§1. Graphs.

A *graph* is a set G with unary operations s, t, called *source*, *target*, respectively, such that

$$ss = st = s, \qquad tt = ts = t.$$

The notation $g : u \to v$ is used to mean $g \in G$ with $u = gs$, $v = gt$. Call $u \in G$ a *vertex* when there exists $g \in G$ with $u = gs$. If u is a vertex then $u = us = ut$. The set of vertices of G will be denoted by Gs; it is the image of the function s (equally, it is the image of the function t).

A pair (g, h) of elements of G is called *compatible* when $gt = hs$. A list $(g_1, g_2, \ldots)$ of elements of G is *compatible* when consecutive pairs of terms are compatible. A list $(g_1, g_2, \ldots)$ is *degenerate* when some term g_i is a vertex.

For $n > 0$, *a path of length* n *in* G is a non-degenerate compatible element $(g_1, g_2, \ldots, g_n)$ of G^n which we denote by $g_1 g_2 \cdots g_n$. *A path of length* 0 *in* G is just a vertex of G. Let G^* denote the set of paths in G. The *length* of a path p is denoted by $|p|$. We regard G^* as a graph with the same vertices as G by defining

$$(g_1 g_2 \cdots g_n)s = g_1 s, \quad (g_1 g_2 \cdots g_n)t = g_n t.$$

A vertex u of G is called *stable* when $gs = u$ implies $g = u$. A *stabilizer* for a vertex u is a path $p : u \to v$ with v stable. The graph G *has stabilizers* when each vertex has a stabilizer.

A graph is said to satisfy the *chain condition* (CC) when every compatible infinite list is degenerate.

Lemma 1. *If a graph satisfies the chain condition CC then it has stabilizers.*

Proof. Suppose u is a vertex with no stabilizer. Then u is not stable so we choose a non-vertex $g_1 : u \to u_1$. Now u_1 can have no stabilizer or else u would. Continuing, we obtain a non-degenerate compatible list

$$u \xrightarrow[g_1]{} u_1 \xrightarrow[g_2]{} u_2 \xrightarrow[g_3]{} \cdots,$$

contradicting CC. $\square$

A vertex u in a graph is called *bounded* when there is a natural number n such that $|p| \leq n$ for all paths $p : u \to v$. The graph satisfies the *boundedness condition* (BC) when every vertex is bounded. Clearly BC implies CC but not conversely.

A graph is called *locally finite* (LF) when, for each vertex u, there are only finitely many elements g with $gs = u$.

We note the following form of König's[†] Lemma.

([†] see e.g. D. E. Knuth "The Art of Computer Programming" (2nd Ed. 1973, Addison-Wesley) Vol. 1 2.3.4.3 p.381)

Lemma 2. CC and LF imply BC.

Proof. Suppose u is a vertex which is not bounded. By LF there is a non-vertex $u \to u_1$ with u_1 not bounded. Continuing, we obtain a non-degenerate compatible list $u \to u_1 \to u_2 \to \ldots$, contradicting CC. □

Suppose G, H are graphs. We use the same symbols s, t for source and target in H as in G. A graph morphism $\rho : G \to H$ is a function such that

$$s\rho = \rho s \quad \text{and} \quad t\rho = \rho t.$$

Graph morphisms preserve vertices: if u is a vertex of G then $u\rho$ is a vertex of H. We say that ρ reflects vertices when $g\rho$ a vertex in H implies g is a vertex in G. We say that ρ is prelifting when $gs = g's$ and $g\rho = g'\rho$ imply $g = g'$. Clearly a prelifting graph morphism reflects vertices but not conversely.

Proposition 3. Suppose $\rho : G \to H$ is a graph morphism. If ρ reflects vertices, H satisfies CC or BC implies G does. If ρ is prelifting, H is LF implies G is. □

Example 4. Put $\mathbb{M} = \{(m,n) \in \mathbb{N}^2 \mid n \leq m\}$ and we identify $n \in \mathbb{N}$ with $(n,n) \in \mathbb{M}$. Regard $\mathbb{M}$ as a graph with $(m,n)s = m$, $(m,n)t = n$. Each vertex of $\mathbb{M}$ is bounded, $\mathbb{M}$ is locally finite, the only stable vertex is 0, and $(m,0)$ is a stabilizer for m. □

§2. Categories.

