Cohomology of groups

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The data for cohomology are a category \( \mathcal{C} \), a simplicial object \( S \) in \( \mathcal{C} \), and, an object \( A \) of \( \mathcal{C} \). This gives rise to a co-simplicial set \( C(S,A) \):

\[
\begin{align*}
C(S_0, A) & \xrightarrow{\partial_0} C(S_1, A) \xrightarrow{\partial_1} C(S_2, A) \xrightarrow{\partial_2} \cdots \\
C(d_0, 1) & \xrightarrow{C(d_1, 1)} C(d_2, 1)
\end{align*}
\]

Any algebraic structure possessed by \( A \) is carried over pointwise to each of the sets \( C(S_n, A) \), and the \( C(d_i, 1) \) become isomorphisms; in particular, if \( A \) is an \( r \)-category then each \( C(S_n, A) \) is.

The 0-dimensional cohomology set is

\[ \mathcal{H}^0(S; A) = \{ f \in C(S_0, A) \mid f d_0 = f d_1 \}. \]

Suppose now that \( A \) is a category in \( \mathcal{C} \). The 1-dimensional cohomology category \( \mathcal{H}^1(S; A) \) is defined as follows:

Objects \( f = (f_0, f_1) \) consist of an object \( f_0 \) of \( C(S_0, A) \) and an arrow \( f_1 : f_0 d_1 \rightarrow f_0 d_0 \) of \( C(S_1, A) \) such that \( f_1 d_0 = (f_1 d_0) \circ (f_1 d_2) \).

\[
\begin{align*}
f_0 d_1 d_2 &= f_0 d_1 f_1 d_2 = f_0 d_0 d_1 = f_0 d_0 d_0 \\
f_1 d_2 &= f_1 d_0 f_0 d_0 d_2 = f_0 d_1 d_0 \\
f_1 d_0 &= f_1 d_0
\end{align*}
\]
an arrow \( r : \mathcal{S} \to \mathcal{S}' \) is an arrow \( r : f_0 \to f'_0 \) in \( \mathcal{C}(\mathcal{S}, \mathcal{A}) \) such that
\[
\begin{align*}
f_0 d_1 & \xrightarrow{f_1} f_0 d_0 \\
rd_1 & \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \\
f_0' d_1 & \rightarrow f_0' d_0.
\end{align*}
\]

Suppose now that \( \mathcal{A} \) is a 2-category in \( \mathcal{C} \).

The 2-dimensional cohomology 2-category \( \mathcal{H}^2(\mathcal{S}; \mathcal{A}) \) is defined as follows:

Objects \( \mathcal{S} = (f_0, f_1, f_2) \) consist of an object \( f_0 \) of \( \mathcal{C}(\mathcal{S}, \mathcal{A}) \), an arrow \( f_1 : f_0 d_1 \to f_0 d_0 \) of \( \mathcal{C}(\mathcal{S}, \mathcal{A}) \), and a 2-cell
\[
\begin{align*}
f_2 & \quad \Rightarrow \quad f_0 d_1 \\
\downarrow f_1 & \quad \Rightarrow \quad \downarrow f_1 \\
f_2 d_1 & \rightarrow f_2 d_0 \\
\downarrow f_0 & \quad \Rightarrow \quad \downarrow f_0 \\
f_2 d_1 & \rightarrow f_2 d_0,
\end{align*}
\]
of \( \mathcal{C}(\mathcal{S}, \mathcal{A}) \) such that
\[
\begin{align*}
\begin{array}{ccc}
\overline{f_1 d_1} & \Rightarrow & \overline{f_2 d_1} \\
\Downarrow f_1 & \Rightarrow & \Downarrow f_2 \\
\Downarrow f_2 & \Rightarrow & \Downarrow f_0 \\
\Downarrow f_1 & \Rightarrow & \Downarrow f_0 \\
\overline{f_1 d_1} & \Rightarrow & \overline{f_2 d_1},
\end{array}
\end{align*}
\]

an arrow \( r : (r_0, r_1) : \mathcal{S} \to \mathcal{S}' \) consists of

an arrow \( r_0 : f_0 \to f'_0 \) and a 2-cell
\[
\begin{align*}
f_0 d_1 & \xrightarrow{f_1} f_0 d_0 \\
rd_1 & \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \\
f_0' d_1 & \rightarrow f_0' d_0
\end{align*}
\]
such that
\[ \begin{array}{c}
\text{a 2-cell } f \xrightarrow{r} f' \text{ is a 2-cell } \xrightarrow{r_0} f_0 \\
\text{in } \mathcal{C}(S, A) \text{ such that }
\end{array} \]

\[ f_0 d_1 \xrightarrow{f_1} f_0 d_0 \]

\[ f_0 d_1 \xrightarrow{(p d_1)} r_0 d_1 \xrightarrow{r_1} r_0 d_0 \]

\[ f_0 d_1 \xrightarrow{f_1} f_0 d_0 \]

\[ f_0 d_1 = f_0 d_1 \]

Suppose now that \( A \) is a 3-category in \( \mathcal{C} \).

