§1. Higher dimensional categories

A partial binary operation $\circ$ on a set $A$ is a partial map from $A \times A$ to $A$ which we write between the arguments. An element $u$ of $A$ is called a $0$-unit when $u \circ a = a$, $b \circ u = b$ for all $a, b$ for which the left-hand sides are defined.

Call a partial binary operation $\circ$ on $A$ a composition on $A$ when it satisfies the axioms:

1. for all $a \in A$, there exists a unique $0$-unit $u$ such that $a \circ u$ is defined, and a unique $0$-unit $v$ such that $v \circ a$ is defined;

2. for all $a, b \in A$, $a \circ b$ is defined if and only if there exists a $0$-unit $u$ such that $a \circ u$ and $u \circ b$ are defined;

3. for all $a, b, c \in A$, $(a \circ b) \circ c = a \circ (b \circ c)$ whenever the left-hand side is defined.

A set together with a composition is called a category. In the notation of (c1) we write $a : u \to v$.

The reader will easily see that this definition agrees with the usual definitions of category when one identifies the elements of the set with arrows and the units with objects.

Suppose $\circ$, $\ast$ are two compositions on a set $A$. 
Call * subordinate to 0, and write * sub 0, when the following axioms hold:

51. every *-unit is a 0-unit;
52. if u, v are 0-units then u * v is a 0-unit provided it is defined;
53. (a * b) * (a' * b') = (a * a') * (b * b') provided the left-hand side is defined.

Following the convention of Roberts [1], we omit brackets when we mean that * should be evaluated before 0; so the expression a * b * c means a * (b * c). It is noted in [1] that 0 is subordinate to itself if and only if 0 is abelian in the sense that a * b = b * a whenever either side is defined.

A 2-category is a set A with two compositions *, o such that * is subordinate to 0. This relates to the usual definition by taking *-units to be objects, 0-units to be arrows (or 1-cells), and elements of A to be 2-cells. The notation

\[
\begin{array}{ccc}
a & \xrightarrow{x} & b \\
\downarrow u & \quad & \quad & \downarrow v \\
\end{array}
\]

can be used to mean a : u \to v, b : u \to v in the category (A, *), and \* : a \to b in the category (A, 0).
An $n$-category is a set $A$ together with a composition $\circ_n$ on $A$ for each natural number $n \geq 0$, where $\circ_m$ is subordinate to $\circ_n$ whenever $m < n$.

A composition on $A$ is discrete when every element of $A$ is a unit.

An $r$-category is an $n$-category $A$ such that $\circ_n$ is discrete. It follows from $S!$ that $\circ_n$ is discrete for all $n > r$.

If $A, B$ have partial binary operations on them, a function $f : A \rightarrow B$ is called a homomorphism when $f$ takes units to units and $f(a \cdot a') = f(a) \cdot f(a')$ whenever the left-hand side is defined. If $A, B$ are categories, a homomorphism is called a functor.

If $A, B$ are $n$-categories, an $n$-functor $f : A \rightarrow B$ is a function which is a homomorphism for respective compositions.

Let $n$-Cat denote the category of $n$-categories and $n$-functors. For each $r$, we have a full subcategory $r$-Cat of $n$-Cat consisting of the $r$-categories. In particular, for $r = 1$, we write $\text{Cat}$ instead of $1$-Cat.
52. Complicial sets.

The category whose objects are the ordered sets $[n] = \{0, 1, \ldots, n\}$ and whose arrows are order-preserving functions will be denoted by $\Delta$. For each $n$ there are $n+1$ monics

$$\varnothing \supseteq \varnothing_1 \supseteq \cdots \supseteq \varnothing_n : [n-1] \rightarrow [n]$$

such that the image of $\varnothing_i$ does not contain $i$; there are $n$ epics

$$\varnothing_0 \subseteq \varnothing_1 \subseteq \cdots \subseteq \varnothing_{n-1} : [n-1] \rightarrow [n-1].$$

A simplicial set is a functor $X : \Delta^\text{op} \rightarrow \text{Set}$. The set $X([n])$ is denoted by $X_n$ and its elements are called elements of $X$ of dimension $n$. We write $d_i : X_n \rightarrow X_{n-1}$ for the $i$-th face function $X(\varnothing_i)$; we write $s_i : X_{n-1} \rightarrow X_n$ for the $i$-th degeneracy function $X(\varnothing_i)$. Call $x \in X_n$ degenerate when it is in the image of some $s_i$.

Let $\text{Simp}^\text{l}$ denote the category $[\Delta^\text{op}, \text{Set}]$ of simplicial sets and simplicial maps (= natural transformations).

A simplicial set with neutrality is a simplicial set $X$ together with a distinguished subset of $X_n$ for each $n \geq 1$ such that each degenerate element is in the distinguished subset of its dimension.
Elements in these distinguished subsets are called neutral elements of $X$. It will be convenient to agree that there are no neutral elements of dimension 0 and that the neutral elements of dimension 1 are precisely the degenerate elements.

Let $\text{NSmpl}$ denote the category of simplicial sets with neutrality, and simplicial maps which preserve neutrality.

The functor $\Delta(-, [n]) : \Delta^\text{op} \to \text{Set}$ represented by $[n]$ is denoted by $\Delta[n]$ and called the standard $n$-simplex. By the Yoneda lemma, a simplicial map $\Delta[n] \to X$ amounts to an element of $X_n$.

For $k \in [n]$, the $k$-horn in $\Delta[n]$ is the subobject $\Lambda^k[n]$ of $\Delta[n]$ consisting of those $\alpha : [m] \to [n]$ whose image does not contain $[n] - \{k\}$. A simplicial map $\Lambda^k[n] \to X$ amounts to $n$ elements $x_i \in X_{n-1}$, $i \neq k$, satisfying the identities:

$$d_i x_j = d_{j-1} x_i \quad \text{for} \ i < j, \ i \neq k, \ j \neq k.$$  

This is called a $k$-horn in $X$.

For $0 < k < n$, we shall define an object $N_k \Delta[n]$ of $\text{NSmpl}$ whose underlying simplicial set is $\Delta[n]$. For $n > 1$, the elements $\alpha : [n] \to [n]$ apart from degeneracies, which we require to be neutral are those $\alpha$ whose images contain $k-1, k, k+1$. 

Write $N_k \Lambda^k[n]$ for the (regular) subobject of $N_k \Delta[n]$ which has underlying simplicial set $\Lambda^k[n]$ and has as neutral elements those of $N_k \Delta[n]$ which are also in $\Lambda^k[n]$.

Let $N'_k \Delta[n]$ denote the object of $\text{NSmpl}$ obtained from $N_k \Delta[n]$ by adding $\partial_0, \partial_{k+1} : [n-1] \to [n]$ as neutral elements.

Let $N''_k \Delta[n]$ denote the object of $\text{NSmpl}$ obtained from $N'_k \Delta[n]$ by adding $\partial_k : [n-1] \to [n]$ as a neutral element.

Recall that an object $C$ is said to be orthogonal to an arrow $f : A \to B$ in a category when, for each arrow $f : A \to C$, there exists a unique arrow $k : B \to C$ with $kf = f$.

A complicial set is an object $C$ of $\text{NSmpl}$ which is orthogonal to the inclusions:

(a) $N_k \Lambda^k[n] \to N_k \Delta[n]$;

(b) $N'_k \Delta[n] \to N''_k \Delta[n]$; $N''_k \Lambda^k[n] \to N''_k \Delta[n]$ for all $n > 0$ and all $0 < k < n$.

Let $\text{Cmpl}$ denote the full subcategory of $\text{NSmpl}$.
consisting of the complicial sets. It follows from the work of Gabriel-Ulmer [1] that the inclusion of Cmpl in NSmpl has a left adjoint \( k \) and that Cmpl is locally presentable.

A \( k \)-horn \( x_i \in C_{n-1}, i \neq k \), \( n \in C \in NSmpl \) will be called complicial when \((C \mu)x_i\) is neutral for all monics \( \mu : [m] \to [n-1] \) and all \( i \) such that the image of \( \beta_i \mu \) contains \( k-1, k, k+1 \).

Conditions (a), (b) above can be reformulated as follows:

(a) if \( x_i \in C_{n-1}, i \neq k \), \( x_i \) is a complicial \( k \)-horn in \( C \) then there exists a unique neutral \( x \in C_n \) with \( d_i x = x_i \) for \( i \neq k \);

(b) if, \( \nu \in (a), x_{\bar{k}-1}, x_{\bar{k}+1} \) are neutral then \( d_k x \) is neutral.

