Cohomology of groups — Feb 1985

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The data for cohomology are a category $\mathcal{C}$, a simplicial object S in $\mathcal{C}$, and, an object A of $\mathcal{C}$. This gives rise to a cosimplicial set $\mathcal{C}(S,A)$:

$$\begin{tikzcd} \mathcal{C}(S_0,A) \arrow[r, shift left, "\mathcal{C}(d_0,1)"] \arrow[r, shift right, "\mathcal{C}(d_1,1)"'] & \mathcal{C}(S_1,A) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{C}(S_2,A) \arrow[r, shift left=3, "\mathcal{C}(d_0,1)"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3, "\mathcal{C}(d_3,1)"'] & \cdots \end{tikzcd}$$

Any algebraic structure possessed by A is carried over pointwise to each of the sets $\mathcal{C}(S_n, A)$, and the $\mathcal{C}(d_i,1)$ become homomorphisms; in particular, if A is an r-category then each $\mathcal{C}(S_n,A)$ is.

The *0-dimensional cohomology set* is

$$\mathcal{H}^0(S;A) = \{ f \in \mathcal{C}(S_0,A) \mid f d_0 = f d_1 \}.$$

Suppose now that A is a category in $\mathcal{C}$. *The* *1-dimensional cohomology category* $\mathcal{H}^1(S;A)$ is defined as follows:

objects $f = (f_0, f_1)$ consist of an object f_0 of $\mathcal{C}(S_0,A)$ and an arrow $f_1 : f_0 d_1 \longrightarrow f_0 d_0$ of $\mathcal{C}(S_1,A)$ such that $f_1 d_1 = (f_1 d_0) * (f_1 d_2)$

$$\begin{tikzcd} f_0 d_1 d_2 = f_0 d_1 d_1 \arrow[rr, "f_1 d_1"] \arrow[dr, "f_1 d_2"'] & & f_0 d_0 d_1 = f_0 d_0 d_0 \\ & f_0 d_0 d_2 = f_0 d_1 d_0 \arrow[ur, "f_1 d_0"'] & \end{tikzcd} \;;$$

an arrow $r : f \to f'$ is an arrow $r : f_0 \to f_0'$ in $\mathcal{C}(S_0, A)$ such that

$$\begin{tikzcd} f_0 d_1 \arrow[r, "f_1"] \arrow[d, "r d_1"'] & f_0 d_0 \arrow[d, "r d_0"] \\ f_0' d_1 \arrow[r, "f_1'"'] & f_0' d_0 \end{tikzcd} .$$

Suppose now that A is a 2-category in $\mathcal{C}$. <u>The</u> 2-<u>dimensional cohomology</u> 2-<u>category</u> $\mathcal{H}^2(S; A)$ is defined as follows:

objects $f = (f_0, f_1, f_2)$ consist of an object f_0 of $\mathcal{C}(S_0, A)$, an arrow $f_1 : f_0 d_1 \to f_0 d_0$ of $\mathcal{C}(S_1, A)$, and a 2-cell

$$\begin{tikzcd} \cdot \arrow[rr, "f_1 d_1"] \arrow[dr, "f_1 d_2"'] & \arrow[d, phantom, "\Downarrow f_2"] & \cdot \\ & \cdot \arrow[ur, "f_1 d_0"'] & \end{tikzcd}$$

of $\mathcal{C}(S_2, A)$ such that

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr] & \cdot \arrow[dl, phantom, "\Downarrow f_2 d_1", near start] \\ \cdot \arrow[r] \arrow[ur, phantom, "\Downarrow f_2 d_3", near start] & \cdot \arrow[u] \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow f_2 d_2", near start] & \cdot \\ \cdot \arrow[r] \arrow[ur] & \cdot \arrow[u] \arrow[ul, phantom, "\Downarrow f_2 d_0", near start] \end{tikzcd} ;$$

an arrow $r = (r_0, r_1) : f \to f'$ consists of an arrow $r_0 : f_0 \to f_0'$ and a 2-cell

$$\begin{tikzcd} f_0 d_1 \arrow[r, "f_1"] \arrow[d, "r_0 d_1"'] \arrow[dr, phantom, "\Downarrow r_1"] & f_0 d_0 \arrow[d, "r_0 d_0"] \\ f_0' d_1 \arrow[r, "f_1'"'] & f_0' d_0 \end{tikzcd}$$

