For Joachim

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BRAIDED MONOIDAL CATEGORIES

by

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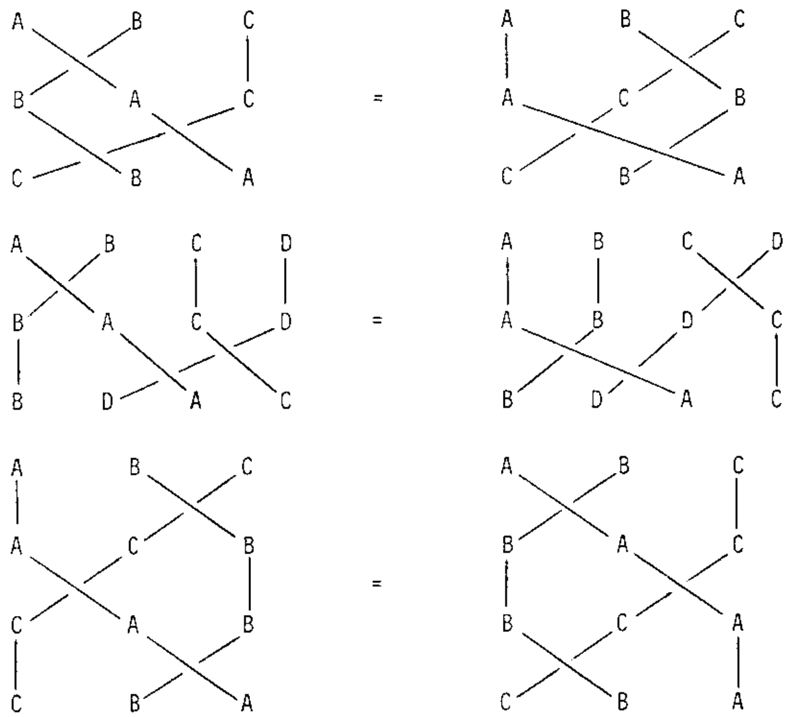

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Introduction

Categories enriched with tensor products, called monoidal categories [11], occur in various branches of mathematics. Large examples such as the categories of abelian groups and of banach spaces are important for studying mathematical structures. Small examples, as found in algebraic topology, are important as mathematical structures in their own right.

Monoidal categories with commutative tensor product deserve special attention as do commutative rings in ring theory. Natural examples of commutativity are not strict in the sense $A \otimes B = B \otimes A$. Rather, natural isomorphisms $c_{A,B}: A \otimes B \to B \otimes A$ exist. With a view to known examples and Mac Lane's coherence theorem [25], it has been consistently felt that the symmetry condition

S. $$c_{BA}c_{AB} = 1_{A\otimes B}$$

should be assumed. Together with a condition expressing $c_{A,B\otimes C}$ in terms of $c_{A,B}$, $c_{A,C}$, which we call B1 (see Section 1 below), this is the notion of a *symmetry* for a monoidal category [11].

The point of the present paper is to show that a somewhat weaker notion, called a *braiding*, admits important new examples (especially in homotopy and cohomology theories), occurs naturally in the theoretical context of further multiplications on a monoidal category, has an adequate coherence theorem, and, is as versatile as symmetry. A braiding consists of natural isomorphisms $c_{A,B}: A \otimes B \to B \otimes A$ satisfying B1 and condition B2 which expresses $c_{A\otimes B,C}$ in terms of $c_{A,C}$, $c_{B,C}$. A symmetry is exactly a braiding which also satisfies the symmetry condition S; but not every braiding is a symmetry.

A *braided monoidal category* is a monoidal category with a chosen braiding. In Section 1 further diagrams B3-B7 are shown to commute. The first three of these show that the identity functor enriches to a monoidal equivalence between the monoidal category and the same category with the reverse tensor product. The last two are essential for later sections.

Section 2 provides a wealth of examples of braidings which are not symmetries. For the usual monoidal structure on the category of graded modules over a commutative ring K, braidings are in bijection with invertible elements k of K; for a symmetry $k^2 = 1$ is required.

Our second example $\mathbb{B}$ is the groupoid whose objects are natural numbers and whose arrows $n \to n$ are elements of the *braid group* $\mathbb{B}_n$. This example is at the heart of the whole theory of braided monoidal categories as appears two sections later.

A non-example suggested by F.E.J. Linton and related to $\mathbb{B}$ shows that B5, B6, B7 do not imply any of B1-B4. Actually, B3, B4, B6 *are* equivalent to B1, B2 (this is the essence of Section 3). Even in the presence of S, conditions B3, B4, B5 do not imply B1 (see Kasangian-Rossi [14]). It is also easy to give examples to show that B1 and B2 are independent and to show that B3, B4, B7 do not imply B1 or B2.

The obvious analogy between homotopies and natural transformations has deeper consequences than one might first expect [29,30]. The analogues of topological groups are categorical groups (or groups in Cat) and these amount to *crossed modules* as occur in homotopy theory [35]. A crossed module is a group homomorphism $\sigma: N \to E$ together with an action of E on N subject to some axioms. A categorical group is an example of a strict monoidal category arising from the crossed module by taking elements e of E as objects, by taking arrows $u: e \to e'$ to be elements u of N with $e = \sigma(u)e'$, and by obtaining the tensor product from the action of E on N. A braiding for this categorical group amounts to a familiar notion in homotopy theory, namely, a *bracket operation* $\{\ ,\ \}: E \times E \to N$ for the crossed module; the properties of this are those of an "abstract commutator".

When the tensor product is strictly associative and is viewed as addition, axioms B1, B2 express bilinearity. This is made precise in the construction of braided monoidal categories from bilinear functions in general and ring multiplications in particular.

The final example in Section 2 points out that the convolution of Day [6], when applied to a small braided (pro-) monoidal category, yields a braided monoidal category of presheaves which is closed, complete and cocomplete.

Section 3 gives a generalization of the argument of Eckmann-Hilton [9] behind the commutativity of the higher homotopy groups. Recall that, if M is a monoid and $f: M \times M \to M$ is a monoid homomorphism such that $f(1,x) = f(x,1) = 1$, then $f(x,y) = xy = yx$ for all $x,y \in M$; so the monoid is commutative. This argument applies to monoids in any category and so to monoids in the category Cat of small categories. However, when M is a monoid in Cat (that is, a strict monoidal category), there is the possibility of considering functors $f: M \times M \to M$ which are not homomorphisms but merely preserve the multiplication (or tensor product) of M up to isomorphism subject to appropriate conditions; that is, we can consider f to be a strong monoidal functor, not necessarily strict. It turns out in this case that $f(x,y) \cong xy$ and a braiding is obtained for M. It is shown more generally that, for any monoidal category V (not necessary strict), an extra "multiplication" on V leads to a braidings also each braiding leads to a multiplication. This gives a natural explanation for briadings as opposed to symmetries.

The free braided monoidal category on the category 1 (with one object and one arrow) is the braid category $\mathbb{B}$ described above. This is the main result of Section 4. It follows from work of Kelly [17] on "clubs" that we have an explicit description of the monad on Cat whose algebras are braided strict monoidal categories and whose pseudo-algebras are braided monoidal categories. In other words $\mathbb{B}$ represents the "theory" in an appropriate sense. The permutation category $\mathbb{P}$ played the corresponding role in the case of symmetric monoidal categories. A coherence result for braided monoidal categories is obtained: a diagram, built up from instances of associativity and the braiding using tensor product and composition, commutes if the associated braids are equal.

When developing the theory of categories enriched over a monoidal category, Eilenberg-Kelly [11] found it necessary to invoke symmetry to define opposite (or dual) enriched categories and to define tensor products of enriched categories. In Section 5 we show that a braiding suffices. The constructions are the same as in [11], however, where Mac Lane's coherence theorem is used in the symmetric case, we make do with B1-B7.

Section 6 gives a treatment of the 3-dimensional cohomology of groups from the viewpoint of categories with structure. The usual interpretation theorem involving exact sequences

$$1 \to M \to N \to E \to G \to 1$$

of groups is shown to be a consequence of a result expressed in terms of 2-categories [21]. Just as sets with a particular kind of structure form a category, categories with a structure form a 2-category: not only are there arrows between objects, but 2-cells between the arrows (because of the natural transformations between functors). Because we are only interested in sets up to isomorphism, we are only interested in categories up to equivalence; hence we are only interested in 2-categories up to *biequivalence* (a concept which ignores the insertion of extra equivalent objects).

A 2-category $\mathcal{H}^3$ is described whose objects (G,M,h) consist of a group G, a G-module M, and, a normalized 3-cocycle $h: G^3 \to M$. This 2-category holds the information of all the cohomology groups $H^3(G,M)$ and $H^2(G,M)$. A very simple construction yields, for each object of $\mathcal{H}^3$, a monoidal category $T(G,N,h)$ for which the associativity comes from h. A more complicated construction involving free groups and coming directly from Eilenberg-Mac Lane leads to a strict monoidal category $S(G,M,h)$ where this time h enters into the tensor product. Yet we prove an equivalence

$$T(G,M,h) \simeq S(G,M,h)$$

of monoidal categories.

Each $S(G,M,h)$ is in fact a categorical group. While this is not true of $T(G,M,h)$ unless $h = 0$, each arrow in the underlying category is invertible and there is an inverse-like operation on the objects.

A category is a *groupoid* when every arrow is invertible. A monoidal category is called *compact* when there is a "duality operation", taking an object A to A^*, suitably axiomatized. It turns out that a monoidal groupoid is compact if and only if, for each object A, there is an object A^* with $A \otimes A^*$ isomorphic to the identity I for tensor product $\otimes$. Alternatively, a monoidal groupoid is compact if and only if it is closed in the sense of [11].

It is shown that T gives a biequivalence between $\mathcal{H}^3$ and the 2-category CMG of compact monoidal groupoids, strong monoidal functors, and, monoidal natural transformations. Hence, a compact monoidal groupoid determines, and is determined up to equivalence by, a group, a module on which it acts, and a 3-cocycle.

It follows that $S: \mathcal{H}^3 \to \mathrm{CMG}$ is also a biequivalence. So every compact monoidal groupoid is equivalent to a categorical group and so essentially amounts to a crossed module. The interpretation theorem for $H^3(G,M)$ (see K. Brown [4]) comes out of this.

In conclusion, Section 7 shows that a compact *braided* monoidal groupoid is classified by a pair of *abelian* groups G,M and a quadratic function $t: G \to M$. Here G is the group of isomorphism classes of objects under tensor product, M is the group of automorphisms of the identity for tensor product under composition; and, t is obtained from the trace of the braiding. Every quadratic function is so obtained. Compact braided monoidal groupoids with isomorphic quadratic functions are equivalent. However, this is not part of a biequivalence of 2-categories; the failure is measured by abelian group extensions.

Compact *symmetric* monoidal groupoids are classified by quadratic functions $t: G \to M$ with $2t(x) = 0$ for all $x \in G$.

This last section uses the work of Eilenberg-Mac Lane [24,10,12] on the cohomology of abelian groups. The connection is that, not only is the 3-cocycle condition a pentagon condition for associativity, but the extra datum for an abelian 3-cocycle amounts precisely to a braiding.

Two previous works had an impact on this paper. Joint work of M. Tierney and the first author on homotopy 3-types showed that arc-connected, simply connected spaces could be represented by what we would call braided categorical groups. It should be noted that the joint work makes use of the 2-groupoid version of a tensor product of Gray [13] and that Gray used the braid groups to prove the pentagon conditions for associativity of his tensor product.

The second work involves categories enriched over bicategories. Results of Street [32] show that operations on enriched categories, if they are to be compatible with modules, must exist already on a suitable base bicategory. In particular, if we are to have tensor products of enriched categories (compare Section 5 here), there should be a global

tensor product on the base bicategory. However, the idea that this global tensor product might lead to symmetry when the base bicategory has one object (and so is merely a monoidal category) was reported to us as occurring in a conversation of A. Carboni, F.W. Lawvere and R.F.C. Walters (Sydney, January 1984); an Eckmann-Hilton argument was envisaged.