Let Cmpl\(_r\) denote the full subcategory of Cmpl consisting of those complicial sets whose elements of dimension \( > r \) are all neutral.
§3. The nerve of a category

Ordered sets are categories: transitivity gives a composition law on the order relation. In this way, $\Delta$ is a full subcategory of $\text{Cat}$. Each category $A$ yields a simplicial set

$\text{Cat}(\text{-}, A) : \Delta^\text{op} \rightarrow \text{Set}$

called the nerve of $A$ and denoted by $\nu A$. The elements of $\nu A$ of dimension $n$ are composable strings of $n$ arrows in $A$. Clearly nerve becomes a functor

$\nu : \text{Cat} \rightarrow \text{Smp}$

which is fully faithful.

It is easy to see that a simplicial set $C$ is isomorphic to the nerve of some category if and only if the following squares are all pullbacks

\[
\begin{array}{ccc}
C_n & \xrightarrow{d_0} & C_{n-1} \\
\downarrow d_n & & \downarrow d_{n-1} \\
C_{n-1} & \xrightarrow{d_0} & C_{n-2}
\end{array}
\]

$n > 1$

Note that $\text{Smp}_1$ is a full subcategory of $\text{Smp}$ since all elements of dimension $> 1$ of an object of $\text{Smp}_1$ are neutral.
Proposition 1. A simplicial set $C$ is (isomorphic to) the nerve of a category if and only if $C$ is in $\text{Cmpl}_1$.

Proof. If $C \in \text{Cmpl}_1$, we must show that the squares above are all pullbacks for $n > 1$. The case $n = 2$ is precisely property (a) with $k = 1$. Assume the result true for $n-1$ and take $x_0, x_n \in C_{n-1}$ with $d_n x_n = d_{n+1} x_0$. From the calculation:

$$d_i d_{n-1} x_n = d_{n-1} d_i x_n = d_{n-2} d_{i-1} x_0 = d_{n-2} d_{i-1} x_0,$$

the inductive hypothesis gives a unique $x_i$ satisfying $d_i x_i = d_{i-1} x_0$ and $d_{n-1} x_i = d_{i-1} x_n$.

Notice that for $i < j$:

$$d_0 d_{i-1} x_i = d_{i-1} d_0 x_i = d_{i-2} d_{i-1} x_0 = d_{i-2} d_{i-1} x_0 = d_i d_{j-1} x_j = d_i d_{j-1} x_j = d_{j-1} d_i x_j = d_{j-1} d_i x_j.$$

So, again by induction, $d_i x_j = d_{i-1} x_j$. By applying (a) to the, say, 1-horn $x_0, x_2, \ldots, x_n$, there exists a unique $x \in C_n$ with $d_i x = x_i$ for $i \neq 1$. By uniqueness of $x_i$, we also have $d_i x = x_i$. So, in fact, $x$ is unique with $d_n x = x_n$. This proves the squares are pullbacks.

Suppose $C$ is isomorphic to the nerve of some category. We must show that the pullback condition implies property (a) (since (b) is taut). Given $x_i, i \neq k$, a simplicial $k$-horn, note that $d_0 x_n = d_{n+1} x_0$. 
So there is a unique \( x \in C_n \) with \( d_0 x = x_0 \), \( d_n x = x_n \). Then
\[
\begin{align*}
d_0 d_i x &= d_i d_0 x = d_i, \quad x_0 = d_0 x_i, \\
d_{n-1} d_i x &= d_i d_{n-1} x = d_i x_n = d_{n-1} x_i \\
\text{so} \quad d_i x &= x_i. \quad \square
\end{align*}
\]

**Corollary 2.** \( \nu : \text{Cat} \cong \text{Cmpl}_1 \). \( \square \)

If \( C \in \text{Cmpl}_1 \), a category \( A \) with \( \nu A \cong C \) is described as follows:

- the set of arrows \( A \) is \( C_1 \),
- \( x, y \in C_1 \) are composable when \( d_1 x = d_0 y \) in which case \( y \circ x \) is \( d_1 z \) where \( z \in C_2 \) with \( d_0 z = x, d_1 z \).
§4. Alternative characterization of complicial sets.

For $0 < k < n$, we shall call an arrow $\mu : [r] \to [m]$ in $\Delta$ a $k$-monic when its image contains $k-1, k, k+1$.

**Lemma 3.** An arrow $\mu : [r] \to [m]$ in $\Delta$ is a $k$-monic if and only if one of the following conditions holds:

(i) $\mu$ is the identity function;

(ii) $\mu = \delta_i \nu$ where $i < k-1$ and $\nu$ is a $(k-1)$-monic;

(iii) $\mu = \delta_i \nu$ where $i > k+1$ and $\nu$ is a $k$-monic.

**Proof.** If $\mu$ is not the identity and monic then $\mu = \delta_i \nu$ for some monic $\nu : [r] \to [m-1]$. If $\mu$ is a $k$-monic then $i \neq k-1, k, k+1$. If $i < k-1$ then $\mu$ is a $k$-monic precisely when $\nu$ is a $(k-1)$-monic. If $i > k+1$, $\mu$ is a $k$-monic precisely when $\nu$ is a $k$-monic. □

For any simplicial set $C$ with neutrality, put:

$$P_n^k = \{ (x, y) \in C_{n-1} \times C_{n-1} \mid d_k x = d_{k-1} y \},$$

$$D_n^k = \{ x \in C_n \mid (C \mu) x \text{ is neutral for all } k \text{-monics } \mu : [m] \to [m+1] \}.$$

It follows from the Lemma 3 that:

$$D_n^k = \{ x \in C_{n+1} \mid x \text{ is neutral, } d_i x \in D_n^k \text{ for } i < k-1, \text{ and } d_i x \in D_n^k \text{ for } i > k+1 \}.$$
Theorem 3. A simplicial set \( C \) with neutrality is a complicial set if and only if, for all \( 0 < k < n \), there is a function
\[ \phi^k : P^k_n \rightarrow C_n \]

satisfying the following conditions:

(i) \( \phi^k(x, y) \) is neutral;

(ii) \[ d_i \phi^k(x, y) = \begin{cases} \phi^k(d_i x, d_i y) & \text{for } i < k-1, \\ \phi^k(d_{i+1} x, d_{i+1} y) & \text{for } i > k+1; \end{cases} \]

(iii) if \( x, y \) are neutral then so is \( d_k \phi^k(x, y) \);

(iv) if \( w \in D^k_n \) then \( w = \phi^k(d_{k-1} w, d_{k+1} w) \).

Furthermore, under these conditions, each \( \phi^k \) is an isomorphism onto \( D^k_n \) and satisfies the further property:

(v) \[ d_k \phi^k(d_k \phi^k(x, y), z) = d_k \phi^k(x, d_k \phi^k(y, z)). \]

Proof. Suppose \( C \) is a complicial set. Using induction on \( n \), we shall define isomorphisms \( \phi^k : D^k_n \rightarrow D^k_n \) satisfying (i) - (iii).

For \( n = 2 \), \( \phi'(x, y) \) satisfying (i) is uniquely determined by property (a) for \( C \), so \( \phi' \) is an isomorphism.
and (ii) and (iii) follows from (b).

Take \((x, y) \in \mathbb{P}^n_2\). Our inductive hypothesis allows us to consider:

\[
\xi_i = \begin{cases} 
\varphi^{k-1}(d_i x, d_i y) & \text{for } i < k-1 \\
x & \text{for } i = k-1 \\
y & \text{for } i = k+1 \\
\varphi^k(d_i x, d_i y) & \text{for } i > k+1.
\end{cases}
\]

The following calculations show that this defines a k-bond.

For \(i < j < k-1\),
\[
d_i \xi_j = d_i \varphi^{k-1}(d_j x, d_j y) = \varphi^{k-2}(d_j d_i x, d_j d_i y) = \varphi^{k-2}(d_j d_i x, d_i d_j y) = d_j \xi_i.
\]

For \(i < j = k-1\),
\[
d_i \xi_j = d_i x = d_{k-2} \varphi^{k-1}(d_i x, d_i y) = d_j \xi_i.
\]

For \(i < j = k+1\) and \(i < k-1\),
\[
d_i \xi_j = d_i y = d_k \varphi^{k-1}(d_i x, d_i y) = d_j \xi_i.
\]

For \(i = k-1 < j = k+1\),
\[
d_i \xi_j = d_{k-1} y = d_k x = d_j \xi_i.
\]

For \(i = k+1 < j\),
\[
d_i \xi_j = d_{k+1} \varphi^k(d_j x, d_j y) = d_j \xi_i.
\]

For \(k+1 < i < j\),
\[
d_i \xi_j = d_i \varphi^k(d_{i-1} x, d_{i-1} y) = \varphi^k(d_i d_{i-1} x, d_i d_{i-1} y).
\]
\[ = \varphi^k(d_{j-2}d_{j-1}x, d_{j-2}d_{j-1}y) = d_{j-1} \varphi^k(d_{j-1}x, d_{j-1}y) = d_{j-1} \mathcal{F}_i. \]

We shall now show that this \( k \)-horn is simplicial.