such that

$$
\begin{tikzcd}
\cdot \arrow[rr, "f_1d_1"] \arrow[dr, "f_1d_2"'] \arrow[dd, "r_0d_1d_2"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow f_2"] \arrow[ddr, phantom, "\Downarrow r_1d_2"] & & \cdot \arrow[dd, "r_0d_0d_0"] \\
& \cdot \arrow[ur, "f_1d_0"'] \arrow[dd] \arrow[dr, phantom, "\Downarrow r_1d_0"] & \\
\cdot \arrow[dr, "f_1'd_2"'] & & \cdot \\
& \cdot \arrow[ur, "f_1'd_0"'] &
\end{tikzcd}
=
\begin{tikzcd}
\cdot \arrow[rr, "f_1d_1"] \arrow[d, "{\substack{r_0d_1d_1 \\ \shortparallel \\ r_0d_1d_2}}"'] \arrow[drr, phantom, "\Downarrow r_1d_1"] & & \cdot \arrow[d, "{\substack{r_0d_0d_1 \\ \shortparallel \\ r_0d_0d_0}}"] \\
\cdot \arrow[rr, "f_1'd_1"] \arrow[dr, "f_1'd_2"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow f_2'"] & & \cdot \\
& \cdot \arrow[ur, "f_1'd_0"'] &
\end{tikzcd}
\;;
$$

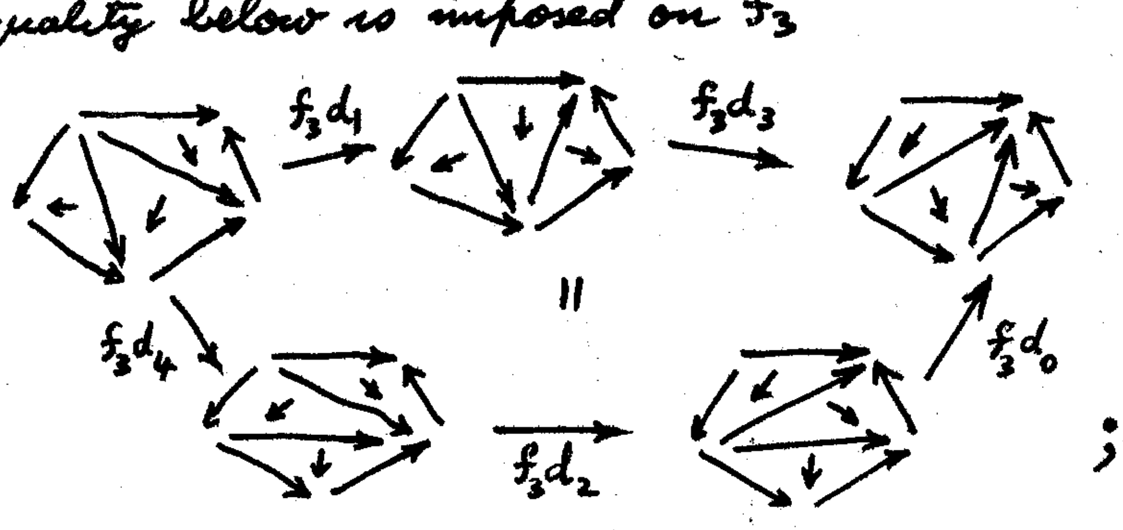
a 2-cell $\begin{tikzcd} f \arrow[r, bend left, "r"] \arrow[r, bend right, "s"'] \arrow[r, phantom, "\Downarrow\rho"] & f' \end{tikzcd}$ is a 2-cell $\begin{tikzcd} f_0 \arrow[r, bend left, "r_0"] \arrow[r, bend right, "s_0"'] \arrow[r, phantom, "\Downarrow\rho"] & f_0' \end{tikzcd}$

in $\mathcal{C}(S_0, A)$ such that

$$
\begin{tikzcd}
f_0d_1 \arrow[r, "f_1"] \arrow[d, bend right=50, "s_0d_1"'] \arrow[d, bend left=30, "r_0d_1"] \arrow[d, phantom, "\Leftarrow \rho d_1"] \arrow[dr, phantom, "\swarrow r_1"] & f_0d_0 \arrow[d, bend left=30, "r_0d_0"] \\
f_0'd_1 \arrow[r, "f_1'"'] & f_0'd_0
\end{tikzcd}
=
\begin{tikzcd}
f_0d_1 \arrow[r, "f_1"] \arrow[d, bend right=30, "s_0d_1"'] \arrow[dr, phantom, "\swarrow s_1"] & f_0d_0 \arrow[d, bend right=50, "s_0d_0"'] \arrow[d, bend left=30, "r_0d_0"] \arrow[d, phantom, "\Leftarrow \rho d_0"] \\
f_0'd_1 \arrow[r, "f_1'"'] & f_0'd_0
\end{tikzcd}
.
$$