The second author would like to thank the "Groupe Interuniversitaire en Etudes Catégoriques" directed by Michael Barr for making possible a six-week visit to Montréal during April-May 1985.

§1. Braidings

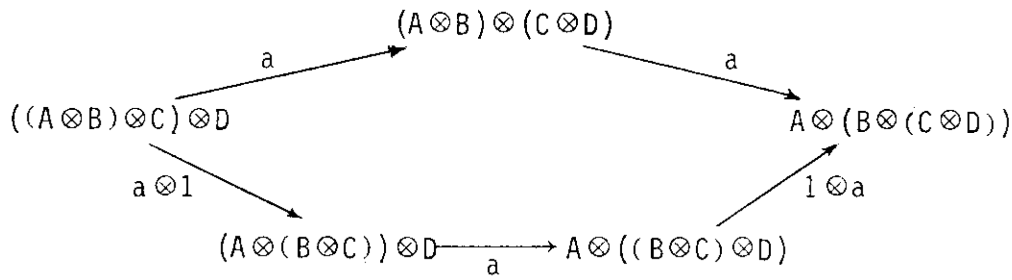
A *monoidal category* $V = (V_0, \otimes, I, r, \ell, a)$ consists of a category V_0, a functor $\otimes: V_0 \times V_0 \to V_0$ (written between the arguments and called the *tensor product* of V), an object I of V_0, and, natural isomorphisms

$$a = a_{ABC} : (A\otimes B)\otimes C \to A\otimes(B\otimes C),$$

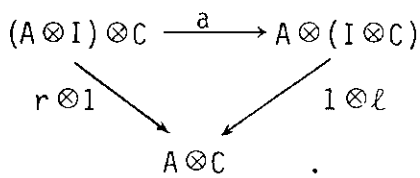
$$r = r_A : A\otimes I \to A, \qquad \ell = \ell_A : I\otimes A \to A;$$

such that the following diagrams commute:

AP.



IT.



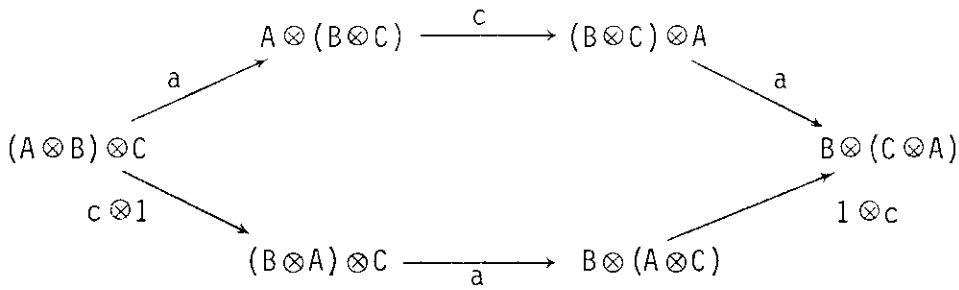
The diagram AP is called the *pentagon for associativity* and the diagram IT is called the *triangle for identities*. This definition is in agreement with that of Eilenberg-Kelly [11 ; p.471] where our AP, IT are called MC3, MC2. Call V *strict* when each a_{ABC}, r_A, ℓ_A is an identity arrow in V .

A *braiding* for V consists of a natural family of isomorphisms

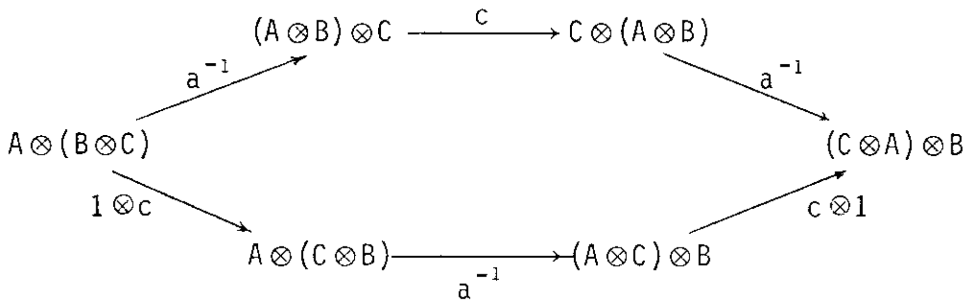
$$c = c_{AB} : A\otimes B \longrightarrow B\otimes A$$

in V such that the following two diagrams commute.

B1.



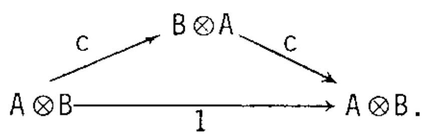
B2.



A monoidal category together with a braiding is called a *braided monoidal category*. Note that B2 amounts to B1 with c_{AB} replaced by c^{-1}_{BA}; so that c^{-1} is a braiding which is generally different from c.

A *symmetry* is a braiding such that the following diagram commutes:

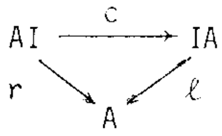
S.



In the presence of S, observe that B2 is the inverse of B1 so that B2 is redundant. Thus this notion of symmetry is exactly that of [11; p.512] (our B1, S are their MC7, MC6); however, Eilenberg-Kelly do not consider general braidings. Examples of braidings which are not symmetries will be given in the next section.

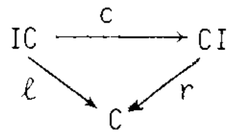
Proposition 1. *In a braided monoidal category, the solid arrows in the following diagrams* B3-B7 *commute*. [The symbol ⊗ has been omitted from the objects to save space.]

B3.

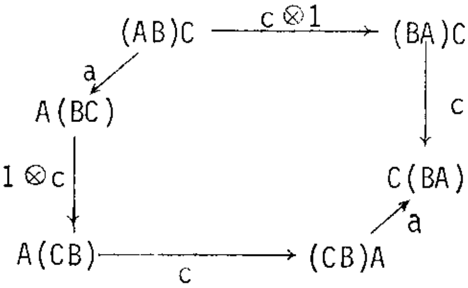

AI
c
IA
r
ℓ
A

,

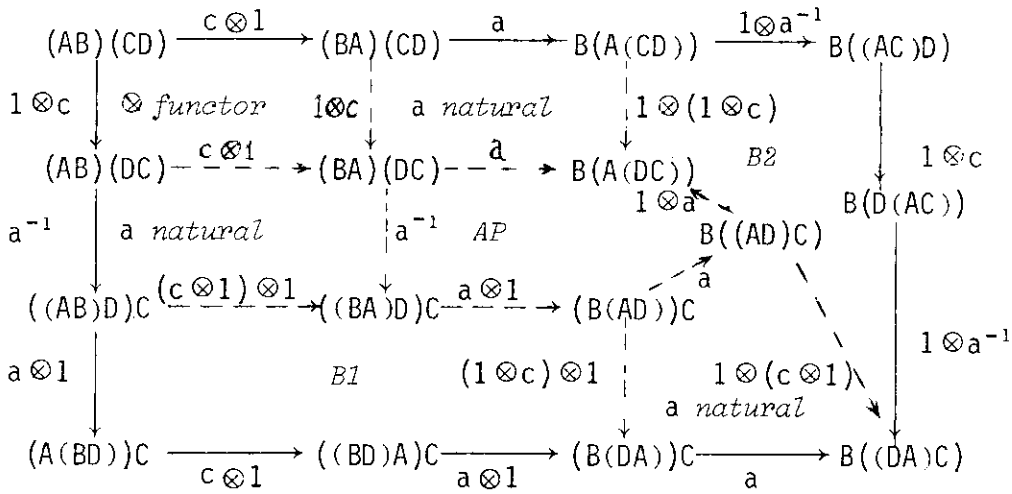
B4.


IC
c
CI
ℓ
r
C

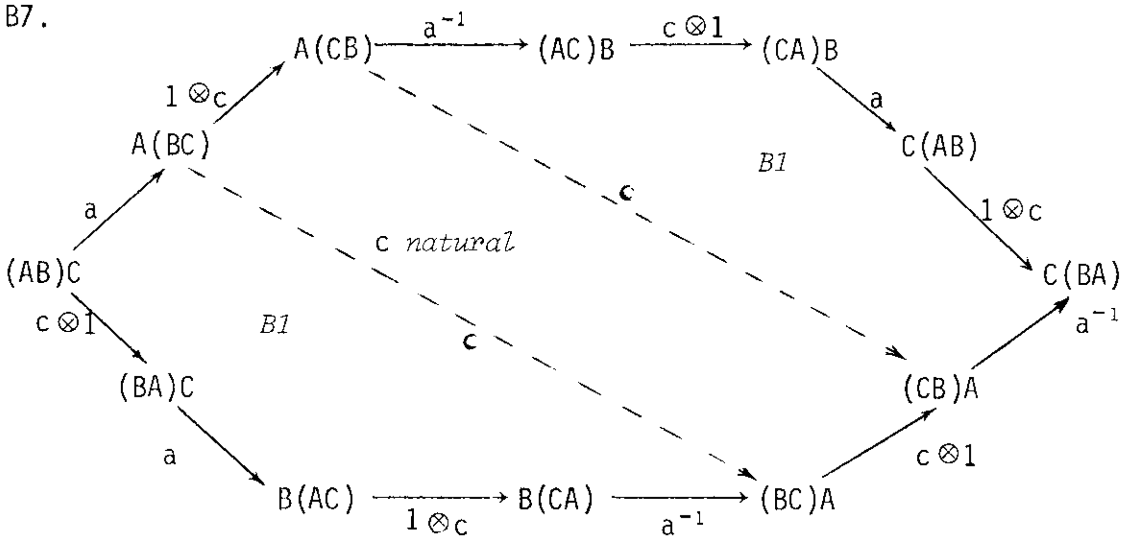
B5.


(AB)C
c ⊗ 1
(BA)C
a
A(BC)
c
1 ⊗ c
C(BA)
a
A(CB)
c
(CB)A

B6.


(AB)(CD)
c ⊗ 1
(BA)(CD)
a
B(A(CD))
1 ⊗ a⁻¹
B((AC)D)
1 ⊗ c
⊗ functor
1⊗c
a natural
1 ⊗ (1 ⊗ c)
(AB)(DC)
c ⊗ 1
(BA)(DC)
a
B(A(DC))
B2
1 ⊗ c
1 ⊗ a
B(D(AC))
a⁻¹
a natural
a⁻¹
AP
B((AD)C)
a
((AB)D)C
(c ⊗ 1) ⊗ 1
((BA)D)C
a ⊗ 1
(B(AD))C
1 ⊗ a⁻¹
a ⊗ 1
B1
(1 ⊗ c) ⊗ 1
1 ⊗ (c ⊗ 1)
a natural
(A(BD))C
c ⊗ 1
((BD)A)C
a ⊗ 1
(B(DA))C
a
B((DA)C)

B7.


A(CB)
a⁻¹
(AC)B
c ⊗ 1
(CA)B
1 ⊗ c
a
A(BC)
B1
C(AB)
c
a
1 ⊗ c
c natural
(AB)C
C(BA)
c ⊗ 1
B1
a⁻¹
c
(BA)C
(CB)A
a
c ⊗ 1
B(AC)
1 ⊗ c
B(CA)
a⁻¹
(BC)A

Proof. B3. Take $B = C = I$ in B1, use the coherence of a, r, ℓ and the invertibility of $c_{A,I}$.

B4. Take $A = B = I$ in B2, use the coherence of a, r, ℓ and the invertibility of $c_{I,C}$.

B5 becomes B7 on replacing the bottom c using B1 and the right-hand c using B2.

B6 and B7 are proved by using the dotted arrows in the diagrams. □

§2. Examples.

1. *Graded modules.*

For a commutative ring K, let V_0 be the category $GMod_K$ of graded K-modules with tensor product given by

$$(A \otimes B)_n = \sum_{p+q=n} A_p \otimes_K B_q \ .$$

The associativity $a : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$ is given by $a((x \otimes y) \otimes z) = x \otimes (y \otimes z)$.