The \textit{3-dimensional cohomology 3-category} \( H^3(S; A) \)

is defined as follows:

An object \( f = (f_0, f_1, f_2, f_3) \) has \( f_0, f_1, f_2 \) as

\( H^3(S; A) \) except that the equality imposed

on \( f_2 \) is replaced by the 3-cell \( f_3 \) and the

equality below is imposed on \( f_3 \).
an arrow \( r = (r_0, r_1, r_2) : f \to f' \) has \( r_0, r_1 \) as in \( H^2(S;A) \) except that the equality imposed on \( r_1 \) is replaced by the 3-cell \( r_2 \) and the equality below is imposed on \( r_2 \)

\[
\text{\( \xymatrix{ \( \text{a 2-cell } f \xymatrix{ \( \xymatrix{ r_0 \ar[r] & f_0 \ar[d] \ar[r] & f' \) \text{ consists of a 2-cell} } \ar[r] & f_0' \) \text{ together with a 3-cell} } \ar[r] \ar[d] & \} \)}
\]

and the equality below is imposed on \( p_1 \)
a 3-cell $f$ is a 3-cell $f'$ such that

\[ f' \mapsto f \mapsto f'' \]
Interpretation for \( C = S^G \) where \( G \) is a group.

The simplicial object \( S \) in \( C \) is factored as

\[
\begin{array}{cccc}
G & \xrightarrow{d_0} & G \times G & \xrightarrow{d_0} & G \times G \times G & \xrightarrow{d_0} & \ldots \\
& & \xrightarrow{d_1} & G \times G & \xrightarrow{d_1} & G \times G \times G & \\
& & & \xrightarrow{d_2} & G \times G \times G & & \\
& & & & \vdots & & \\
\end{array}
\]

where \( G \) acts on itself by multiplication and

\[
d_\nu(x_0, \ldots, x_\nu) = (x_0, \ldots, x_\nu, \ldots, x_\nu).
\]

The object \( A \) is taken to be an abelian group in \( C \); that is, a \( \mathbb{Z}G \)-module. Let \( K(A, n) \) denote the \( n \)-category in \( C \) which has \( A \) as its object of \( n \)-cells and only one \( m \)-cell for all \( m < n \); the compositions are addition in \( A \).

The \( n \)-categories

\[
\mathbb{H}^n(G; A) = \mathbb{H}^n(S; K(A, n))
\]

will be examined for \( n = 0, 1, 2, 3 \).

\[
\mathbb{H}^0(G; A) = \{ f \in \mathbb{S}(G, A) \mid f(y) = f(x) \text{ for all } x, y \in G \}
\]

\[
= \{ f \in \mathbb{S}(G, A) \mid f \text{ constant and } f(xy) = x f(y) \}
\]

\[
= \{ a \in A \mid a = xa \text{ for all } x \in G \}
\]

\[
= A^G, \text{ the set of } G\text{-invariants in } A.
\]

Notice that this set has an abelian group structure as a subgroup of \( A \).
The category \( K(A,1) \) has one object and elements of \( A \) as arrows. So \( \mathcal{H}^1(G;A) \) is the category whose objects are \( f: G \times G \to A \) in \( \mathcal{C} \) such that

\[
f(x, z) = f(y, z) + f(x, y),
\]
and whose arrows \( r: f \to f' \) are \( r: G \to A \) in \( \mathcal{C} \) such that

\[
r(y) + f(x, y) = f'(x, y) + r(x).
\]

Let \( G_e \) denote the chaotic category on \( G \) in \( \mathcal{C} \); that is, arrows of \( G_e \) are pairs \((x, y): x \to y\) of elements of \( G \). Clearly,

\[
\mathcal{H}^1(G;A) = \text{Cat}(\mathcal{C})(G_e, K(A,1)).
\]

Notice that \( \mathcal{H}^1(G;A) \) is a groupoid and has a canonical structure of abelian group in the category of categories (properties it inherits from \( K(A,1) \) pointwise).

