Hall's Marriage Theorem

For sets $A$, $B$, each relation $R \subseteq A \times B$ can be identified with a union preserving function $\wp A \rightarrow \wp B$ between the power sets taking $S \subseteq A$ to the subset $R(S) = \{ b \in B : (a, b) \in R \text{ for some } a \in S \}$
of $B$. In particular, $R(\emptyset) = \emptyset$. Each function $f : A \rightarrow B$ has a graph $f_*$ which is the relation
$\{(a, f(a)) : a \in A\} \subseteq A \times B$.

**Theorem** If $R \subseteq A \times B$ is a relation between finite sets such that $\# S \leq \# R(S)$ for all $S \subseteq A$ then there exists an injective function $f : A \rightarrow B$ with $f_* \subseteq R$.

**Proof** (Halmos 1958?) If $A = \emptyset$ then the unique function $f : A \rightarrow B$ will do.

The property of the relation $R$ in the Theorem gives that $R(S)$ is non-empty for each singleton subset $S$. In particular, if $A$ is a singleton $\{a\}$, there is an element $b \in B$ with $(a, b) \in R$; so we can define $f$ by $f(a) = b$.

We have proved the result when $A$ has cardinality 0 or 1. The proof is completed by induction as follows. One of the two possibilities (a) or (b) must hold:

- (a) there exists a subset $\emptyset \subset S \subset A$ with $\# S = \# R(S)$, or
- (b) $\# S < \# R(S)$ for all $\emptyset \subset S \subset A$.

First consider case (a); so we have $T = R(S)$ with $\# T = \# S$. Let $R'$ be the restriction of $R$ to a relation between $S$ and $T$. Clearly $R'$ satisfies the hypothesis of the Theorem. Since $\# S < \# A$, we inductively obtain an injective (and hence bijective) function $g : S \rightarrow T$ with $g_* \subseteq R'$. In fact, $R' = g_*$ for all $a \in S$; for, if $\# R([a]) > 1$ for some $a \in S$ then
$$\# T = \# R(S) = \# \bigcup_{a \in S} R([a]) > \# g(S) = \# T,$$
a contradiction. Let $R''$ be the restriction of $R$ to a relation between the complements $\neg S$, $\neg T$ of $S$, $T$ in $A$, $B$, respectively. We shall show that the relation $R'' \subseteq \neg S \times \neg T$ satisfies the hypothesis of the Theorem. Take $X \subseteq \neg S$ and let $Y = \{ a \in S : g(a) \in R(X) \}$ so that $R(X \cup Y) = R(X) \cup R(Y) = R''(X) \cup R(Y)$. Since $X, Y$ are disjoint and $R''(X), R(Y)$ are disjoint, we have
$$\# X + \# Y = \# (X \cup Y) \leq \# R(X \cup Y) = \# R''(X) + \# R(Y).$$
But $Y \subseteq S$, so $\# Y = \# R(Y)$ and can be cancelled from the previous inequality to give $\# X \leq \# R''(X)$, as required. Since $\# \neg S < \# A$, induction provides an injective function $h : \neg S \rightarrow \neg T$ with $h_* \subseteq R''$.

Define a function $f : A \rightarrow B$ to agree with $g$ on $S$ and to agree with $h$ on $\neg S$. Since $g, h$ are injective and have disjoint images, $f$ is injective. Also $f_* \subseteq R$ follows from $g_* \subseteq R'$, $h_* \subseteq R''$.

It remains to deal with case (b). Select any $a \in A$ and $b \in R([a])$. Put $C = A \setminus \{a\}$, $D = B \setminus \{b\}$. Let $R'' \subseteq C \times D$ be the restriction of $R$ to a relation between $C, D$. If $\emptyset \subset X \subset C$ then (b) implies that $\# X \leq \# R(X) - 1 \leq \# R''(X)$, so $R'$ satisfies the hypothesis of the Theorem. Since $\# C = \# A - 1$, induction produces an injective function $f : C \rightarrow D$ with $f_* \subseteq R'$. Extend $f$ to $f : A \rightarrow B$ by defining $f(a) = b$. This $f$ is as required.qed