Braidings $c : A \otimes B \longrightarrow B \otimes A$ for this monoidal structure on $GMod_K$ are in bijection with invertible elements k of K via the formula

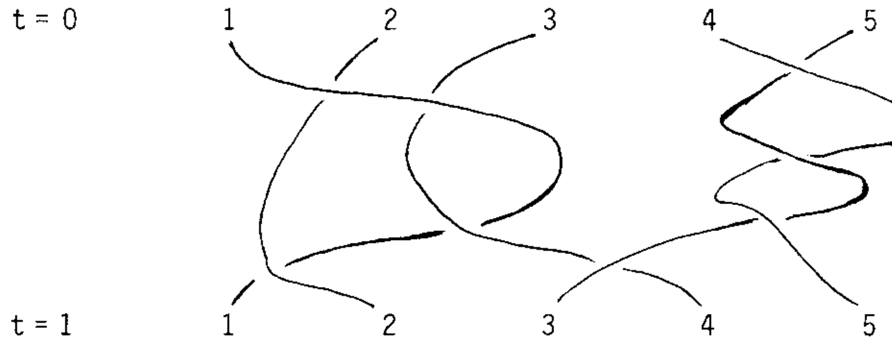
$$c(x \otimes y) = k^{pq}(y \otimes x) \quad \text{where } x \in A_p,\ y \in B_q \ .$$

The proof can be extracted from [11 ; pp.558-559] where it is shown that symmetries are in bijection with elements k of K satisfying $k^2 = 1$.

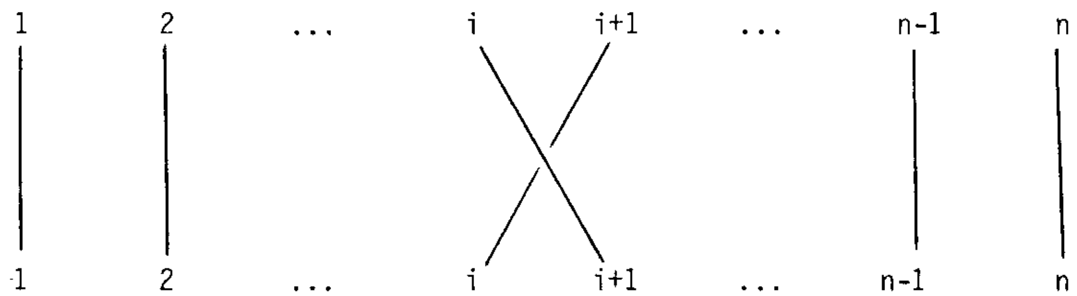
Note that, by taking k non-invertible and defining c as above, we still obtain a natural c satisfying B1,B2. Hence the requirement that a braiding be an isomorphism is independent of the other requirements.

2. *The braid category* $\mathbb{B}$.

Let P denote a Euclidean plane with n distinct collinear points distinguished and labelled $1,2,\ldots,n$. Let $\binom{P}{n}$ denote the space of subsets of P of cardinality n. *The braid group* $\mathbb{B}_n$ *on* n *strings* is the fundamental group of $\binom{P}{n}$. A loop $\omega : [0,1] \longrightarrow \binom{P}{n}$ at the point $\{1,2,\ldots,n\}$ of this space can be depicted by a diagram in Euclidean space of the form



where a horizontal cross-section by P at level $t \in [0,1]$ intersects the curves in the subset $\omega(t)$ of P. Let τ_i be the homotopy class of the loop depicted by the following diagram



for $i=1,\ldots,n-1$. A presentation of $\mathbb{B}_n$ is given by the generators $\tau_1, \tau_2, \ldots, \tau_{n-1}$ and relations

BG1. $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$ for $i=1,\ldots,n-2$.

BG2. $\tau_i \tau_j = \tau_j \tau_i$ for $|i-j| > 1$, $i,j=1,\ldots,n-1$.

For details see [1], [3], [5].

There are relations between the various $\mathbb{B}_n$. There are group homomorphisms

$$h : \mathbb{B}_m \longrightarrow \mathbb{B}_{m+n} , \qquad k : \mathbb{B}_n \longrightarrow \mathbb{B}_{m+n}$$

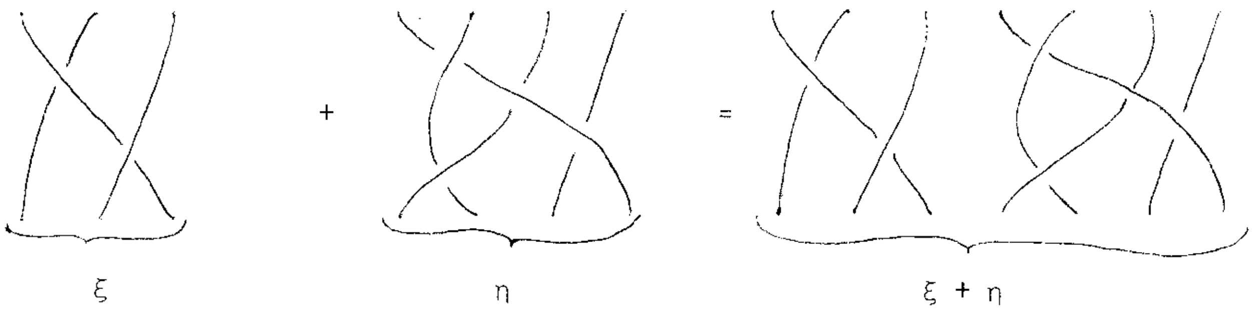
given by : $h(\tau_i) = \tau_i$ for $i=1,\ldots,m-1$,

$k(\tau_i) = \tau_{m+i}$ for $i=1,\ldots,n-1$.

Note that τ_m is not in the image of h or k. By BG2, elements in the image of h commute with elements in the image of k; hence the function

$$+ \ : \ \mathbb{B}_m \times \mathbb{B}_n \longrightarrow \mathbb{B}_{m+n}$$

given by $\xi + \eta = h(\xi)\,k(\eta)$ is a group homomorphism called *addition of braids*. Pictorially, addition of braids amounts to juxtaposition of diagrams.



The *braid category* $\mathbb{B}$ is the coproduct of the $\mathbb{B}_n$ as one-object categories. More explicitly, the objects of $\mathbb{B}$ are the natural numbers 0,1,2,..., the homsets are given by

$$\mathbb{B}(m,n) = \begin{cases} \mathbb{B}_n & \text{when } m = n, \\ \emptyset & \text{when } m \neq n, \end{cases}$$

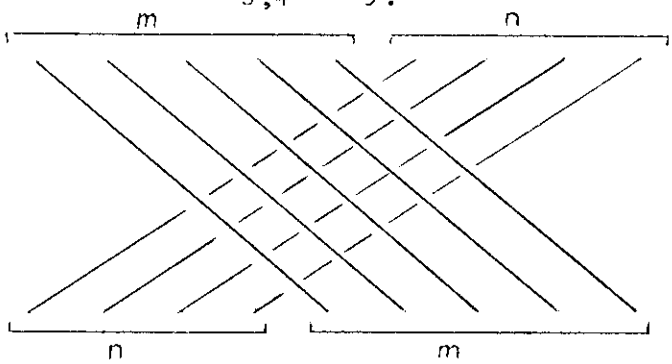
and, the composition is multiplication in the braid groups.

The tensor product $+ : \mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B}$ is given on objects by addition of natural numbers and on arrows by addition of braids. This defines a strict monoidal structure on $\mathbb{B}$ with identity object $I = 0$.

This brings us to the definition of a braiding

$$c = c_{m,n} : m+n \longrightarrow n+m$$

for the strict monoidal category $\mathbb{B}$. The idea is illustrated by the following diagram for $c_{5,4} \in \mathbb{B}_9$.



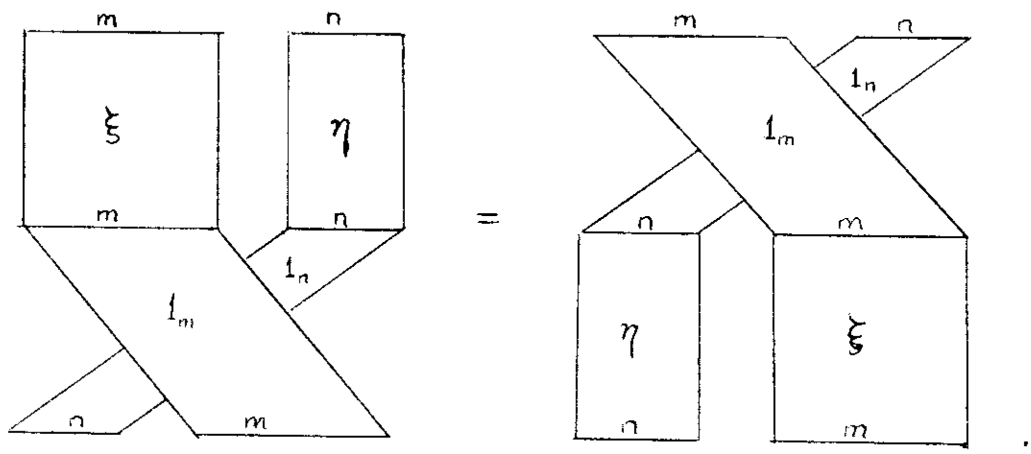
To describe this algebraically, put

$$\gamma = \tau_n \tau_{n-1} \cdots \tau_2 \tau_1 \in \mathbb{B}_{n+1} \quad \text{and}$$
$$\gamma^{(p)} = 1_p + \gamma + 1_{m-p-1} \in \mathbb{B}_{m+n} \quad \text{for} \quad p=0,1,\ldots,m-1.$$

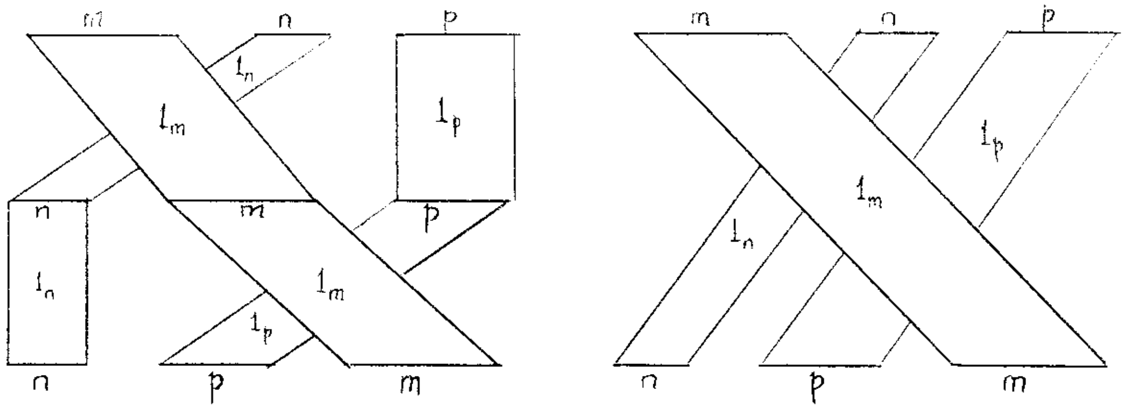
Then

$$c_{m,n} = \gamma^{(0)} \gamma^{(1)} \cdots \gamma^{(m-1)} \in \mathbb{B}_{m+n} .$$

Naturality of $c_{m,n}$ is proved pictorially by:



Axiom B1 is proved pictorially by:



For axiom B2, view the last picture from behind the page. Algebraic proofs seem to add nothing to our discussion and so will not be included. This braiding is not a symmetry since $c_{1,1} = \tau_1 \in \mathbb{B}_2$ and $\tau_1 \tau_1 \neq 1_2$.

