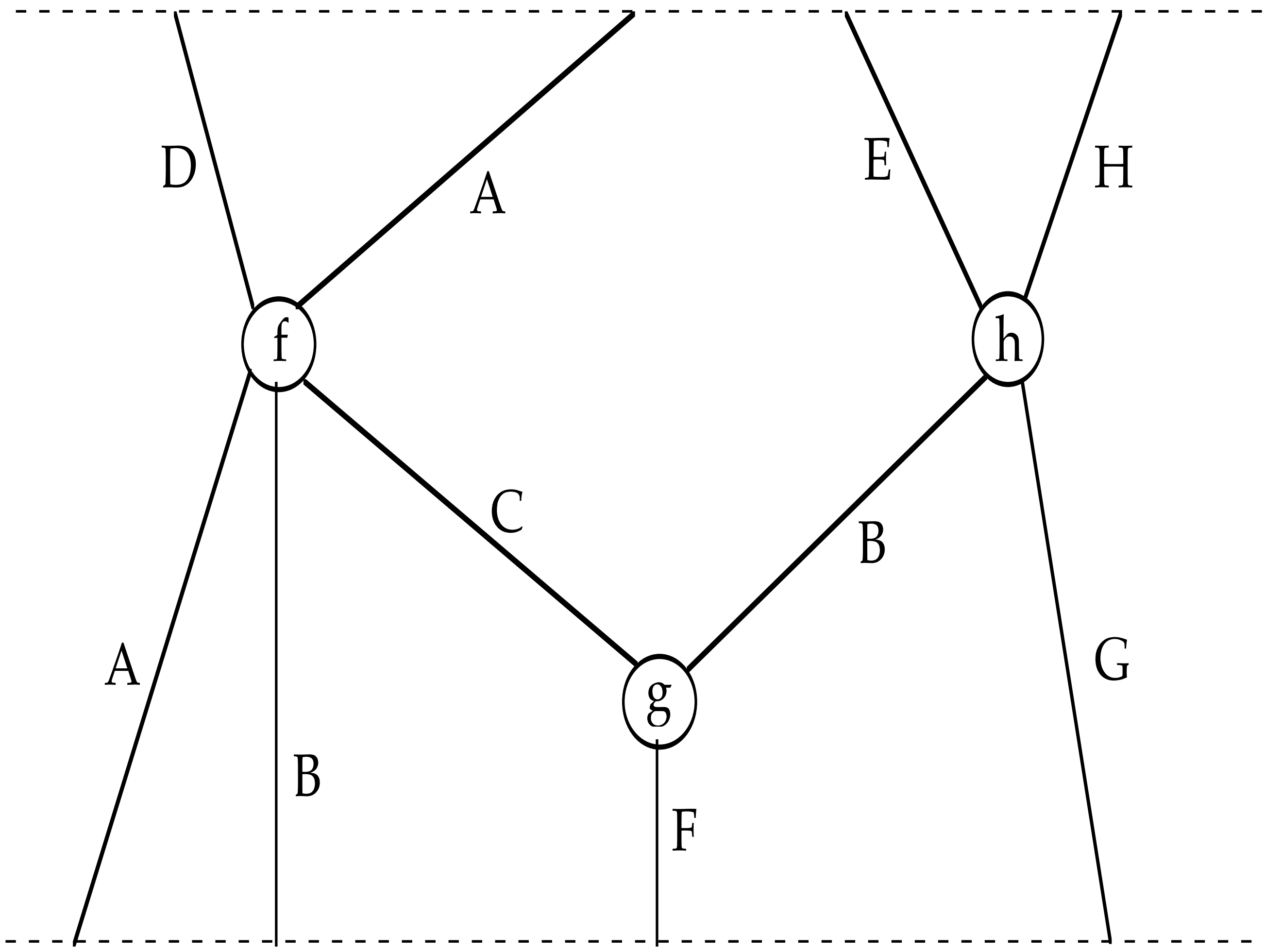
$$f \otimes f' : A \otimes A' \longrightarrow B \otimes B'$$

is depicted horizontally from left to right (electronics term: *in parallel*):



The unit for the tensor product is denoted by I . An arrow $f : I \longrightarrow A \otimes B$ would be depicted by:





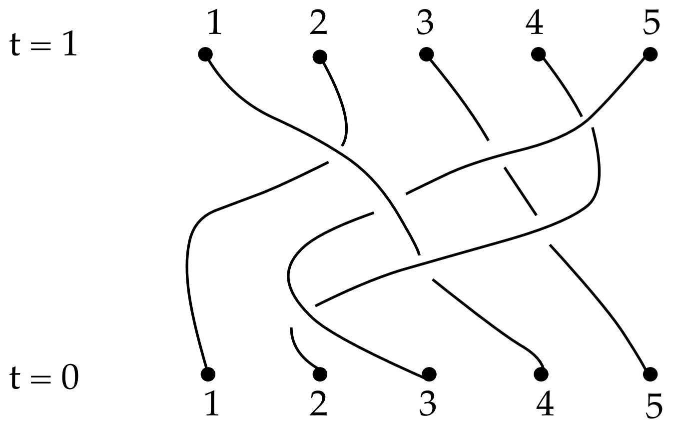
The *value* of the above diagram is a certain arrow

$$\mathbf{A \otimes B \otimes F \otimes G \longrightarrow D \otimes A \otimes E \otimes H.}$$

Theorem (Joyal-Street) *The value of a progressive plane string diagram in a monoidal category is deformation invariant.*

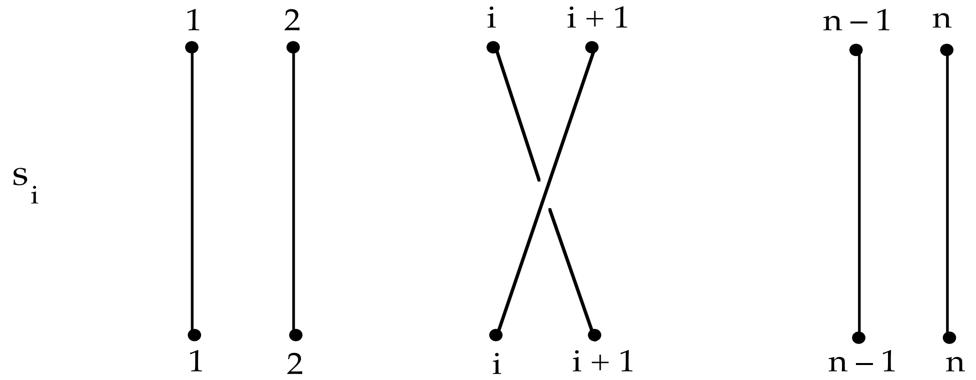
Example of a monoidal category

Let $\mathbf{B}_n$ be the Artin n string braid group. Here is an element of $\mathbf{B}_5$.



A presentation for $\mathbf{B}_n$ is given by the generators $s_1, \ldots, s_{n-1}$ and the relations

(A1) $s_i\, s_{i+1}\, s_i = s_{i+1}\, s_i\, s_{i+1}$ for $1 \le i \le n\text{-}2$,
(A2) $s_i\, s_j = s_j\, s_i$ for $1 \le i < j\text{-}1 \le n\text{-}2$.



The *braid category* **B** is the disjoint union of the $\mathbf{B}_n$. More explicitly, the objects of **B** are the natural numbers 0, 1, 2, . . . , the homsets are given by

$$\mathbf{B}(m, n) = \begin{cases} \mathbf{B}_n & \text{when } m = n \\ \varnothing & \text{otherwise} \end{cases} ,$$

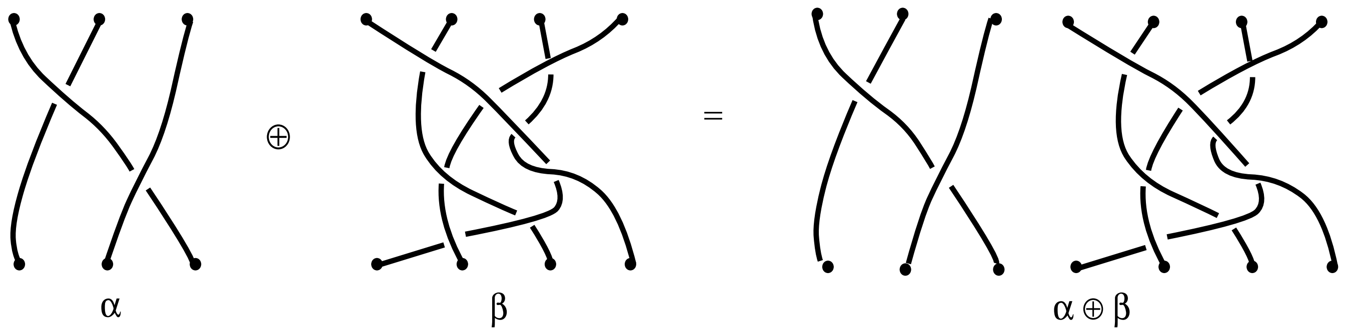
and composition is the multiplication of the braid groups.