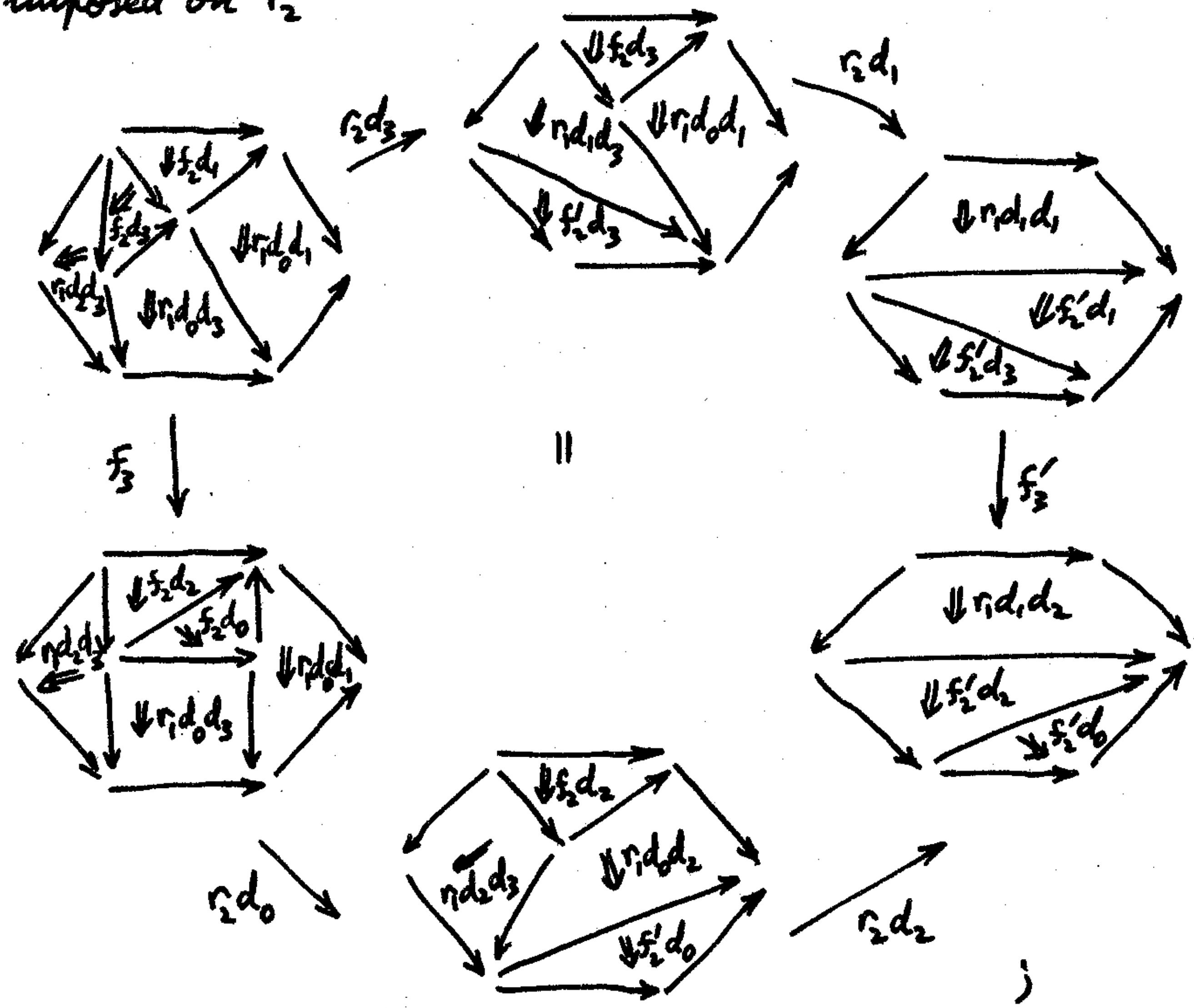
Suppose now that A is a 3-category in $\mathcal{C}$. The <u>3-dimensional cohomology</u> 3-<u>category</u> $\mathcal{H}^3(S;A)$ is defined as follows:

an object $f = (f_0, f_1, f_2, f_3)$ has f_0, f_1, f_2 as in $\mathcal{H}^2(S;A)$ except that the equality imposed on f_2 is replaced by the 3-cell f_3 and the equality below is imposed on f_3