Consider \( \mu : [m] \rightarrow [n-1] \) monic and \( 0 \leq i \leq n \) such that
\( d_i \mu \) is a \( k \)-monic. By Lemma 3, either \( i \leq k-1 \) and \( \mu \) is a \((k-1)\)-monic, or \( i > k+1 \) and \( \mu \) is a \( k \)-monic.

In the former case, \( (C \mu)_i = (C \mu) \varphi^{k-1}(d_{i-1}x, d_{i-1}y) \)
which is neutral since induction gives \( \varphi^{k-1}(d_{i-1}x, d_{i-1}y) \in \mathcal{D}_{n-1}^{k-1} \). In the latter case, \( (C \mu)_i = (C \mu) \varphi^k(d_{i-1}x, d_{i-1}y) \)
which is neutral since \( \varphi^k(d_{i-1}x, d_{i-1}y) \in \mathcal{D}_{n-1}^k \).

By property (a) for \( C \), there exists a unique neutral \( \varphi^k(x, y) \in \mathcal{C}_n \) with \( d_i \varphi^k(x, y) = \mathcal{F}_i \) for \( i \neq k \). Thus
(i), (ii) are satisfied, and (iii) follows from (b) for \( C \) and induction. The uniqueness property of \( \varphi^k(x, y) \)
precisely yields that \( \varphi^k \) is an isomorphism.

Properties (c), (iv) follow since \( \varphi^k : P_n \rightarrow \mathcal{D}_n^k \).

Conversely, suppose \( \varphi^k : P_n \rightarrow \mathcal{C}_n \) exist
satisfying (c) – (iv). Suppose \( \mathcal{F}_i \in \mathcal{C}_{n-1} \), \( i \neq k \), form a simplicial \( k \)-horn. Put \( \mathcal{F} = \varphi^k(3_{k-1}, 3_{k+1}) \) which
is neutral by (c). By (i), \( d_{k-1} \mathcal{F} = 3_{k-1} \), \( d_{k+1} \mathcal{F} = 3_{k+1} \).
For \( i < k-1 \), \( \mathcal{F}_i \in \mathcal{D}_{n-1}^{k-1} \) and, by (ii), \( d_i \mathcal{F} = \varphi^{k-1}(d_{i-1} \mathcal{F}, d_{i+1} \mathcal{F}) = \varphi^{k-1}(d_{i-1} \mathcal{F}_i, d_{i+1} \mathcal{F}_i) = \mathcal{F}_i \) by (iv). For \( i > k+1 \),
Lemma 4. (a) If $x \in D_{n+1}^k$ and $d_{n+1} x \in D_1^k$ then $x \in D_{n+1}^k$.
(b) If $x \in D_1^k$ and $d_n x$, $d_{n+1} x \in D_n^k$ then $x \in D_n^{k+1}$.
(c) If $C$ is a complicial set then $x \in D_{n+1}^k$, $d_{n+1} x$, $d_n x, d_{n+1} x \in D_n^k$.

Proof. The cases $n = 2$ all become clear when one notes that $D_2^k$ is the set of neutral elements of $C_2$, $D_2^k$ is the set of neutral $x \in C_2$ with $d_3 x$ neutral, and $D_3^k$ is the set of neutral $x \in C_2$ with $d_0 x$ neutral.

(a) Assume (a) for $n-1$. Take $x \in D_{n+1}^k$ with $d_{n+1} x \in D_n^k$.
To show $x \in D_{n+1}^k$, since $x$ is neutral, we must see that:
(i) $i < k$ implies $d_i x \in D_n^k$;
(ii) $i > k+2$ implies $d_i x \in D_n^k$.

To prove (i), note that the case $i = k+1$ is given so we may assume $i < k+1$. Observe then that $d_i x \in D_n^{k-1}$ (since $x \in D_n^k$) and $d_{k+2} d_i x = d_k d_{k+1} x \in D_n^{k-1}$ (since $d_{k+1} x \in D_n^k$), so by induction $d_i x \in D_n^k$. To prove (ii), suppose $i > k+2$ and observe that $d_0 x \in D_n^k$ (since $x \in D_n^k$) and $d_{k+2} d_i x = d_{k+1} d_k x \in D_n^k$ (since $d_k x \in D_n^k$), so by induction...
\( d_i x \in D_m^{k+1} \) as required.

(b) Assume (b) for \( n-1 \). Take \( x \in D_{n+1}^k \) with \( d_k x, d_{k+1} x \in D_n^{k-1} \). To prove \( x \in D_n^{k-1} \) we must see:

(i) \( i < k-2 \) implies \( d_i x \in D_{n+1}^{k-2} \) and

(ii) \( i > k \) implies \( d_i x \in D_n^{k-1} \).

If \( i < k-2 \) then \( d_i x \in D_{n+1}^{k-1} \) (since \( x \in D_{n+1}^k \) and \( d_k x \in D_{n+1}^{k-1} \)), and \( d_i d_{i+1} x = d_i d_{i+1} x \in D_{n+1}^{k-2} \) (since \( d_k x \in D_{n+1}^{k-1} \)), so by induction \( d_i x \in D_{n+1}^{k-2} \);

thus (i) is true. If \( i > k+1 \) then \( d_i x \in D_{n+1}^k \) (since \( x \in D_{n+1}^k \) and \( d_k x \in D_{n+1}^{k-1} \)), and \( d_{k+1} d_i x = d_{k+1} d_i x \in D_{n+1}^{k-1} \) (since \( d_k x \in D_{n+1}^{k-1} \)), so by induction \( d_i x \in D_{n+1}^{k-2} \). If \( i = k+1 \) then \( d_i x \in D_n^{k-1} \) is given, so (ii) does hold.

(c) Assume (c) for \( n-1 \). Take \( x \in D_{n+1}^k \) with \( d_k x, d_{k+1} x \in D_n^{k+1} \). Using Theorem 3, we have that \( d_k x \) is neutral, so in order to show \( d_k x \in D_n^k \) we must show:

(i) \( i < k-2 \) implies \( d_i d_k x \in D_n^{k-2} \) and

(ii) \( i > k \) implies \( d_i d_k x \in D_n^{k-1} \).

If \( i < k-2 \) then \( d_i d_k x = d_i d_k x \in D_n^{k-2} \) (since \( x \in D_{n+1}^k \), \( d_k x \in D_{n+1}^{k-1} \)), and \( d_k d_{k+1} x = d_k d_{k+1} x \in D_{n+1}^{k-2} \) (since \( d_k x \in D_{n+1}^{k-1} \)), so by induction
induction, $d_{k+1}x \in D_{n-1}^k$ as required for (i). To prove (ii),
suppose $i > k$ and we must show $d_i d_k x = d_k d_{i+1} x \in D_{n-1}^{k-1}$.
But $d_{i+1} x \in D_{n+1}^k$ (since $x \in D_{n+1}^k$), $d_{i+1} x = d_i d_k x$
and $d_{i+1} d_k x = d_i d_{k+1} x$ both lie in $D_{n-1}^{k-1}$ (since $d_{i+1} x, d_k x$
$\in D_{n-1}^{k-1}$); so by induction $d_i d_{i+1} x \in D_{n-1}^{k-1}$, as required.

(d) Take $x \in D_{n+1}^k$ with $d_k x, d_{k+1} x \in D_{n+1}^k$. By theorem,
$d_k x$ is neutral. If $i < k$, then $d_i x \in D_{n-1}^{k-1}$, $d_k d_i x = d_i d_{k+1} x \in D_{n-1}^{k-1}$, $d_k^2 d_i x = d_i d_{k+1} x \in D_{n-1}^{k-1}$, so
by induction $d_i x \in D_{n-1}^{k-1}$; so $d_i d_k x \in D_{n-1}^{k-1}$. If $i > k+1$
then $d_{i+1} x \in D_{n+1}^k$, $d_{k+1} d_{i+1} x = d_{k+1} d_{i+1} x \in D_{n-1}^{k-1}$, $d_k d_{i+1} x$
$= d_i d_{k+1} x \in D_{n-1}^{k-1}$, so by induction $d_i d_k x = d_k d_{i+1} x$
$\in D_{n-1}^{k-1}$. Thus $d_k x \in D_{n+1}^k$. ☐

For $C \in NSmp^l$ and $0 < k \leq n$, there is a
simplicial set $C(n, k)$ defined as follows:

\[ C(n, k)_0 = C_{n-1}^k, \quad C(n, k)_1 = C_n, \]
\[ C(n, k)_m = \left\{ x \in D_{m+n-1}^k \mid d_{k+1} x, \ldots, d_{k+m} x \in C(n, k)_{m-1} \right\} \]

for $m \geq 2$,
the face functions $C(n, k)_m \rightarrow C(n, k)_{m-1}$ are $d_{k+1}, \ldots, d_{k+m}$ of $C$,
the degeneracies $C(n, k)_m \rightarrow C(n, k)_{m-1}$ are $s_{k+1}, \ldots, s_{k+m-2}$ of $C$.
Notice that $C(n, k)_2 = D_{n+1}^k$, and, from Lemma 4(a),
we have:

\[ C(n,k) \subset \bigcap_{l=0}^{m-2} D_{m+n-l} \quad \text{for } m \geq 2. \]

**Theorem 5.** If \( C \) is a simplicial set then, for all \( 0 < k \leq n \), the simplicial set \( C(n,k) \) is the nerve of a category whose set of arrows is \( C_n \) and whose composition \( \cdot \) is given by:

\[ y \cdot x = d_{k+1}^*e_k^*(x,y). \]

**Proof.** We begin by proving the following statement by induction on \( m \):

\[ \text{if } w \in D_{m+n}^{k+l}, d_{k+l-1}w, d_{k+l}w \in C(n,k)_m \text{ for some } 0 \leq k \leq m, \text{ then } w \in C(n,k)_{m+1}. \]

For \( m = 1 \) we must have \( l = 0 \) and the statement is clear from Lemma 4(b).