The category **B** is equipped with a strictly associative tensor structure defined by *addition of braids*

$$\oplus : \mathbf{B}_m \times \mathbf{B}_n \longrightarrow \mathbf{B}_{m+n}$$

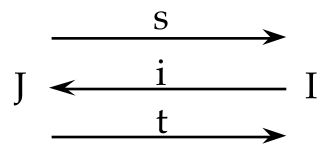
which is algebraically described by

$$s_i \oplus s_j = s_i \, s_{m+j} \, .$$



Model category for cubical set

A *cointerval* in a monoidal category $\mathcal{V}$ is a diagram

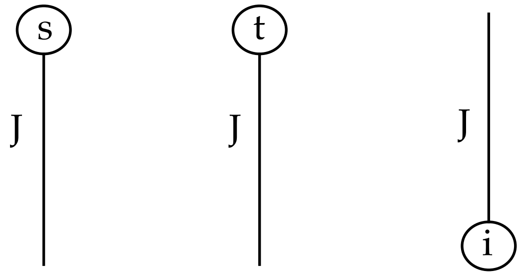


$s\,i = 1_I = t\,i$

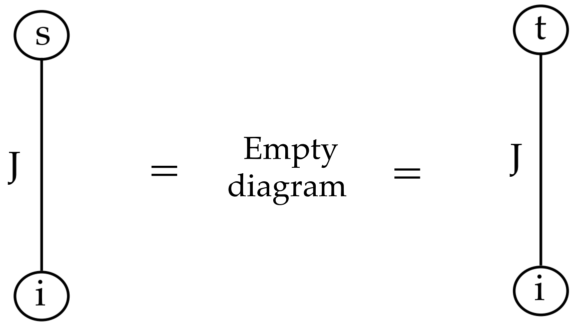
where I is the unit for the tensor product.

Can we find a model for the free monoidal category containing a generic cointerval?

This will be a monoidal category generated by a single object J and three arrows depicted diagrammatically by

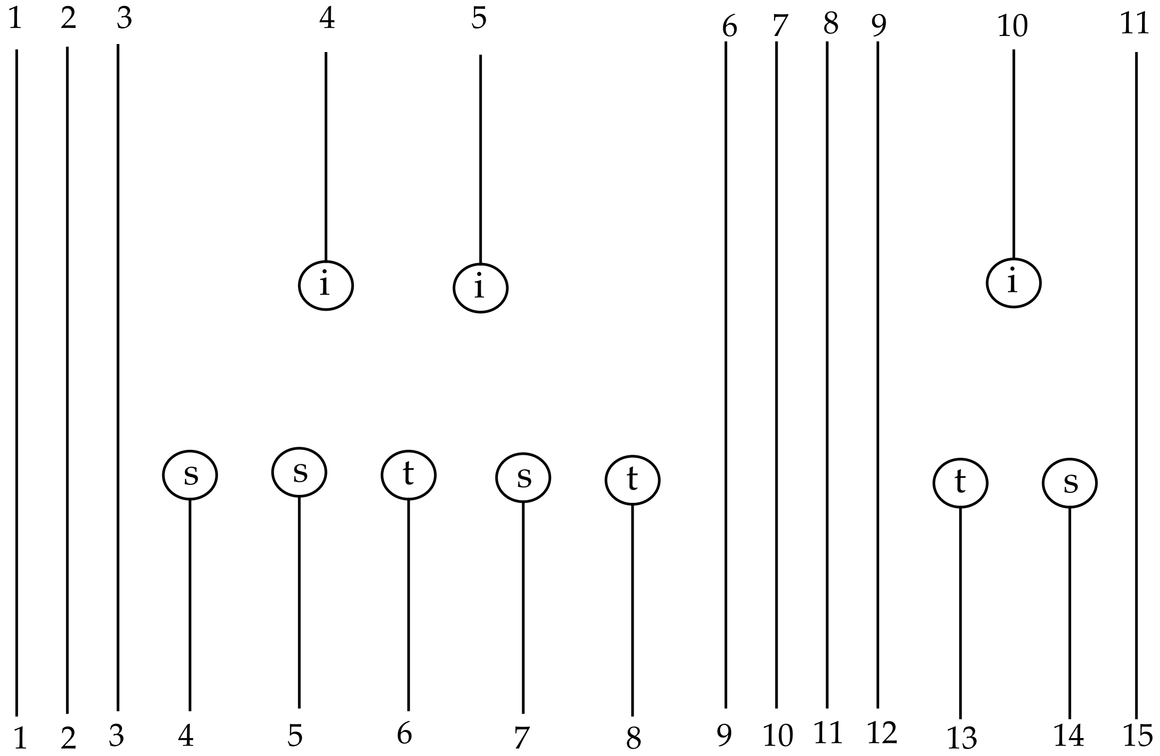


subject to the two relations



Objects will be tensor powers $J^{\otimes n} = J \otimes J \otimes \ldots \otimes J$ (n terms) of J.

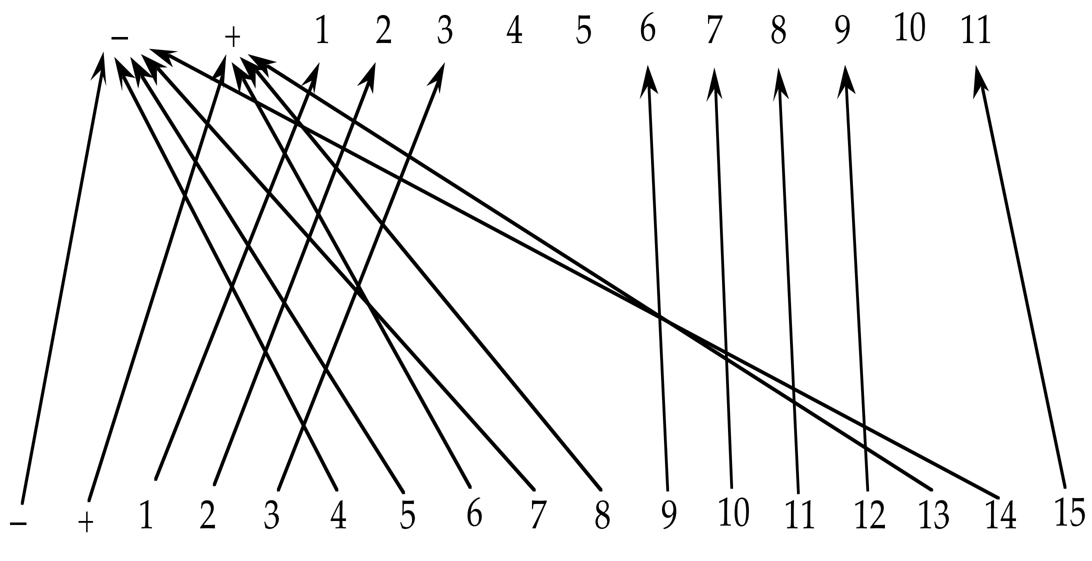
A typical arrow $J^{\otimes 15} \longrightarrow J^{\otimes 11}$ is depicted below.



This diagram can be interpreted as a function
ξ : < 15 > $\longrightarrow$ < 11 > where

$<k> \; = \; \{-, +, 1, 2, \ldots, k\}$

as follows.



So our model category $\mathbb{I}$ has objects the bi-pointed sets $< k >$ and arrows

$$\xi \; : \; < m > \longrightarrow < n >$$

those functions which preserve $-, +$ and have

$$i < j \quad \text{iff} \quad \xi(i) < \xi(j)$$

whenever $\xi(i), \xi(j) \in \{1, 2, \ldots, n\}$. The tensor product is given by

$$< m > \otimes < n > \; = \; < m + n >$$

$$(\xi \otimes \zeta)(i) = \begin{cases} \xi(i) & \text{for } 0 < i \leq m \\ \zeta(i) & \text{for } m < i \leq m + n \end{cases}$$

The cointerval in $\mathbb{I}$ is

$$< 1 > \; \begin{array}{c} \xrightarrow{\;\;s\;\;} \\ \xleftarrow{\;\;i\;\;} \\ \xrightarrow[\;\;t\;\;]{} \end{array} \; < 0 >$$

which is generic in the sense that the tensor-preserving functors T from $\mathbb{I}$ into any monoidal category $\mathcal{V}$ are in natural bijection with cointervals in $\mathcal{V}$. The bijection takes T to the image of the generic cointerval under T.

A cubical set, as used in algebraic topology, is precisely a functor

$$X \; : \; \mathbb{I} \longrightarrow \mathbf{Set} \, .$$

Braided monoidal categories

A *braiding* for a monoidal category is a natural family

$$c_{A,B} : A \otimes B \xrightarrow{\cong} B \otimes A$$

of isomorphisms compatible with the tensor product in the sense that the following two diagrams commute.

$$A \otimes B \otimes C \xrightarrow{c_{A \otimes B, C}} C \otimes A \otimes B, \quad A \otimes B \otimes C \xrightarrow{1 \otimes c_{B,C}} A \otimes C \otimes B \xrightarrow{c_{A,C} \otimes 1} C \otimes A \otimes B$$

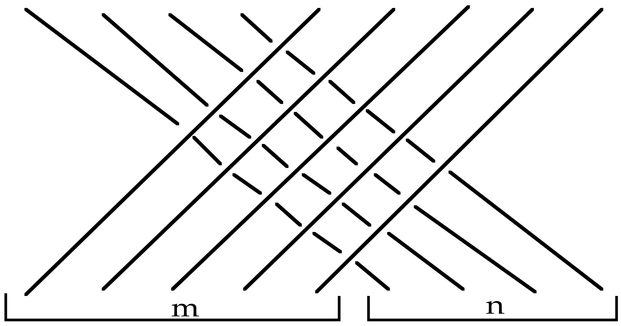
$$A \otimes B \otimes C \xrightarrow{c_{A, B \otimes C}} B \otimes C \otimes A, \quad A \otimes B \otimes C \xrightarrow{c_{A,B} \otimes 1} B \otimes A \otimes C \xrightarrow{1 \otimes c_{A,C}} B \otimes C \otimes A$$

A *braided monoidal category* is a monoidal category with a selected braiding.