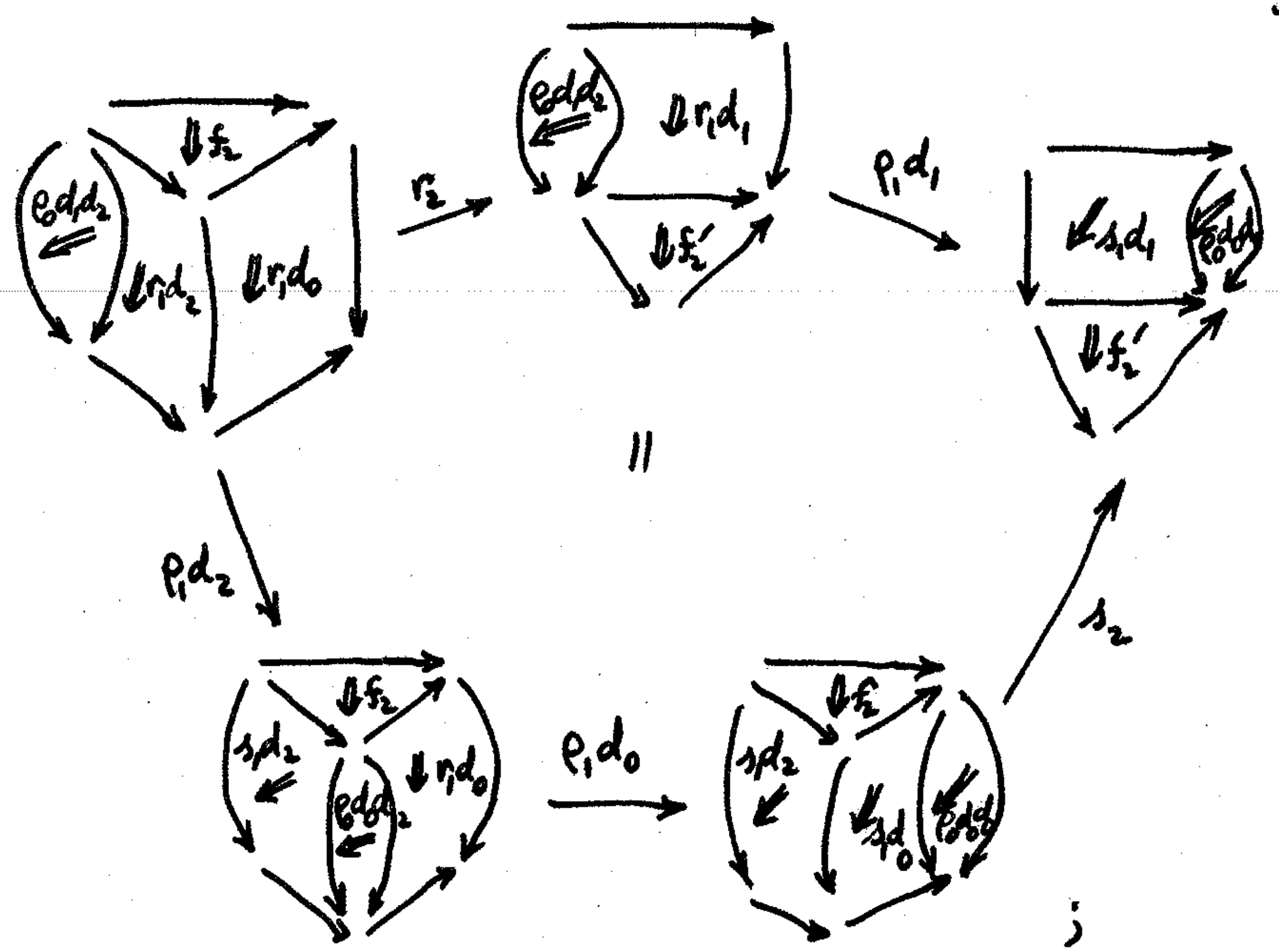
;

an arrow $r = (r_0, r_1, r_2) : f \to f'$ has r_0, r_1 as in $\mathcal{H}^2(S;A)$ except that the equality imposed on r_1 is replaced by the 3-cell r_2 and the equality below is imposed on r_2



a 2-cell $f \underset{s}{\overset{r}{\Downarrow \rho}} f'$ consists of a 2-cell $f_0 \underset{s_0}{\overset{r_0}{\Downarrow \rho_0}} f'_0$ together with a 3-cell

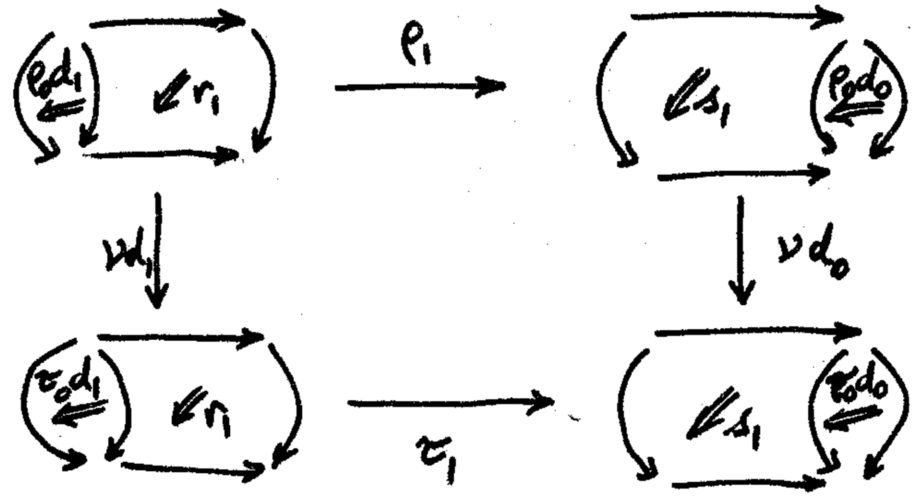
and the equality below is imposed on ρ_1



;

a 3-cell $f \overset{r}{\underset{s}{\Downarrow\rho \overset{\nu}{\Rrightarrow} \Downarrow\tau}} f'$ is a 3-cell $f_0 \overset{r_0}{\underset{s_0}{\Downarrow\rho_0 \overset{\nu}{\Rrightarrow} \Downarrow\tau_0}} f_0'$

such that



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<u>Interpretation for</u> $\mathcal{C} = \mathcal{S}^G$ <u>where</u> G <u>is a group</u>.

