THE GEOMETRY OF TENSOR CALCULUS II

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Abstract. This paper defines and proves the correctness of the appropriate string diagrams for various kinds of monoidal categories with duals.

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Introduction

While this paper is the second in a couplet, our intention is to make it as independent as possible of the detailed constructions in the first paper [JS2]. Of course, some familiarity with the spirit and applications of the first paper would be a help and this can be obtained from [JS4], [St2] and [JSV]. We shall provide the definitions and results of [JS2] as they are needed here; we hope these extra explanations, uncluttered by detailed proofs, will be a useful adjunct.

As foreshadowed in [JS2], this paper gives the precise modification of the string diagrams in monoidal (also called "tensor") categories required to accommodate the existence of dual objects. This modification should be geometrically natural and should lead to a geometric model of the free monoidal structures. A monoidal category in which each object has both a left and a right dual is called *autonomous* (also called "rigid"). For the case of autonomous monoidal categories, a preprint [JS0] has been available for many years on request. That preprint expresses its results in terms of piecewise-linear strings. In view of our move to smooth strings in [JS2], we have decided to continue here with smooth strings. This allows us to make use of some results from Morse theory to avoid detailed combinatorial case analyses.

Besides the autonomous case, we deal with the diagrams for *pivotal, spherical and tortile* monoidal categories.

CHAPTER 1: Progressive Diagrams and Surgery

This chapter reviews relevant definitions and results from [JS2], and proves some simple results on surgery of string diagrams.

Section 1.1 Progressive plane diagrams

A *graph with boundary* $\Gamma = (\Gamma, \Gamma_0, \partial\Gamma)$ consists of a compact Hausdorff topological space Γ, a discrete closed subset $\Gamma_0 \subseteq \Gamma$ such that the complement $\Gamma_1 = \Gamma - \Gamma_0$ is a topological sum of open intervals (called *edges of* Γ) and circles (called *circles of* Γ), and a subset $\partial\Gamma \subseteq \Gamma_0$ such that each $x \in \partial\Gamma$ has a sufficiently small connected neighbourhood V in Γ for which $V - \{x\}$ has a single connected component. The elements of Γ_0, $\partial\Gamma$, $\Gamma_0 - \partial\Gamma$ are called the *nodes, outer nodes, inner nodes* of Γ, respectively. (We often ignore the outer nodes: each outer node can be identified with the unique edge whose closure it is in.) An *isomorphism* $f : \Gamma \longrightarrow \Omega$ *of graphs with boundary* is a homeomorphism inducing bijections on the inner and on the outer nodes.

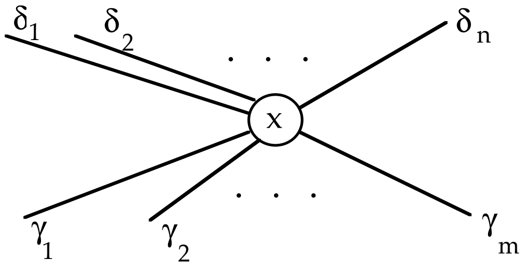
Let $a < b$ be real numbers. A *progressive plane graph between the levels* a *and* b is a graph Γ with boundary embedded in $\mathbb{R} \times [a, b]$ such that $\partial\Gamma = \Gamma \cap (\mathbb{R} \times \{a, b\})$ and the second projection $pr_2 : \mathbb{R} \times [a, b] \longrightarrow [a, b]$ is injective on each connected component of $\Gamma - \Gamma_0$. It follows that Γ can have no circles or circuits. A *valuation* $v : \Gamma \longrightarrow \mathcal{V}$ of such a graph in a monoidal category $\mathcal{V}$ is a pair of functions

$$v_0 : \Gamma_1 \longrightarrow \mathrm{obj}\,\mathcal{V}, \qquad v_1 : \Gamma_0 - \partial\Gamma \longrightarrow \mathrm{arr}\,\mathcal{V},$$

such that, for all inner nodes x of Γ,

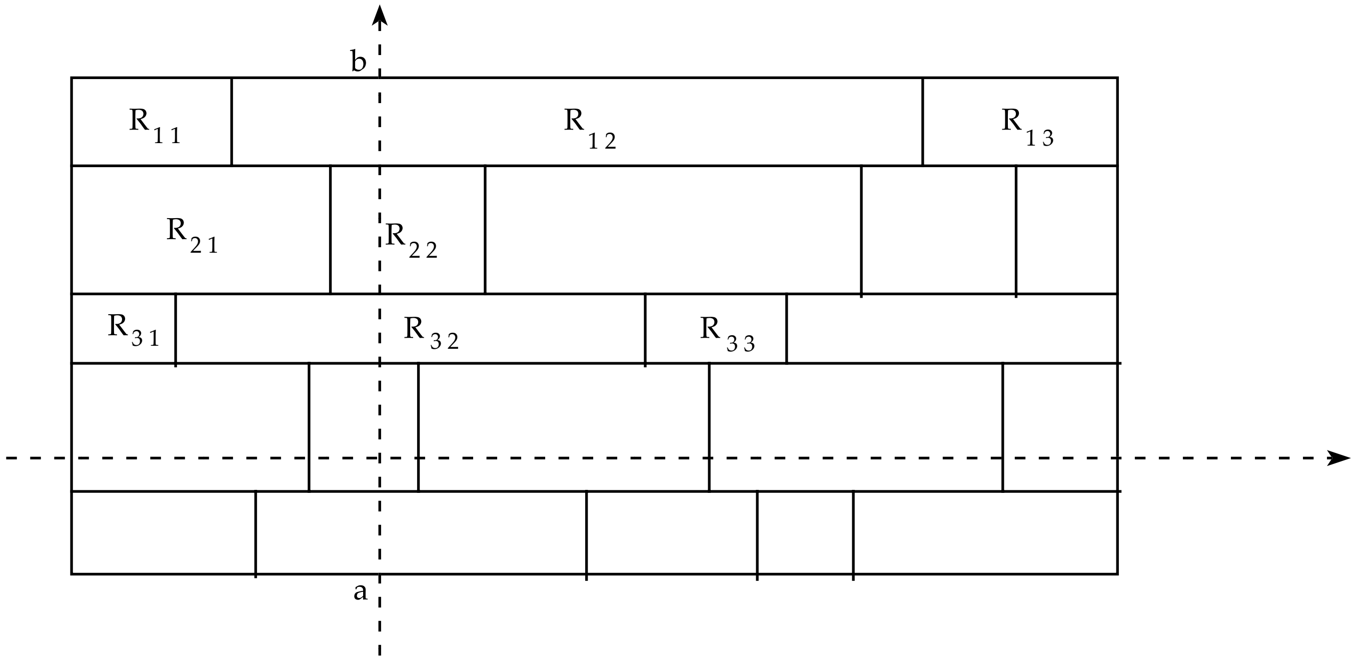
$$v_1(x) : v_0(\gamma_1) \otimes \ldots \otimes v_0(\gamma_m) \longrightarrow v_0(\delta_1) \otimes \ldots \otimes v_0(\delta_n)$$

where $\gamma_1, \ldots, \gamma_m, \delta_1, \ldots, \delta_n$ are the edges whose closures contain x and arranged so that their relationship to the canonical orientation of $\mathbb{R} \times [a, b]$ is as shown in the diagram below.