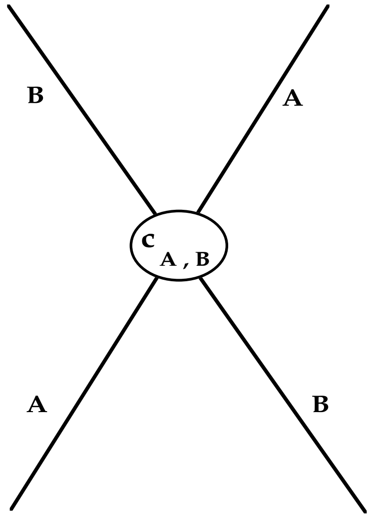
Example The braid category $\mathbf{B}$ is braided monoidal. A braiding is given by the elements

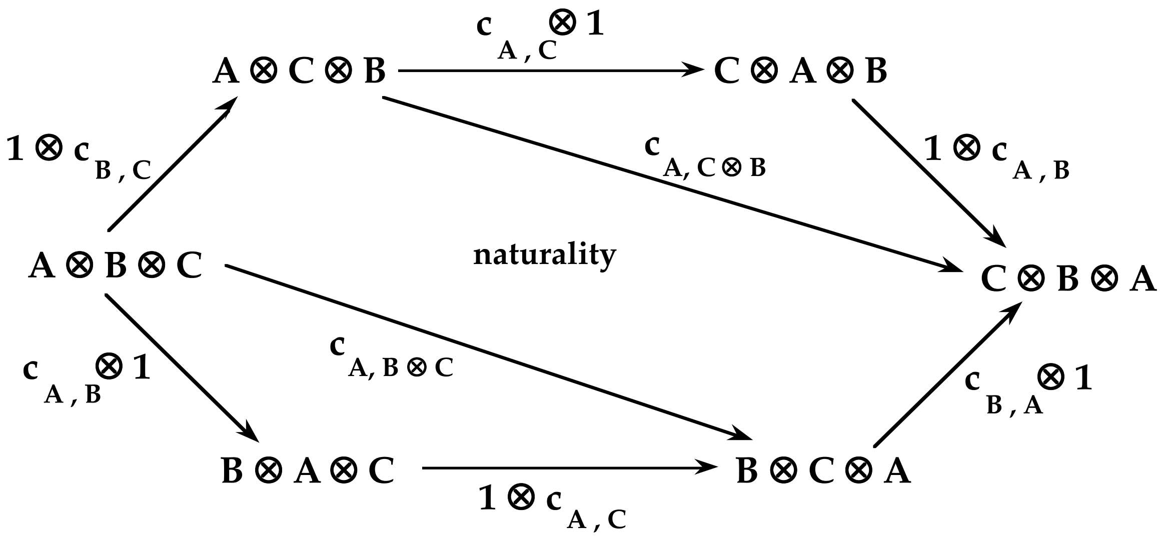
$$c = c_{m,n} : m + n \longrightarrow n + m$$

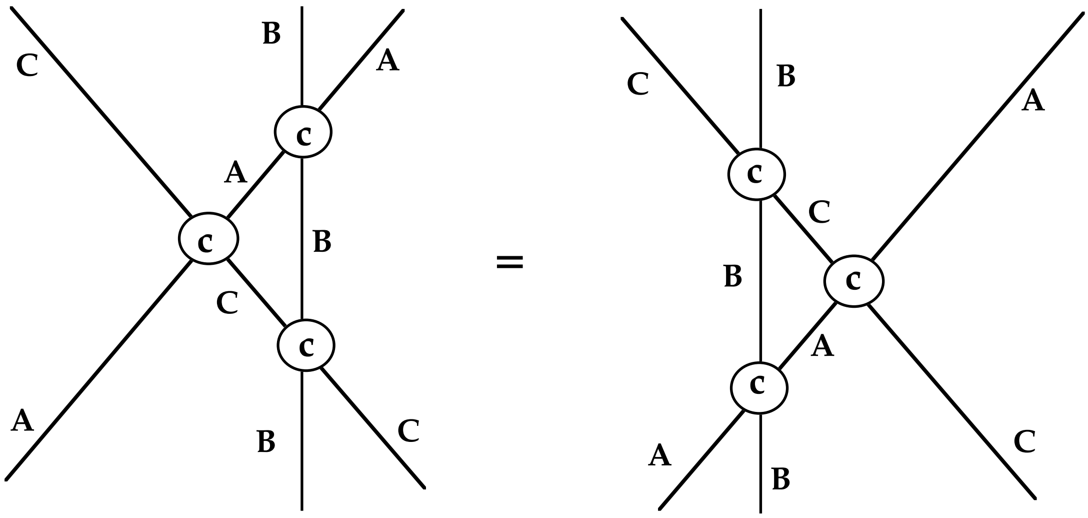
illustrated by the following figure.



Theorem [JS] *The braid category* $\mathbf{B}$ *is the free braided monoidal category generated by a single object.*


B
A
$c_{A,B}$
A
B


A ⊗ C ⊗ B
$c_{A,C} \otimes 1$
C ⊗ A ⊗ B
$1 \otimes c_{B,C}$
$c_{A, C \otimes B}$
$1 \otimes c_{A,B}$
A ⊗ B ⊗ C
naturality
C ⊗ B ⊗ A
$c_{A,B} \otimes 1$
$c_{A, B \otimes C}$
$c_{B,A} \otimes 1$
B ⊗ A ⊗ C
B ⊗ C ⊗ A
$1 \otimes c_{A,C}$


C
B
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c
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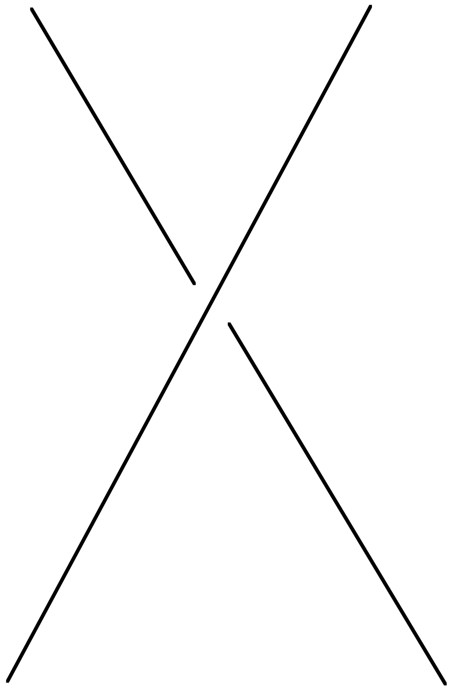
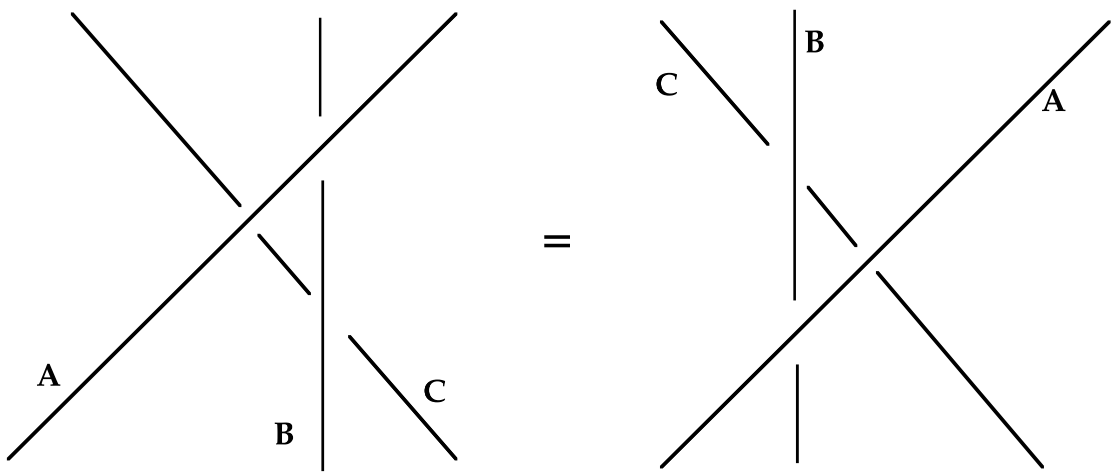
B
C