A category is a graph A together with a function $*$, called composition, which assigns to each compatible pair (a,b) an element $a * b$ with source as and target bt such that

$$(as) * a = a = a * (at)$$

$$(a * b) * c = a * (b * c) \text{ for } (a,b,c) \text{ compatible.}$$

The composite of a compatible element $(a_1, a_2, \ldots, a_n)$ of A^n is inductively defined by the equation

$$a_1 * a_2 * \cdots * a_n = (a_1 * a_2 * \cdots * a_{n-1}) * a_n$$

which is interpreted as a_1 when $n = 1$. Then, for $1 \leq m < n$,

$$a_1 * a_2 * \cdots * a_n = (a_1 * \cdots * a_m) * (a_{m+1} * \cdots * a_n).$$

(See Section 5 Proposition 5.5 for a proof of this.)

A decomposition of $a \in A$ of length n is a path of length n in the underlying graph of A whose composite in A is a. Each element in the path is called a component of the decomposition. An element of A is called decomposable when it admits a decomposition of length 2.

A category A is called free when each vertex is indecomposable and each non-vertex has a unique decomposition with indecomposable components. Then each $a \in A$ has a *length* $|a|$ which is 0 when a is a vertex and is the length of the unique decomposition otherwise.

For each graph G, the graph G^* becomes a category by defining composition of paths to be concatenation:

$$(g_1 g_2 \cdots g_m) * (h_1 h_2 \cdots h_n) = g_1 g_2 \cdots g_m h_1 h_2 \cdots h_n .$$

Proposition 1. *For each graph G, the category G^* of paths in G is free and contains G as the subgraph of indecomposables.* $\square$

Notice that the length of a path in G is precisely its length as an element of the free category G^*.

For categories A, X a *functor* $\varphi : A \to X$ is a graph morphism such that

$$(a * b)\varphi = (a\varphi) * (b\varphi) \text{ for all compatible pairs } (a,b) \text{ in } A.$$

Proposition 2. *If G is the subgraph of indecomposables of a free category A then each graph morphism $\psi : G \to X$ into a category X extends to a unique functor $\varphi : A \to X$.*

Proof. In order for φ to be a functor extending ψ we are forced to define $a\varphi = g_1\psi * \dots * g_n\psi$ where $g_1 \cdots g_n$ is the decomposition of a with indecomposable $g_1, \dots, g_n$. Then φ is a functor by uniqueness of the decomposition. $\square$

Example 3. The set $\mathbb{N}$ of natural numbers becomes a category with $ns = nt = 0$ and $m * n = m + n$. The only vertex is 0, the only indecomposables are $0, 1$. The category is free with $|n| = n$. Any free category A has a *length functor* $|\ | : A \to \mathbb{N}$ taking each $a \in A$ to its length $|a|$. $\square$

Suppose G is a graph and $\pi : G^* \to A$ is a functor into a category A. The *reuniting condition* (RC) for the pair G, π is as follows:

for all $g : u \to v$, $g' : u \to v'$ *in* G, *there exist paths* $q : v \to w$, $q' : v' \to w$ *in* G *with* $(gq)\pi = (g'q')\pi$

$$\begin{tikzcd} u \arrow[r, "g"] \arrow[d, "g'"'] & v \arrow[d, dashed, "q"] \\ v' \arrow[r, dashed, "q'"'] & w \end{tikzcd}$$

Lemma 4. *If the functor $\pi : G^* \to A$ satisfies the reuniting condition RC and u is a bounded vertex of G then, for all paths $p : u \to v$, $p' : u \to v'$ in G there exist paths $q : v \to w$, $q' : v' \to w$ in G with $(pq)\pi = (p'q')\pi$.*