A *progressive plane diagram* (Γ, v) in a monoidal category $\mathcal{V}$ consists of a progressive plane graph Γ together with a specified valuation v. We consider the valuation to assign to an outer node the same object of $\mathcal{V}$ as is assined to the edge whose closure the outer node is in. The *domain* $\mathrm{dom}(\Gamma, v) \in \mathrm{obj}\,\mathcal{V}$ of the diagram is the tensor product of the values of the outer nodes on the line $y = a$ taken in left-to-right order; the *codomain* $\mathrm{cod}(\Gamma, v) \in \mathrm{obj}\,\mathcal{V}$ is similarly obtained from the outer nodes on the line $y = b$.

Each progressive plane diagram (Γ, v) has a *value* $v(\Gamma) : \mathrm{dom}(\Gamma, v) \longrightarrow \mathrm{cod}(\Gamma, v)$ which is an arrow in $\mathcal{V}$ obtained as follows. Cover Γ in $\mathbb{R} \times [a, b]$ by rectangles R_{ij} of the form shown in the diagram below such that the horizontal lines meet no nodes of Γ, the vertical lines do not meet Γ, and, for each rectangle R_{ij}, the diagram $\Gamma \cap R_{ij}$ is either *prime* (that is, it is connected and contains precisely one inner node) or *invertible* (that is, contains no inner nodes).



Then we use composition to define

$$v(\Gamma) = v(\Gamma \cap R_1) \circ v(\Gamma \cap R_2) \circ v(\Gamma \cap R_3) \circ \ldots$$

and tensor product to define

$$v(\Gamma \cap R_i) = v(\Gamma \cap R_{i1}) \otimes v(\Gamma \cap R_{i2}) \otimes v(\Gamma \cap R_{i3}) \otimes \ldots$$

where the value of a prime diagram with node x is just $v_1(x)$, and the value of an invertible diagram with edges $\gamma_1, \ldots, \gamma_m$ (ordered from left to right in the plane) is the identity arrow of the tensor product $v_0(\gamma_1) \otimes \ldots \otimes v_0(\gamma_m)$. It is shown in [JS2] that this definition does not depend on the choice of rectangles. In fact, [JS2] shows that the value is invariant under "deformation" in the sense to be explained below.

Remark 1.1.1 In the definitions of valuation and value we have written as if $\mathcal{V}$ were a strict monoidal category. All that is required if $\mathcal{V}$ is not strict is to make a choice from the "clique" of bracketings of the source tensor product and from that for target to obtain a unique arrow as the value.

Let Γ be a graph with boundary. A *deformation of progressive plane graphs* (between levels a and b) is a continuous function

$$h : \Gamma \times [0,1] \longrightarrow \mathbb{R} \times [a,b]$$

such that, for all $t \in [0,1]$, the function $h(-,t) : \Gamma \longrightarrow \mathbb{R} \times [a,b]$ is an embedding whose image is a progressive plane graph $\Gamma(t)$ between the levels a and b. A valuation on any $\Gamma(t_0)$ transports across the canonical h-induced isomorphism $\Gamma(t_0) \cong \Gamma \cong \Gamma(t)$ to a valuation on each graph Γ in which case h is called a *deformation of progressive plane diagrams.*

Theorem 1.1.2 [JS2] *If* $h : \Gamma \times [0,1] \longrightarrow \mathbb{R} \times [a,b]$ *is a deformation of progressive plane diagrams in a monoidal category* $\mathcal{V}$ *then*

$$v(\Gamma(0)) = v(\Gamma(1)).$$

We now consider *surgery* of plane diagrams. An *incision* in a progressive plane graph Γ is a standard rectangle $R = [\alpha,\beta]\times [c,d] \subseteq \mathbb{R}\times[a,b]$ whose horizontal sides $[\alpha,\beta]\times\{y\}$ contain no inner nodes of Γ and whose vertical sides $\{x\}\times [c,d]$ do not meet Γ.

Theorem 1.1.3 *Suppose* R *is an incision for two progressive plane diagrams* (Γ, v) *and* (Ω, w). *If the embedded graphs* Γ, Ω *are equal outside* R, *if the valuations* v, w *agree on nodes outside* R, *and if* $v(\Gamma\cap R) = w(\Omega\cap R)$, *then* $v(\Gamma) = w(\Omega)$.

Proof Deform the common part of the diagrams outside R so that the two whole lines $x = c$ and $x = d$, containing the horizontal sides of R, contain no nodes. Then there are arrows f, g such that $v(\Gamma) = f\circ v(\Gamma\cap[c,d])\circ g$ and $w(\Omega) = f\circ v(\Omega\cap[c,d])\circ g$. But also there are arrows h, k such that $v(\Gamma\cap[c,d]) = h\otimes v(\Gamma\cap R)\otimes k$ and $w(\Omega\cap[c,d]) = h\otimes w(\Omega\cap R)\otimes k$, and we have the required result.**qed**

Section 1.2 Progressive 3D diagrams

Let $a < b$ be real numbers. The *front projection* $fr : \mathbb{R}^2\times[a,b]\longrightarrow\mathbb{R}\times[a,b]$ is given by $fr(x,y,z) = (x,z)$. A *progressive polarised (smooth) 3D graph between the levels* a *and* b is a smooth embedded graph Γ with boundary in $\mathbb{R}^2\times[a,b]$ such that

(i) $\partial\Gamma = \Gamma\cap(\mathbb{R}^2\times\{a,b\})$ and the edges meet $\mathbb{R}^2\times\{a,b\}$ transversally,

(ii) the second projection $pr_2 : \mathbb{R}^2\times[a,b]\longrightarrow[a,b]$ is a smooth embedding on each connected component of $\Gamma - \Gamma_0$, and

(iii) for any pair of distinct edges with a common node x in their closures, the unit tangents at x to the front projections of the two edges are distinct.

It follows here too that Γ can have no circles or circuits. A *valuation* $v : \Gamma\longrightarrow\mathcal{V}$ of such a graph in a *braided* monoidal category $\mathcal{V}$ is defined exactly as in the plane case except that here we apply the front projection to get into $\mathbb{R}\times[a,b]$ and so obtain a partition of the edges at each node into two linearly ordered sets. A *progressive polarised (smooth) 3D diagram* (Γ, v) in a braided monoidal category $\mathcal{V}$ consists of a progressive plane graph Γ together with a specified valuation v.