Recall that a function \( k: G \to A \) satisfying

\[
k(xy) = xk(y) + k(x)
\]

is called a crossed homomorphism. If \( k \) is such then \( f(y, z) = k(z) - k(y) \) defines an object \( f \) of \( \mathcal{H}^1(G;A) \). Conversely, if \( f \in \mathcal{H}^1(G;A) \) then \( k(x) = f(e, x) \) defines a crossed homomorphism \( k \).
Hence, $H^1(G; A)$ is isomorphic to the category whose objects are crossed homomorphisms $k: G \to A$ and whose arrows $c: k \to k'$ are elements $c \in A$ for which $yc + k(y) = k'(y) + c$. The usual 1-dimensional cohomology group $H^1(G; A)$ with coefficients in $A$ is thus the set of connected components (= isomorphism classes) of $H^1(G; A)$ with the abelian group structure inherited from that on $H^1(G; A)$. Two crossed homomorphisms $k$, $k'$ are isomorphic iff their difference $k' - k$ is an inner crossed homomorphism $k_c$ (i.e. $k_c(x) = xc - c$).

**Proposition 1.** $H^1(G; A)$ is equivalent to the full subcategory of $\mathcal{C}^A$ consisting of those $(T, A \times T \xrightarrow{\pi} T)$ such that $\pi_1: T \to 1$ is epic and $(\pi_2): A \times T \to T \times T$ is invertible.

**Proof.** A functor $f: G \to K(A, 1)$ in $\mathcal{C}$ factors as $G \xrightarrow{f} E \xrightarrow{\pi} K(A, 1)$ where $f$ is initial and $\pi$ is a discrete opfibration. Since $K(A, 1)$ has only one object, $E$ amounts to an object $(T, u)$ on which $A$ acts. In fact, $T$ is obtained by inverting certain arrows in $f/K(A, 1)$.

Alternatively, corresponding to the crossed homomorphism $k: G \to A$ one has the $A$-object $(T, u)$ with $T = A$ as set, with $G$-action $G \times T \xrightarrow{m} T$ given by $m(x, a) = xa + k(x)$, and, with $A$-action just addition. □
Recall that there is a short exact sequence
\[ 0 \rightarrow A \xrightarrow{i} A \times G \xrightarrow{\pi} G \rightarrow 1 \]
in which the middle object is called the semidirect product of $A$ and $G$. The underlying set of $A \times G$ is $A \times G$ and the multiplication is 
\[ (a, x)(b, y) = (a + x \cdot b, xy). \]

**Proposition 2.** There is an equivalence of categories as in the following commutative diagram.

\[ \begin{CD} (SG)^A @>\sim>> \mathcal{D}^{A \times G} \\
\downarrow \text{forget} @VVV \downarrow \text{forget} \\
\mathcal{D}^A @>\sim>> \mathcal{D}^A \times \mathcal{D}^G \end{CD} \]

**Proof.** Objects $T$ of $(SG)^A$ are $G$-sets which are also $A$-sets for which the $A$-action is a $G$-map. Define an action of $A \times G$ on $T$ by $(a, x)t = a(x \cdot t)$. \( \square \)

**Corollary.** $H^1(G; A)$ is equivalent to the full subcategory of $\mathcal{D}^{A \times G}$ consisting of those $A \times G$-sets whose action restricts along $\zeta : A \rightarrow A \times G$ to the action of $A$ on $A$ by addition. \( \square \)

The 2-category $K(A, 2)$ has one object, one arrow, and 2-cells are elements of $A$. So the
2-category $H^2(G; A)$ is as follows. Objects are arrows $f : G \times G \times G \to A$ in $C$ satisfying

$$f(w, x, y) + f(w, y, z) = f(x, y, z) + f(w, x, z).$$

An arrow $r : f \to f'$ is a function $r : G \times G \to A$ in $C$ satisfying

$$r(x, y) + r(y, z) + f(x, y, z) = f(x, y, z) + r(x, z).$$

A 2-cell $\frac{f}{r}$ is an arrow $p : G \to A$ in $C$ satisfying

$$p(x) + r(x, y) = s(x, y) + p(y).$$

Clearly,

$$H^2(G; A) = Bicat_c(G_c, K(A, 2)),$$

the 2-category of morphisms of bicategories $G_c \to K(A, 2)$, transformations, and modifications all internal to the category $C$.