3. *Ribbons and braids.*

Fred Linton provided the following example (after a lecture on the material of this paper [33]) which shows that B3-B7 of Proposition 1 do not imply B1 or B2. The monoidal category $\mathbb{L}$ is defined similarly to $\mathbb{B}$ except that the arrows are braids on ribbons (instead of on strings) and it is permissible to twist the ribbons. Each ribbon has two edges which act as strings so there is a faithful strict monoidal functor $\mathbb{L} \longrightarrow \mathbb{B}$ taking n to 2n. Let $c_{m,n} : m+n \longrightarrow n+m$ be defined for $\mathbb{L}$ as for $\mathbb{B}$ except that each ribbon is also given one twist through 2π. Since B1,B2 involve an odd number of c's whereas B5-B7 each involve an even number of c's, and, since B1-B7 all hold when the twists are ignored (by Example 2), we see that $c_{m,n}$ is a natural isomorphism satisfying B5-B7 but not B1 or B2.

4. *Crossed modules with bracket operations.*

Crossed modules appeared in the work of J.H.C. Whitehead [35] on homotopy theory. J-L. Verdier (1965) observed that they amount to groups in the category of groupoids. The underlying category of any group in the category Cat of categories is automatically a groupoid (who first observed this we do not know!), so *crossed modules are precisely groups in* Cat. Regarding the group multiplication as a tensor product, we obtain a strict monoidal category. The extra structure required for a braiding occurs naturally in homotopy theory as the Samelson bracket [34 ; p.467]. The details follow.

A *crossed semi-module* $(N,E,\partial,*)$ consists of monoids N,E, a monoid homomorphism $\partial : N \to E$, and, a function $* : E \times N \to N$ (written between the arguments), satisfying the following axioms:

$$e * 1 = 1, \qquad e * (uv) = (e*u)(e*v),$$

$$1 * u = u, \qquad (ef) * u = e * (f * u),$$

$$\partial(e*u)e = e\partial(u), \qquad (\partial(u) * v)u = uv.$$

The first four of these say that $*$ is an *action* of E on N (that is, $E \longrightarrow \mathrm{End}(N)$, $e \longmapsto e * -$, is a monoid homomorphism) and the last two say the action behaves like an *abstract conjugation* "$e * u = e u e^{-1}$". A crossed module is a crossed semi-module in which N,E are groups [4 ; p.102].

A *bracket operation* for a crossed semi-module is a function $\{\ ,\ \} : E \times E \longrightarrow N$ into the invertible elements of N satisfying the following axioms

$$\partial\{e,f\}fe = ef, \qquad \{1,g\} = \{f,1\} = 1,$$

$$\{ef,g\} = (e * \{f,g\})\{e,g\}, \qquad \{e,gf\} = \{e,f\}(f * \{e,g\}),$$

$$\{\partial(u),f\}(f*u) = u, \qquad \{e,\partial(v)\}v = e * v.$$

This operation should be thought of as an *abstract commutator*.

Each crossed semi-module $(N,E,\partial,*)$ yields a strict monoidal category V as follows. The objects of V_0 are the elements of E. An arrow $u : e \longrightarrow e'$ in V_0 is an element u of N with $e = \partial(u)e'$. Composition in V_0 is multiplication in N. The tensor product is given by

$$(e \xrightarrow{u} e') \otimes (f \xrightarrow{v} f') = (ef \xrightarrow{u(e' * v)} e'f').$$

Braidings for this V precisely amount to bracket operations via the formula

$$c_{e,f} = \{e,f\} : ef \longrightarrow fe.$$

In the very special case where N,E are commutative monoids and $\partial, *$ are trivial (that is, $\partial(u) = 1$ and $e * u = u$ for all $c \in E, u \in N$), a bracket operation is a function $\{,\} : E \times E \longrightarrow N$ into the invertible elements of N satisfying the conditions

$$\{1,g\} = \{f,1\} = 1,$$
$$\{ef,g\} = \{e,g\}\{f,g\}, \qquad \{e,gf\} = \{e,f\}\{e,g\} .$$

These conditions are precisely those for a *bilinear map* $E \times E \longrightarrow N$ when the more familiar additive notation is used in the commutative monoids.

When $E = N = \mathbb{Z}$ and the bilinear map $\mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ is multiplication, the associated braided monoidal category is an algebraic model of the homotopy 3-type of the 2-sphere (as in unpublished work of Joyal-Tierney).

When E is the additive group $\mathbb{Z}$ and N is the multiplicative monoid of a commutative ring K, each invertible element k of K gives a bracket $\mathbb{Z} \times \mathbb{Z} \longrightarrow K$ with $\{p,q\} = k^{pq}$. The associated braided monoidal *additive* category leads to Example 1 via convolution as described in Example 5 below.

5. *Convolution.*

Convolution of monoidal structures was discussed by Day [6] : we centre attention on pages 17-29 of that paper.

Let U be a cocomplete symmetric (braided would do here but that requires ⊗ and $(\ )^{op}$ of U-categories discussed later) monoidal closed category. The notion of *promonoidal* U-*category* P is defined on pages 17 and 18 of [6] (although the word "premonoidal" was used there). A *braiding* for a promonoidal U-category is defined just as Day defines "symmetry" on page 23, except that PC3 is deleted and PC4 (the analogue of B1) is augmented by the obvious analogue of B2.

If P is a small braided monoidal U-category then the U-functor U-category [P,U] with the convolution structure is a cocomplete braided monoidal closed category. (All the diagrams need to prove this already appear in Day [6].)

Each braided monoidal U-category gives rise to a braided promonoidal U-category [6 ; p.26]. So all our small examples of braided monoidal categories can be convoluted to give more examples. We point out the particular example [$\mathbb{B}$,Set] as worthy of detailed study (elsewhere).

§3. Multiplications on monoidal categories.

For monoidal categories V, V', a *monoidal functor* $\Phi = (\phi, \tilde{\phi}, \phi^0) : V \to V'$ consists of a functor $\phi : V_0 \longrightarrow V_0'$, a natural transformation $\tilde{\phi} = \tilde{\phi}_{AB} : \phi A \otimes \phi B \longrightarrow \phi(A \otimes B)$, and, an arrow $\phi^0 : I \longrightarrow \phi I$, satisfying axioms MF1, MF2, MF3 [11; p.473]. Call Φ *strong* when $\tilde{\phi}$, ϕ^0 are invertible. Call Φ *strict* when $\tilde{\phi}$, ϕ^0 are identities.

Let MC_L denote the 2-category of (small) monoidal categories, monoidal functors, and, monoidal natural transformations. The 2-category of more interest here is the sub-2-category MC of MC_L with the same objects, with strong monoidal functors as arrows, and, with monoidal natural transformations as 2-cells. There is also the 2-category MC_S of strict monoidal categories, strict monoidal functors, and, monoidal natural transformations; so MC_S is the 2-category of monoids in the 2-category Cat of categories.

The 2-category MC_L admits products and the projections are strict monoidal functors; so the sub-2-categories MC, MC_S are closed under formation of products. The product of V, V' in MC_L has $(V \times V')_0 = V_0 \times V_0'$, $(A,A') \otimes (B,B') = (A \otimes B, A' \otimes B')$, $I = (I,I)$. The terminal object MC_L is the category $\mathbb{1}$ with one object and one arrow enriched with its unique monoidal structure.

A binary operation on an object V of MC is a strong monoidal functor $\Phi : V \times V \longrightarrow V$. For reference we give axiom MF3 in this case.

MF3.
$$\begin{array}{ccc} (\phi(A,A')\phi(B,B'))\phi(C,C') & \xrightarrow{\quad a \quad} & \phi(A,A')(\phi(B,B')\phi(C,C')) \\ \downarrow{\scriptstyle \tilde{\phi} \otimes 1} & & \downarrow{\scriptstyle 1 \otimes \tilde{\phi}} \\ \phi(AB,A'B')\phi(C,C') & & \phi(A,A')\phi(BC,B'C') \\ \downarrow{\scriptstyle \tilde{\phi}} & & \downarrow{\scriptstyle \tilde{\phi}} \\ \phi((AB)C,(A'B')C') & \xrightarrow[\phi(a,a)]{\qquad} & \phi(A(BC),A'(B'C')) \end{array}$$

A *pseudo-identity* for the binary operation in MC (as in any 2-category) is a nullary operation $\mathbb{1} \longrightarrow V$ which acts as an identity for Φ up to an invertible 2-cell. Up to isomorphism there is only one arrow $\mathbb{1} \longrightarrow V$ in MC, namely, the strong monoidal functor taking the one object of $\mathbb{1}$ to I. Hence Φ admits a pseudo-identity if and only if there exist isomorphisms

$$\lambda_A : A \cong \phi(I,A), \qquad \rho_A : A \cong \phi(A,I)$$

such that $\lambda_I = \phi^0 = \rho_I$ and the following commute.

PI.

$$\begin{array}{ccc} AB & \xrightarrow{\lambda} & \phi(I,AB) \\ {\scriptstyle \lambda\otimes\lambda}\downarrow & & \uparrow{\scriptstyle \phi(\ell_I,1)} \\ \phi(I,A)\phi(I,B) & \xrightarrow[\tilde{\phi}]{} & \phi(II,AB) \end{array} \quad , \quad \begin{array}{ccc} AB & \xrightarrow{\rho} & \phi(AB,I) \\ {\scriptstyle \rho\otimes\rho}\downarrow & & \uparrow{\scriptstyle \phi(1,r_I)} \\ \phi(A,I)\phi(B,I) & \xrightarrow[\tilde{\phi}]{} & \phi(AB,II). \end{array}$$

A binary operation Φ together with λ,ρ as above will be called a *multiplication on* V.

The following result generalizes the fact that the binary operation of a commutative monoid is a monoid homomorphism.

Proposition 2. *If* c *is a braiding for a monoidal category* V *then a multiplication* (Φ,λ,ρ) *on* V *is defined by*

$$\phi = -\otimes- : V_0 \times V_0 \longrightarrow V_0 ,$$

$$\begin{array}{ccc} (A\otimes A')\otimes(B\otimes B') & \xrightarrow{\tilde{\phi} = m} & (A\otimes B)\otimes(A'\otimes B') \\ {\scriptstyle (1\otimes a^{-1})a}\downarrow & & \downarrow{\scriptstyle (1\otimes a^{-1})a} \\ A\otimes((A'\otimes B)\otimes B') & \xrightarrow[1\otimes(c_{A',B}\otimes 1)]{} & A\otimes((B\otimes A')\otimes B'), \end{array}$$

$$\lambda_A = \ell_A^{-1} : A \longrightarrow I\otimes A, \qquad \textit{and}, \qquad \rho_A = r_A^{-1} : A \longrightarrow A\otimes I.$$

Proof. Conditions MF1, MF2, PI follow from B3, B4 of Proposition 1 whereas MF3 comes from B6. □