Each progressive polarised 3D diagram (Γ, v) has a *value* $\bar{v}(\Gamma)$ that is an arrow in an extension $\overline{\mathcal{V}}$ of $\mathcal{V}$ which needs some explanation. To begin with, the outer nodes of Γ in the plane $z = a$ have no canonical order so that we have no guide as to how to form their tensor product to obtain $dom(\Gamma, v)$; these outer nodes on $z = a$ form a finite set of points in the plane. This leads us to consider the topological space C of *configurations of distinct points in* $\mathbb{R}^2$; a point of C is a finite subset S of points in $\mathbb{R}^2$; this is the topological sum, for all $n \geq 0$, of the usual configuration spaces C_n of n-element subsets of $\mathbb{R}^2$. We write $\rho(n)$ for the finite set of points $1, 2, \ldots, n$ on the x-axis. Note that the braid category $\mathbb{B}$ is the groupoid whose objects are the natural numbers and whose arrows $m\longrightarrow n$ are the arrows $\rho(m)\longrightarrow\rho(n)$ in the fundamental groupoid $\pi_1 C$ of C (of course, there are no such arrows unless $m = n$). If we have a family of objects A_s of $\mathcal{V}$ with $s\in S\in C$ then each arrow $\alpha : \rho(m) \longrightarrow S$ in $\pi_1 C$ determines a canonical isomorphism

$$\langle\alpha\rangle : A_{\alpha(1)} \otimes \ldots \otimes A_{\alpha(m)} \longrightarrow A_1 \otimes \ldots \otimes A_m$$

where $\underline{\alpha} : \{1, 2, \ldots, m\} \longrightarrow S$ is the bijection determined by the path α.

Now we can define the category $\overline{\mathcal{V}}$. The objects are families $(A_s \mid s\in S)$ of objects A_s of $\mathcal{V}$ indexed by finite subsets $S\in C$ of $\mathbb{R}^2$. An arrow $[\alpha, f, \beta] : (A_s \mid s\in S) \longrightarrow (B_t \mid t\in T)$ is an equivalence class of triplets (α, f, β) consisting of arrows $\alpha : \rho(m) \longrightarrow S$, $\beta : \rho(n) \longrightarrow T$ in $\pi_1 C$ and an arrow $f : A_{\alpha(1)} \otimes \ldots \otimes A_{\alpha(m)} \longrightarrow B_{\beta(1)} \otimes \ldots \otimes B_{\beta(n)}$ in $\mathcal{V}$; two such triplets (α, f, β), (α', f', β') are equivalent when $\langle\beta\rangle \circ f \circ \langle\alpha\rangle^{-1} = \langle\beta'\rangle \circ f' \circ \langle\alpha'\rangle^{-1}$. It is worthwhile noting that an arrow in $\overline{\mathcal{V}}$ is uniquely determined by a triple (α, f, β) for which the domains M, N of the arrows $\alpha : M \longrightarrow S$, $\beta : N \longrightarrow T$ are arbitrary subsets of the x-axis with the same cardinalities as S, T, respectively; for, there are canonical arrows $\rho(m) \longrightarrow M$, $\rho(n) \longrightarrow N$ obtained by sliding points along the x-axis as on an abacus. Regarding each object A of $\mathcal{V}$ as a family indexed by $\rho(1)$, we obtain an inclusion $\mathcal{V} \longrightarrow \overline{\mathcal{V}}$ which is an equivalence of categories.

The sets of outer nodes of Γ in the planes $\mathbb{R}^2\times\{a\}$, $\mathbb{R}^2\times\{b\}$ determine elements $\mathrm{dom}\Gamma, \mathrm{cod}\Gamma\in C$, respectively. Thus we have objects

$$\mathrm{dom}(\Gamma, v) = (v_0(p) \mid p\in \mathrm{dom}\Gamma), \quad \mathrm{cod}(\Gamma, v) = (v_0(p) \mid p\in \mathrm{cod}\Gamma)$$

of the category $\overline{\mathcal{V}}$.

We shall define the value of (Γ, v) as an arrow

$$\bar{v}(\Gamma) : \mathrm{dom}(\Gamma, v) \longrightarrow \mathrm{cod}(\Gamma, v)$$

in $\overline{\mathcal{V}}$. We build up to this in four steps.

(i) Suppose the diagram is *invertible;* that is, has no inner nodes. Then we obtain an arrow $\gamma : \mathrm{dom}\Gamma \longrightarrow \mathrm{cod}\Gamma$ in $\pi_1 C$ whose underlying bijection $\underline{\gamma}$ is compatible with the valuation. The identity arrows $v_0(\underline{\alpha}(i)) \longrightarrow v_0(\underline{\gamma}(\underline{\alpha}(i)))$ for $\alpha : \rho(n) \longrightarrow \mathrm{dom}\Gamma$ and $i\in\rho(n)$, determine $\bar{v}(\Gamma) : \mathrm{dom}(\Gamma, v) \longrightarrow \mathrm{cod}(\Gamma, v)$ in $\overline{\mathcal{V}}$.

(ii) Suppose the diagram (Γ, v) is *prime*; that is, it is connected, it has precisely one inner node, and the restriction of the front projection to Γ is an injective function. We put $\bar{v}(\Gamma) = v(p)$ where p is the inner node. This is an arrow of $\mathcal{V}$ and hence of $\overline{\mathcal{V}}$.

(iii) Suppose there is a finite set of disjoint standard[1] rectangles $R_1, R_2, \ldots, R_r$ in the plane $\mathbb{R}^2$ such that Γ is covered by the rectangles $R_h\times [a, b]$, $h = 1, 2, \ldots, r$, and that the part (Γ_h, v_h) of (Γ, v) in each rectangle R_h is either invertible or prime. Take a path $\alpha : \rho(m) \longrightarrow \mathrm{dom}\Gamma$ such that $i \leq j$, $\underline{\alpha}(i)\in R_h$, $\underline{\alpha}(j)\in R_k$ imply $h \leq k$; and take a similar path $\beta : \rho(n) \longrightarrow \mathrm{cod}\Gamma$. Paths $\alpha_h : M_h \longrightarrow \mathrm{dom}\Gamma_h$, $\beta_h : N_h \longrightarrow \mathrm{cod}\Gamma_h$ are then obtained by restriction of α, β. The value $\bar{v}_h(\Gamma_h) : \mathrm{dom}\Gamma_h \longrightarrow \mathrm{cod}\Gamma_h$ is uniquely determined by a triplet of the form (α_h, f_h, β_h) for all $h = 1, 2, \ldots, r$. The arrow $\bar{v}(\Gamma) : \mathrm{dom}(\Gamma, v) \longrightarrow \mathrm{cod}(\Gamma, v)$ in $\overline{\mathcal{V}}$ is defined to be $[\alpha, f, \beta]$ where $f = f_1 \otimes f_2 \otimes \ldots \otimes f_r$.

(iv) In general, there exists a partition $a = u_0 < u_1 < u_2 < \ldots < u_k = b$ such that the restriction (Γ_i, v_i) of (Γ, v) to each layer $\mathbb{R}^2\times [u_{i-1}, u_i]$ satisfies the hypothesis of (iii). Define the value of (Γ, v) to be the composite

[1] *Standard rectangles* are products $[\xi, \zeta] \times [\xi', \zeta']$ of closed intervals.

$$\bar{v}(\Gamma) \;=\; \bar{v}_k(\Gamma_k) \circ \ldots \circ \bar{v}_2(\Gamma_2) \circ \bar{v}_1(\Gamma_1)$$

in $\overline{\mathcal{V}}$.

The value so defined is independent of the choices involved [JS2].