Enter 3 Dimensions



Braid relation, Yang-Baxter equation, or Reidemeister move III



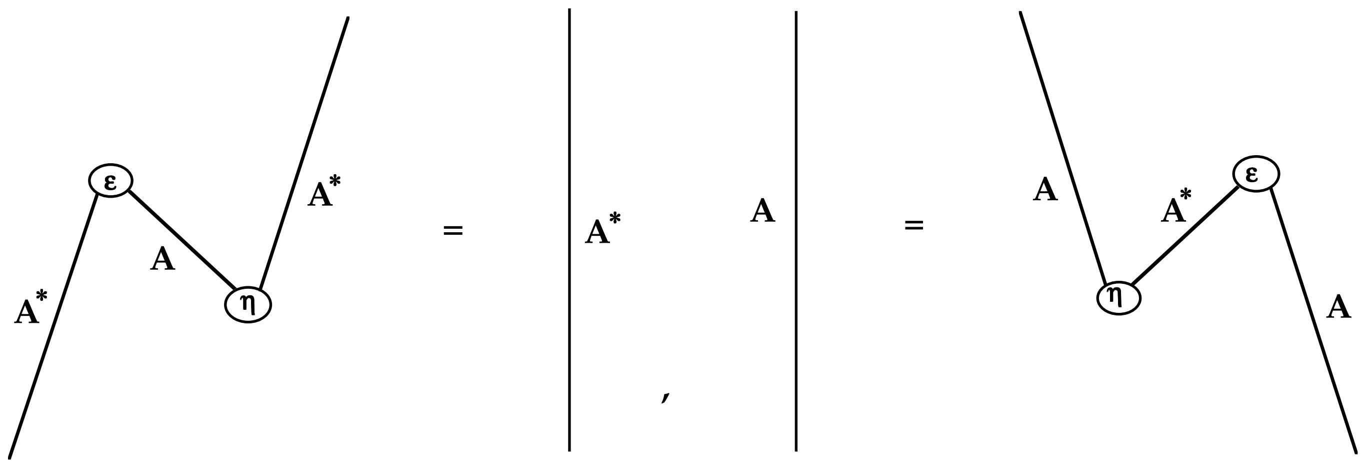
Theorem **(Joyal-Street)** *The value of a progressive 3D string diagram in a braided monoidal category is deformation invariant.*

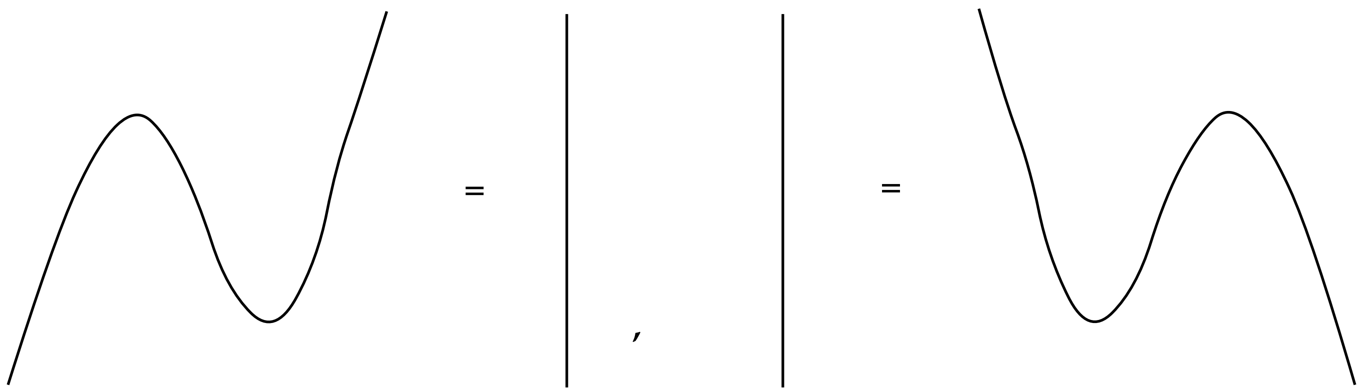
Duality in monoidal categories

A *left dual* for an object A of a monoidal category consists of an object A^* together with arrows

$$\varepsilon : A^* \otimes A \longrightarrow I\,, \qquad\qquad \eta : I \longrightarrow A \otimes A^*$$

such that





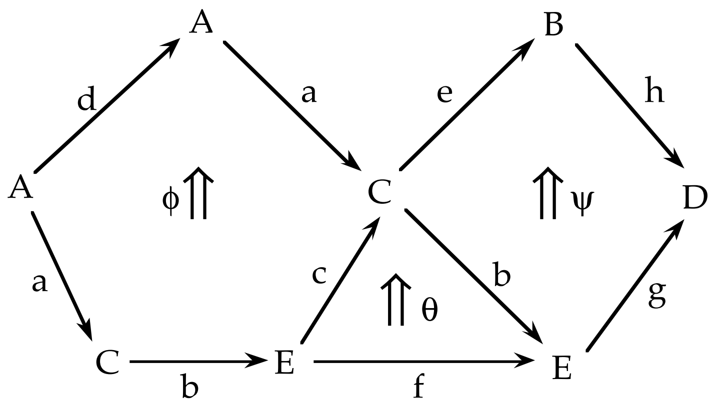
For monoidal categories with duality on both sides, this leads to string diagrams in the plane which have winding,

and, for braided monoidal categories with duality, this leads to tangles (these include both braids and links).

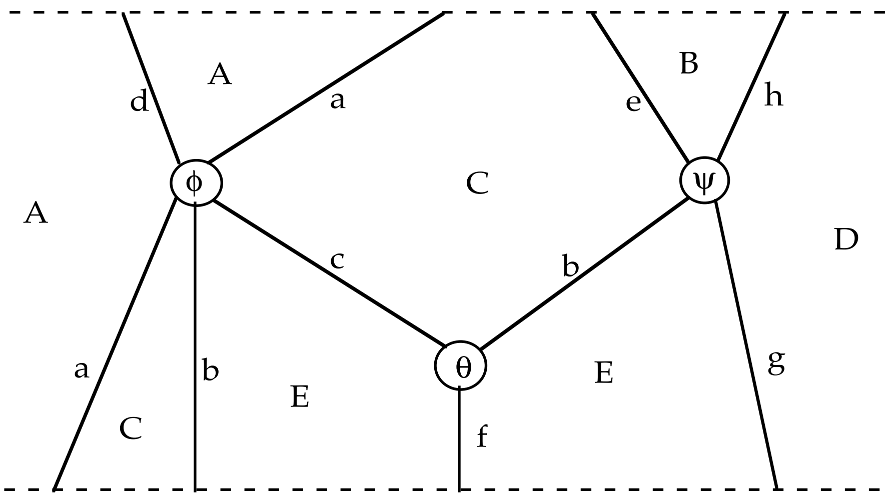
Again, each such diagram has a deformation invariant value.

String diagrams for monoidal categories are in fact appropriate for bicategories in the sense of Bénabou. A monoidal category is a bicategory with one object (in the same way as a monoid is a category with one object). What are called arrows of the monoidal category are called 2-cells in the bicategory; what are called objects of the monoidal category are called 1-cells in the bicategory; the one object (or 0-cell) of the bicategory never rates a mention in the monoidal category. However, in the string diagram, we should really think of this single 0-cell as labelling the plane regions between the strings.

The more usual diagrams for bicategories have been called *pasting diagrams* . The passage from pasting diagrams to string diagrams is via planar Poincaré duality. For example, consider the pasting diagram below.

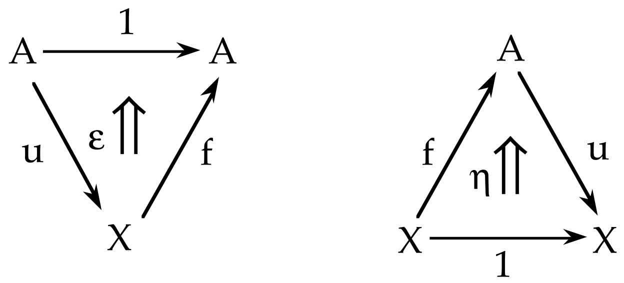


The corresponding string diagram is obtained by replacing 2-cells by nodes, 1-cells by edges, and 0-cells by plane regions, while preserving the incidence relations.

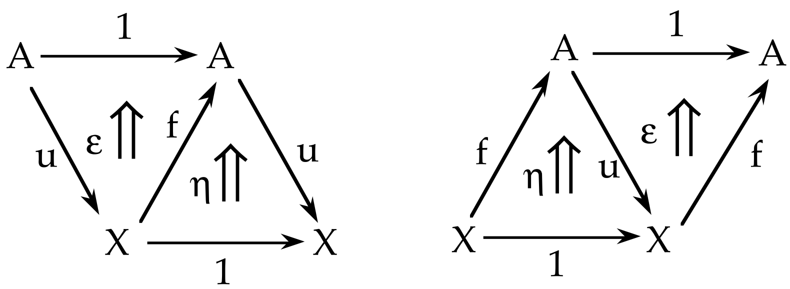


String diagrams have an advantage over pasting diagrams especially when identity 1-cells are involved.