The simplicial object S in $\mathcal{C}$ is fixed as

$$G \underset{d_1}{\overset{d_0}{\leftleftarrows}} G \times G \underset{d_2}{\overset{d_0}{\Lleftarrow}} G \times G \times G \underset{d_3}{\overset{d_0}{\Lleftarrow}} \cdots$$

where G acts on itself by multiplication and

$$d_r(x_0, \ldots, x_n) = (x_0, \ldots, \hat{x}_r, \ldots, x_n).$$

The object A is taken to be an abelian group in $\mathcal{C}$; that is, a $\mathbb{Z}G$-module. Let $K(A,n)$ denote the n-category in $\mathcal{C}$ which has A as its object of n-cells and only one m-cell for all $m < n$; the compositions are addition in A.

The n-categories

$$\mathcal{H}^n(G;A) = \mathcal{H}^n(S; K(A,n))$$

will be examined for $n = 0, 1, 2, 3$.

$$\begin{aligned}
\mathcal{H}^0(G;A) &= \{ f \in \mathcal{C}(G,A) \mid f(y) = f(x) \text{ for all } x, y \in G \} \\
&= \{ f \in \mathcal{S}(G,A) \mid f \text{ constant and } f(xy) = x f(y) \} \\
&= \{ a \in A \mid a = xa \text{ for all } x \in G \} \\
&= A^G, \text{ the set of } G\text{-invariants in } A.
\end{aligned}$$

Notice that this set has an abelian group structure as a subgroup of A.

The category $K(A,1)$ has one object and elements of A as arrows. So $\mathcal{H}^1(G;A)$ is the category whose objects are $f : G \times G \longrightarrow A$ in $\mathcal{C}$ such that

$$f(x,z) = f(y,z) + f(x,y),$$

and, whose arrows $r : f \longrightarrow f'$ are $r : G \longrightarrow A$ in $\mathcal{C}$ such that

$$r(y) + f(x,y) = f'(x,y) + r(x).$$

Let G_c denote the chaotic category on G in $\mathcal{C}$; that is, arrows of G_c are pairs $(x,y) : x \longrightarrow y$ of elements of G. Clearly,

$$\mathcal{H}^1(G;A) = \mathcal{C}at(\mathcal{C})(G_c\,, K(A,1)).$$

Notice that $\mathcal{H}^1(G;A)$ is a groupoid and has a canonical structure of abelian group in the category of categories (properties it inherits from $K(A,1)$ pointwise).

Recall that a function $k : G \longrightarrow A$ satisfying

$$k(xy) = x\,k(y) + k(x)$$

is called a <u>crossed homomorphism</u>. If k is such then $f(y,z) = k(z) - k(y)$ defines an object f of $\mathcal{H}^1(G;A)$. Conversely, if $f \in \mathcal{H}^1(G;A)$ then $k(x) = f(e,x)$ defines a crossed homomorphism k.

Hence, $\mathcal{H}^1(G;A)$ is isomorphic to the category whose objects are crossed homomorphisms $k: G \to A$ and whose arrows $c: k \to k'$ are elements $c \in A$ for which $yc + k(y) = k'(y) + c$. The usual 1-dimensional cohomology group $H^1(G;A)$ with coefficients in A is thus the set of connected components (= isomorphism classes) of $\mathcal{H}^1(G;A)$ with the abelian group structure inherited from that on $\mathcal{H}^1(G;A)$. Two crossed homomorphisms k, k' are isomorphic iff their difference $k' - k$ is an inner crossed homomorphism k_c (i.e. $k_c(x) = xc - c$).

Proposition 1. $\mathcal{H}^1(G;A)$ is equivalent to the full subcategory of $\mathcal{C}^A$ consisting of those $(T, A \times T \xrightarrow{u} T)$ such that $T \to 1$ is epic and $\binom{u}{pr_2}: A \times T \to T \times T$ is invertible.

Proof. A functor $f: G_c \to K(A,1)$ in $\mathcal{C}$ factors as $G_c \xrightarrow{j} E \xrightarrow{p} K(A,1)$ where j is initial and p is a discrete opfibration. Since $K(A,1)$ has only one object, E amounts to an object (T, u) on which A acts. In fact, T is obtained by inverting certain arrows in $f/K(A,1)$.