Proof. The proof is by induction on the bound n on the lengths of paths out of u. For $n = 0$, there is nothing to prove (u is stable). Suppose the Lemma is true for vertices of bound $< n$, and suppose u has bound n. If one of the paths $p : u \to v$, $p' : u \to v'$ has length 0 the result is clear. Otherwise $p = g p_1$, $p' = g' p_1'$ where $g : u \to u_1$, $g' : u \to u_1'$ in G. Apply RC to g, g' to obtain paths $u_1 \to w_1$, $u_1' \to w_1$. Since u_1, u_1', w_1 have bound $< n$, the inductive hypothesis can be applied three times as indicated by the following diagram

$$\begin{tikzcd}
u \arrow[r, "g"] \arrow[d, "g'"'] & u_1 \arrow[r, "p_1"] \arrow[d] & v \arrow[d] \\
u_1' \arrow[r] \arrow[d] & w_1 \arrow[r] \arrow[d] & w_2 \arrow[d] \\
v' \arrow[r] & w_2' \arrow[r] & w
\end{tikzcd} \quad . \square$$

<u>Corollary 5.</u> *<u>If</u>* $\pi : G^* \to A$ *<u>satisfies the reuniting condition RC and</u>* $p : u \to v$, $p' : u \to v'$ *<u>are stabilizers for a bounded vertex</u>* u *<u>of</u>* G *<u>then</u>*

$$v = v' \qquad \underline{\text{and}} \qquad p\pi = p'\pi .$$

<u>Proof</u>. Choose paths $q : v \to w$, $q' : v' \to w$ as in Lemma 2.4. But v, v' are stable so q, q' have length 0. So $v = w = v'$ and $p\pi = p'\pi$. $\square$

Call G, $\pi : G^* \to A$ <u>perfect</u> when G has stabilizers and, if $p : u \to v$, $p' : u \to v'$ are stabilizers for the same vertex u, then $v = v'$ and $p\pi = p'\pi$. (We do not ask $p = p'$.) In the case where $A = 1$, so that π is the unique functor $G \to 1$ we merely say G is perfect. Lemma 1.1 and Corollary 1.5 have the following consequence.

<u>Proposition 6.</u> *<u>If</u>* G *<u>satisfies BC and</u>* $\pi : G^* \to A$ *<u>satisfies RC then</u>* (G, π) *<u>is perfect</u>*. $\square$

§3. 2-Graphs and 2-categories.

A *2-graph* is a set G with two graph structures s_0, t_0 and s_1, t_1 interrelated by the conditions

$$s_0 s_1 = s_0 = s_1 s_0 = t_1 s_0, \qquad t_0 = s_1 t_0 = t_1 t_0.$$

It is easily deduced that $s_0 t_1 = s_0$ and $t_0 t_1 = t_0 = t_0 s_1$. The notations

$$\begin{tikzcd} x \arrow[r, bend left, "u"{name=U}] \arrow[r, bend right, "v"'{name=V}] \arrow[from=U, to=V, phantom, "\Downarrow g"] & y \end{tikzcd}, \qquad g : u \Longrightarrow v, \quad \text{and} \quad g : u \Longrightarrow v : x \longrightarrow y$$

are used to mean $g \in G$, $u = g s_1$, $v = g t_1$, $x = g s_0$, $y = g t_0$. The graph (G, s_i, t_i) is denoted G_i. Vertices of G_i are called *i-vertices* and compatible elements of G_i^n are called *i-compatible*. Notice that the set $G s_1$ of 1-vertices becomes a graph using s_0, t_0 as source and target. Elements of G which are not 1-vertices will be called *solid*.

A *2-graph morphism* $\varphi : G \longrightarrow H$ is a function which is a graph morphism $G_i \longrightarrow H_i$ for $i = 0, 1$. So $g : u \Rightarrow v : x \longrightarrow y$ in G yields $g\varphi : u\varphi \Rightarrow v\varphi : x\varphi \longrightarrow y\varphi$ in H.