Remark 1.2.1 In order to obtain a value of (Γ, v) which is actually an arrow $v(\Gamma)$ in $\mathcal{V}$, we need to *anchor* the diagram by providing arrows $\alpha : \rho(m) \longrightarrow \mathrm{dom}\Gamma$, $\beta : \rho(n) \longrightarrow \mathrm{cod}\Gamma$; then $v(\Gamma)$ is determined by the equation

$$\bar{v}(\Gamma) \;=\; [\alpha, v(\Gamma), \beta]. \quad //$$

A *deformation of progressive polarised (smooth) 3D graphs* is a continuous function

$$h : \Gamma \times [0,1] \longrightarrow \mathbb{R}^2 \times [a,b]$$

such that, for all $t \in [0,1]$, the function $h(-,t) : \Gamma \longrightarrow \mathbb{R}^2 \times [a,b]$ is a smooth embedding whose image $\Gamma(t)$ is a progressive polarised 3D graph, and, for each edge γ of Γ (smoothly parametrized by $\gamma(s)$, $s \in [0,1]$)

$$\frac{d}{ds} h(\gamma(s), t)$$

is a continuous function on $[0,1] \times [0,1]$. As in the plane case this leads to the notion of *deformation of progressive polarised 3D diagrams.*

Theorem 1.2.2 [JS2] *If* $h : \Gamma \times [0,1] \longrightarrow \mathbb{R}^2 \times [a,b]$ *is a deformation of progressive polarised 3D diagrams in a braided monoidal category* $\mathcal{V}$ *then the square*

$$\begin{array}{ccc} (\, v(p) \mid p \in \mathrm{dom}\,\Gamma(0)\,) & \xrightarrow{\;\;\cong\;\;} & (\, v(p) \mid p \in \mathrm{dom}\,\Gamma(1)\,) \\ {\scriptstyle \bar{v}(\Gamma(0))}\big\downarrow & & \big\downarrow{\scriptstyle \bar{v}(\Gamma(1))} \\ (\, v(q) \mid q \in \mathrm{cod}\,\Gamma(0)\,) & \xrightarrow{\;\;\cong\;\;} & (\, v(q) \mid p \in \mathrm{cod}\,\Gamma(1)\,) \end{array}$$

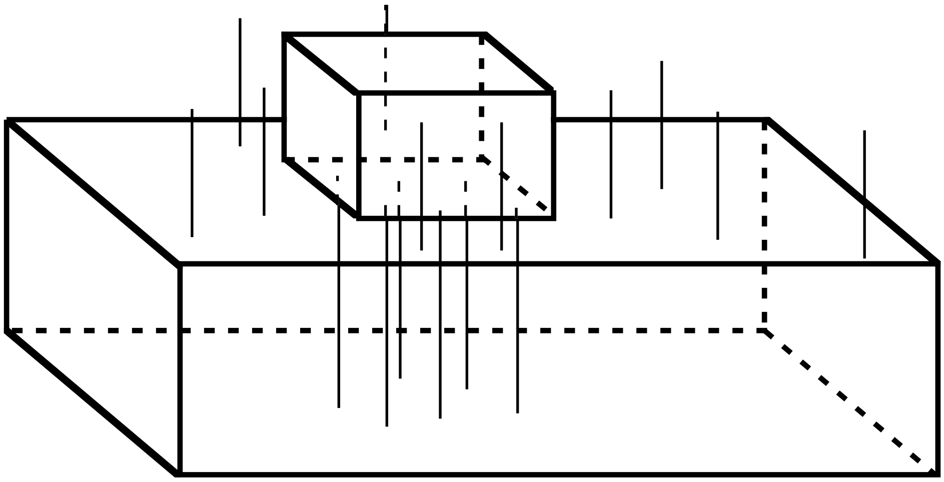
in $\overline{\mathcal{V}}$ *commutes, where the horizontal isomorphisms are induced by the paths* $t \longmapsto \mathrm{dom}\ \Gamma(t)$, $t \longmapsto \mathrm{cod}\ \Gamma(t)$ *in the configuration space* C.

We now consider *surgery* of 3D diagrams. An *incision* in a progressive polarised 3D graph Γ is a standard rectangle $R = [\alpha,\beta] \times [\gamma,\delta] \times [c,d] \subseteq \mathbb{R}^2 \times [a,b]$ whose horizontal sides $[\alpha,\beta] \times [\gamma,\delta] \times \{z\}$ contain no inner nodes of Γ and whose vertical sides $\{x\} \times [\gamma,\delta] \times [c,d]$, $\{y\} \times [\gamma,\delta] \times [c,d]$ do not meet Γ.

Theorem 1.2.3 *Suppose* R *is an incision for two progressive polarised 3D diagrams* (Γ, v) *and* (Ω, w). *If the embedded graphs* Γ, Ω *are equal outside* R, *if the valuations* v, w *agree on nodes outside* R, *and if* $\bar{v}(\Gamma \cap R) = \bar{w}(\Omega \cap R)$, *then* $\bar{v}(\Gamma) = \bar{w}(\Omega)$.

Proof Using a small deformation, it is possible to ensure that the planes containing the horizontal sides of R contain no inner nodes of Γ, and therefore none of those of Ω either. Then there are arrows f, g such that $v(\Gamma) = f \circ v(\Gamma \cap [c,d]) \circ g$ and $w(\Omega) = f \circ v(\Omega \cap [c,d]) \circ$

g. So the problem reduces to the case where $[c,d] = [a,b]$. Using a small deformation, we can arrange that the tangents to the edges which meet the horizontal planes $z = a$, $z = b$ have vertical tangents at the points of intersection. Now raise the rectangle R above the layer $\mathbb{R}^2 \times [a,b]$ so that it appears as a chimney. Extend the edges of the domain of the part of the diagram in R as straight lines below the chimney. Extend the edges of the codomain of the part of the diagram outside R as straight lines above the layer.



This reduces the problem to the case where the two graphs Γ, Ω consist, outside the rectangle R, of the same collection of straight line edges. For this case the result follows directly from the definition of value. **qed**

CHAPTER 2: Autonomous Monoidal Categories

Section 2.1 Coherence for autonomous monoidal categories

Let $\mathcal{V}$ denote a monoidal category with tensor product functor $\otimes : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ and unit object I. We avail ourselves of a coherence theorem [JS1; Corollary 1.4] to write as if the tensor product were strictly associative and the unit were a strict unit. We now recall some definitions from [JS1; Section 7].