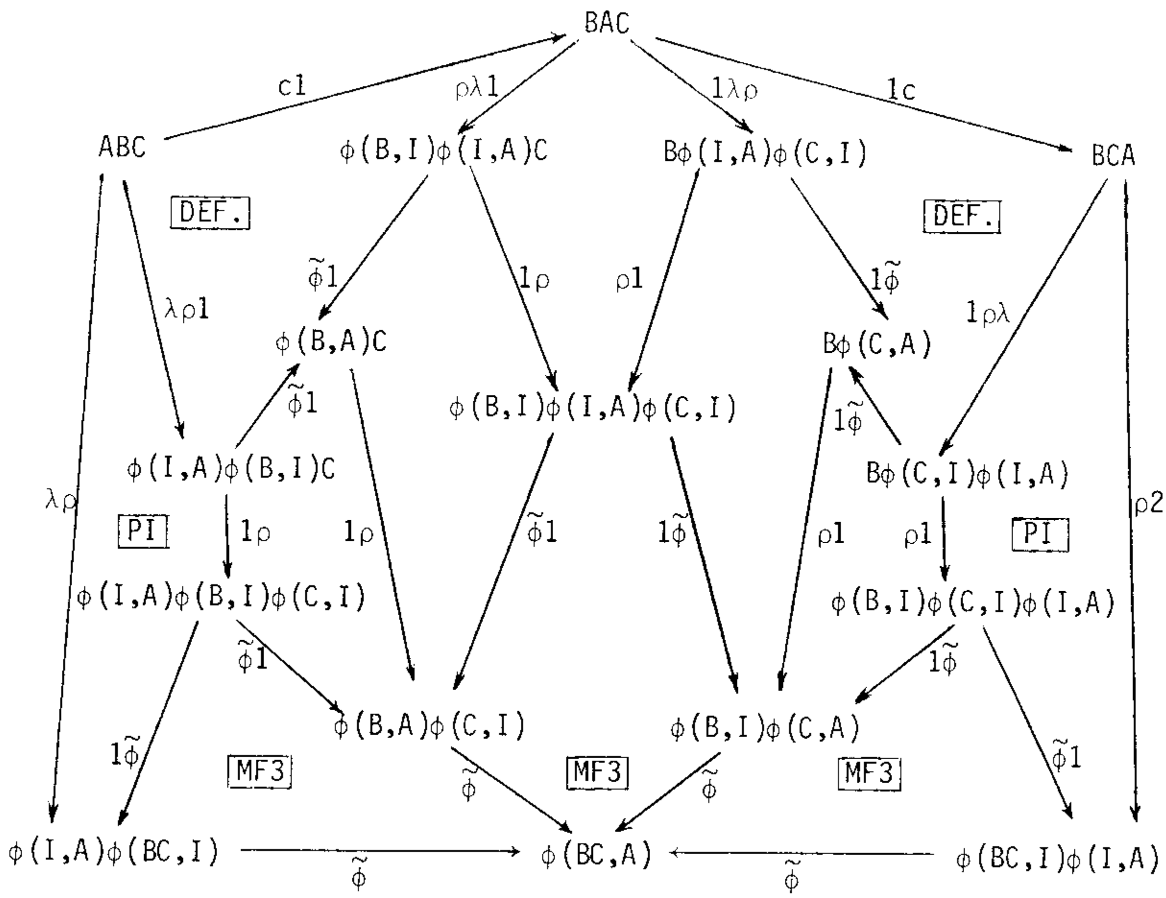
The following result generalizes the fact that, for a monoid in the category of monoids, the two binary operations agree and the monoid is commutative (Eckmann-Hilton [9]).

<u>Proposition 3</u>. *For any multiplication* (Φ,λ,ρ) *on a monoidal category* V, *the following diagram defines a braiding* c *for* V.

$$
\begin{array}{ccc}
A\otimes B & \xrightarrow{\quad c_{A,E}\quad} & B\otimes A \\
\downarrow{\scriptstyle \lambda\otimes\rho} & & \downarrow{\scriptstyle \rho\otimes\lambda} \\
\phi(I,A)\otimes\phi(B,I) & & \phi(B,I)\otimes\phi(I,A) \\
\downarrow{\scriptstyle \tilde{\phi}} & & \downarrow{\scriptstyle \tilde{\phi}} \\
\phi(I\otimes B,A\otimes I) & & \phi(B\otimes I,I\otimes A) \\
& {\scriptstyle \phi(\ell,r)}\searrow \quad \phi(B,A) \quad \swarrow{\scriptstyle \phi(r,\ell)} &
\end{array}
$$

The multiplication obtained from this c *via Proposition 2 is isomorphic (in the obvious sense) to* (Φ,λ,ρ). *If* c' *is any braiding for* V *and* (Φ,λ,ρ) *is obtained from* c' *via Proposition 2 then* c = c'.

<u>Proof</u>. Since each arrow in the definition of c is a natural isomorphism, it remains to prove B1,B2. The following diagram proves B1 for V strict monoidal. We leave it to the reader to modify the diagram in the general case.



The last two sentences of the Proposition are straightforward. □

For braided monoidal categories V, V', a monoidal functor $\Phi : V \longrightarrow V'$ is said to be *braided* when the following diagram commutes.

$$\begin{array}{ccc} \phi A \otimes \phi B & \xrightarrow{\ c\ } & \phi B \otimes \phi A \\ {\scriptstyle \tilde{\phi}}\downarrow & & \downarrow{\scriptstyle \tilde{\phi}} \\ \phi(A \otimes B) & \xrightarrow[\phi c]{} & \phi(B \otimes A) \end{array}$$

Let BMC_L denote the 2-category of braided monoidal categories, braided monoidal functors, and, monoidal natural transformations. Restricting to braided strong monoidal functors, we have the 2-category BMC. Further restricting to braided strict monoidal categories and braided strict monoidal functors, we have the 2-category BMC_S. Of these, the one of most interest is BMC.

The results of this section can be summarized as an equivalence between the 2-category BMC and the appropriate 2-category Mult(MC) of monoidal categories with multiplication.

§4. Coherence for braidings.

One form of the coherence theorem for monoidal categories is that every monoidal category V is equivalent in MC to a strict monoidal category (= monoid in Cat) [25], [2], [27]. However, it is *not* true that every braided (or even symmetric) monoidal category is equivalent in BMC to a commutative monoid in Cat. The reason is that $c : A \otimes A \longrightarrow A \otimes A$ is generally not the identity of $A \otimes A$ and this distinction is preserved by equivalence.

Recall from Kelly [17] that the free symmetric strict monoidal category on $\mathbb{1}$ is the category $\mathbf{P}$ of finite cardinals and permutations. The analogue for braided strict monoidal categories comes out of the next result.

Theorem 4. *For each braided monoidal category* V, *evaluation at* $1 \in \mathbf{B}$ *is an equivalence of categories*

$$\mathrm{BMC}(\mathbf{B}, V) \simeq V_0 .$$

If V *is strict monoidal, this restricts to an isomorphism of categories*

$$\mathrm{BMC}_S(\mathbf{B}, V) \cong V_0 .$$

Proof. Let M be a strict monoidal category with $V \simeq M$ in MC. Clearly the braiding on V transports to a braiding on M such that $V \simeq M$ lifts to BMC. There is a commutative diagram of functors

$$\begin{array}{ccc} \mathrm{BMC}(\mathbb{B}, V) & \xrightarrow{\ \mathrm{ev}_1\ } & V_0 \\ {\scriptstyle\simeq}\big\downarrow & & \big\downarrow{\scriptstyle\simeq} \\ \mathrm{BMC}(\mathbb{B}, M) & \xrightarrow[\ \mathrm{ev}_1\ ]{} & M_0 \end{array}$$

So it suffices to prove the Theorem for V strict monoidal.

For each object A of V_0, we shall describe the unique braided strict monoidal functor $\Phi : \mathbb{B} \rightarrow V$ with $\phi(1) = A$. To preserve tensor product, we are forced to put $\phi(n) = A^n$ (where again we put $A \otimes B = AB$). To give ϕ on arrows we must define a monoid homomorphism $\phi : \mathbf{B}_n \longrightarrow V_0(A^n, A^n)$ for each n. Since Φ is to be braided and $c_{1,1} = \tau_1 : 2 \longrightarrow 2$ in $\mathbf{B}$, we are forced to have $\phi(\tau_1) = c_{A,A} : A^2 \longrightarrow A^2$. But then the equality $\tau_i = 1_{i-1} + \tau_1 + 1_{n-i-1} : (i-1)+2+(n-i-1) \rightarrow (i-1)+2+(n-i-1)$ in $\mathbb{B}$ forces the definition

$$\phi(\tau_i) = 1_{A^{i-1}} c_{A,A} 1_{A^{n-i-1}} : A^{i-1}AAA^{n-i-1} \longrightarrow A^{i-1}AAA^{n-i-1} .$$

To see that this gives the desired monoid homomorphism we must see that the relations of the braid group are preserved: BG1 follows from B7 of Proposition 1 and BG2 from functoriality of $\otimes$ in V. Naturality of the equality $\phi(m)\phi(n) = \phi(m+n)$ in $m,n \in \mathbf{B}$ follows from the definition of addition of braids (look at the images of h,k separately). Hence we have a strict monoidal functor $\Phi : \mathbf{B} \longrightarrow V$. Properties B1,B2 in $\mathbf{B}$ and the fact that $n = 1+\ldots+1$ show that each $c_{m,n}$ is built up from $c_{1,1} = \tau_1 : 2 \longrightarrow 2$ using the monoidal structure. Thus $c_{A,A} = \phi(\tau_1)$ implies Φ is braided.

Now we show that evaluation at 1 is fully faithful. Take braided strong (not necessarily strict) monoidal functors $\Phi, \Psi : \mathbf{B} \longrightarrow V$ and an arrow $f : \phi(1) \rightarrow \psi(1)$ in V_0. Let $\tilde{\phi}^n : \phi(1)^n \rightarrow \phi(n)$ be defined inductively by $\tilde{\phi}^0 = \phi^0 : I \rightarrow \phi(0)$ and

$$\tilde{\phi}^{n+1} = (\phi(1)^n \phi(1) \xrightarrow{\tilde{\phi}^n 1} \phi(n)\phi(1) \xrightarrow{\tilde{\phi}} \phi(n+1)).$$

In order to have a monoidal natural transformation $\alpha : \Phi \longrightarrow \Psi$ with $\alpha_1 = f$, we are forced to define α_n by the commutative diagram

$$\begin{array}{ccc} \phi(1)^n & \xrightarrow{\quad f^n \quad} & \psi(1)^n \\ {\scriptstyle\tilde{\phi}}\big\downarrow\cong & & \cong\big\downarrow{\scriptstyle\tilde{\psi}} \\ \phi(n) & \xrightarrow[\quad \alpha_n \quad]{} & \psi(n) \end{array} .$$

The naturality of α follows from the naturality of $\tilde{\alpha},\tilde{\psi}$ and the braidedness of Φ,Ψ (it suffices to see that $\psi(\tau_i)\alpha_n = \alpha_n\phi(\tau_i)$). That α is monoidal follows from commutativity of the diagram

$$\begin{array}{ccc} \phi(1)^m\phi(1)^n & \xrightarrow{\quad \tilde{\phi}^m\tilde{\phi}^n \quad} & \phi(m)\phi(n) \\ \Big\| & & \big\downarrow{\scriptstyle\tilde{\phi}} \\ \phi(1)^{m+n} & \xrightarrow[\quad \tilde{\phi}^{m+n} \quad]{} & \phi(m+n) \end{array}$$

and the similar one for Ψ (these commutativities can be proved by induction directly or by appeal to [22] for coherence of monoidal functors). □

A slight modification of the work of Kelly [17] on "clubs" gives the 2-monadicity of BMC_s over Cat and that the 2-monad is determined by the free object on $\mathbb{1}$. The second sentence of Theorem 4 precisely states that $\mathbb{B}$ is the free braided strict monoidal category on $\mathbb{1}$. Again from Kelly [15], [17], [18] we know that the 2-category of braided monoidal categories and braided strict monoidal functors is 2-monadic over Cat. In this sense, the free braided monoidal category $\mathcal{B}$ on $\mathbb{1}$ is such that the category of braided strict monoidal functors $\mathcal{B} \longrightarrow V$ is isomorphic to V_0 2-naturally in braided monoidal V. Let $\Gamma : \mathcal{B} \longrightarrow \mathbb{B}$ correspond to $1 \in \mathbb{B}$ under this isomorphism.