A *2-category* is a 2-graph A together with category structures $*_0, *_1$ on the graphs A_0, A_1, respectively, such that the following conditions hold:

$a t_0 = a' s_0$ implies $(a *_0 a') s_1 = (a s_1) *_0 (a' s_1)$ and $(a *_0 a') t_1 = (a t_1) *_0 (a' t_1)$; and

$a t_0 = a' s_0$, $a t_1 = b s_1$, $a' t_1 = b' s_1$ imply $(a *_1 b) *_0 (a' *_1 b') = (a *_0 a') *_1 (b *_0 b')$.

The last equation is called the *middle-four-interchange law*.

Notice that the composition $*_0$ enriches $A s_1$ with a category structure.

Proposition 1 *In a 2-category, if $a t_0 = b s_0$ then*

(1) $\quad (a *_0 b s_1) *_1 (a t_1 *_0 b) = a *_0 b = (a s_1 *_0 b) *_1 (a *_0 b t_1)$.

If $a t_1 = b s_1 = x$ and x is a 0-vertex then

(2) $\quad a *_0 b = a *_1 b = b *_0 a$.

Proof. The middle-four-interchange law rewrites the left-hand side of (1) as $(a *_1 a t_1) *_0 (b s_1 *_1 b)$ which is $a *_0 b$ since $a t_1$, $b s_1$ are 1-vertices; a symmetric argument applies to the right-hand side. The first equation of (2) comes from the first equation of (1) since $b s_1$, $a t_1$ are 0-vertices. Also $b t_0 = a s_0 = x$ so (1) applies with a, b interchanged; the second equation of this gives the second equation of (2). $\Box$

An element a of a 2-graph is called *central* when $a s_0 = a s_1 = a t_1$. Pictorially,

$$\begin{tikzcd} x \arrow[r, bend left, "x"{name=U}] \arrow[r, bend right, "x"'{name=V}] \arrow[from=U, to=V, phantom, "\Downarrow a"] & x \end{tikzcd}.$$

For central a, b, 0-compatibility of (a, b) is the same as 1-compatibility; in this case, we call (a, b) merely *compatible*.

Proposition 2. *If a, b form a compatible pair of central elements in a 2-category then*

$$a *_1 b = b *_1 a = a *_0 b = b *_0 a.$$

Proof. Proposition 1 equation (2) holds for a, b and for b, a. $\Box$

A *2-functor* $\varphi : A \longrightarrow B$ is a function which is a functor $A_i \longrightarrow B_i$ for $i = 0$ and 1.

§4. Derivation schemes.

A *derivation scheme* is a 2-graph D together with a category structure $\circ$ on the graph D_{s_1} of 1-vertices. Each 2-category has an underlying derivation scheme with 0-composition of 1-vertices.

An *elementary derivation* in D is a 0-compatible triple (x, e, y) where x, y are 1-vertices, and, if e is a 1-vertex then x, y are 0-vertices. Each $e \in D$ will be identified with the elementary derivation (es_0, e, et_0). So D is regarded as a subset of the set $D\mathcal{E}$ of elementary derivations. The 2-graph structure on $D\mathcal{E}$ defined by the equations

$$(x,e,y)s_0 = xs_0\,, \qquad (x,e,y)t_0 = yt_0\,,$$
$$(x,e,y)s_1 = x\circ(es_1)\circ y, \quad (x,e,y)t_1 = x\circ(et_1)\circ y,$$

is such that the 1-vertices of $D\mathcal{E}$ are precisely the 1-vertices of D. Thus $D\mathcal{E}$ becomes a derivation scheme for which $D\mathcal{E}s_1 = Ds_1$ as categories.

The elementary derivation (x,e,y) can be depicted

$$\xrightarrow{\;x\;}\ \begin{array}{c}\xrightarrow{\;u\;}\\ \Downarrow e\\ \xrightarrow[\;v\;]{}\end{array}\ \xrightarrow{\;y\;}$$

so that we call x the *left whisker* and y the *right whisker* of (x,e,y).