Assume the result for \( m-1 \) and take \( w \in D_{m+n}^{k+l} \) with \( d_{k+l-1}w, d_{k+l}w \in C(n,k)_m \). We shall show that, for all \( 0 \leq r \leq m-1 \), we have \( w \in D_{m+n}^{k+r} \) and \( d_{k+r}w, d_{k+r+1}w \in C(n,k)_m \).

This must be done by induction on \( r \) in two steps: one with \( r \leq l \) and the other with \( l \leq r \). The starting point for both these inductions is \( r = l \) in which case the result is given. Suppose \( r \leq l \) and we already know \( w \in D_{m+n}^{k+r} \) with \( d_{k+r}w, d_{k+r+1}w \in C(n,k)_m \).

By Lemma 4(b) & (c), we have \( w \in D_{m+n}^{k+r} \). To prove
\(d_{k+1}w \in C(n, k)\) we use our inductive hypothesis; for
\(m-1\) which gives the result provided \(d_{k+1}w \in D_{m+n-1}^{k+1}\)
\(d_{k+1}d_{k+1}w \in C(n, k)_{m-1}\) and \(d_{k+1}d_{k+1}w \in C(n, k)_{m-1}\). The first of these follows since \(w \in D_{m+n}^{k+1}\); the second follows from \(d_{k+1}d_{k+1}w = d_{k+1}d_{k+1}w\) and \(d_{k+1}w \in C(n, k)_m\); the third follows from \(d_{k+1}d_{k+1}w = d_{k+1}d_{k+1}w\) and \(d_{k+1}w \in C(n, k)_m\). To prove \(d_{k+1}w \in C(n, k)\) we again use induction, the inductive hypothesis; so we must see that \(d_{k+1}w \in D_{m+n}^{k+1}\)
\(d_{k+1}d_{k+1}w \in C(n, k)_{m-1}\) and \(d_{k+1}d_{k+1}w \in C(n, k)_{m-1}\). The first follows from Lemma 4(c) since \(w \in D_{m+n}^{k+1}\)
\(d_{k+1}w \in D_{m+n}^{k+1}\). The second follows from \(d_{k+1}d_{k+1}w = d_{k+1}d_{k+1}w\) since we already have proved \(d_{k+1}w \in C(n, k)_m\). The third follows from
\(d_{k+1}d_{k+1}w = d_{k+1}d_{k+1}w\) and \(d_{k+1}w \in C(n, k)_m\).

Now suppose \(l < r\) and we already have \(w \in D_{m+n}^{k+1}\) with \(d_{k+1}w \in C(n, k)_m\). By Lemma 4(a), we have \(w \in D_{m+n}^{k+1}\). To prove \(d_{k+1}w \in C(n, k)_m\) we must show that \(d_{k+1}w \in D_{m+n-1}^{k-1}\), \(d_{k+1}w \in D_{m+n-1}^{k-1}\), \(d_{k+1}w \in D_{m+n-1}^{k-1}\), \(d_{k+1}w \in D_{m+n-1}^{k-1}\), \(d_{k+1}w \in D_{m+n-1}^{k-1}\). The first is true since
\(w \in D_{m+n}^{k+1}\). The second and third follow from
\[ d_{k+r-2} d_{k+r-1} w = d_{k+r} d_{k+r-2} w, \quad d_{k+r} d_{k+r+1} w = d_{k+r} d_{k+r} w \]

and \( d_{k+r-2} w, d_{k+r} w \in C(n, k)_m \). To prove \( d_{k+r-1} w \in C(n, k)_m \) we must show that \( d_{k+r-1} w \in D^{k+r-1}_{m+n-1} \)

\( d_{k+r-2} d_{k+r-1} w \in C(n, k)_{m-1} \) and \( d_{k+r} d_{k+r-1} w \in C(n, k)_{m-1} \)

The first follows from Lemma 4(b) since \( w \in D^{k+r-1}_{m+n-1} \)

\( d_{k+r-2} w, d_{k+r} w \in C(n, k)_m \). The second follows from

\( d_{k+r-2} d_{k+r-1} w = d_{k+r-2} d_{k+r-2} w \) and \( d_{k+r} d_{k+r} w \in C(n, k)_m \)

The third follows from \( d_{k+r} d_{k+r-1} w = d_{k+r-1} d_{k+r+1} w \)

since we already have proved \( d_{k+r+1} w \in C(n, k)_m \).

Thus we have completed our two step induction on \( r \). In particular, we can take \( r = 0 \) and obtain

\( w \in D^k_{m+n} \). Allowing \( r \) to run from 0 to \( m-1 \), we

also get \( d_{k-1} w, \ldots, d_{k+m} w \in C(n, k)_m \) hence

\( w \in C(n, k)_{m+1} \). This completes the induction on \( m \).

Now we turn to a proof of the theorem. Consider \( C(n, k) \in NSmpl \) by taking all elements of degree \( > 1 \)

to be neutral (as they are also in the sense of \( C \)).

To see that \( C(n, k) \) is the nerve of a category we must see, by Proposition 1, that \( C(n, k) \) is a complicial set

\( C(n, k) \).

For this we shall show that Theorem 3 applies to \( C(n, k) \).

Suppose \( x, y \in C(n, k)_{m-1} \), with \( d_{k+l} x = d_{k+l-1} y \).
Then \( w = q^{k+l}(x, y) \in D_{m+n-1} \) with \( d_{k+l-1}w = x \), \( d_{k+l+1}w = y \in C(n, k)_{m-1} \). So, by the first part of this proof, \( w \in C(n, k)_m \). So we have functions:

\[
q^{k+l} : P_{m+n-2}^{k+l} C(n, k) \times C(n, k)_{m-1} \rightarrow C(n, k)_m
\]

for \( 0 \leq l \leq m-2 \), as required for Theorem 3 applied to \( C(n, k) \). Properties (i) - (iii) of Theorem 3 are clear since they hold in \( C \).

For any \( w \in C(n, k)_m \) we have \( w \in D^{k+l} \) for all \( l = 0, \ldots, m-2 \). So \( w = q^{k+l}(d_{k+l-1}w, d_{k+l+1}w) \). This proves property (iv) of Theorem 3.

The final assertion of Theorem 5 is now clear as is property (v) of Theorem 3. \( \square \)

**Corollary 6.** A simplicial set \( C \) with neutrality is a complicial set if and only if, for all \( 0 < k < n \), the diagram

\[
D_k \xrightarrow{d_{k-1}} C_n \xrightarrow{d_k} C_{n-1}
\]

can be completed to the nerve of a category (whose set of arrows is \( C_n \) and composition is denoted \( \ast \)) such that the following conditions hold:

(i) composites of neutral elements are neutral;

(ii) for \( i < k-1 \), \( d_i : (C_n, \ast) \rightarrow (C_{n-1}, \ast) \) is a functor.
(iii) For $i > k+1$, $d_{i-1}: (C_n, *) \to (C_{n-1}, *)$ is a functor. \[ \square \]
85. The 2-category underlying a complicial set.

**Lemma 7.** The following identities hold in a complicial set \( C \):
\[
\begin{align*}
  w * (u * v) &= (\varphi^{k+1}(d^k_2 u, d^k_1 w) * u) * (w * v), \\
  (y * z) * x &= (y * x) * (z * \varphi^{-1}(d^k_2 x, d^k_1 y)).
\end{align*}
\]

In particular, if \( d^k_{k+1} w \) is a unit for \( * \) and \( d^k_{k-1} x \) is a unit for \( \ast \), then these reduce to the identities:
\[
\begin{align*}
  w * (u * v) &= u * (w * v), \\
  (y * z) * x &= (y * x) * z.
\end{align*}
\]

**Proof.** Put \( a = \varphi^{k+1}(\varphi^{-1}(v, u), \varphi^{-1}(v, w)) \). By Theorem 3(iii):
\[
d_{k-1} d_{k-1} a = d_{k-1} d_{k-2} a = d_{k-1} \varphi^{k+1}(v, u) = \varphi^{k}(d_{k-1} v, d_{k-1} w) \in D^k_n;
\]
while, for \( i < k-1 \), \( d_i d_{k-1} a \in D^k_n \).