Theorem 4 implies that $\Gamma : \mathcal{B} \longrightarrow \mathbb{B}$ *is an equivalence in* BMC. The objects of $\mathcal{B}$ are the *integral shapes* of Kelly-Mac Lane [20]; they include $I, 1$ and $T \otimes S$ for any integral shapes T, S. The arrows are built up from the basic a,r,ℓ,c using $\otimes$ and composition. The faithfulness of Γ implies the following coherence result:

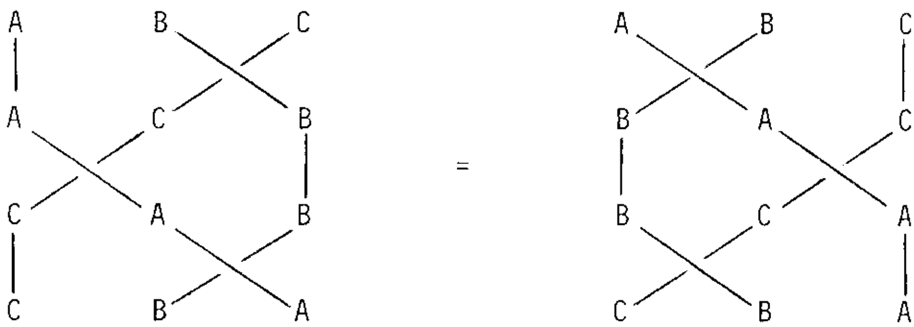
to test whether a diagram built up from a,r,ℓ,c *commutes in all braided monoidal categories it suffices to see that each leg of the diagram has the same underlying braid.*

For example, the following equalities of braids reprove B5, B6, B7 of Proposition 1.

B5.

B6.

B7.



§5. Categories enriched over braided monoidal categories.

Categories with homs enriched in a monoidal category V were defined by Eilenberg-Kelly [11 ; pp.495-496]; they are more briefly called V-*categories*. There is a 2-category (= hypercategory) V-Cat of (small) V-categories, V-functors and V-natural transformations. We write $V\text{-Cat}^{co}$ for the 2-category obtained from V-Cat by reversing 2-cells (but not 1-cells).

Proposition 5. *Suppose* V *is a braided monoidal category. For each* V-*category* A *a* V-*category* A^{op} *is defined by the following data:*

(i) objects of A^{op} are those of A ;

(ii) $A^{op}(A,B) = A(B,A)$;

(iii) $j : I \longrightarrow A^{op}(A,A)$ is $j : I \longrightarrow A(A,A)$; and,

(iv)
$$\begin{array}{ccc} A^{op}(B,C)\otimes A^{op}(A,B) & \xrightarrow{\ M\ } & A^{op}(A,C) \\ \| & & \| \\ A(C,B)\otimes A(B,A) & & A(C,A) \\ & {}_{c}\searrow \qquad \nearrow_{M} & \\ & A(B,A)\otimes A(C,B)\ . & \end{array}$$

The assignment $A \longmapsto A^{op}$ *is the object function of an isomorphism of 2-categories*

$$(\quad)^{op} : V\text{-Cat}^{co} \longrightarrow V\text{-Cat}.$$

Proof. Compare [11 ; p.514]. The only difference is that we cannot appeal to MacLane's coherence to prove commutativity of the top hexagon in the diagram on p.515 of [11]. However, the hexagon does commute by Proposition 1, B5. □

A word of warning: $(A^{op})^{op} \neq A$. The isomorphism $(\quad)^{op}$ is not involutory unless the braiding is a symmetry. However, since $(\quad)^{op}$

is an isomorphism *the principle of duality does apply to the general braided case.*

<u>Proposition 6</u>. *Suppose* V *is a braided monoidal category and* A,B *are* V*-categories. The following data define a* V*-category* C *denoted by* $A \otimes B$:

(i) *objects of* C *are ordered pairs* (A,B) *of objects* A,B *of* A, B, *respectively;*

(ii) $C((A,B),(A',B')) = A(A,A') \otimes B(B,B')$;

(iii)
$$\begin{array}{ccc} I & \xrightarrow{\quad j \quad} & C((A,B),(A,B)) \\ \downarrow{\scriptstyle \ell^{-1}} & & \| \\ I \otimes I & \xrightarrow[\quad j \otimes j \quad]{} & A(A,A) \otimes B(B,B); \text{ and,} \end{array}$$

(iv)
$$\begin{array}{ccc} C((A',B'),(A'',B'')) \otimes C((A,B),(A',B')) & \xrightarrow{M} & C((A,B),(A'',B'')) \\ \| & & \| \\ (A(A',A'') \otimes B(B',B'')) \otimes (A(A,A') \otimes B(B,B')) & & A(A,A'') \otimes B(B,B'') \\ {\scriptstyle m} \searrow & & \nearrow {\scriptstyle M \otimes M} \\ & (A(A',A'') \otimes A(A,A')) \otimes (B(B',B'') \otimes B(B,B')) & \end{array}$$

where m *is the "middle-four interchange" appearing in Proposition 2. Taking the definitions precisely as in Eilenberg-Kelly* [11 ; p.519], V-Cat *becomes a braided monoidal 2-category.*

<u>Proof</u>. The only difference here from [11] is that we cannot appeal to MacLane's coherence to prove commutativity of the top hexagon in the bottom diagram of p.518. The hexagon does commute in our case too by Proposition 1, B6. □

Opposite monoids and tensor products of monoids in V are special cases of the above results since a *monoid in* V is a one-object V-category. A monoid A in a braided V is *commutative* when the following diagram commutes.

$$\begin{array}{ccc} A \otimes A & \xrightarrow{c_{A,A}} & A \otimes A \\ & {\scriptstyle M}\searrow \quad \swarrow{\scriptstyle M} & \\ & A & \end{array}$$

Proposition 6 allows us to define *monoidal* V-*categories* and *braidings* thereon.

§6. Cohomology of groups.

Let $H^3(G,M)$ denote the 3-dimensional cohomology group of the group G with coefficients in the G-module M (by which we mean a module over the group ring $\mathbb{Z}G$). The purpose of this section is to show that the usual interpretation (see Brown [4 ;p.103] and the references there) of $H^3(G.M)$ in terms of crossed modules can be viewed as a combination of an easy interpretation in terms of compact monoidal groupoids and a coherence result which allows the replacement, up to equivalence, of a pseudo-structure by a genuine one.

The 2-category H^3 is defined as follows. An object $(G,M.h)$ consists of a group G, a G-module M and a function $h : G^3 \longrightarrow M$

$$h(x,1,y) = 0$$

$$uh(x,y,z) + h(u,xy,z) + h(u,x,y) = h(u,x,yz) + h(ux,y,z)$$

(that is, h is a *normalized* 3-*cocycle*). An arrow $(g,p,k) : (G,M,h) \longrightarrow (G',M',h')$ consists of a group homomorphism $g : G \longrightarrow G'$ (so that M' becomes a G-module), a G-module homomorphism $p : M \longrightarrow M'$ and a function $k : G^2 \longrightarrow M'$ such that

$$k(x,1) = k(1,y) = 0$$

$$ph(x,y,z) + k(xy,z) + k(x,y) = k(x,yz) + (gx)k(y,z) + h'(gx,gy,gz)$$

(that is, $ph-h'g^3$ is *the coboundary* of a normalized k). A 2-cell $\theta : (g,p,k) \Rightarrow (g_1,p_1,k_1) : (G,M,h) \rightarrow (G',M',h')$ is a function $\theta : G \rightarrow M'$ such that

$$\theta(1) = 0$$

$$\theta(xy) + k(x,y) = k_1(x,y) + \theta(x) + g_1(x)\theta(y)$$

(that is, $k - k_1$ is *the coboundary* of a normalized θ). Compositions are defined in the obvious way; however, if there is any doubt, we are about to define an embedding T of H^3 into the 2-category MC (see

Section 3) of monoidal categories and the compositions are preserved by T.

For $(G,M,h) \in H^3$, the monoidal category $V = T(G,M,h)$ is defined as follows:

- objects are elements of G;
- $V_0(x,y) = \begin{cases} M & \text{for } x = y, \\ 0 & \text{for } x \neq y; \end{cases}$
- composition is addition in M;
- tensor product is given by

$$(x \xrightarrow{\mu} x) \otimes (y \xrightarrow{\nu} y) = (xy \xrightarrow{\mu + x\nu} xy);$$

- the associativity isomorphism is

$$h(x,y,z) : (xy)z \longrightarrow x(yz);$$

- the identity element of G acts as a strict identity object of V.

For each arrow $(g,p,k) : (G,M.h) \to (G',M',h')$, there is a monoidal functor $\Phi = T(g,p,k) : T(G,M,h) \to T(G',M',h')$ defined as follows:

$$\phi(x \xrightarrow{\mu} x) = (g(x) \xrightarrow{p(\mu)} g(x));$$

$$\tilde{\phi}_{x,y} = k(x,y) : g(x)g(y) \longrightarrow g(xy);$$

$$\phi^0 = 0 : 1 \longrightarrow g(1).$$

The monoidal natural transformation $T\theta : T(g,p,k) \Rightarrow T(g_1,p_1,k_1)$ is given by $(T\theta)_x = \theta(x) : g(x) \longrightarrow g_1(x)$.

A monoidal category V is called *compact* when, for all objects A, there exist an object A^* and arrows $n_A : I \longrightarrow A^* \otimes A$, $e_A : A \otimes A^* \to I$ such that the following composites are identities.

$$A \xrightarrow{r^{-1}} A\otimes I \xrightarrow{1\otimes n} A\otimes(A^*\otimes A) \xrightarrow{a^{-1}} (A\otimes A^*)\otimes A \xrightarrow{e\otimes 1} I\otimes A \xrightarrow{\ell} A$$

$$A^* \xrightarrow{\ell^{-1}} I\otimes A^* \xrightarrow{n\otimes 1} (A^*\otimes A)\otimes A^* \xrightarrow{a} A^*\otimes(A\otimes A^*) \xrightarrow{1\otimes e} A^*\otimes I \xrightarrow{r} A^*$$

(This differs from the definition in the literature [15 , 7 , 19] in that we do not require symmetry.)

Proposition 7. *If* V *is a monoidal category such that, for all objects* A, *there exist an object* A^* *and an invertible arrow* $e_A : A\otimes A^* \longrightarrow I$ *then* V *is compact.*

Proof. Define $n_A : I \longrightarrow A^*\otimes A$ to be the composite

$$I \xrightarrow[e_{A^*}^{-1}]{} A^*A^{**} \xrightarrow[1\otimes\ell^{-1}]{} A^*(IA^{**}) \xrightarrow[1\otimes(e_A^{-1}\otimes 1)]{} A^*((AA^*)A^{**}) \xrightarrow[1\otimes a]{} A^*(A(A^*A^{**}))$$

$$\xrightarrow[1\otimes(1\otimes e_{A^*})]{} A^*(AI) \xrightarrow[1\otimes r]{} A^*A.$$

The two diagrams which complete the proof will be omitted. □

A *groupoid* is a category in which all arrows are invertible. So a monoidal groupoid is compact if and only if, for each object A, there exists an object A^* with $A\otimes A^* \longrightarrow I$.

Let CMG denote the full sub-2-category of MC consisting of the compact monoidal groupoids. Notice that T actually lands in CMG.

Recall that a 2-functor $F : K \longrightarrow L$ is called a *biequivalence* [31] when it induces an equivalence

$$K(K,K') \longrightarrow L(FK,FK')$$

on hom-categories for all K,K', and, for all objects L of L, there exist an object K of K and an equivalence $FK \simeq L$ in L .

Proposition 8. *The 2-functor* $T : H^3 \longrightarrow$ CMG *is a biequivalence.*

Proof. A monoidal natural transformation $T(g,p,k) \Rightarrow T(g_1,p_1,k_1)$ is precisely a 2-cell $(g,p,k) \Rightarrow (g_1,p_1,k_1)$ in H^3. Every monoidal functor is isomorphic to a *normal* one (that is, one with ϕ^0 an identity) and a normal monoidal functor $T(G,M,h) \longrightarrow T(G',M',h')$ is precisely an arrow $(G,M,h) \longrightarrow (G',m',h')$ in H^3. So T induces equivalences on hom-categories.