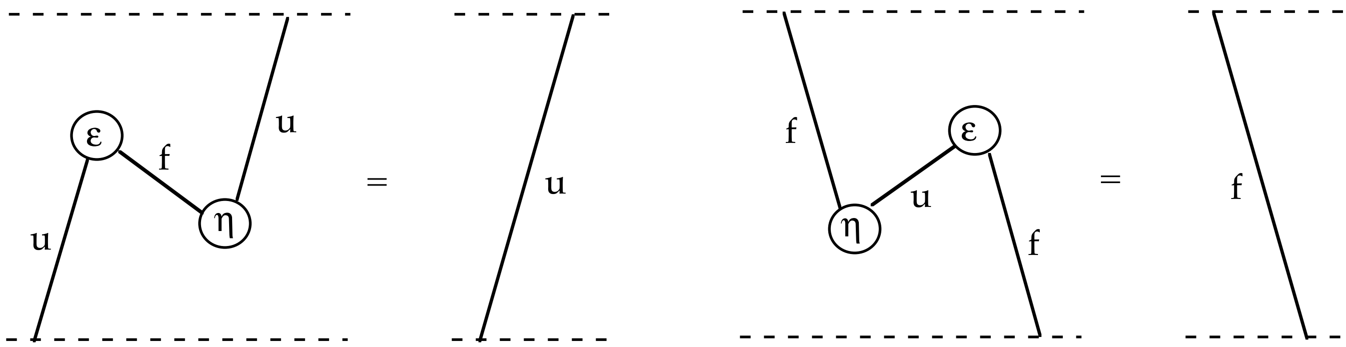
Identity arrows occur in some of the basic concepts in bicategories. As an example, consider a pair of *adjoint arrows* $u : A \longrightarrow X$, $f : X \longrightarrow A$ in a bicategory. This means that there are 2-cells ε, η



(called the *counit* and *unit*) satisfying the two conditions that the pasting composites



are equal to the identity 2-cells of u, f, respectively. In terms of string diagrams, these conditions become the following two equations between values.



Adjoints in bicategories generalize duals in monoidal categories and lead to diagrams with winding as before; but now 2D regions are labelled by objects.

We shall later consider diagrams for higher adjoints.

Alternative view of braidings

Commutativity can be expressed by saying the operation is a homomorphism.

An abelian monoid "is" a monoidal category with one object; that is, a bicategory with one object and only an identity arrow.

A braided monoidal category is a monoidal category for which the tensor product preserves the tensor product up to coherent natural isomorphism. That is, it is a monoidal bicategory with one object. That is, it is a tricategory with one object (= 0-cell) and only an identity arrow (= 1-cell).

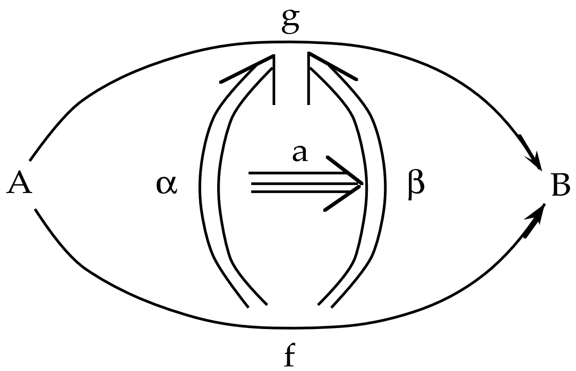
Diagrams for n-th order categories belong in n-dimensional Euclidean space.

This is the explanation of why diagrams for monoidal categories are 2D and those for braided monoidal categories are 3D.

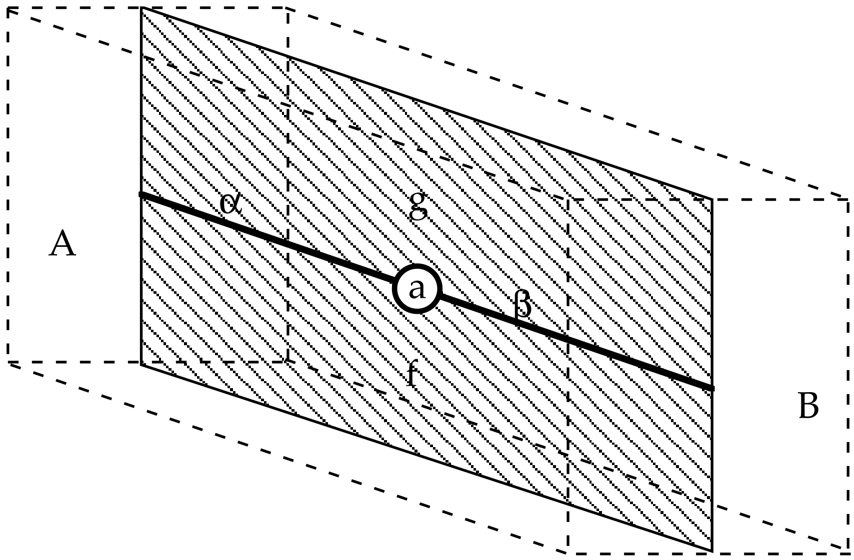
Symmetric monoidal categories are one object, one arrow, one 2-cell tetracategories. Diagrams for tetracategories belong in 4D. In fact, diagrams for symmetric monoidal categories belong in 4 and all higher dimensions: they are combinatorial.

Surfaces in 3D and tricategories

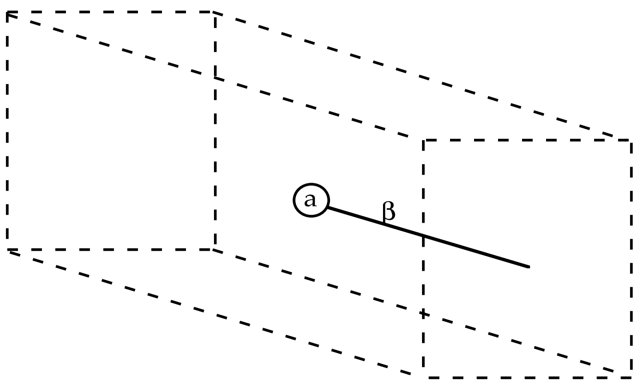
The starting point is a 3-dimensional generalization of the Penrose notation. A 3-cell in a tricategory



transformations via 3D Poincaré duality to

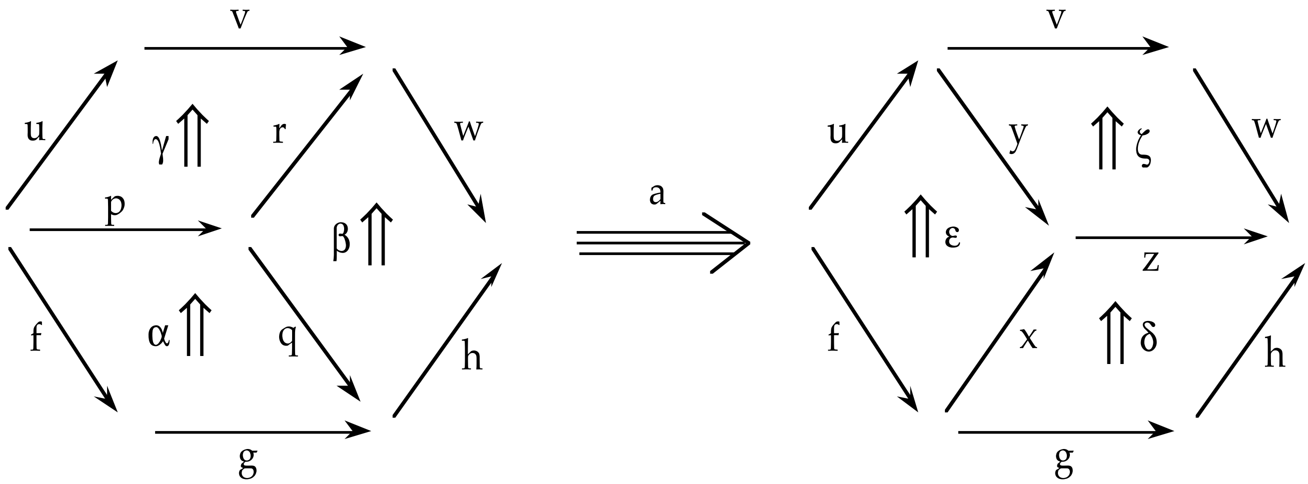


Consider the case where both α and g are identities; the picture has one single 3D region A and no specific distinguished plane.

