A 2-categorical version of a result mentioned by Lack on Wednesday 21 January 1998

We work in a 2-category which admits the following comma square.

\[
\begin{array}{ccc}
P & \xrightarrow{v} & B \\
\downarrow{u} & \searrow{\lambda} & \downarrow{q} \\
A & \Rightarrow & C \\
\end{array}
\]

**Proposition** If \(v\) has a right adjoint \(r\) with identity counit and with unit \(\eta : 1_P \Rightarrow r\) \(v\) then the following is a comma square.

\[
\begin{array}{ccc}
P & \xrightarrow{v} & B \\
\downarrow{u} & \searrow{\Omega} & \downarrow{u r = t} \\
A & \Rightarrow & A \\
\end{array}
\]

**Proof** One adjunction triangle for \(v \dashv r\) gives \(v \eta = 1\) and hence the equality \((\ast)\) below.

\[
\begin{array}{ccc}
P & \xrightarrow{1} & P \\
\downarrow{v} & \searrow{\eta} & \downarrow{r} \\
B & \Rightarrow & B \\
\end{array}
\quad \quad = \quad \quad
\begin{array}{ccc}
P & \xrightarrow{p u} & C \\
\downarrow{v} & \searrow{\lambda} & \downarrow{q} \\
B & \Rightarrow & B \\
\end{array}
\]

Another consequence of \(v \dashv r\) is that, for all arrows \(f : X \to P\), \(b : X \to B\), there is a bijection between 2-cells \(\rho : f \Rightarrow r b\) and 2-cells \(\beta : v f \Rightarrow b\) determined by the equations

\[
\beta = v \rho, \quad \rho = r \beta \circ \eta f.
\]

By the 2-cell property of the comma object \(P\), the 2-cells \(\rho : f \Rightarrow r b\) are in bijection with pairs of 2-cells \(\alpha : u f \Rightarrow t b\), \(\beta : v f \Rightarrow b\) such that the square \((\ast\ast)\) below commutes.

\[
\begin{array}{ccc}
p u f & \xrightarrow{\lambda f} & q v f \\
\downarrow{p} & \searrow{\alpha} & \downarrow{q} \\
p t b & \Rightarrow & q b \\
\end{array}
\]

The bijection is determined by the equations \(\alpha = u \rho\), \(\beta = v \rho\). It follows that, for each 2-cell \(\beta : v f \Rightarrow b\), there exists a unique 2-cell \(\alpha : u f \Rightarrow t b\) such that \((\ast\ast)\) commutes.

Now we prove the 2-cell \(u \eta\) has the claimed comma property. Take any 2-cell \(\alpha : a \Rightarrow
t b. By the comma property of $\lambda$, there exists a unique arrow $f : X \rightarrow P$ such that

![Diagram]

So $\alpha$ satisfies (***) with $\beta = 1$. By (**), the 2-cell $u \eta f$ also satisfies (*) as the $\alpha$ with $\beta = 1$. So $\alpha = u \eta f$, as required. If $u \eta f = u \eta g$ for some $g$, the equality (*) gives $\lambda f = \lambda g$, so, by the comma property of $\lambda$, we obtain $f = g$. So $f$ is unique with $\alpha = u \eta f$.

The further 2-cell property required of $u \eta$ is immediate. Q.E.D.