Alternatively, corresponding to the crossed homomorphism $k: G \to A$ one has the A-object (T, u) with $T = A$ as sets, with G-action $G \times T \xrightarrow{m} T$ given by $m(x,a) = xa + k(x)$, and, with A-action just addition. $\square$

Recall that there is a short exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} A \rtimes G \xrightarrow{\pi} G \longrightarrow 1$$

in which the middle object is called *the semidirect product* of A and G. The underlying set of $A \rtimes G$ is $A \times G$ and the multiplication is

$$(a, x)(b, y) = (a + xb, xy).$$

Proposition 2. *There is an equivalence of categories as in the following commutative diagram.*

$$\begin{tikzcd}
(\mathcal{S}^{G})^{A} \arrow[rr, "\simeq"] \arrow[dr, "\text{forget}"'] & & \mathcal{S}^{A \rtimes G} \arrow[dl, "\mathcal{S}^{\iota}"] \\
& \mathcal{S}^{A} &
\end{tikzcd}$$

Proof. Objects T of $(\mathcal{S}^{G})^{A}$ are G-sets which are also A-sets for which the A-action is a G-map. Define an action of $A \rtimes G$ on T by $(a, x)t = a(xt)$. $\square$

Corollary. $\mathcal{H}^1(G; A)$ *is equivalent to the full subcategory of* $\mathcal{S}^{A \rtimes G}$ *consisting of those* $A \rtimes G$*-sets whose action restricts along* $\iota : A \to A \rtimes G$ *to the action of* A *on* A *by addition.* $\square$

The 2-category $K(A, 2)$ has one object, one arrow, and, 2-cells are elements of A. So the

2-category $\mathcal{H}^2(G;A)$ is as follows. Objects are arrows $f : G \times G \times G \longrightarrow A$ in $\mathcal{C}$ satisfying

$$f(w,x,y) + f(w,y,z) = f(x,y,z) + f(w,x,z).$$

An arrow $r : f \to f'$ is a function $r : G \times G \to A$ in $\mathcal{C}$ satisfying

$$r(x,y) + r(y,z) + f(x,y,z) = f'(x,y,z) + r(x,z).$$

A 2-cell
$$\begin{tikzcd} f \arrow[r, bend left, "r"] \arrow[r, bend right, "s"'] \arrow[r, phantom, "\Downarrow\rho"] & f' \end{tikzcd}$$
is an arrow $\rho : G \to A$ in $\mathcal{C}$ satisfying

$$\rho(x) + r(x,y) = s(x,y) + \rho(y).$$

Clearly,

$$\mathcal{H}^2(G;A) = \mathrm{Bicat}_{\mathcal{C}}(G_c, K(A,2)),$$

the 2-category of morphisms of bicategories $G_c \longrightarrow K(A,2)$, transformations, and modifications all internal to the category $\mathcal{C}$.

Recall that a function $k : G \times G \to A$ satisfying $k(x,y) + k(xy,z) = x\,k(y,z) + k(x,yz)$ is called a <u>factor set</u>. If k is such then

$$f(x,y,z) = x\,k(x^{-1}y, y^{-1}z)$$

defines an object of $\mathcal{H}^2(G;A)$. Conversely, if f is an object of $\mathcal{H}^2(G;A)$ then a factor set is defined by

$$k(x,y) = f(e,x,xy).$$

Hence, $\mathcal{H}^2(G;A)$ is isomorphic to the 2-category whose objects are factor sets k, whose arrows $h : k \to k'$ are functions $h : G \to A$ satisfying

$$h(x) + x\,h(y) + k(x,y) = k'(x,y) + h(xy),$$

and, whose 2-cells

$$\begin{tikzcd} k \arrow[r, bend left, "h"] \arrow[r, bend right, "l"'] \arrow[r, phantom, "\Downarrow c"] & k' \end{tikzcd}$$

are elements c of A satisfying

$$c + h(x) = l(x) + xc \,.$$

Notice that $\mathcal{H}^2(G;A)$ has every 2-cell and every arrow invertible. It is also an abelian group in the category of 2-categories under pointwise addition. The abelian group $H^2(G;A)$ consists of the equivalence classes (or isomorphism classes) of objects of $\mathcal{H}^2(G;A)$.

Proposition 3. *Regard* $K(A,1)$ *as an abelian group in* $\mathcal{C}at(\mathcal{C})$ *so that* $\mathcal{C}at(\mathcal{C})^{K(A,1)}$ *is the 2-category of categories* T *in* $\mathcal{C}$ *together with a* $K(A,1)$*-action* $K(A,1) \times T \xrightarrow{u} T$. *Then* $\mathcal{H}^2(G;A)$ *is biequivalent to the full sub-2-category of* $\mathcal{C}at(\mathcal{C})^{K(A,1)}$ *consisting of those objects* (T, u) *such that* $\pi(T) \cong 1$ *and*

$$K(A,1) \times T \xrightarrow{\binom{u}{pr_2}} T \times T$$

is an equivalence of categories in $\mathcal{S}$.