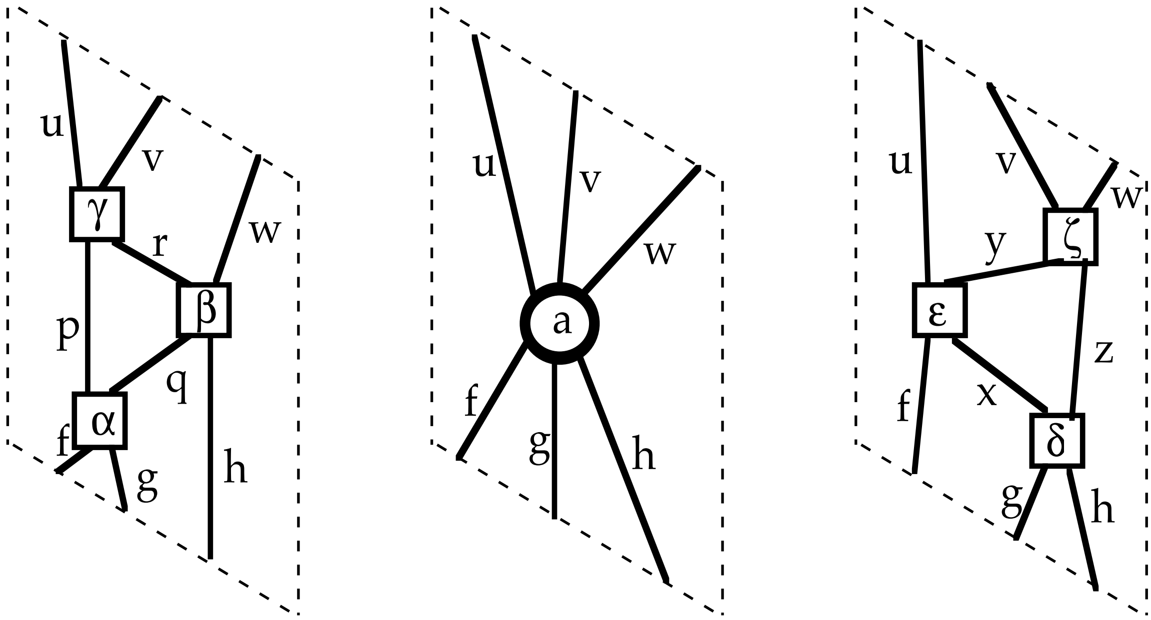


Cube Example

Pasting version



Movie version

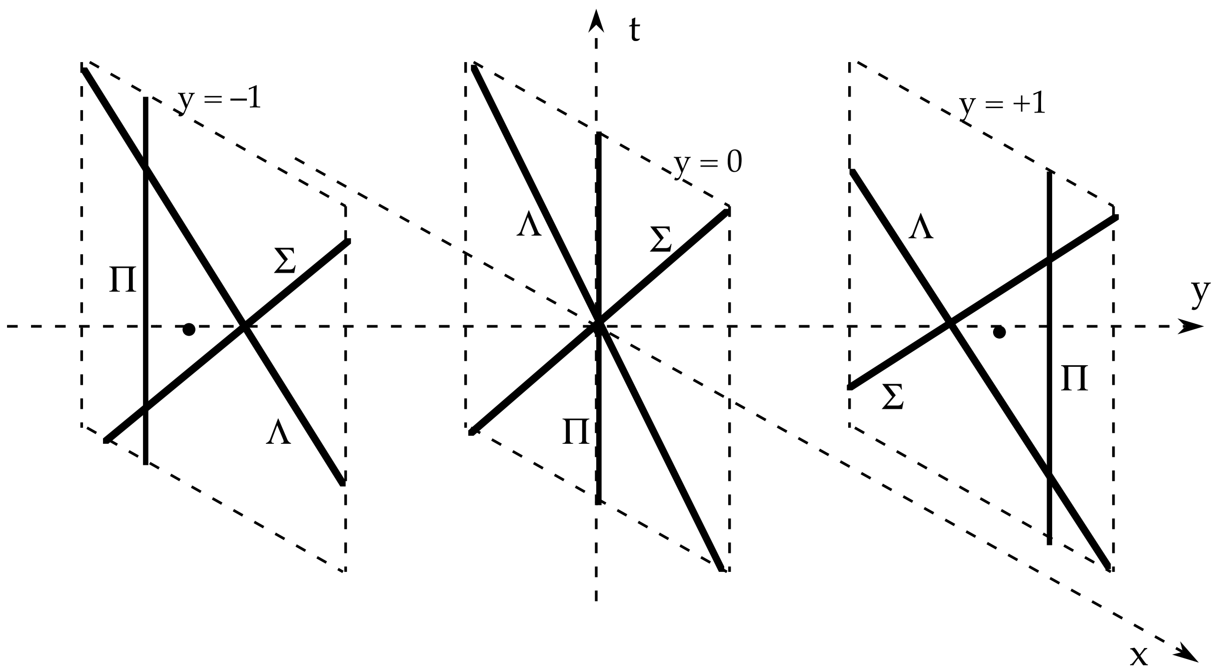


3D version

Take the following three planes in xyt-space:

$$\Lambda : x + y + t = 0$$
$$\Pi : x - y = 0$$
$$\Sigma : x + y - t = 0 \quad .$$

Then the 1-cells u, r, x, h label parts of the plane Λ, the 1-cells v, p, z, g label parts of the plane Π, and the 1-cells w, q, y, h label parts of the plane Σ. The 2-cells α, ζ label parts of the line $\Pi \cap \Sigma$, the 2-cells β, ε label parts of the line $\Sigma \cap \Lambda$, and the 2-cells γ, δ label parts of the line $\Lambda \cap \Pi$. Of course, the 3-cell a labels the point $\Lambda \cap \Pi \cap \Sigma$.



This relates to the Zamolodchikov tetrahedra equations.