We write $\mathcal{V}(A, B)$ for the set of arrows in $\mathcal{V}$ from A to B. An arrow $\varepsilon : A \otimes B \longrightarrow I$ is called a *pairing of* A, B. The pairing is *exact* when the function

$$\varepsilon^{\#} : \mathcal{V}(X, B \otimes Y) \longrightarrow \mathcal{V}(A \otimes X, Y)$$

is invertible where, for all arrows $f : X \longrightarrow B \otimes Y$, the arrow $\varepsilon^{\#}(f) : A \otimes X \longrightarrow Y$ is the composite

$$A \otimes X \xrightarrow{1_A \otimes f} A \otimes B \otimes Y \xrightarrow{\varepsilon \otimes 1_Y} Y .$$

In this case, there is a unique arrow $\eta : I \longrightarrow B \otimes A$ defined by the equation $\varepsilon^{\#}(\eta) = 1_A$. Then the following two *adjunction triangles* commute.

$$\begin{tikzcd} A \arrow[rr, "1_A \otimes \eta"] \arrow[dr, "1_A"'] & & A \otimes B \otimes A \arrow[dl, "\varepsilon \otimes 1_A"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, "\eta \otimes 1_B"] \arrow[dr, "1_B"'] & & B \otimes A \otimes B \arrow[dl, "1_B \otimes \varepsilon"] \\ & B & \end{tikzcd}$$

Indeed, a pairing $\varepsilon : A \otimes B \longrightarrow I$ is exact if and only if there exists an arrow $\eta : I \longrightarrow B \otimes A$ such that the two adjunction triangles commute. Symmetrically then we have a bijection

$$\eta^{\#} : \mathcal{V}(Y \otimes B, X) \longrightarrow \mathcal{V}(Y, X \otimes A).$$

In particular, the homsets of the form $\mathcal{V}(B, X)$ are determined up to isomorphism by those of the form $\mathcal{V}(I, X \otimes A)$.

A pair (η, ε) is said to be an *adjunction* between the objects A and B when they satisfy the two adjunction triangles. When such a pair exists we say that A is *left adjoint* or *left dual* to B; we write

$$(\eta, \varepsilon) : A \dashv B .$$

We also say B is *right dual* to A. We call η the *unit* and ε the *counit* of the adjunction.

A monoidal category $\mathcal{V}$ is called *left (right) autonomous* when every object has a left (right) dual. It is *autonomous* when it is both left and right autonomous. When $\mathcal{V}$ is left autonomous, a choice of adjunction

$$(\eta_A, \varepsilon_A) : A^* \dashv A$$

for each object A determines a fully faithful functor

$$(\ )^* : \mathcal{V}^{op} \longrightarrow \mathcal{V}$$

given on arrows $f : A \longrightarrow B$ by taking $f^* : B^* \longrightarrow A^*$ to be the value of $\varepsilon_B{}^{\#}$ at the composite

$$I \xrightarrow{\eta_A} A \otimes A^* \xrightarrow{f \otimes 1_{A^*}} B \otimes A^*.$$

The functor $(\ )^*$ is an equivalence of categories if and only if $\mathcal{V}$ is autonomous.

Example 2.1.1 Let H be a Hopf algebra over a commutative ring k. The category $\mathrm{Pr}_k(H)$ of (left) H-modules, which are finitely generated projective as k-modules, is autonomous monoidal in such a way that the forgetful functor $\mathrm{Pr}_k(H) \longrightarrow \mathrm{Mod}_k$ into the category of k-modules is monoidal. The left H-action on $M^* = \mathrm{Hom}_k(M, k)$ is given by

$$(x \bullet \phi)(m) = \phi(\nu(x) \bullet m)$$

for all $x \in H$, $\phi \in M^*$ and $m \in M$. //

Recall the definition of *monoidal functor* $F : \mathcal{V} \longrightarrow \mathcal{W}$ between monoidal categories $\mathcal{V}$, $\mathcal{W}$. These are the "tensor functors" of [JS1,2] and the "strong monoidal functors" of [EK]. Such a monoidal functor F consists of a functor $F : \mathcal{V} \longrightarrow \mathcal{W}$ together with an isomorphism $\phi_0 : I \longrightarrow FI$ and a family of natural isomorphisms $\phi_{2,\,A,B} : FA \otimes FB \longrightarrow F(A \otimes B)$ satisfying three coherence conditions. The monoidal functor is called *strict* when ϕ_0 and each $\phi_{2,\,A,B}$ is an identity.

Monoidal functors $F : \mathcal{V} \longrightarrow \mathcal{W}$ preserve duals. More precisely, if $\varepsilon : A \otimes B \longrightarrow I$ is an exact pairing from A to B then the pairing from FA to FB given by the composite

$$FA \otimes FB \xrightarrow{\phi_{2,A,B}} F(A \otimes B) \xrightarrow{F\,\varepsilon} FI \xrightarrow{\phi_0^{-1}} I$$

is exact.

Suppose $F, G : \mathcal{V} \longrightarrow \mathcal{W}$ are monoidal functors. A natural transformation $\theta : F \longrightarrow G$ is called *monoidal* when the following diagrams commute.

$$\begin{tikzcd}
FA \otimes FB \arrow[r, "\phi_{2,\,A,\,B}"] \arrow[d, "\theta_A \otimes \theta_B"'] & F(A \otimes B) \arrow[d, "\theta_{A \otimes B}"] \\
GA \otimes GB \arrow[r, "\phi_{2,\,A,\,B}"'] & G(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\phi_{0,\,I}"] \arrow[dr, "\phi_{0,\,I}"'] & FI \arrow[d, "\theta_I"] \\
 & GI
\end{tikzcd}$$

If a monoidal functor $F : \mathcal{V} \longrightarrow \mathcal{W}$ is an equivalence then it is a *monoidal equivalence* in the sense that there exists a monoidal functor $H : \mathcal{W} \longrightarrow \mathcal{V}$ and invertible monoidal natural transformations $HF \longrightarrow 1_{\mathcal{V}}$ and $FH \longrightarrow 1_{\mathcal{W}}$.

We regard the opposite $\mathcal{V}^{\mathrm{op}}$ of a monoidal category $\mathcal{V}$ as monoidal with tensor product $A \otimes B$ in $\mathcal{V}^{\mathrm{op}}$ given by $B \otimes A$ in $\mathcal{V}$. If $\mathcal{V}$ is left autonomous then the functor $(\ )^* : \mathcal{V}^{\mathrm{op}} \longrightarrow \mathcal{V}$ is monoidal: there are canonical isomorphisms

$$(A \otimes B)^* \cong B^* \otimes A^*, \quad I^* \cong I .$$

When $\mathcal{V}$ is autonomous it follows that $(\ )^* : \mathcal{V}^{\mathrm{op}} \longrightarrow \mathcal{V}$ is a monoidal equivalence.

In a left autonomous monoidal category $\mathcal{V}$, the families of arrows $\varepsilon_A : A^* \otimes A \longrightarrow I$,

$\eta_A : I \longrightarrow A \otimes A^*$, for $A \in \mathcal{V}$, are *monoidal dinatural transformations.* For ε this means that the following three diagrams commute (and for η there are three symmetrically obtained diagrams). The unnamed isomorphisms are the canonical ones.

$$\begin{tikzcd}
 & B^* \otimes B \arrow[dr, "\varepsilon_B"] & \\
B^* \otimes A \arrow[ur, "1 \otimes f"] \arrow[dr, "f^* \otimes 1"'] & & I \\
 & A^* \otimes A \arrow[ur, "\varepsilon_A"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
I^* \otimes I \arrow[r, shift left, "\varepsilon_I"] \arrow[r, shift right, "\cong"'] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \otimes B)^* \otimes (A \otimes B) \arrow[rr, "\varepsilon_{A \otimes B}"] \arrow[d, "\cong"] & & I \\
(B^* \otimes (A^* \otimes A)) \otimes B \arrow[r, "(1 \otimes \varepsilon_A) \otimes 1"] & (B^* \otimes I) \otimes B \arrow[r, "\cong"] & B^* \otimes B \arrow[u, "\varepsilon_B"']
\end{tikzcd}$$

Definition 2.1.2 An autonomous monoidal category $\mathcal{V}$ is called *strict* when $\mathcal{V}$ is a strict monoidal category and the monoidal functor $(\ )^* : \mathcal{V}^{\mathrm{op}} \longrightarrow \mathcal{V}$ is a strict monoidal isomorphism.