The structure of "compact monoidal groupoid" transports across equivalences of categories. Since every category is equivalent to a skeletal category, each object of CMG is equivalent to a $V \in$ CMG such that $A \longrightarrow B$ in V_0 implies $A = B$. Let G denote the group of objects of V: the multiplication is tensor product $A \otimes B$ and the inverse of A is A^*. Let M denote group $V_0(I,I)$ of endo-arrows of I under composition. Since $(f,g) \longrightarrow r_I(f \otimes g)r_I^{-1}$ defines a homomorphism $M \times M \longrightarrow M$ taking $(1,f)$ and $(f,1)$ to f, it follows that M is an *abelian* group and $fg = r_I(f \otimes g)r_I^{-1}$ (Eckmann-Hilton [9] again). We have homomorphisms $\lambda_A, \rho_A : M \longrightarrow V_0(A,A)$ given by the composites

$$\lambda_A : V_0(I,I) \xrightarrow{-\otimes A} V_0(I \otimes A, I \otimes A) \xrightarrow{V_0(\ell^{-1},\ell)} V_0(A,A)$$

$$\rho_A : V_0(I,I) \xrightarrow{A \otimes -} V_0(A \otimes I, A \otimes I) \xrightarrow{V_0(r^{-1},r)} V_0(A,A)$$

which are dinatural [26;p.214] in A with inverses given by the composites

$$V_0(A,A) \xrightarrow{-\otimes A^*} V_0(A \otimes A^*, A \otimes A^*) \xrightarrow{V_0(e^{-1},e)} V_0(I,I)$$

$$V_0(A,A) \xrightarrow{A^* \otimes -} V_0(A^* \otimes A, A^* \otimes A) \xrightarrow{V_0(n,n^{-1})} V_0(I,I) .$$

It follows that $A \cdot f = \lambda_A^{-1}(\rho_A(f))$ defines an action of G on M so that M becomes a G-module. Let W denote the underlying category of $T(G,M,0)$ and define a functor $\phi : W_0 \longrightarrow V_0$ by $\phi(A) = A$ and

$\phi(A \xrightarrow{f} A) = \lambda_A(f) : A \longrightarrow A$ where $f \in M$. Then ϕ is an isomorphism of categories. There is a unique normal (that is, each r_A, ℓ_A an identity) monoidal structure W on W_0 for which ϕ underlies a monoidal functor $\Phi : W \longrightarrow V$. The associativity isomorphism for W gives a normalized 3-cocycle $h : G^3 \longrightarrow M$. Clearly $T(G,M,h) = W \cong V$ in CMG. □

Consider the following classical construction from the cohomology of groups [23,4].

Let G be a group and let F denote the free group on the underlying pointed set of G so that the inclusion $\sigma : G \longrightarrow F$ of generators is a function satisfying $\sigma(1) = 1$. There is a unique group homomorphism $\omega : F \longrightarrow G$ satisfying $\omega\sigma(x) = x$. Let R denote the free group on the square of the underlying pointed set of G so that the inclusion of generators gives a function $\tau : G^2 \longrightarrow R$ with $\tau(x,1) = \tau(1,y) = 1$ for all x,y G. There is a unique group homomorphism $\kappa : R \longrightarrow F$ satisfying

$$\kappa\tau\,(x,y) \;=\; \sigma(x)\sigma(y)\sigma(xy)^{-1} .$$

This gives a short-exact sequence of groups

$$1 \longrightarrow R \xrightarrow{\kappa} F \xrightarrow{\omega} G \longrightarrow 1 .$$

If M is a G-module, we can extend this to an exact sequence of groups

$$1 \xrightarrow{0} M \xrightarrow{\iota} K \xrightarrow{\partial} F \xrightarrow{\omega} G \longrightarrow 1$$

by taking $K = M \times R$, $\iota(\mu) = (\mu,1)$, $\partial(\mu,\rho) = \kappa(\rho)$. Since M is abelian and R is free, each $x \in G$ and function $h : G^3 \longrightarrow M$ determine a unique homomorphism $\eta_x : K \longrightarrow K$ satisfying

$$\eta_x(\mu,\tau(y,z)) \;=\; (x\mu + h(x,y,z),\ \tau(x,y)\tau(xy,z)\tau(x,yz)^{-1}).$$

provided $h(x,1,z) = h(x,y,1) = 0$. If h is a 3-cocycle, we deduce the identity

$$\eta_u \eta_x = \gamma(u,x)\eta_{ux}$$

where $\gamma(u,x)$ is the inner automorphism of K given by conjugation with $(0,\tau(u,x))$. It follows that each η_x is an automorphism of K, that an action of F on K is defined by the equation

$$\sigma(x)*_h(\mu,\rho) = \eta_x(\eta,\rho) ,$$

and that $(K,N,\partial,*_h)$ is a crossed module.

For $(G,M,h) \in H^3$, let $S(G,M,h)$ denote the strict monoidal category obtained as in Section 2 Example 4 from the crossed module $(K,N,\partial,*_h)$ classically constructed above. Since $(K,N,\partial,*_h)$ is a crossed module and not merely a crossed semi-module, $S(G,M,h)$ is a group in Cat.

An explicit description of $S(G,M,h)$ is as follows:

- objects are elements of F,
- arrows $\mu\colon w \to w'$ have $\omega(w) = \omega(w')$ and $\mu \in M$,
- composition is addition in M, and,
- tensor product is determined by

$$(w \xrightarrow{\mu} \sigma(x))\otimes(\sigma(y)\sigma(z) \xrightarrow{\nu} \sigma(yz)) = (w\sigma(y)\sigma(z) \xrightarrow{\mu+x\nu+h(x,y,z)} \sigma(x)\sigma(yz))$$

where $\omega(w) = x$.

Notice that an arrow $\mu\colon w \to w'$ determines a unique element $\rho \in R$ with $w = \kappa(\rho)w'$ (since $ww'^{-1} \in \ker \omega = \operatorname{im} \kappa$) and hence a unique element $(\mu,\rho) \in K$ with $w = \partial(\mu,\rho)w'$. Notice also that the underlying category of $S(G,M,h)$ is independent of both the action of G on M and the cocycle h.

There is a monoidal functor $\Sigma\colon T(G,M,h) \to S(G,M,h)$ which is *strict* (despite the fact that $T(G,M,h)$ is *not* strict unless $h = 0$). The underlying functor σ of Σ takes an object $x \in G$ to the object $\sigma(x) \in F$ and takes an arrow $\mu\colon x \to x$ to $\mu\colon \sigma(x) \to \sigma(x)$. The reader should draw the diagram MF3 for Σ to understand how h enters into this.

Proposition 9. $\Sigma\colon T(G,N,h) \to S(G,N,h)$ *is a monoidal equivalence.*

Proof. Since the underlying categories are groupoids, it suffices that the underlying functor σ of Σ should be an equivalence of categories. That σ is fully faithful is obvious. For each $w \in F$ we have $0\colon \sigma(\omega(w)) \to w$ in the groupoid $S(G,M,h)$; so σ is surjective up to isomorphism on objects. $\square$

Corollary 10. *Every compact monoidal groupoid is equivalent in* CMG *to a group in* Cat. $\square$

The above Corollary is an immediate consequence of Propositions 8 and 9. However, it is one of a class of coherence results which assert that certain pseudo-algebraic structures on categories are equivalent to strict such structures. The first step of proving these results is to form a free structure on the set of objects of the category with the pseudo-structure. In the present case this is mirrored by the passage from G to F.

The classical construction of a 3-cocycle from a crossed module $(N,E,\partial,*)$ is also contained in the above results. To see this, let V denote the group in Cat obtained from the crossed module as in Example 4 of Section 2. Propositions 8 and 9 yield an equivalence

$$\Phi: S(G,M,h) \to V$$

for some $(G,M,h) \in H^3$. This h is the desired cocycle. More explicitly, G and M are the cokernel and kernel of $\partial: N \to E$, the action of G on M is induced by that of E on N, and, the cocycle h is obtained as follows. We have the exact sequence

$$1 \longrightarrow M \xrightarrow{\iota} N \xrightarrow{\partial} E \xrightarrow{\omega} G \longrightarrow 1$$

of groups. Since w is surjective we can choose a function $\sigma: G \to E$ with $\sigma(1) = 1$ and $\omega\sigma = 1_G$. Since $\sigma(x)\sigma(y)$ and $\sigma(xy)$ are in the same fibre of w, we can choose a function $\tau: G^2 \to N$ with

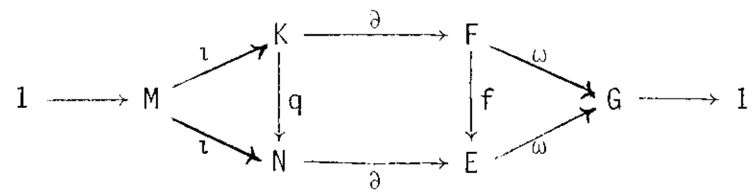
$$\tau(x,1) = 1 = \tau(1,y)$$

$$\sigma(xy) = \partial(\tau(x,y))\sigma(x)\sigma(y) .$$

Then $\tau(x,y)\tau(xy,z)$ and $(\sigma(x) * \tau(y,z))\tau(x,yz)$ are in the same fibre of ∂ so there is a unique function $h: G^3 \to M$ determined by the equation

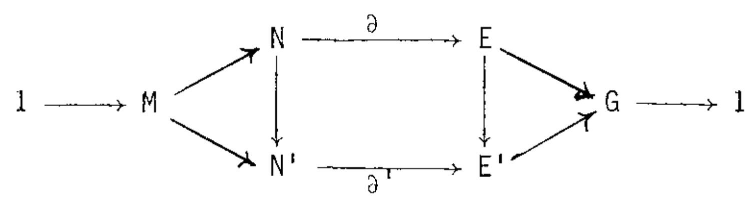
$$\tau(x,y)\tau(xy,z) = \iota(h(x,y,z))(\sigma(x) * \tau(y,z))\tau(x,yz).$$

Then h is a normalized 3-cocycle and the equivalence Φ is obtained from the commutative diagram



where $f(\sigma(x)) = \sigma(x)$, $q(\mu,\tau(x,y)) = \iota(\mu)\tau(x,y)$. It is important to observe that Φ: $S(G,M,h) \to V$ is a homomorphism of groups in Cat; that is, *strict* monoidal (although its inverse equivalence cannot generally be chosen to be). This is a phenomenon of coherence theory not particular to this case!

To extract the interpretation theorem of Brown [4; p.103], let M be a G-module. There is a category CrMod(G,M) of crossed modules $(N,E,\partial,*)$ for which G,M are the cokernel, kernel of ∂ and $*$ induces the action of G on M. The arrows are commutative diagrams



of groups in which the vertical homomorphisms are compatible with the actions $*,*'$.

For a category A recall that $\pi_0 A$ denotes the set of undirected path components of objects of A. If a functor $A \to B$ has an adjoint then it induces a bijection $\pi_0 A \cong \pi_0 B$.

Write $H^3(G,M)$ for the subcategory of H^3 consisting of the objects of the form (G,M,h) with the given G, M, and, the arrows (g,p,k) with g,p identity functions. Notice that $H^3(G,M)$ is a groupoid and

$$H^3(G,M) \;=\; \pi_0\mathcal{H}^3(G,M).$$

Corollary 11. *The functor* $\mathcal{H}^3(G,M) \to \mathrm{CrMod}(G,M)$ *taking* (G,M,h) *to* $(K,F,\partial,*_h)$ *has a right adjoint and so induces a bijection*

$$H^3(G,M) \;\cong\; \pi_0\mathrm{CrMod}(G,M).$$

Proof. For a crossed module $(N,E,\partial,*)$, let h' be the 3-cocycle obtained from choices of σ,τ as above. The arrow $(q,f)\colon (K,F,\partial,*_{h'}) \to (N,E,\partial,*)$ in $\mathrm{CrMod}(G,M)$ acts as a component of a counit for the adjunction. □

§7. Cohomology of abelian groups

Suppose G,M are abelian groups. Then M can be regarded as a trivial G-module (via the action $x\mu = \mu$ for $x \in G$, $\mu \in M$) and the cohomology groups $H^n(G,M)$ can be considered. However, it is argued by Eilenberg-Mac Lane [24,10,12] that these groups are inappropriate here since they take no account of the commutativity of G, and so should be replaced by groups $H^n_{ab}(G,M)$. We shall describe $H^3_{ab}(G,M)$.