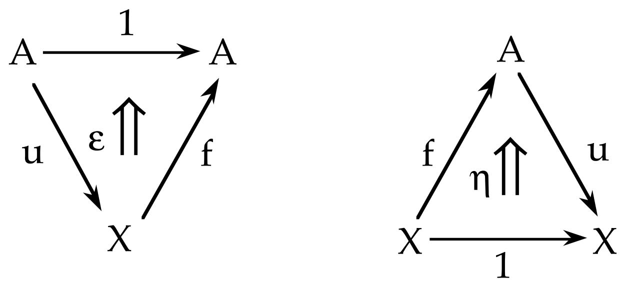
Lax adjunctions in tricategories

A *lax adjunction* in a tricategory consists of

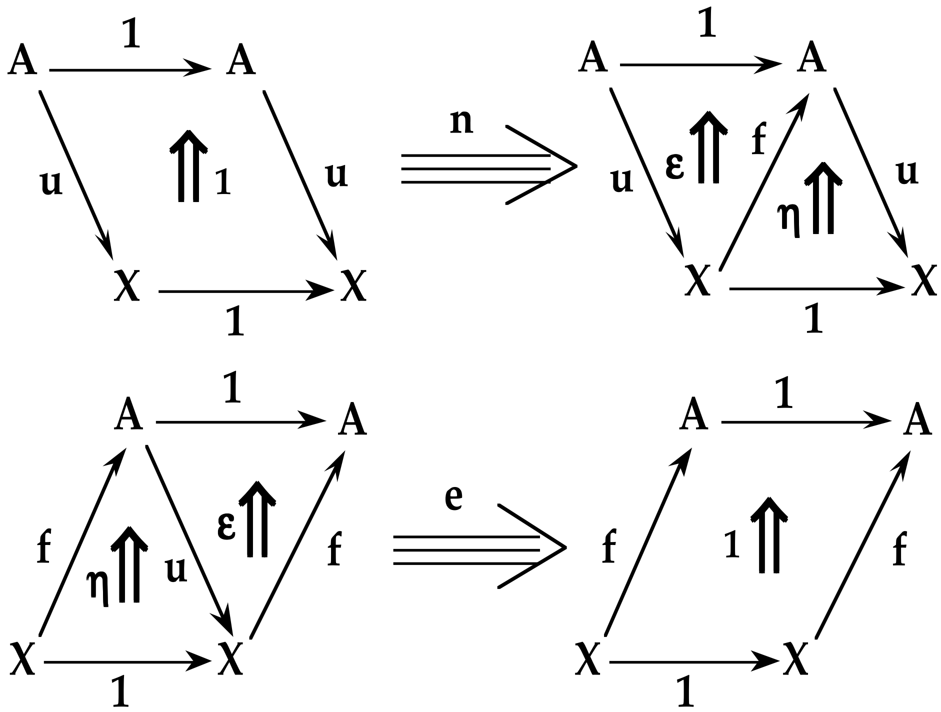
objects A , X ,

arrows u : A ⟶ X , f : X ⟶ A ,

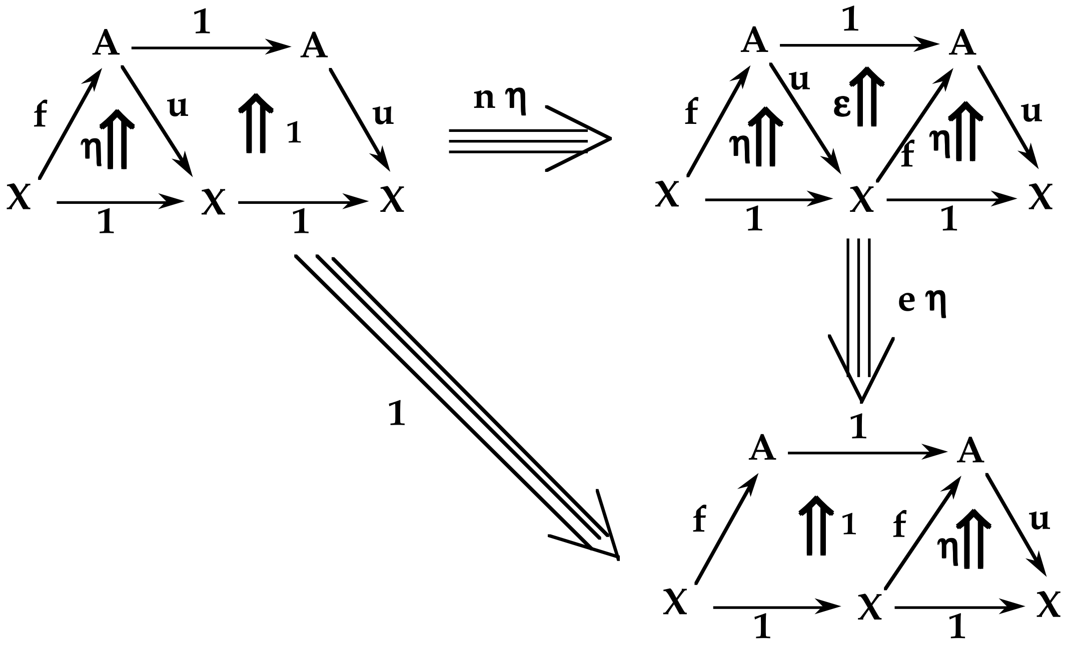
2-cells ε, η

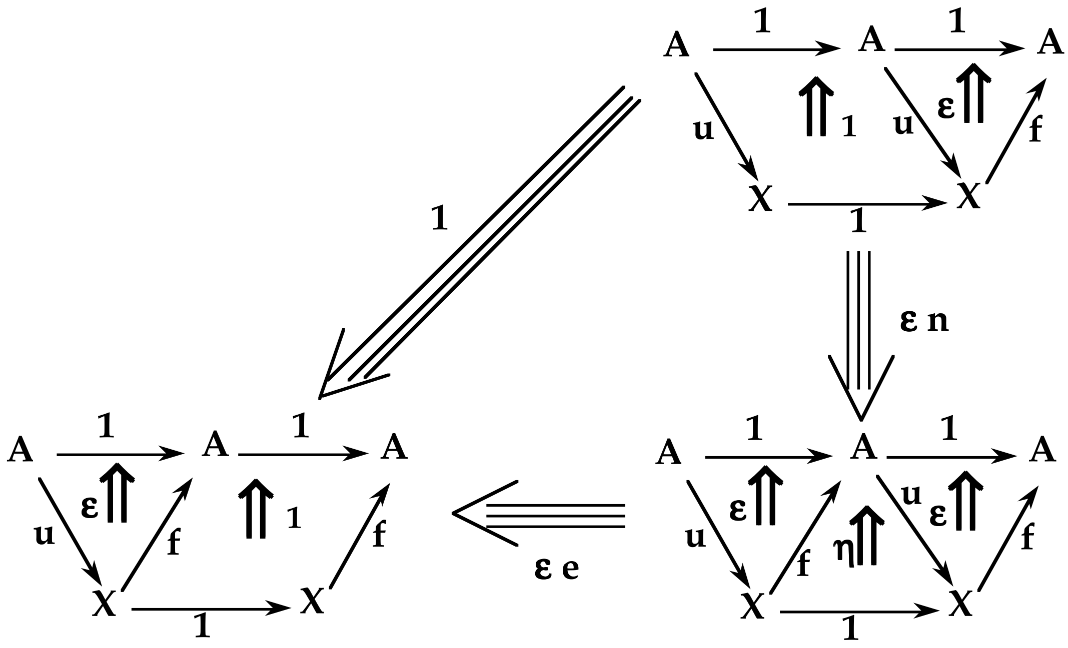


and 3-cells n, e

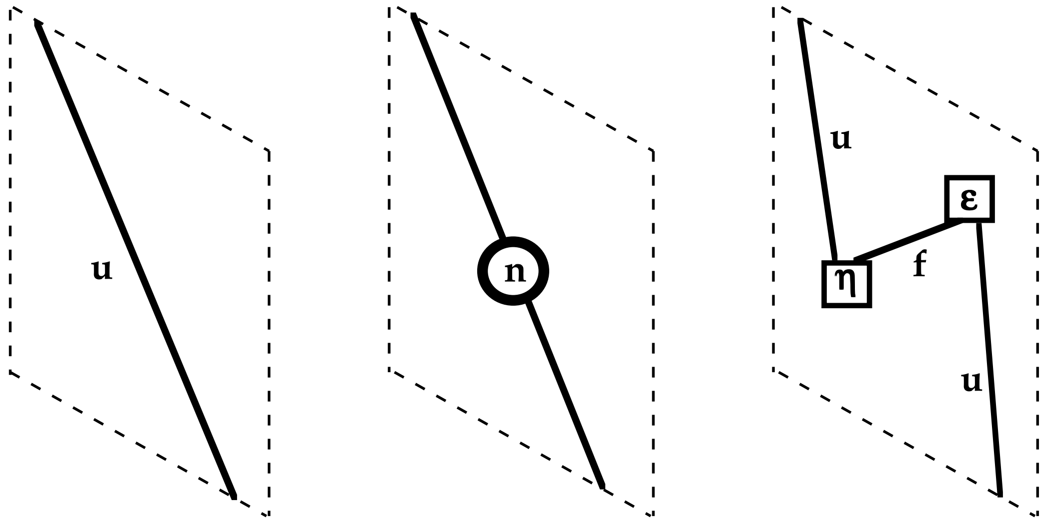


satisfying the following two conditions:

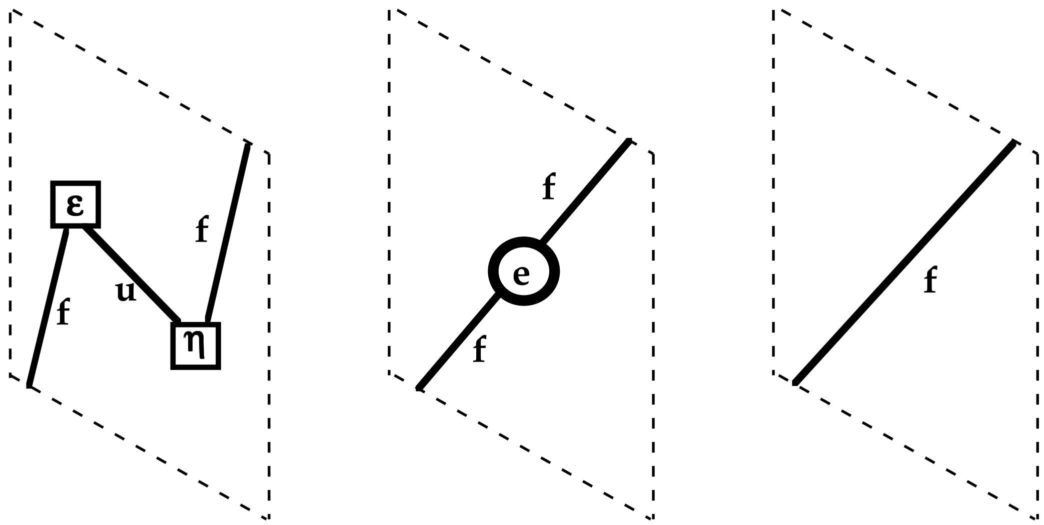

n η
e η


ε n
ε e

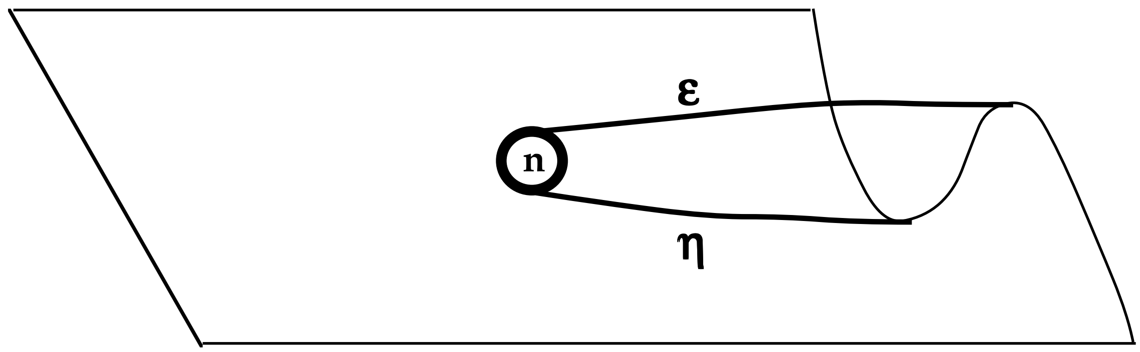
Movie for n



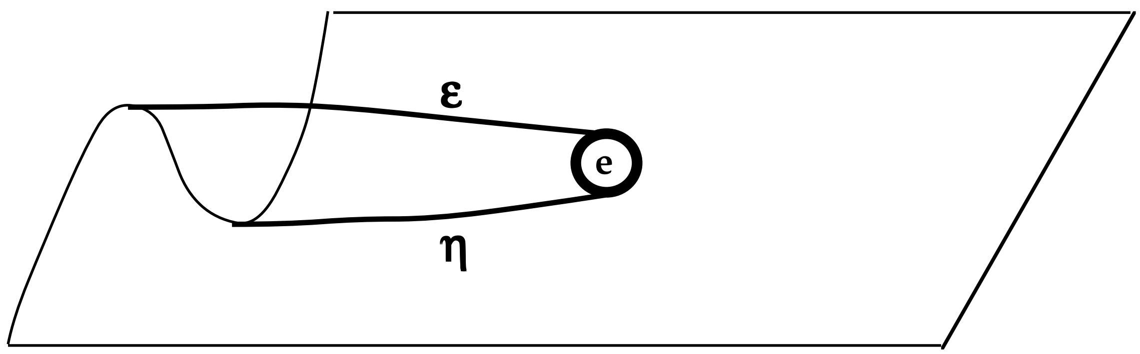
Movie for e



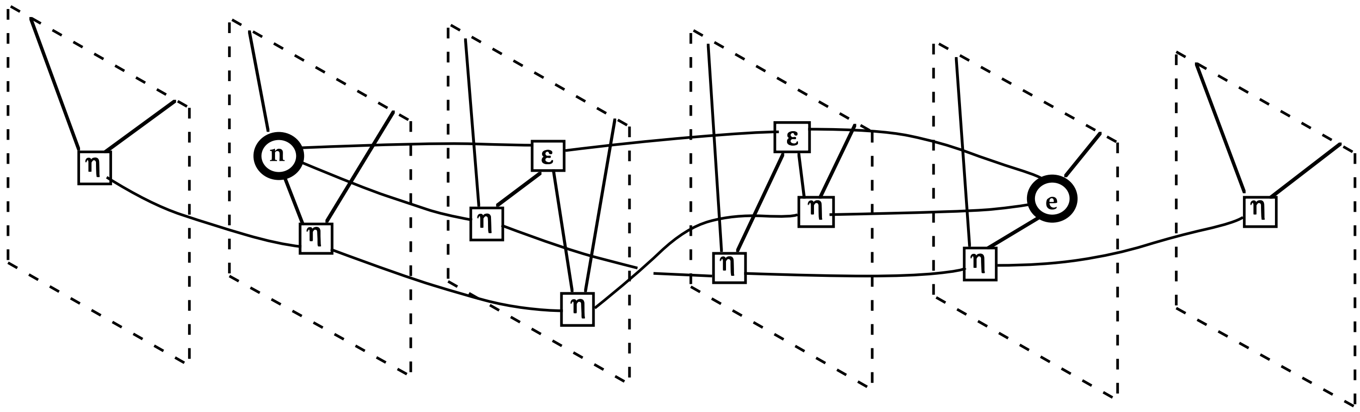
Surface for n



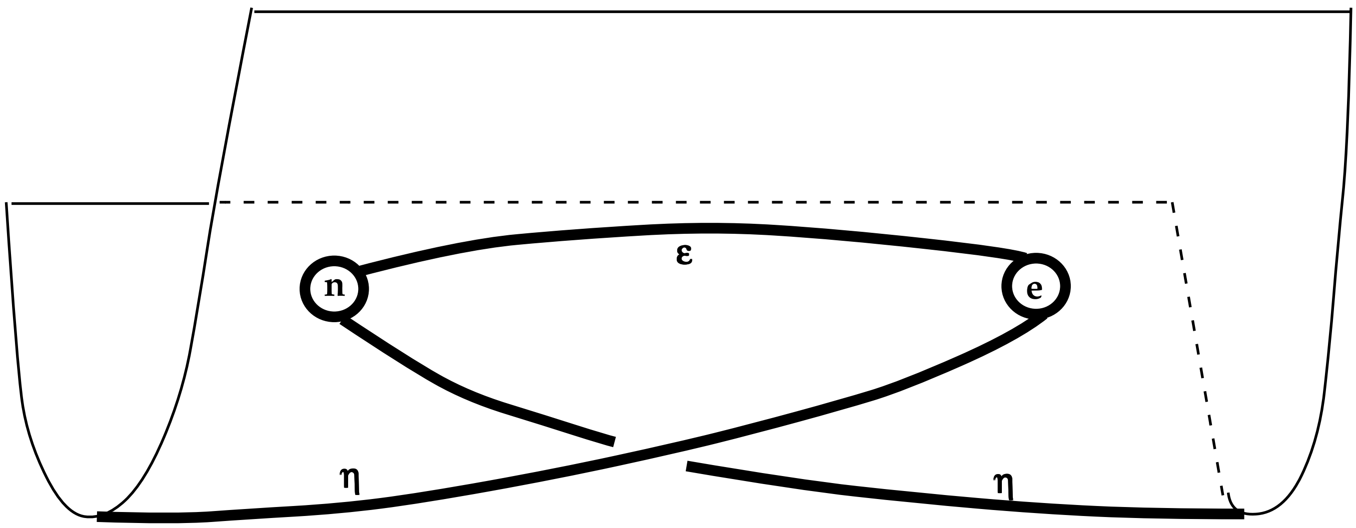
Surface for e



Movie for lax adjunction axiom



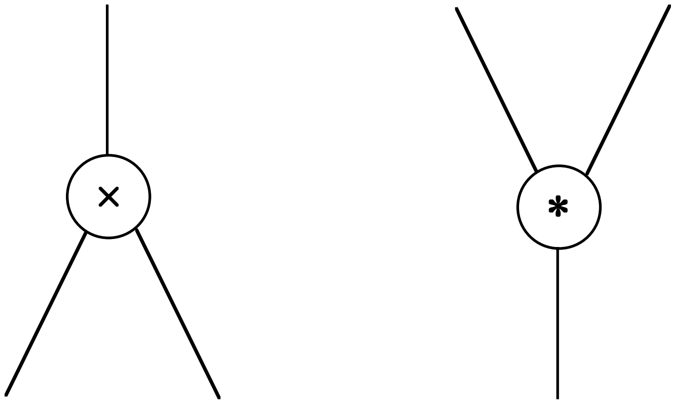
Surface diagram for axiom



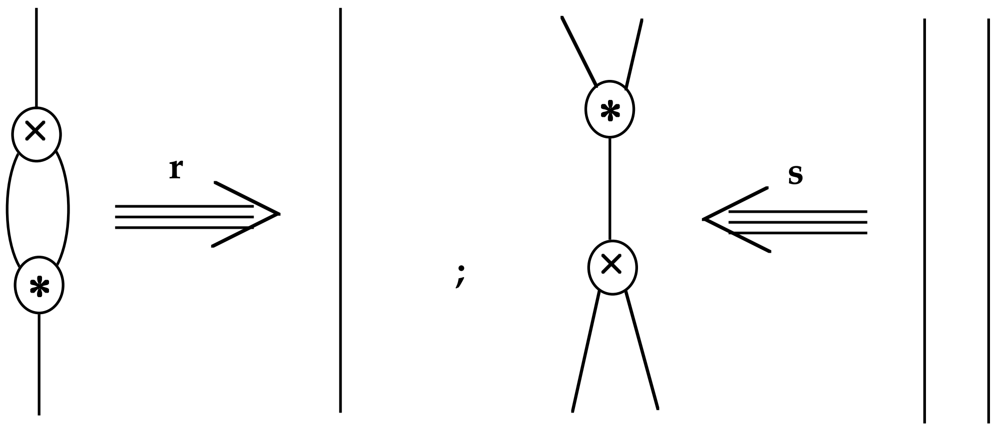
Example from Blute-Cockett-Seely-Trimble

They expressed the logic of their “weakly distributive categories” in terms of string diagrams and then used rewrite rules on them to find normal forms.

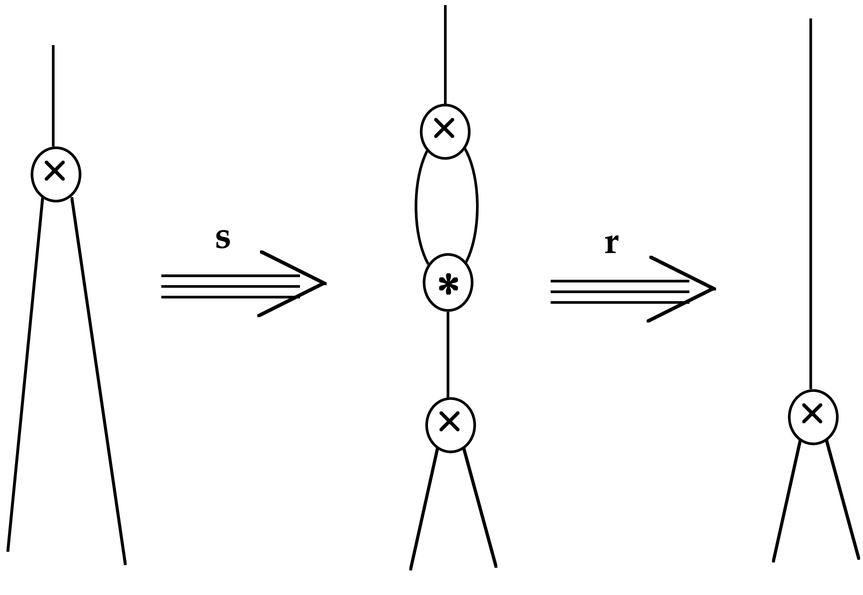
data



rewrites



axiom



is the identity

“The Cockett Pocket” (Verity)