Is every object f of $\mathcal{H}^2(G;A)$ equivalent to a normalized object g?

$\cancel{f(x\,y) + f(x\,y)}$

$$\begin{matrix} w\,x\,y\,z \\ f(x\,y\,z) + f(w\,x\,z) = f(w\,y\,z) + f(w\,x\,y) \end{matrix}$$

$\underline{w=x}$

$$f(x\,y\,z) + f(x\,x\,z) = f(x\,y\,z) + f(x\,x\,y)$$

$$f(x\,x\,y) = b(x) \text{ for all } y.$$

$\underline{y=z}$

$$f(x\,y\,y) + \cancel{f(w\,x\,y)} = f(w\,y\,y) + \cancel{f(w\,x\,y)}$$

$$f(x\,y\,y) = c(y) \text{ for all } x.$$

$$b(x) = f(x\,x\,x) = c(x) \text{ for all } x.$$

$$g(x\,y\,z) = f(x,y,z) - b(y)$$

$$g(x\,x\,z) = f(x\,x\,z) - b(x) = 0$$

$$g(x\,y\,y) = f(x\,y\,y) - b(y) = 0$$

$$\left.\begin{matrix} f(x\,y\,z) + f(w\,x\,z) = f(w\,y\,z) + f(w\,x\,y) \\ -b(y) \qquad -b(x) \qquad\quad -b(y) \qquad -b(x) \end{matrix}\right] \checkmark$$

So g is an object of $\mathcal{H}^2(G;A)$.

__Proof.__ Every object of $\mathcal{H}^2(G;A)$ is isomorphic to an object f with $f(x,x,y) = f(x,y,y) = f(x,y,x) = 0$. From such an f define T to be the category whose objects are elements x of G, whose arrows $a : x \longrightarrow y$ are $a \in A$, and whose composition is

$$\begin{tikzcd} x \arrow[r, "a"] \arrow[rr, bend right, "a+b+f(x,y,z)"'] & y \arrow[r, "b"] & z \end{tikzcd} .$$

This becomes a category in $\mathcal{C}$ by defining

$$G \times T \longrightarrow T$$

$$(w\,,\ x \xrightarrow{a} y) \longmapsto (wx \xrightarrow{wa} wy).$$

There is an action of $K(A,1)$ on T in $\mathcal{C}$ given by

$$K(A,1) \times T \xrightarrow{\ u\ } T$$

$$(\bullet \xrightarrow{c} \bullet\,,\ x \xrightarrow{a} y) \longmapsto (x \xrightarrow{c+a} y).$$

Notice that T is a connected groupoid. The functor

$$K(A,1) \times T \xrightarrow{\binom{u}{\pi_2}} T \times T$$

is fully faithful. Thus (T, u) is as advertised. The converse is left to the reader. $\square$

The general setting for 1-dimensional cohomology

$\mathcal{E}$ is a respectable category such as a topos.

A is a category in $\mathcal{E}$.

Each epic $e : V \twoheadrightarrow U$ determines a category

$$er(e) : V \times_U V \rightrightarrows V$$

which is the "equivalence relation determined by e"; it is a poset in $\mathcal{E}$.

$$\boxed{\mathcal{H}^1_{\mathcal{E}}(U, A) = \operatorname*{colim}_{V \xrightarrow{e} U} \mathcal{C}at(\mathcal{E})(er(e), A)}$$

Čech cocycle category.

Also, $(\mathrm{Tors}_{\mathcal{E}} A)U$ is the full subcategory of $\mathcal{E}^{A^{op} \times U}$ (= discrete fibrations from U to A) consisting of those
$$\begin{tikzcd} & E \arrow[dl, "p"'] \arrow[dr] & \\ A & & U \end{tikzcd}$$
for which there exists $e : V \twoheadrightarrow U$ such that e^*E:

$$\begin{tikzcd} & & e^*E \arrow[dl] \arrow[dr] & \\ & E \arrow[dl] \arrow[dr] \arrow[rr, phantom, "pb"] & & V \arrow[dl, "e"] \\ A & & U & \end{tikzcd}$$

is representable. ["Torsors are locally convergent modules".]

<u>Theorem:</u> $\mathcal{H}^1_{\mathcal{E}}(U, A) \simeq (\mathrm{Tors}_{\mathcal{E}} A)U$.