Proposition 2.1.3 *Each autonomous monoidal category is monoidally equivalent to a strict one.*

Proof For any strict monoidal category $\mathcal{V}$, consider the strict autonomous monoidal category $\mathrm{Adj}\ \mathcal{V}$ defined as follows. An object $A = (A_n, \varepsilon^A{}_n)_{n \in \mathbb{Z}}$ consists of an object A_n of $\mathcal{V}$ for each integer $n \in \mathbb{Z}$ and an exact pairing $\varepsilon^A{}_n : A_{n+1} \otimes A_n \longrightarrow I$ (and we write $\eta^A{}_n$ for the corresponding unit). An arrow $f = (f_n)_{n \in \mathbb{Z}} : A \longrightarrow B$ consists of arrows $f_n : A_n \longrightarrow B_n$ in $\mathcal{V}$ for n even and $f_n : B_n \longrightarrow A_n$ in $\mathcal{V}$ for n odd such that the following diagrams commute.

$$\begin{tikzcd}
 & A_{n+1} \otimes A_n \arrow[dr, "\varepsilon^A_n"] & \\
B_{n+1} \otimes A_n \arrow[ur, "f_{n+1} \otimes 1"] \arrow[dr, "1 \otimes f_n"'] \arrow[rr, phantom, "n \text{ even}"] & & I \\
 & B_{n+1} \otimes B_n \arrow[ur, "\varepsilon^B_n"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A_{n+1} \otimes A_n \arrow[dr, "\varepsilon^A_n"] & \\
A_{n+1} \otimes B_n \arrow[ur, "1 \otimes f_n"] \arrow[dr, "f_{n+1} \otimes 1"'] \arrow[rr, phantom, "n \text{ odd}"] & & I \\
 & B_{n+1} \otimes B_n \arrow[ur, "\varepsilon^B_n"'] &
\end{tikzcd}$$

The tensor product for $\mathrm{Adj}\ \mathcal{V}$ is given by

$$(A \otimes B)_n = \begin{cases} A_n \otimes B_n & \text{for } n \text{ even} \\ B_n \otimes A_n & \text{for } n \text{ odd} \end{cases}$$

with $(\varepsilon^{A \otimes B})_n$ equal to the composite

$$B_{n+1} \otimes A_{n+1} \otimes A_n \otimes B_n \xrightarrow{1 \otimes \varepsilon^A_n \otimes 1} B_{n+1} \otimes B_n \xrightarrow{\varepsilon^B_n} I$$

for n even and equal to the composite

$$A_{n+1} \otimes B_{n+1} \otimes B_n \otimes A_n \xrightarrow{1 \otimes \varepsilon^B_n \otimes 1} A_{n+1} \otimes A_n \xrightarrow{\varepsilon^A_n} I$$

for n odd. There is a left dual of $A \in \mathrm{Adj}\,\mathcal{V}$ given by $A^* = (A_{n+1}, \varepsilon^A_{n+1})_{n \in \mathbb{Z}}$ with exact pairing $\varepsilon^A : A^* \otimes A \longrightarrow I$ having n-th component equal to $\varepsilon^A_n : A_{n+1} \otimes A_n \longrightarrow I$ for n even and $\eta^A_n : I \longrightarrow A_n \otimes A_{n+1}$ for n odd. Clearly $\mathrm{Adj}\,\mathcal{V}$ is strict autonomous. There is a fully faithful strict-monoidal functor $E : \mathrm{Adj}\,\mathcal{V} \longrightarrow \mathcal{V}$ given by $EA = A_0$, $Ef = f_0$. Clearly E is an equivalence if and only if $\mathcal{V}$ is autonomous.$_{\mathbf{qed}}$

In a strict autonomous monoidal category $\mathcal{V}$, we write D for the isomorphism of categories $(\ )^* : \mathcal{V}^{op} \longrightarrow \mathcal{V}$ and D^{-1} for its inverse as a functor $\mathcal{V}^{op} \longrightarrow \mathcal{V}$. We write D^n for the n-fold composite of D with itself for $n \geq 0$ and of D^{-1} with itself for $n < 0$. For n even, we have the monoidal isomorphism $D^n : \mathcal{V} \longrightarrow \mathcal{V}$ and, for n odd, we have the monoidal isomorphism $D^n : \mathcal{V}^{op} \longrightarrow \mathcal{V}$.

CHAPTER 3: Pivotal Categories

Section 3.1 Coherence for pivotal categories

Definition 3.1.1 A *pivotal category* $\mathcal{V}$ consists of a left autonomous monoidal category $\mathcal{V}$ with a monoidal natural isomorphism

$$i_A : A \xrightarrow{\;\cong\;} A^{**}$$

such that the following diagram commutes.

$$\begin{tikzcd} A^{*} \arrow[r, "i_{A^*}"] \arrow[dr, "1_{A^*}"'] & A^{***} \arrow[d, "i_A^{\;*}"] \\ & A^{*} \end{tikzcd}$$

The condition that i_A be monoidal means that i_I is the canonical isomorphism $I \cong I^{**}$ and $i_{A\otimes B}$ is the composite of $i_A \otimes i_B$ with the canonical isomorphism $A^{**} \otimes B^{**} \cong (A \otimes B)^{**}$. Every object of $\mathcal{V}$ is isomorphic to a left dual (namely, A^{**}), so $(\;)^* : \mathcal{V}^{op} \longrightarrow \mathcal{V}$ is an equivalence of categories. It follows that every pivotal category is autonomous.

Example 3.1.2 Let H be a Hopf algebra over a commutative ring $\mathcal{k}$. We write $\delta : H \longrightarrow H \otimes H$ for the comultiplication, $\varepsilon : H \longrightarrow \mathcal{k}$ for the counit, and $\nu : H \longrightarrow H$ for the antipode. Recall that an element $u \in H$ is called *group-like* when

$$\delta(u) = u \otimes u\,.$$

The group-like elements u satisfy the condition $\varepsilon(u) = 1$ and form a group under multiplication in H with inverse given by $u^{-1} = \nu(u)$. (They are linearly independent when $\mathcal{k}$ is a field.) Suppose we have a group-like element $u \in H$ satisfying

$$\nu^2(x)\, u = u\, x \quad \text{for all } x \in H.$$

Then the autonomous monoidal category $Pr_{\mathcal{k}}(H)$ (Example 2.1.1) becomes pivotal on defining $i_M : M \longrightarrow M^{**}$ by $i_M(m)(\phi) = \phi(u\, m)$ for all $m \in M$ and $\phi \in M^*$. //

Definition 3.1.3 A pivotal category is called *strict* when it is a strict autonomous monoidal category and each i_A is an identity.