For \( i > k+2 \), \( d_i d_{k-1} a \in D^k_n \), so \( d_{k-1} a \in D^k_{n+1} \). But
\[
\begin{align*}
  d_{k-2} d_{k-1} a &= d_{k-2} d_{k-1} a = d_{k-2} \varphi^{k+1}(v, w) \text{ and } d_{k+2} d_{k-1} a = d_{k+2} d_{k+3} a \\
  &= d_{k+2} \varphi^{-1}(u, \varphi^{k+1}(d_{k+1} v, d_{k+1} w)).
\end{align*}
\]
Thus \( d_{k+1} a = \varphi^{k+1}(d_{k+1} \varphi^{k+1}(v, w), d_{k+1} \varphi^{k+1}(u, \varphi^{k+1}(d_{k+1} v, d_{k+1} w))) \), so
\[
\begin{align*}
  d_{k+1} \varphi^{k+1}(d_{k+1} \varphi^{k+1}(v, w), d_{k+1} \varphi^{k+1}(u, \varphi^{k+1}(d_{k+1} v, d_{k+1} w))) &= d_{k+1} d_{k+2} a = d_{k+1} \varphi^{k+1}(d_{k+1} \varphi^{k+1}(v, w), d_{k+1} \varphi^{k+1}(u, \varphi^{k+1}(d_{k+1} v, d_{k+1} w))),
\end{align*}
\]
which is the first equation.
The second equation follows similarly starting with
\[ b = \varphi^{k+1}(\varphi^k(x, y), \varphi^{k-1}(z, y)). \]

If \( d_{k+1} w \) is a unit for \( \varphi^k \) then \( \varphi^k(d_{k+1} d_{k+1} w) \) is a unit for \( \varphi^k \). If \( d_{k+2} x \) is a unit for \( \varphi^k \) then \( \varphi^k(d_{k+2} x, d_{k+2} y) \) is a unit for \( \varphi^k \) so the right-hand sides simplify as asserted in these cases. \( \square \)

Suppose \( C \) is a complicial set. There is a shift function \( \text{sh} : C_2 \to C_2 \) given by \( \text{sh} x = d_1 d_2 (x, s_0 d_2) \). This function is idempotent and its image is the set of \( x \) in \( C_2 \) with \( d_2 x \) degenerate. Notice that:

\[ \text{sh}(a, b) = s_0 d_1 \varphi'(a, b), \quad \text{and} \]

\[ \text{sh}(y * x) = y * \text{sh} x \text{ if } d_2 y \text{ is degenerate.} \]

The latter uses Lemma 7 (first particular case).

Let \( K_n = \text{Smp}(\Delta[n], C) \). More explicitly, \( K_n \) is the subset of \( C_{n+1} \) consisting of those \( (z_0, z_1, \ldots, z_n) \) satisfying \( d_i z_i = d_{i+1} z_i \) for \( i < j \). This is the simplicial kernel of \( d_0, \ldots, d_{n+1} : C_n \to C_{n+1} \).

There is a canonical function \( d : C_n \to K_n \) given by \( d y = (d_0 y, \ldots, d_n y) \).
Lemma 8. Let $A$ denote the image of $\theta : C_2 \to C_2$ for a coplactic set $C$.

(i) The following square is a pullback:

\[
\begin{array}{c}
C_2 \\
\downarrow \theta \\
A \\
\downarrow \\
C_1 \times C_1
\end{array}
\]

where the bottom function takes $x$ to $(d_0 x, d_1 x)$ and the right function takes $(y_0, y_1, y_2)$ to $(y_2 \cdot y_0, y_1)$.

(ii) The function $d : C_3 \to K_3$ restricted to the neutral elements in $C_3$ is the equalizer of the following two functions from $K_3$ to $A$:

\[
\begin{array}{c}
K_3 \\
\downarrow k_1 \\
D_3 \\
\downarrow k_2 \\
C_2 \\
\downarrow \\
A
\end{array}
\]

where $k_1 (y_0, y_1, y_2, y_3) = \varphi(y_0, y_2)$, $k_2 (y_0, y_1, y_2, y_3) = \varphi^2(y_1, y_3)$.

Proof. (i) Take $x \in A$, $(y_0, y_1, y_2) \in K_2$ with $d_0 x = y_0 \cdot y_0$

$\frac{d}{dx} y_1 = y_1$. But $z = x \cdot \varphi(y_0, y_2)$. Then:

$\frac{d}{dx} z = d_0 \cdot d_0 \cdot \varphi(y_0, y_2, x) = d_0 \cdot \varphi(y_0, y_2) = y_0,$

$\frac{d}{dx} \frac{d_1 z}{y_1} = \frac{d_1 \varphi(y_0, y_2, x)}{y_1} = d_1 x = y_1,$
\[ d_2 y = d_1 d_2 \varphi'(\varphi(y_0, y_2), x) = d_1 \varphi'(y_0, d_2 x) = y_2 \text{ since } x \in A. \]

Also, \[ s_0 d_1 \varphi'(y_0, y_2) = x. \]

It remains to show that \( y \) is unique. Suppose \[ s_0 d_1 \varphi'(y_0, y_2) = x. \]

Let \( y' := x \) and \( d_1 y' = y_2 \). Now \( \varphi^2(y', s_0 y_2) = D_3 \) since it

and \( d_1 \varphi^2(y', s_0 y_2) = s_0 y_2 \) are neutral. Hence

\[ \begin{align*}
\varphi^2(y', s_0 y_2) &= \varphi'(d_2 \varphi^2(y', s_0 y_2), d_2 \varphi^2(y', s_0 y_2)) \\
&= \varphi'(\varphi'(y_0, y_2), s_0 y_2) \\
&= \varphi'(\varphi'(y_0, y_2), x).
\end{align*} \]

So \[ y' = d_1 \varphi^2(y', s_0 y_2) = d_1 \varphi'(\varphi'(y_0, y_2), x) = y. \]

(ii) Take \((y_0, y_1, y_2, y_3) \in K_3^2\). Consider \( a = \varphi^2(\varphi(y_1, y_3), \varphi(s_0 y_0, y_2)) \) which is defined if and only if

\[ d_2 \varphi^2(y_1, y_3) = d_1 \varphi'(s_0 y_0, y_2). \]

Then \( d_2 a \) is a neutral element of \( C_3 \) with \( d_1 d_2 a = y_1 \).

Conversely, suppose \( w \in C_3 \) is a neutral element of \( C_3 \) with \( d_1 w = y_1 \). But \( b = \varphi^2(w, s_0 y_2) \). It is easily checked that \( b \in D_3 \), so \( b = \varphi^2(d_1 b, d_2 b) \). But \( d_1 b = \varphi^2(y_1, y_3) \) and it can be seen that \( d_2 b \in D_3 \). So

\[ d_1 b = \varphi'(d_0 d_2 b, d_2 d_3 b) = \varphi'(d_2 d_0 b, d_2 d_3 b) = \varphi'(s_0 y_0, y_2). \]

So \( b = a \).

We have shown that there is a unique neutral \( w \in C_3 \) with \( d_1 w = y_1 \) if and only if \( d_2 \varphi^2(y_1, y_3) = \).
The corresponding faces of the sides of the last equation are equal, so by (1) above, the last equation holds if and only if it does after \( \text{sh} \) is applied. So there is a unique neutral \( w \in C_3 \) with 
\[
d_2 w = y, \quad \text{if and only if} \quad \text{sh}(y_2 \star y_1) = \text{sh}(y_2 \star y_0) = \text{sh}(y_2 \star y_0).
\]

**Corollary 9.** If \( x, y, z \in C_2 \) with \( d_2 x, d_0 y \) degenerate then 
\[
\text{sh}(y \star (z \star x)) = \text{sh}((y \star z) \star x)
\]

**Proof.** Put \( c = \varphi^2(\varphi'(x, z), \varphi^{2}(z, y)). \) Then \( d_2 c \) is neutral, 
\[
d_0 d_2 c = d_0 d_0 c = d_0 \varphi'(x, \varphi'(d_0 x, d_0 y)) = x \quad \text{since } d_0 y \text{ is degenerate},
\]
\[
d_2 d_2 c = d_1 d_2 c = d_1 \varphi'(x, z),
\]
\[
d_2 d_2 c = d_2 d_3 c = d_2 \varphi'(z, y), \quad \text{and}
\]
\[
d_2 d_2 c = d_2 d_4 c = d_2 \varphi'(d_2 x, d_2 y) = y \quad \text{since } d_2 x \text{ is degenerate}.
\]
The result follows on evaluating the two functions in Lemma 8 (ii) at the faces of \( d_2 c. \)

**Theorem 10.** For each complicial set \( C \), there is a 2-category defined as follows:

- the set of 2-cells is \( A = \{ x \in C_2 \mid d_2 x \text{ is degenerate} \}; \)
- the vertical composition \( y \circ x \) is defined when \( d_2 x = d_0 y \) and is given by \( y \circ x = y \star x; \)
- the horizontal composite \( y \star x \) is defined when \( d_1 d_2 x = d_0 d_0 y \) and is given by \( y \star x = y \star (\varphi(d_2 x, d_0 y) \star x). \)

The underlying category of this 2-category is isomorphic...
vi $s_0: C_1 \rightarrow A$ to the category $(C_1, \ast)$ whose nerve is $C(1,1)$.