Let $\mathcal{X}$ be a locally small fibration over $\mathcal{E}$ (recall Bénabou's lectures). Think of $\mathcal{X}$ as a category of structures in the mathematics based on the set theory $\mathcal{E}$. Objects of the fibre $\mathcal{X}_W$

are families of structures of type $\mathcal{X}$ indexed by W.

$$\begin{tikzcd} \mathcal{E}/W \arrow[rr, "X"] \arrow[dr, "bo"'] & & \mathcal{X} \\ & \mathcal{E}/\!/[X] \arrow[ur, "ff"'] & \end{tikzcd}$$

There exists, for each $X \in \mathcal{X}_W$, a category

$$[X] : \; ? \rightrightarrows W$$

in $\mathcal{E}$ which should be thought of as the full subcategory of $\mathcal{X}$ consisting of the "objects in the family X".

Call $\mathcal{X}$ a <u>stack</u> when it satisfies the descent condition (recall Joyal and Tierney's talks). Equivalently, $\mathcal{X}$ is a stack when it admits colimits weighted by torsors.

<u>**Theorem.**</u> <u>*If $\mathcal{X}$ is a stack and $X \in \mathcal{X}_W$ then $\mathrm{Tors}_{\mathcal{E}}\{[X]\}$ is the locally small category fibred over $\mathcal{E}$ of objects of $\mathcal{X}$ locally isomorphic to a member of the family X.*</u>

Let A be a $\mathbb{Z}G$-module. So A is an abelian group in $\mathcal{S}^G$. Define an abelian group $A^{G^2} = \{k : G\times G \to A \text{ in } \mathcal{S}\}$ in $\mathcal{S}^{G^{op}\times G}$ by:

$$(x\cdot k)(y,z) = x\,k(y,z)$$
$$(k\cdot x)(y,z) = k(x,y) + k(xy,z) - k(x,yz).$$

Let $\overline{G}$ be the category whose objects are $(u,v)\in G^2$ and $\overline{G}((u,v),(x,y)) = \begin{cases} 1 & \text{when } uv = xy \\ 0 & \text{otherwise} \end{cases}$

Notice that $\overline{G}$ is a category in $\mathcal{S}^{G^{op}\times G}$ with

$$w\cdot(u,v) = (wu, v)$$
$$(u,v)\cdot w = (u, vw)$$
$$(u,v)\longrightarrow(x,y) \implies \begin{array}{l} (wu,v)\longrightarrow(wx,y) \\ (u,vw)\longrightarrow(x,yw). \end{array}$$

Consider the category

$$\mathrm{Cat}(\mathcal{S}^{G^{op}\times G})(\overline{G}, K(A^{G^2},1)).$$

A functor $f : \overline{G} \longrightarrow K(A^{G^2},1)$

$$\begin{tikzcd}
\{(u,v)\to(x,y)\to(z,w)\} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & \{(u,v,x,y) \mid uv = xy\} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "f"] & G\times G \arrow[d] \\
A^{G^2}\times A^{G^2} \arrow[r, shift left=2, "\mu_1"] \arrow[r, "+"] \arrow[r, shift right=2, "\mu_2"'] & A^{G^2} \arrow[r, shift left] \arrow[r, shift right] & 1
\end{tikzcd}$$

$$f(u,v,u,v) = 0$$
$$f(u,v,z,w) = f(u,v,x,y) + f(x,y,z,w)$$
$$t\cdot f(u,v,x,y) = f(tu,v,tx,y),\ (f\cdot t)(u,v,x,y) = f(u,vt,x,yt)$$

$$\bar{f}(x,y,z) = x\cdot m(y)\cdot z$$

$$G\times G\times G \longrightarrow \{(u,v,x,y) \mid uv = xy\}$$

$$(a,b,c) \longrightarrow (a,\ a^{-1}b,\ bc^{-1},\ c)$$

$$\underline{av = xc}$$

$$\cancel{v = a^{-1}xc}$$

$$a^{-1}b \qquad bc^{-1}$$

$$m(x) = f(1,x,x,1)$$

G^2 $\quad m(1)=0$

$$t\cdot m(x) = f(t,x,tx,1)$$

$$t\cdot m(x)\cdot s = f(t,\ \underset{\underset{u}{\shortparallel}}{xs},\ \underset{\underset{v}{\shortparallel}}{ts},\ s)$$

$$f(t,u,v,s) = t\cdot m(us^{-1})\cdot s$$

$$u\, m(vw^{-1})w = u\, m(vy^{-1})y + x\, m(yw^{-1})w$$

$$w = 1,$$

$$u\, m(v) = u\, m(vy^{-1})y + x\, m(y)$$

$$(u,v) \longrightarrow (x,y) \longrightarrow (z,1)$$

$$\boxed{uv = xy} = z$$

$$vy^{-1} = u^{-1}x$$

$$u\, m(v) = u\, m(u^{-1}x)y + x\, m(y)$$

$$\shortparallel$$

$$u\, m(u^{-1}xy) \qquad m(xy) = m(x)y + x\, m(y)$$