Proposition 3.1.4 *Each pivotal category is monoidally equivalent to a strict one.*

Proof The proof is a "mod 2" version of the proof of Proposition 2.1.3. For any strict monoidal category, we define a pivotal category PAdj $\mathcal{V}$ as follows. An object is a quadruplet $A = (A_0, A_1, \varepsilon^{A}{}_0, \varepsilon^{A}{}_1)$ where $\varepsilon^{A}{}_0 : A_0 \otimes A_1 \longrightarrow I$, $\varepsilon^{A}{}_1 : A_1 \otimes A_0 \longrightarrow I$ are exact pairings. An arrow $f = (f_0, f_1) : A \longrightarrow B$ consists of arrows $f_0 : A_0 \longrightarrow B_0$, $f_1 : B_1 \longrightarrow A_1$ such that the two conditions hold as for arrows in Adj $\mathcal{V}$ taken for n modulo 2. The monoidal structure is given as for Adj $\mathcal{V}$ taken for n modulo 2. Clearly PAdj $\mathcal{V}$ is a strict pivotal category. There is a fully faithful strict-monoidal functor $E : \mathrm{PAdj}\,\mathcal{V} \longrightarrow \mathcal{V}$ given by $EA = A_0$, $Ef = f_0$. It is routinely verified that E is an equivalence if and only if $\mathcal{V}$ is pivotal.**qed**

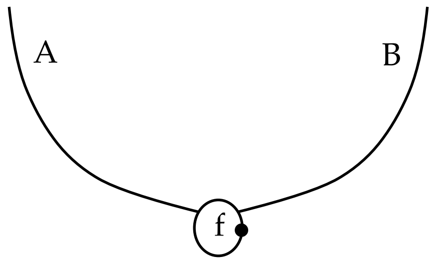
In a pivotal category $\mathcal{V}$ there is a certain amount of *cyclic symmetry* which we now explain. For objects $A, B \in \mathcal{V}$, there is a bijection

$$\tau_{A,B} : \mathcal{V}(I, A \otimes B) \xrightarrow{\cong} \mathcal{V}(I, B \otimes A)$$

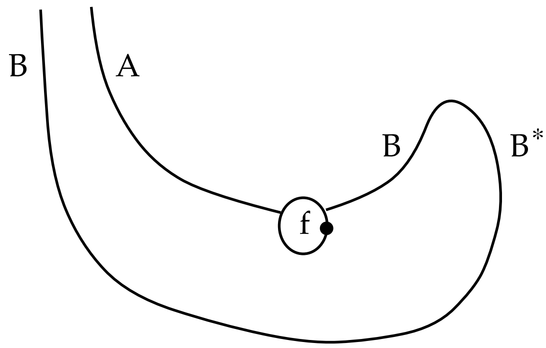
taking the arrow $f : I \longrightarrow A \otimes B$ to the composite

$$I \xrightarrow{\eta_B} B \otimes B^* \xrightarrow{1 \otimes f \otimes 1} B \otimes A \otimes B \otimes B^* \xrightarrow{1 \otimes 1 \otimes \varepsilon_{B^*}} B \otimes A .$$

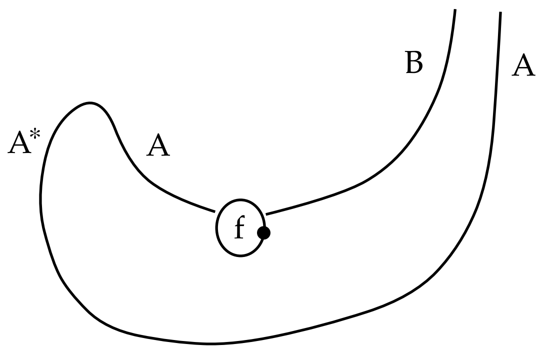
Diagrammatically, if we draw f as



(with the significance of the marker on the node to be explained later) then $\tau_{A,B}(f)$ is the value of the following plane diagram in which the string labelled B has been "dragged around under the node".

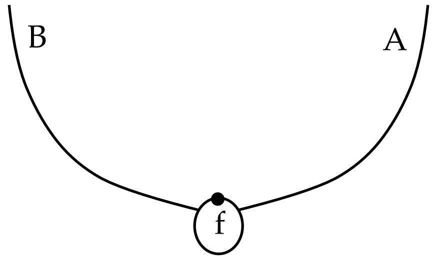


This raises the question as to whether dragging the string labelled A around under the node, as in the diagram



,

would lead to the same value which we could then depict by



which differs from the picture for f in that the labels on the strings are switched and the marker is cranked around through a quarter turn. This possible alternative to $\tau_{A,B}(f)$ is the composite

$$I \xrightarrow{\eta_{A^*}} A^* \otimes A \xrightarrow{1 \otimes f \otimes 1} A^* \otimes A \otimes B \otimes A \xrightarrow{\varepsilon_A \otimes 1 \otimes 1} B \otimes A\,.$$

The next Proposition shows that this composite is indeed equal to $\tau_{A,B}(f)$.

Proposition 3.1.5 *In any pivotal category, the following diagram commutes for all arrows* $f : I \longrightarrow A \otimes B$.

$$\begin{tikzcd}
& B \otimes B^* \arrow[r, "1 \otimes f \otimes 1"] & B \otimes A \otimes B \otimes B^* \arrow[dr, "1 \otimes 1 \otimes \varepsilon_{B^*}"] & \\
I \arrow[ur, "\eta_B"] \arrow[dr, "\eta_{A^*}"'] & & & B \otimes A \\
& A^* \otimes A \arrow[r, "1 \otimes f \otimes 1"'] & A^* \otimes A \otimes B \otimes A \arrow[ur, "\varepsilon_A \otimes 1 \otimes 1"'] &
\end{tikzcd}$$

Proof Put $g = \varepsilon_A{}^{\#}(f) : A^* \longrightarrow B$ which is the composite

$$A^* \xrightarrow{1_A \otimes f} A^* \otimes A \otimes B \xrightarrow{\varepsilon_A \otimes 1_B} B$$

so that f is the composite

$$I \xrightarrow{\eta_A} A \otimes A^* \xrightarrow{1 \otimes g} A \otimes B\,.$$

Now $g^* : B^* \longrightarrow A$ is defined to be the composite

$$B^* \xrightarrow{\eta_A \otimes 1} A \otimes A^* \otimes B^* \xrightarrow{1 \otimes g \otimes 1} A \otimes B \otimes B^* \xrightarrow{1 \otimes \varepsilon_{B^*}} A$$

which, from our formula for f in terms of g, is equal to the composite

$$B^* \xrightarrow{f \otimes 1} A \otimes B \otimes B^* \xrightarrow{1 \otimes \varepsilon_{B^*}} A\,.$$

Therefore the three inner regions of the following diagram commute.