Recall that a function $k : G \times G \to A$ satisfying $k(x, y) + k(x, y, z) = xk(x, y, z) + k(x, yz)$ is called a factor set. If $k$ is such then

$$f(x, y, z) = x \cdot k(x', y', y'z)$$

defines an object of $H^2(G; A)$. Conversely, if $f$ is an object of $H^2(G; A)$ then a factor set is defined by

$$k(x, y) = f(e, x, xy).$$
Hence, $H^2(G;A)$ is isomorphic to the 2-category whose objects are factor sets $k$, whose arrows $h : k \to k'$ are functions $h : G \to A$ satisfying
\[ h(\alpha) + x h(\gamma) + k(x, \gamma) = h'(x, \gamma) + h(\gamma), \]
and, whose 2-cells $\mu$ are elements $c$ of $A$ satisfying
\[ c + h(\alpha) = l(\alpha) + mc. \]

Notice that $H^2(G;A)$ has every 2-cell and every arrow invertible. It is also an abelian group in the category of 2-categories under pointwise addition. The abelian group $H^2(G;A)$ consists of the equivalence classes (or isomorphism classes) of objects of $H^2(G;A)$.

**Proposition 3.** Regard $K(A,1)$ as an abelian group in $\text{Cat}(C)$ so that $\text{Cat}(C)$ is the 2-category of categories $T$ in $C$ together with a $K(A,1)$-action $K(A,1) \times T \to T$. Then $H^2(G;A)$ is biequivalent to the full sub-2-category of $\text{Cat}(C)$ consisting of those objects $(T, \mu)$ such that $\pi(T) \equiv 1$ and
\[ K(A,1) \times T \xrightarrow{(\mu)} T \times T \]
is an equivalence of categories in $\mathcal{Q}$. 

Is every object of $\mathcal{H}^2(G; A)$ equivalent to a normalized object $g$?

\[
2 \times 2
\]

\[
f(x \cdot y \cdot z) + f(x \cdot w \cdot y) = f(w \cdot y \cdot z) + f(w \cdot x \cdot y)
\]

\[
\Rightarrow
\]

\[
f(x \cdot y \cdot y) + f(x \cdot x \cdot y) = f(x \cdot y \cdot y) + f(x \cdot x \cdot y)
\]

\[
f(x \cdot x \cdot y) = b(x) \text{ for all } y.
\]

\[
f(x \cdot y \cdot y) + f(x \cdot x \cdot y) = f(x \cdot y \cdot y) + f(x \cdot x \cdot y)
\]

\[
f(x \cdot y \cdot y) = b(y) \text{ for all } x.
\]

\[
b(x) = f(x \times x \times x) = c(x) \text{ for all } x.
\]

\[
g(x \cdot y \cdot z) = f(x, y, z) - b(y)
\]

\[
g(x \cdot x \cdot z) = f(x \cdot x \cdot y) - b(x) = 0
\]

\[
g(x \cdot y \cdot y) = f(x \cdot y \cdot y) - b(y) = 0
\]

\[
f(x \cdot y \cdot z) + f(w \cdot x \cdot y) = f(w \cdot y \cdot z) + f(w \cdot x \cdot y)
\]

\[
\Rightarrow
\]

So $g$ is an object of $\mathcal{H}^2(G; A)$. 
Proof. Every object \( \xi \) of \( \mathcal{H}^2(G; A) \) is isomorphic to an object \( \xi \) with \( f(x, x, y) = f(x, y, y) = f(x, y, x) = 0 \). From such an \( \xi \) define \( T \) to be the category whose objects are elements \( x \) of \( G \), whose arrows \( a : x \to y \) are \( a \in A \), and whose composition is:

\[
\begin{array}{c}
\xymatrix{\ast & y & \ast \\
\ast & y & \ast \\
& & \ast \\
a+b+f(x,y,y)
}
\end{array}
\]

This becomes a category in \( G \) by defining:

\[
G \times T \to T
\]

\[
(w, x \overset{a}{\to} y) \mapsto (wx \overset{wa}{\to} wy).
\]

There is an action of \( K(A, 1) \) on \( T \) via \( G \) given by:

\[
K(A, 1) \times T \to T
\]

\[
(\cdot \overset{c+\alpha}{\to} \cdot, x \overset{a}{\to} y) \mapsto (x \overset{c+a}{\to} y).
\]

Notice that \( T \) is a connected groupoid. The functor

\[
K(A, 1) \times T \to T \times T
\]

\[
(\mu, 1_T)
\]

is fully faithful. Thus \( (T, \mu) \) is as advertised.

The converse is left to the reader. \( \square \)
The general setting for 1-dimensional cohomology

$E$ is a respectable category such as a topos.

$A$ is a category in $E$.

Each $e: V \to U$ determines a category

$\text{er}(e): V \times Y \to V$

which is the "equivalence relation" determined by $e$; it is a poset in $E$.

$$\mathcal{H}_E^1(U, A) = \text{colim}_V \text{cat}(\text{er}(e)(\text{er}(e), A))$$

Each cocycle category.