Proof. The following calculation shows that $\ast$ on $C_2$ restricts to $A$:
\[ d_2(y \ast x) = d_2 d_1 q(x, y) = d_2 d_1 q'(x, y) = d_1 q'(d_1 x, d_1 y) = d_1 y \ast d_1 x. \]

Given the first sentence of the Theorem, the second follows from the calculation: for $a, b \in C_1$ with $d_1 a = d_0 b$, we have $s_0 b \ast s_0 a = s_0 b \ast (s_0 a) = s_0 b \ast q'(a, b)^{-1} = d_2 q'(a, b) \ast s_0 b = s_0 a \ast s_0 b$.

Take $x, y \in A$ with $d_1 x = d_0 y$ and put $a = d_0 x$, $b = d_1 x$, $e = d_0 y$, $f = d_1 y$. We shall prove:
\[ (s_0 f \ast x) \circ (y \ast s_0 a) = y \ast x = (y \ast s_0 b) \circ (s_0 e \ast x). \]

First note that $y \ast s_0 a = y \ast (q'(a, f) \ast s_0 a) = y \ast q'(a, f)$
Also note that $q'(b, s) \ast x = sh(q'(b, s) \ast x) \ast q'(a, f)$ since both sides of the equation have the same faces and
\[ sh(sh(q'(b, s) \ast x) \ast q'(a, f)) = sh(q'(b, s) \ast x) \ast sh(q'(a, f)) \]
\[ = sh(q'(b, s) \ast x) \ast s_0 (f \ast a) = sh(q'(b, s) \ast x) \]
so Lemma 8(i) applies. Thus:
\[ (s_0 f \ast x) \circ (y \ast s_0 a) = (s_0 f \ast (q'(b, s) \ast x)) \ast (y \ast q'(a, f)) \]
\[ = y \ast (s_0 f \ast (q'(b, s) \ast x)) \ast q'(a, f)) = y \ast (sh(q'(b, s) \ast x) \ast q'f) \]

\[ = y \ast (q'(b, s) \ast x) = y \ast x, \] where Lemma 7 was used at the second step.
For the other equality, we have:
\[(y * s_0 b) * (s_0 e * x) = \left( y * q'(b, s) \right) * \left( s_0 e * q'(b, e) * x \right)\]
\[= \left( y * q'(b, s) / 2 \right) * sh(q'(b, e) * x) = sh\left( \left( y * q'(b, s) / 2 \right) * (q'(b, e) * x) \right)\]
\[= sh\left( \left( y * q'(b, s) / 2 \right) * q'(b, e) * x \right) = sh\left( \left( y * q'(b, s) / 2 \right) * (s_0 e * q'(b, e)) * x \right)\]
\[= \left( y * s_0 e / 2 \right) * q'(b, s) * x = \left( y * s_0 e / 2 \right) * (q'(b, s) * x) = y * x,\]
where the sixth step uses Lemma 7 and the seventh uses Corollary 9.

Take \( x \in A \) and \( u, t \in C \), with \( d, t = d_0 u, d, u = d_0 d_0 \)
and put \( a = d_0 x, b = d_1 x \). Then
\[(x * s_0 u) * s_0 t = x * s_0 (u * t).\]

To see this, consider \( y = q'(q'(t, b * u), q'(u, b)) \). Since
\[d_2 y \text{ is neutral we have } d_2 y = q'(d_0 d_2 z, d_2 d_2 z) = q'(d_1 d_0 z, d_2 d_2 z) = q'(d, q'(t, u), d_2 q'(u, b)) = q'(u * t, b).
\]
So \( (x * s_0 u) * s_0 t = (x * q'(u, b)) / 2 q'(t, b * u) = x * (q'(u, b) * q'(t, b * u)) = x * d_2 z = x * q'(u * t, b) = x * s_0 (u * t).\)

Take also \( v \in C \), with \( d_0 v = d, d_1 x \). Then
\[(s_0 v) * x * s_0 u = (s_0 v) * (x * s_0 u).\]
To see this, we have:
\[(s_0v) \ast (x \ast s_0u) = s_0v \ast (x \ast \phi'(a, b)) \]
\[= (s_0v)^\ast_2 \phi'(b, v \ast u, v) \ast (x \ast \phi'(a, b)) \]
\[= (s_0v)^\ast_2 \phi'(b, v \ast x) \ast \phi'(b, v \ast u, v) \ast \phi'(a, b)) \]
\[= (s_0v)^\ast_2 \phi'(b, v \ast x) \ast \phi'(a, u \ast v) \ast \phi'(a, b) \]
\[= ((s_0v)^\ast_2 \phi'(b, v \ast x)) \ast \phi'(a, u \ast v) \ast \phi'(a, b) \]
\[= ((s_0v)^\ast_2 x) \ast s_0u. \]

Take also \( w \in C \), with \( s_0w = 1, v \). Then
\[(s_0w) \ast ((s_0v) \ast x) = s_0(w \ast v) \ast x. \]

To see this, we have:
\[(s_0w) \ast ((s_0v) \ast x) = (s_0w)^\ast_2 \phi'(v \ast b, w) \ast ((s_0v)^\ast_2 \phi'(b, v) \ast x)) \]
\[= s_0w^\ast_2 \phi'(v \ast b, w) \ast s_0v^\ast_2 \phi'(b, v) \ast x \]
\[= s_0(w \ast v)^\ast_2 \phi'(v \ast b, w) \ast (\phi'(b, v) \ast x) \]
\[= s_0(w \ast v)^\ast_2 (\phi'(b, w \ast v) \ast x) \]
\[= s_0(w \ast v)^\ast_2 x. \]

Associativity of \( \ast \) and the middle-four-interchange law \( S3 \) (of \( \mathcal{E}1 \)) are immediate consequences of the above properties. The units for \( \ast \) are elements of the form \( s_0s_0a \) where \( a \in C_0 \). The remaining axioms for a 2-category are easily deduced. \( \square \)
It is obvious that each arrow $f: C \to C'$ in $\text{Cmpl}$ induces a 2-functor $A \to A'$ which is the restriction of $f_2: C_2 \to C'_2$ (since the $\mathcal{2}$-category structure is defined in terms of the complicial structure). Thus we have a functor $\text{Cmpl} \to \text{2-Cat}$.

**Theorem 11.** The restriction of the above functor $\text{Cmpl} \to \text{2-Cat}$ to $\text{Cmpl}_2$ is an equivalence of categories $\text{Cmpl}_2 \simeq \text{2-Cat}$.

**Proof.** Let $(A, \ast, 0)$ be a 2-category. Define a simplicial set $C$ as follows. Let $C_0$ denote the set of $\ast$-units, let $C_1$ denote the set of $0$-units, and let $d_0, d_1: C_1 \to C_0$ denote the source and target functions for $\ast$-composition. Let $K_2$ denote the simplicial kernel of $d_0, d_1$. Let $C_2 = \{ (a; x_0, x_1, x_2) \mid a \in A, (x_0, x_1, x_2) \in K_2, a: x_2 \ast x_0 \Rightarrow x_1 \}$. Let $d_0, d_1, d_2: C_2 \to C_1$ be the functions which take $(a; x_0, x_1, x_2)$ to $x_0, x_1, x_2$, respectively. Let $K_3$ denote the simplicial kernel of $d_0, d_1, d_2$. Let $p: C_2 \to A$ be the function which takes $(a; x_0, x_1, x_2)$ to $a$. Let $C_3 = \{ (u_0, u_1, u_2, u_3) \in K_3 \mid pu_2 = pu_0, pu_2 = pu_0, pu_3 = pu_1 \}$, the projections $K_3 \to C_2$ restrict to give $d_0, d_1, d_2, d_3: C_3 \to C_2$. The remaining $C_n (n \geq 3)$ are taken to be the simplicial kernels $K_n$ of the $d_0, \ldots, d_{n-1}: C_{n-1} \to C_{n-2}$; so $C$ is to be
the coskeleton of the truncated simplicial set defined up to dimension 3. The degeneracies are clear-cut: $s_0 : C_0 \to C_1$ is the inclusion; $s_i : C_i \to C_{i+1}$ are given by $s_i x = u$, $s_{i+1} x = v$ where $p u, p v$ are 0-units, $d_2 u, d_0 v$ are $*$-units, and $d_0 u = d_1 v = d_2 v = x$.