$$\begin{tikzcd}
 & B \otimes B^* \arrow[r, "1\otimes f \otimes 1"] \arrow[rrd, "1\otimes g^*"'] & B \otimes A \otimes B \otimes B^* \arrow[rd, "1\otimes 1\otimes \varepsilon_{B^*}"] & \\
I \arrow[ru, "\eta_B"] \arrow[rd, "\eta_{A^*}"'] & & & B \otimes A \\
 & A^* \otimes A \arrow[r, "1\otimes f \otimes 1"'] \arrow[rru, "g\otimes 1"] & A^* \otimes A \otimes B \otimes A \arrow[ru, "\varepsilon_A \otimes 1 \otimes 1"'] &
\end{tikzcd}$$

qed

To explain why this leads to "cyclic" symmetry and to express the coherence of the bijections $\tau_{A,B}$ we need some formalism. Suppose we are given a set Σ and "conjugation symbols" $\tau_{u,v} : u\,v \longrightarrow v\,u$ for all words u, v in the alphabet Σ. Let $\mathrm{Conj}(\Sigma)$ denote the category whose objects are words in the alphabet Σ and whose arrows are generated by the conjugation symbols subject to the following relations.

$$\begin{tikzcd}
u\,v \arrow[r, "\tau_{u,v}"] \arrow[rd, "1"'] & v\,u \arrow[d, "\tau_{v,u}"] \\
 & u\,v
\end{tikzcd}
\qquad
\begin{tikzcd}
u\,v\,w \arrow[r, "\tau_{u,vw}"] \arrow[rd, "\tau_{uv,w}"'] & v\,w\,u \arrow[d, "\tau_{v,wu}"] \\
 & w\,u\,v
\end{tikzcd}$$

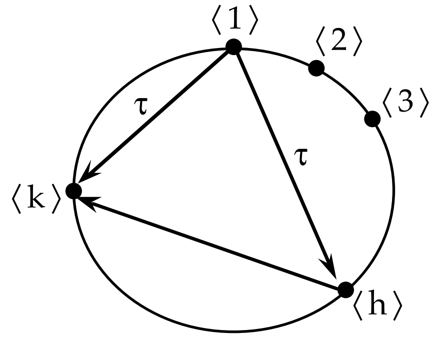
It follows that $\tau_{u,v}$ is an identity arrow when either u or v is the empty word.

Proposition 3.1.6 *In the category* $\mathrm{Conj}(\Sigma)$, *there is a unique arrow between any two conjugate words with distinct letters.*

Proof Let $\Sigma = \{1, 2, \ldots, n\}$, and, for $k \in \Sigma$, let

$$\langle k \rangle = k \;\; k{+}1 \;\; k{+}2 \;\ldots\; n{-}1 \;\; n \;\; 1 \;\; 2 \;\; 3 \;\ldots\; k{-}1$$

be the cyclic permutation of $1, 2, \ldots, n$ starting with k. It suffices to show that, for all $k \in \Sigma$, the unique arrow in this $\mathrm{Conj}(\Sigma)$ from $\langle 1 \rangle$ to $\langle k \rangle$ is $\tau_{u,v}$ where $u = 1\;2\;3 \ldots k{-}1$ and $v = k\;\; k{+}1\;\; k{+}2 \ldots n{-}1\;\; n$. This is done by induction on the minimum length of the path of conjugation symbols representing the arrow. Any other arrow $\langle 1 \rangle \longrightarrow \langle k \rangle$ must factor through some $\tau : \langle 1 \rangle \longrightarrow \langle h \rangle$ where, by induction, the arrow $\langle h \rangle \longrightarrow \langle k \rangle$ must also be a conjugation symbol τ.



The triangle created is then a defining relation for $\mathrm{Conj}(\Sigma)$ and so commutes. So the full subcategory of Conj(Σ) consisting of the objects $\langle k \rangle$ is equivalent to the terminal category $\mathbb{1}$ (and so is a *clique* in the sense of [JS2; p.58]). **qed**

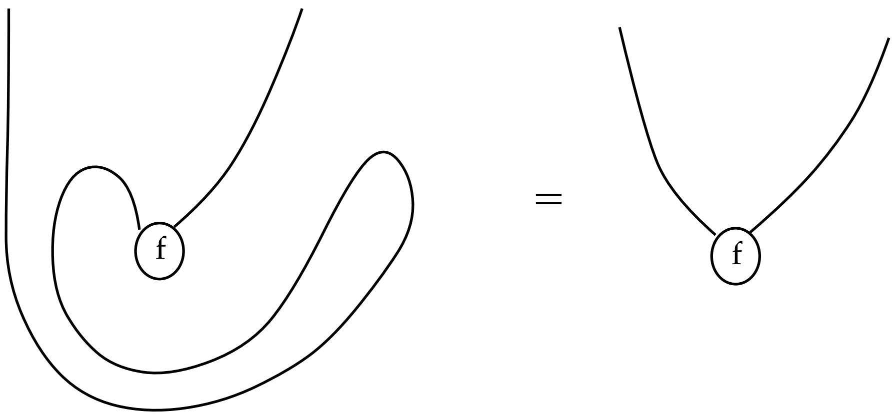
Suppose $\mathcal{V}$ is a strict pivotal category. Let Σ be the set of objects of $\mathcal{V}$. We shall define a functor $T : \mathrm{Conj}(\Sigma) \longrightarrow \mathrm{Set}$ into the category of sets. For each object A of $\mathcal{V}$, put

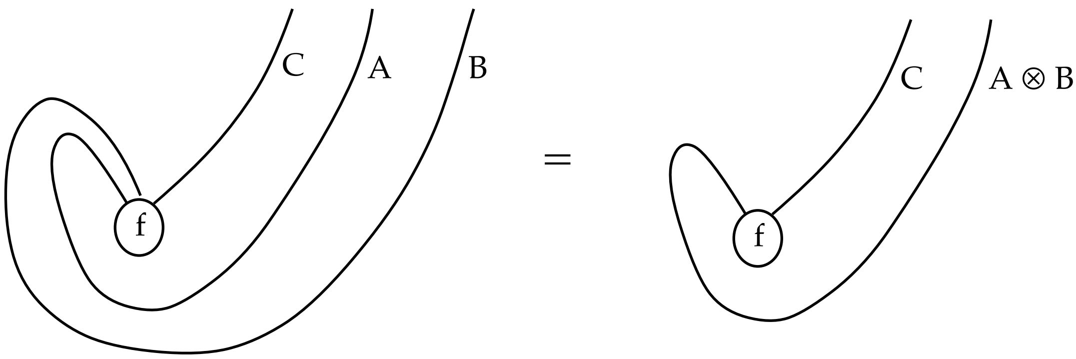
$$TA = \mathcal{V}(I, A)$$

which defines T on words of length 1. For an arbitrary word $u = A_1 A_2 \dots A_m$ of objects of $\mathcal{V}$, we define $Tu = T(\otimes u)$ where $\otimes u = A_1 \otimes A_2 \otimes \dots \otimes A_m \in \mathcal{V}$. For each generating arrow $\tau_{u,v} : u\,v \longrightarrow v\,u$ of $\mathrm{Conj}(\Sigma)$, we define

$$T\tau_{u,v} = \tau_{A,B} : \mathcal{V}(I, A \otimes B) \xrightarrow{\cong} \mathcal{V}(I, B \otimes A)$$

where $A = \otimes u$ and $B = \otimes v$. That T respects the two defining relations for $\mathrm{Conj}(\Sigma)$ is proved by the following two diagrammatic identities (the first of which uses the two descriptions of τ available from Proposition 3.1.5; the second uses the monoidalness of ε, η).





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