Also, $(\text{Tors}_E(A))U$ is the full subcategory of $\mathcal{E}^A_{U \times U}$ (= discrete fibrations from $U$ to $A$) consisting of those $\xymatrix{ A \ar[r]^E \ar[d] & U \ar[d] \\ E \ar[r] & V }$ for which there exists $e: V \to U$ such that $e^*E$ is representable. ["Torsors are locally convergent modules."]

**Theorem:** \( \mathcal{H}_E^1(U, A) \cong (\text{Tors}_E(A))U \).

Let $X$ be a locally small fibration over $E$ (recall Bénabou's lectures). Think of $X$ as a category of structures in the mathematician-based

...
There exists, for each $X \in \mathcal{E}_W$, a category $[X] : \mathcal{X} \rightarrow \mathcal{W}
$ which should be thought of as the full subcategory of $\mathcal{X}$ consisting of the “objects in the family $X$.”

Call $X$ a stack when it satisfies the descent condition (recall Joyal and Tierney’s talks). Equivalently, $X$ is a stack when it admits colimits that are weighted by torsors.

**Theorem.** If $X$ is a stack and $X \in \mathcal{E}_W$ then $\text{Tors}_{\mathcal{E}}[X]_{\mathcal{E}}$ is the locally small category fibred over $\mathcal{E}$ of objects of $X$ locally isomorphic to a member of the family $X$. 
Let $A$ be a $\mathbb{Z}$-module. So $A$ is an abelian group in $\mathcal{S}$. Define an abelian group $A^{G^2} = \{ k : G \times G \to A \mid k \in \mathcal{S} \}$ in $\mathcal{S}^{G \times G}$ by:

$$(x \cdot k)(y, z) = x \cdot k(y, z)$$

$$(k \cdot x)(y, z) = k(x, y) + k(xy, z) - k(x, yz).$$

Let $\overline{G}$ be the category whose objects are $(u, v) \in G^2$ and $\overline{G}(u, v), (x, y)) = \begin{cases} 1 & \text{when } uv = xy \\ 0 & \text{otherwise} \end{cases}$

Notice that $\overline{G}$ is a category in $\mathcal{S}^{G \times G}$ with

$$\overline{G} \cdot (u, v) = (u \cdot e, v)$$

$$(u, v) \cdot w = (u, v \cdot w)$$

$$(u, v) \longrightarrow (x, y) \quad \Longrightarrow \quad (u, v, w) \longrightarrow (x, yw).$$

Consider the category

$$\text{Cat}(\mathcal{S}^{G \times G})(\overline{G}, K(A^{G^2}, 1)).$$

A functor $f : \overline{G} \longrightarrow K(A^{G^2}, 1)$

$$f((u, v) \cdot (x, y), (u, v, w)) = (u, v, x, y) | uv = xy$$

$$\begin{array}{ccc}
A^{G^2} \times A^{G^2} & \xrightarrow{+} & A^{G^2} \\
\downarrow & \searrow f & \downarrow \\
A^{G^2} & \longrightarrow & 1
\end{array}$$

$$f(u, v, u, v) = f(u, v, x, y) + f(x, y, z, w)$$

$$f(u, v, x, y) = f(tu, v, tx, v) \cdot (f, x)(u, v, x, y) = f(u, v, x, y)$$
\[ f(x, y, z) = x \cdot m(y) \cdot z \]

\[
\begin{align*}
G \times G \times G & \rightarrow \{ (u, v, x, y) \mid uv = xy \} \\
(a, b, c) & \mapsto (a, ab, bc, c)
\end{align*}
\]

\[
\begin{align*}
a \cdot v = xc \\
\Rightarrow \\
bc^{-1}
\end{align*}
\]

\[
m(x) = f(1, x, x, 1) \\
t \cdot m(x) = f(t, x, tx, 1) \\
t \cdot m(x) \cdot s = f(t, xs, ts, s)
\]

\[
f(t, u, v, s) = t \cdot m(u s^{-1}) \cdot s
\]

\[
u \cdot m(vw^{-1})w = u \cdot m(vy^{-1})y + x \cdot m(yw^{-1})w
\]

\[
w = 1,
\]

\[
u \cdot m(v) = u \cdot m(vy) + x \cdot m(y)
\]

\[
(u, v) \rightarrow (x, y) \rightarrow (z, 1)
\]

\[
[uv = xy = z]
\]

\[
v y^{-1} = u^{-1} x
\]

\[
u \cdot m(v) = u \cdot m(u^{-1} x y) + x \cdot m(y)
\]

\[
u \cdot m(u^{-1} x y) = m(z) y + x \cdot m(y)
\]