Now we make $C$ into an object of $NSmp_{\leq 1}$ by declaring $(a; x_0, x_1, x_2) \in C_2$ to be neutral when $a$ is a unit for $x_2$, and declaring all elements of $C_n$ to be neutral for $n > 2$. It is clear that the assignment $A \mapsto C$ is the object function of a natural functor $\nu : 2-Cut \to NSmp_{\leq 1}$. We must check that $\nu$ lands in $Cmp_{\leq 2}$.

For each $0 < k < n$, there is a unique $k$-monic $\mu : [2] \to [n]$; so $D_k$ is the set of $x \in C_n$ for which $p(C(\mu)) x$ is a 0-unit for this particular $\mu$.

We must verify conditions (a), (b) of §2. For $n = 2$ a complicial 1-horn $x_0, x_2, x_2 \in C_1$ gives $x = (x_2 \star x_0 \circ x_0, x_2 \star x_0, x_2)$ as the unique neutral as required for (a) and (b).

Consider the case $n = 3$, $k = 1$. Take $x_0, x_2, x_3 \in C_2$ forming a complicial 1-horn. Then $p x_3$ is a 0-unit. Let $x_1 = (p x_2, d_2 x_2 \star p x_0, d_0 x_0, d_1 x_2, d_1 x_3)$. Then
\[ x = (x_0, x_1, x_2, x_3) \in C_3 \text{ is the required element for (a) and (b).} \]

Similarly for \( n = 3, k = 2 \), a complicial \( k \)-horn \( x_0, x_1, x_2, x_3 \in C_2 \) has \( p x_0 \) as a 0-unit. Let 
\[ x_2 = (p x_1 \cdot p x_3 \cdot d_0 x_1, d_1 x_0, d_3 x_1, d_2 x_3), \]
Then \( x = (x_0, x_1, x_2, x_3) \in C_3 \) gives (a), (b).

Since \( d_0, d_1, d_2, d_3 : C_2 \to C_2 \) are jointly monic and \( C \) is coskeletal after dimension 3, it follows that the arrows \( d_i : C_n \to C_{n-1}, i \neq k \), are jointly monic for \( n > 3 \). Thus any \( k \)-horn in \( C_n \) (whether or not it is complicial) comes from a unique element of \( C_m \). So (a) holds and (b) is vacuous.

Thus \( C \in Cmpl_2 \) as asserted.

It is easily seen that the composite
\[ \text{a-Cat} \to Cmpl_2 \to \text{a-Cat} \]
is isomorphic to the identity.

To see that the composite
\[ Cmpl_2 \to \text{a-Cat} \to Cmpl_2 \]
is the identity, the only point remaining after Lemma 8 is that, for \( C \in Cmpl_2 \) and \( (A, *, 0) \in \text{a-Cat} \) as in Theorem 10, the functions \( d : C_n \to K_m \) are
Isomorphisms for \( n \geq 3 \). In fact, we shall prove by induction that, for \( n \geq 3 \), the following hold:

(i) \( d_0, d_1, d_2, d_3 : C_n \rightarrow C_{n-1} \) are jointly monic;

(ii) for all \((y_0, y_1, \ldots, y_n) \in K_n \) the equation

\[
d_2 \varphi^2(y_1, y_2) = d_1 \varphi'(d_0 d_2 y_0 \ast y_0, y_1) \quad \text{holds};
\]

(iii) \( d : C_n \rightarrow K_n \) is epic.

Notice that (i) holds even for \( n = 3 \) by Lemma 8(ii), so this begins the inductive proof of (i). To see that the equation of (ii) holds, notice first:

\[
d_0 d_2 \varphi^2(y_1, y_2) = d_1 d_0 \varphi^2(y_1, y_2) = d_1 \varphi'(d_0 y_1, d_0 y_2)
\]

\[
= d_1 \varphi'(d_0 y_1, d_2 y_0) = d_0 d_2 \varphi^2(y_0, d_0 d_2 y_0) = d_0 (d_0 d_2 y_0 \ast y_0)
\]

\[
= d_0 d_0 \varphi'(d_0 d_2 y_0 \ast y_0, y_2) = d_0 d_1 \varphi'(d_0 d_2 y_0 \ast y_0, y_2);
\]

\[
d_1 d_2 \varphi^2(y_1, y_2) = d_1 d_1 \varphi^2(y_1, y_2) = d_1 y_1 = d_1 y_2 = d_1 d_2 \varphi'(d_0 d_2 y_0 \ast y_0, y_2) = d_1 d_1 \varphi'(d_0 d_2 y_0 \ast y_0, y_2);
\]

\[
d_2 d_2 \varphi^2(y_1, y_2) = d_2 d_2 \varphi^2(y_1, y_2) = d_2 y_1 = d_1 d_2 \varphi'(d_0 d_2 y_0 \ast y_0, y_2) = d_2 d_2 \varphi'(d_0 d_2 y_0 \ast y_0, y_2).
\]

However, \( d_2 d_2 \varphi^2(d_0 d_2 y_0 \ast y_0, y_2) = d_1 d_2 \varphi'(d_0 d_2 y_0 \ast y_0, d_2 y_2) = d_1 \varphi'(d_0 d_2 y_0 \ast y_0, d_2 y_2) \). These are equal when \( n = 4 \) (see proof of Lemma 8(ii)); for \( n > 4 \), they are equal.

By (ii) applied to \((d_0 y_4, d_1 y_4, \ldots, d_{n-1} y_4) \in K_{n-1} \) (induction).
To prove (iii), take \((y_0, y_1, \ldots, y_n) \in K_n\). By (ii), we can put \(a = \varphi^2(\varphi(\varphi(y_1, y_2)), \varphi(\varphi(d_1 y_1, y_2), y_3))\) and \(d_0 a = y_0, d_1 a = y_1, d_2 a = y_2, d_3 a = y_3\). We also want \(d_i a = y_i\) for \(i > 3\). By (i) it suffices to check \(d_j d_i a = d_j y_i\) for \(i \leq 4\), but \(d_j d_i a = d_{i-1} d_j a = d_{i-1} d_j y_i = d_j y_i\). So \(a = (y_0, y_1, \ldots, y_n)\).

To prove (i) for \(n > 3\), suppose \(\varphi^2(\varphi(\varphi(\varphi(\varphi(\varphi(w, x), d w))))\) as in the proof of Lemma 8 (ii). The only 2-monic \(\mu: [2] \rightarrow [n+1] \) is \(\mu = \Theta_4 \Theta_0 \Theta_{n-1} \Theta_{n-2} \cdots \Theta_4 \Theta_3\) so \((C \mu) b = d_3 d_4 \cdots d_{n-2} d_{n-1} d_0 d_1 b = d_3 d_4 \cdots d_{n-2} d_{n-1} d_0 d_1 d_2 d_3 w = d_3 d_4 \cdots d_{n-2} d_{n-1} s_0 d_0 d_1 d_2 d_3 w = s_0 d_0 d_1 d_2 d_3 w\) which is degenerate. So \(b = D^2\). The only 1-monic \(\nu: [2] \rightarrow [n]\) is \(\nu = \Theta_{n-1} \cdots \Theta_4 \Theta_3\) so \((C \nu) d_2 b = d_3 d_4 \cdots d_{n-1} d_3 b = d_3 d_4 \cdots d_{n-1} d_0 d_1 b = d_3 d_4 \cdots d_{n-1} \varphi^2(d_0 d_1 d_2 d_3 w) = d_3 d_4 \varphi^3(d_0 d_1 d_2 d_3 w, d_4, d_{n-1}, d_3, d_2, d_1, d_0) = d_3 d_4 \varphi^3(d_0 d_1, d_2 d_3, d_4, d_{n-1}, d_3, d_2, d_1, d_0) = d_3 d_4 \cdots d_{n+1}, d_3 \varphi w\) which is degenerate. So \(d_3 b \in D^1\). So \(d_3 b = \varphi^1(d_0 d_1 d_2 d_3 b) = \varphi^1(d_0 d_1 b, d_2 d_3 b) = \varphi^1(s_0 d_0 d_1 w, d_0 w, d_2 d_3 w)\). So \(w = d_2 b\) is uniquely determined by \(d_0 w, d_1 w, d_2 w, d_3 w\).

**Corollary 12.** If \(C \in Cmpl_2\) then \(d^i C_n \rightarrow C_{n+1}, i = 0, \ldots, n\), form the simplicial kernel of \(d_j: C_{n+1} \rightarrow C_n, j = 0, \ldots, n-1\), for \(n > 3\), and \(d_0, d_1, d_2, d_3: C_n \rightarrow C_{n+1}\) are jointly monic for \(n > 3\).
§6. More $\mathcal{S}$-categories associated with a complicial set.