An *abelian* 3-*cocycle* for G with coefficients in M is a pair (h,c) where $h: G^3 \to M$ is a normalized 3-cocycle

$$h(x,0,y) = 0$$

$$h(x,y,z) + h(y,x+y,z) + h(u,x,y) = h(u,x,y+z) + h(y+x,y,z)$$

and $c: G^2 \to M$ is a function satisfying

$$h(y,z,x) + c(x,y+z) + h(x,y,z) = c(x,z) + h(y,x,z) + c(x,y)$$

$$-h(z,x,y) + c(x+y,z) - h(x,y,z) = c(x,z) - h(x,z,y) + c(y,z).$$

For any function $k: G^2 \to M$ satisfying

$$k(x,0) = k(0,y) = 0,$$

the *coboundary* of k is the abelian 3-cocycle $\partial(k) = (h,c)$ defined by the equations

$$h(x,y,z) = k(y,z) - k(x+y,z) + k(x,y+z) - k(x,y)$$

$$c(x,y) = k(x,y) - k(y,x).$$

Then $H^3_{ab}(G,M)$ is the abelian group of abelian 3-cocycles modulo the coboundaries.

A function $t: G \to M$ is called *quadratic* when it satisfies the conditions

$$t(-x) = t(x)$$

$$t(x+y+z) + t(x) + t(y) + t(z) = t(y+z) + t(z+x) + t(x+y).$$

The *trace* of an abelian 3-cocycle (h,c) is the function $t: G \to M$ given by $t(x) = c(x,x)$. A calculation shows that traces are quadratic functions.

Theorem 12. (Eilenberg-Mac Lane [24,10,12]). *Trace induces an isomorphism between the group* $H^3_{ab}(G.M)$ *and the group of quadratic functions from* G *to* M.

Proof. For fixed M, trace is natural in abelian groups G. As ~~additive functors~~ from the category of abelian groups to its dual, $H^3_{ab}(-,M)$ and quadratic functions into M preserve filtered colimits. Every abelian group G is a filtered colimit of finitely generated abelian groups. Every finitely generated abelian group is a finite direct sum of cyclic groups. So it suffices to verify the isomorphism when G is cyclic.

false! They are quadratic functors.

If G is the integers under addition, then $H^3(G,M) = 0$. If $G = \{0,1,\ldots,n-1\}$ with addition modulo n, then

$$H^3(G,M) \cong \{\mu \in M \mid n\mu = 0\}$$

where the coset of the 3-cocycle h given by

$$h(x,y,z) = \begin{cases} 0 & \text{for } y+z < n \\ x\mu & \text{for } y+z \geq n \end{cases}$$

corresponds to μ.

Suppose $t: G \to M$ is a quadratic function for G cyclic as above. Put $\nu = t(1)$. By induction using the quadratic property we see that $t(x) = x^2\nu$. Define $c(x,y) = xy\nu$. If G is infinite $(0,c)$ is an abelian 3-cocycle whose trace is t. If G has order n, notice that $2n\nu = n^2\nu = 0$ in order for t to be well defined. Let h be the 3-cocycle defined in terms of $\mu = n\nu$ as above. Then (h,c) is an abelian 3-cocycle with trace t. This proves surjectivity.

Suppose (h',c') is an abelian 3-cocycle with $c(x,x) = 0$. If G is infinite take h to be the zero 3-cocycle. If G is of order n take h to be the 3-cocycle of the form above for some μ such that $h - h'$ is a coboundary. Then $(h,c) - (h',c')$ is a coboundary for some c with $c(x,x) = 0$. Using $c(1,1) = 0$ one deduces $c(1,y) = 0$ and then $c(x,y) = 0$ by induction. Then $h = 0$. So (h',c') represents the zero element of $H^3_{ab}(G.M)$. This proves injectivity. □

The remarkable observation connecting braidings with cohomology is the following.

Proposition 13. *To say that* c *is a braiding for the monoidal category* $T(G,M,h)$ *is precisely to say* (h,c) *is an abelian 3-cocycle.*

Proof. The properties of c in the definition of abelian 3-cocycle precisely amount to B1 and B2. □

The 2-category H^3_{ab} is defined as follows. An object (G,M,h,c) consists of abelian groups G,M together with an abelian 3-cocycle (h,c). An arrow (g,p,k): (G,M,h,c) $\to$ (G',M',h',c') is an arrow (g,p,k): (G.M.h) $\to$ (G',M',h') in H^3 (with trivial module actions) such that

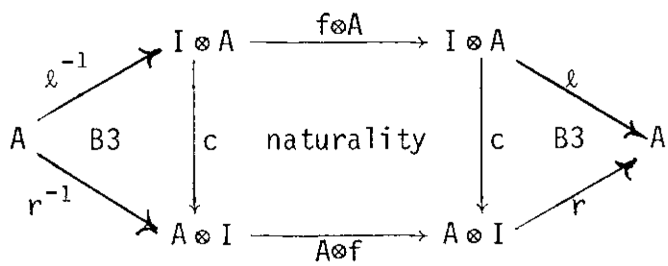
$$pc(x,y) + k(x,y) = k(y,x) + c(g(x),g(y)).$$

A 2-cell $\theta:(g,p,k) \Rightarrow (g_1,p_1,k_1)$ is precisely a 2-cell in H^3. Compositions are as in H^3.

Let CBMG denote the full sub-2-category of BMC consisting of the compact braided monoidal groupoids. Write T(G,M,h,c) for the compact monoidal groupoid T(G,M,h) enriched with the braiding c (Proposition 13). Notice that, for an arrow (g,p,k) in H^3_{ab}, the monoidal functor T(g,p,k) is braided. Hence we have a 2-functor T: $H^3_{ab} \to$ CBMG.

<u>Proposition 14</u>. *The 2-functor* T: $H^3_{ab} \to$ CBMG *is a biequivalence.*

<u>Proof</u>. That T induces equivalences on hom-categories follows as in the proof of Proposition 8. Proceed as in that proof where this time V is braided. For any f: I $\to$ I, we have the commutative diagram



So this time the functions λ_A, ρ_A are equal and hence the action of G on M is trivial. The braiding c on V yields a function

$d\colon G^2 \to M$ given by

$$d(A,B) = \lambda^{-1}_{A\,B}(c(A,B))$$

and hence $(G,M,h,d) \in \mathcal{H}^3_{ab}$ with $T(G,M,h,d) \cong V$ in CBMG. □

Let Quad denote the category whose objects are quadratic functions $t\colon G \to M$ between abelian groups, and, whose arrows are commutative squares

$$\begin{array}{ccc} G & \xrightarrow{\;t\;} & M \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p} \\ G' & \xrightarrow[\;t'\;]{} & M' \end{array}$$

with g,p homomorphisms of groups. As with any category we can regard Quad as a 2-category with only identity 2-cells.

Trace defines a 2-functor $tr\colon \mathcal{H}^3_{ab} \to$ Quad taking (G,m,h,c) to the trace of (h,c), taking (g,p,k) to (g,p), and, taking θ to an identity.

For abelian groups G,M, we write $\underset{\sim}{\text{Ext}}(G,M)$ for the groupoid whose objects are short exact sequences $0 \to M \to E \to G \to 0$ for abelian groups and whose arrows are commutative diagrams

$$\begin{array}{ccccccc} & & & E & & & \\ & & \nearrow & \big\downarrow & \searrow & & \\ 0 \longrightarrow & M & & & & G & \longrightarrow 0 \\ & & \searrow & & \nearrow & & \\ & & & E_1 & & & \end{array}$$

of homomorphisms of groups. The abelian group $\text{Ext}(G,M)$ of homological algebra is the group $\pi_0\underset{\sim}{\text{Ext}}(G,M)$ of isomorphism classes of objects of $\underset{\sim}{\text{Ext}}(G,M)$.

Proposition 15. *The underlying functor of* tr: $H^3_{ab} \to$ Quad *is surjective on objects, full and conservative (= reflects isomorphisms). Furthermore, the fibres of the functor*

$$\text{tr}: H^3_{ab}((G,M,h,c),(G',M',h',c')) \to \text{Quad}(t,t')$$

on hom-categories (where t,t' *are the traces of* (h,c), (h',c')*) are all equivalent to the category* $\underset{\sim}{\text{Ext}}(G,M')$.

Proof. Theorem 12 immediately yields surjectivity on objects. If $(g,p)\colon (G,M,t) \to (G',M',t')$ is an arrow in Quad with t,t' as stated above, then (ph,pc), $(h'g^3,c'g^2)$ are both abelian 3-cocycles for G with coefficients in M' having the same trace $pt = t'g$. By Theorem 12 the abelian 3-cocycles differ by a coboundary and so we have $(g,p,k)\colon (G,M,h,c) \to (G',M',h',c')$ on H^3_{ab} with $\text{tr}(g,p,k) = (g,p)$. So tr is full on arrows.

If (g,p,k) is any arrow of H^3_{ab} with $\text{tr}(g,p,k) = (g,p)$ invertible then g,p are isomorphisms of groups and $(g^{-1},p^{-1},-k)$ is an inverse for (g,p,k). So the underlying functor of tr is conservative.

Now take any $(g,p)\colon t \to t'$ in Quad. The fibre over (g,p) of the functor in the second sentence of the Proposition has objects $k\colon G^2 \to M'$ such that $(g,p,k)\colon (G,N,h,c) \to (G',M',h',c')$ is an arrow of H^3_{ab}, and, has arrows $\theta\colon k \to k_1$ functions $\theta\colon G \to M'$ satisfying $\theta(1) = 0$ and $k_1(x,y) - k(x,y) = \theta(x+y) - \theta(x) - \theta(y)$.

Let $H^2_{ab}(G,M')$ denote the category whose objects f are *abelian 2-cocycles* (that is, functions $f: G^2 \to M'$ such that $f(x,0) = 0$, $f(x,y) = f(y,x)$ and $f(x+y,z) + f(x,y) = f(x,y+z) + f(y,z)$) and whose arrows $\theta: f \to f_1$ are functions $\theta: G \to M'$ such that $\theta(1) = 0$ and $f_1(x,y) - f(x,y) = \theta(x+y) - \theta(x) - \theta(y)$.

Since tr is full on arrows, there does exist k_0 in the fibre category. Any other object k of the fibre gives $f = k - k_0 \in H^2_{ab}(G,M')$ and arrows $\theta: k \to k_1$ of the fibre are precisely arrows $\theta: k - k_0 \to k_1 - k_0$ in $H^2_{ab}(G,M')$. Hence the fibre is isomorphic to $H^2_{ab}(G,M')$.

The equivalence of categories

$$H^2_{ab}(G,M') \simeq \underset{\sim}{\mathrm{Ext}}(G,M')$$

comes from the standard interpretation of H^2 for group cohomology. Each f is a *factor set* and so gives a short exact sequence of groups

$$0 \to M' \to E \to G \to 0$$

(inducing the trivial action of G on M') and E is *abelian* since $f(x,y) = f(y,x)$. □

<u>Corollary 16</u>. *There is a 2-functor* $K: \mathrm{CBMG} \to \mathrm{Quad}$ *satisfying, and determined up to isomorphism by, the condition that* $KT \cong \mathrm{tr}$. *Furthermore,* K *is surjective up to isomorphism on objects, full on arrows, and, any arrow which is inverted by* K *is an equivalence.* □

This means, in a sense, that a quadratic function is a "complete invariant" for a compact braided monoidal groupoid. Each compact braided monoidal groupoid V can be assigned a quadratic function KV; each quadratic function t has the form KV for some V; and, if $KV \cong KV'$ then $V \simeq V'$.

Finally note that each compact braided monoidal groupoid V is equivalent in CBMG to a braided group in Cat (in fact, one of the form $S(G,M,h)$ with a braiding) by Propositions 14 and 9, and, it should be remembered from Section 2 Example 4 that a braided group in Cat amounts to a crossed module with bracket operation.

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