For $C \in \mathcal{NS}mpl$ and $0 < k \leq n$, there is a simplicial set $C[n, k]$ defined as follows:

$C[n, k]_0 = C_{n-1}$, \hspace{1cm} $C[n, k]_1 = C_n$

$C[n, k]_2 = \{ x \in C_{n+1} \mid d_i x \in D_{n} \quad \text{for} \quad i < k-1, \quad d_i x \in D_n \quad \text{for} \quad i > k+1 \}$

$C[n, k]_m = \{ x \in D_{m+n-1} \mid d_{k-1} x, \ldots, d_{k+m-1} x \in C[n, k]_{m-1} \quad \text{for} \quad m \geq 2 \}$

The faces of elements of dimension $m$ are given by $d_{k-1}, \ldots, d_{m+k-1}$ of the degeneracies on elements of dimension $m$ are $s_{k-1}, \ldots, s_{m+k-2}$ of $C$.

Notice that $C(n, k)$ (see §4) is a subsimplicial set of $C[n, k]$.

Declare $x \in C[n, k]_m$, $m \geq 2$, to be neutral when it is a neutral element of $C$ of dimension $m+n-1$. Thus $C[n, k] \in \mathcal{NS}mpl$ and all elements of dimension $> 2$ are neutral.

**Theorem 13.** If $C$ is a complicial set then, for all $0 < k \leq n$, $C[n, k]$ is an object of $\mathcal{Cmpl}_2$.

**Proof.** As in the proof of Theorem 5, we begin by proving
the following statement by induction on $m$:

$$d_{k+1} w, d_{k+1} w, d_{k+1} w \in C[n, k]_m$$

for some $0 \leq l \leq m-1$ then $w \in C[n, k]_{m+1}$.

The statement is clear for $m = 1$, $l = 0$.

Consider the case $m = 2$, $l = 0$. If $w \in D_{n+2}$ and $d_{k-1} w, d_{k+1} w \in C[n, k]_2$, we must see that $d_{k-1} w, d_{k+1} w \in C[n, k]_2$. Since $w \in D_{n+2}$, we have $d_{k+1} w \in C[n, k]_2$. So it remains to see that $d_{k+1} w \in C[n, k]_2$. For $i < k-1$, $d_i w \in D_{n+1}$, and $d_{k-1} d_i w = d_i d_{k+1} w, d_{k-1} d_i w \in D_{n+1}$; so, by Lemma 4(d), $d_i d_{k+1} w = d_{k-1} d_i w \in D_{n+1}$. For $i > k$, $d_i w \in D_{n+1}$ and $d_{k-1} d_i w = d_i d_{k+1} w, d_{k+1} d_i w = d_{k+1} d_i w \in D_{n+1}$; so, by Lemma 4(d), $d_i d_{k+1} w = d_{k+1} d_i w \in D_{n+1}$.

Consider the case $m = 2$, $l = 1$. If $w \in D_{n+2}$ and $d_{k-1} w, d_{k+1} w \in C[n, k]_2$, we must see that $d_{k-1} w, d_{k+1} w \in C[n, k]_2$. Since $w \in D_{n+2}$, we have $d_{k+1} w \in C[n, k]_2$. So it remains to see that $d_{k+1} w \in C[n, k]_2$. For $i < k-1$, $d_i w \in D_{n+1}$, and $d_{k-1} d_i w = d_i d_{k+1} w, d_{k+1} d_i w = d_i d_{k+1} w \in D_{n+1}$; so, by Lemma 4(d), $d_i d_{k+1} w = d_{k-1} d_i w \in D_{n+1}$. For $i > k$, $d_i w \in D_{n+1}$ and $d_{k-1} d_i w = d_i d_{k+1} w, d_{k+1} d_i w = d_{k+1} d_i w \in D_{n+1}$, $d_{k+2} d_{k+1} w = d_{k+2} d_{k+1} w \in D_{n+1}$; so, by Lemma 4(e), $d_{k+2} d_{k+1} w = d_{k+1} d_i w \in D_{n+1}$.
With this start for the induction, the proof of the statement is word for word the same as the proof of the statement at the beginning of the proof of Theorem 5 with the only exception being that \( C[n, k] \) should be replaced by \( C[n, k]_l \).

Now we turn to a proof of the theorem. We shall show that Theorem 3 applies to \( C[n, k] \).

Suppose \( x, y \in C[n, k]_m \) with \( d_{k+l} x = d_{k+l-1} y \). Then \( w = q^{k+l} (x, y) \in D_{m-l} \) with \( d_{k+l} w = x, d_{k+l+1} w = y \in C[n, k]_m \). By the first part of this proof, \( w \in C[n, k]_m \), so we have functions:

\[
q^k : P_{m+n-2} \cap (C[n, k]_{m-1} \times C[n, k]_{m-1}) \rightarrow C[n, k]_m,
\]

\( 0 \leq l \leq m-2 \),

as required for Theorem 3 applied to \( C[n, k] \). Properties (i)-(iii) of Theorem 3 are clear since they hold in \( C \).

Property (iv) amounts to the following statement:

If \( w \in C[n, k]_m \), \( 0 \leq l \leq m-2 \), and \( (C[n, k]_{\mu}) w \) is neutral for the unique \((l+1)\)-monic \( \mu : [2] \rightarrow [m] \) then \( w \in D_{m+n-1} \).

We again must resort to induction. For \( m = 2 \) and for \( m \geq 2, l = 0 \) the statement is clear. If \( m = 3, l = 1 \), the unique 2-monic \( \mu : [2] \rightarrow [3] \) is \( \varphi_0 \), and \( (C[n, k]_{\mu}) w = d_{k-1} w \). If \( w \in C[n, k]_3 \) and \( d_{k-1} w \) is neutral then \( d_{k-1} w \in D_{n+1} \), so \( w \in D_{n+2} \) by Lemma 4(2). This proves the statement for \( m = 3, l = 1 \).
Assume the statement for $m-1 \geq 3$ and take $w \in C[n, k]_{m-1}$ with $(C[n, k]_{m-1})^* w$ neutral for the $(k+1)$-monic $\mu : [2] \to [m]$. Since $w \in D^k_{m-1}$, it is neutral. To prove $w \in D^{k+l}_{m-1}$, we must prove that $d_i^k w \in D^{k+l}_{m-1}$ for $i < k+l-1$, and that $d_i^{k+l} w \in D^{k+l}_{m-1}$ for $i > k+l$. For $i < k+l-1$, $(C[n, k])^* w = (C[n, k])^* d_i^k w$ where $\nu : [2] \to [m-1]$ is the unique $l$-monic, so $d_i^k w \in D^{k+l}_{m-1}$ by induction provided $d_i^k w \in C[n, k]_{m-1}$. For $i > k+l$, $(C[n, k])^* w = (C[n, k])^* d_i^{k+l} w$ where $\nu$ is the unique $(l+1)$-monic; so $d_i^{k+l} w \in D^{k+l}_{m-1}$ by induction, provided $d_i^{k+l} w \in C[n, k]_{m-1}$.

In fact, if $x \in C[n, k]_{m-1}$ then all $d_i x \in C[n, k]_{m-1}$ so the provisions in the above proof can be eliminated. For $m = 3$ we have $x \in C[n, k]_{m-1}$ then certainly $d_{k-1} x, d_k x, d_{k+1} x, d_{k+2} x \in C[n, k]_{2}$. For $i > k+2$, $x \in D^k_{m-1}$ implies $d_i x \in D^k_{m-i} \subset C[n, k]_{2}$. For $i < k-1$, $d_i^k x \in D_{m+1}$ and $d_{k-2} d_i x = d_{k-2} d_{k-1} x \in D^{k-1}_{m-2}$ (since $x \in D^k_{m-1}$, $d_{k-1} x \in D^k_{m-1}$); so, by Lemma 4(a), $d_i x \in D^k_{m-i} \subset C[n, k]_{2}$. Now assume the result for $m-1 \geq 3$ and take $x \in C[n, k]_{m-1}$. Then we must see $d_i x \in C[n, k]_{m-1}$ for $i < k-1$ and $i > k+m-1$. Note that these cases were left untouched in $m = 3$ case we deduced that $x \in D^k_{m-1}$ for these $i$. $d_i x \in D^k_{m-i-2}$. For $i < k-1$, $d_{k-1} d_i x = d_i d_k x$, $d_{k+1} d_i x = d_i d_{k+1} x$, ..., $d_k d_i x = d_i d_k x \in C[n, k]_{m-1}$ by induction. For $i > k+m-1$, $d_{k+1-i} d_i x = d_{k+1-i} d_{k-1} x$, ..., $d_k d_i x = d_{k-i} d_{k+1} x \in C[n, k]_{m-2}$ by induction. So $d_i x \in C[n, k]_{m-1}$ as required.