Elements e of D are viewed as *rules* which, from the assumption es_1, derives the conclusion et_1. We view (x,e,y) as the rule e applied with prefix x and suffix y. In this context it is natural to consider the result of applying one elementary derivation after another.

A *derivation* in D is a path in the graph $(D\mathcal{E})_1 = (D\mathcal{E}, s_1, t_1)$. So a derivation d is either a 1-vertex of D or a path $d = d_1 d_2 \ldots d_n$ of elementary derivations $d_i = (x_i, e_i, y_i)$. The compatibility condition for a path amounts to the equations

$$x_i \circ (e_i t_1)\circ y_i \;=\; x_{i+1}\circ(e_{i+1}s_1)\circ y_{i+1}.$$

The non-degeneracy condition for a path amounts to the e_i all solid. The *length of a derivation* d is the length of the path in $(D\mathcal{E})_1$ and is denoted by $\|d\|$.

We obtain a derivation scheme $D\partial$ whose elements are derivations d in D, whose sources and targets are given by

$$d s_0 = d_i s_0\,, \quad d t_0 = d_i t_0 \text{ for any } i = 1,\ldots,n,$$
$$d s_1 = d_1 s_1\,, \quad d t_1 = d_n t_1,$$

and $D\partial s_1 = D s_1$ as categories.

Various compositions are now available in $D\partial$. We have the composition $\circ$ of 1-vertices since these are the same as for D; this is the derivation scheme aspect of $D\partial$. Also $(D\partial)_1 = (D\mathcal{E})_1^*$ is a free category and we shall write $*_1$ for concatenation of derivations

$$(d_1 \ldots d_n) *_1 (d'_1 \ldots d'_{n'}) \;=\; d_1 \ldots d_n\, d'_1 \ldots d'_{n'}$$

when $d_n t_1 = d'_1 s_1$.

The composition $\circ$ of Ds_1 can be extended to a composition $d *_0 d'$ of 0-compatible pairs (d, d') in $D\partial$ <u>provided one of</u> d, d' <u>is a 1-vertex</u>. Suppose h, k are 1-vertices of D and $d = d_1 \dots d_n$ is a derivation with $d_i = (x_i, e_i, y_i)$ elementary. If $h t_0 = d s_0$ and $d t_0 = k s_0$, define

$$h *_0 d = (h *_0 d_1, \dots, h *_0 d_n), \qquad d *_0 k = (d_1 *_0 k, \dots, d_n *_0 k)$$

where $h *_0 d_i = (h \circ x_i, e_i, y_i)$, $d_i *_0 k = (x_i, e_i, y_i \circ k)$.
We have the following associativity properties

$$(0) \qquad \begin{aligned} (h *_0 d) *_0 k &= h *_0 (d *_0 k) \\ h' *_0 (h *_0 d) &= (h' \circ h) *_0 d \\ (d *_0 k) *_0 k' &= d *_0 (k \circ k'). \end{aligned}$$

Also, if h, k are 0-vertices then $h *_0 d = d = d *_0 k$.

However, there is an obstruction to extending this composition $*_0$ to all 0-compatible pairs (d, d') in such a way as to obtain a 2-category structure on $D\partial$. The problem is that there are two unequal choices to make for $d *_0 d'$; if we are to have a 2-category these two must be equal by Proposition 3.1. This means we need to pass to a quotient of $D\partial$ to obtain a 2-category.

For $e : u \Rightarrow v$, $e' : u' \Rightarrow v'$ and 1-vertices x, y in D, consider the square

$$\begin{tikzcd}
x \circ u \circ y \circ u' \circ z \arrow[r, Rightarrow, "{(x \circ u \circ y, e', z)}"] \arrow[d, Rightarrow, "{(x, e, y \circ u' \circ z)}"'] & x \circ u \circ y \circ v' \circ z \arrow[d, Rightarrow, "{(x, e, y \circ v' \circ z)}"] \\
x \circ v \circ y \circ u' \circ z \arrow[r, Rightarrow, "{(x \circ v \circ y, e', z)}"'] & x \circ v \circ y \circ v' \circ z .
\end{tikzcd}$$

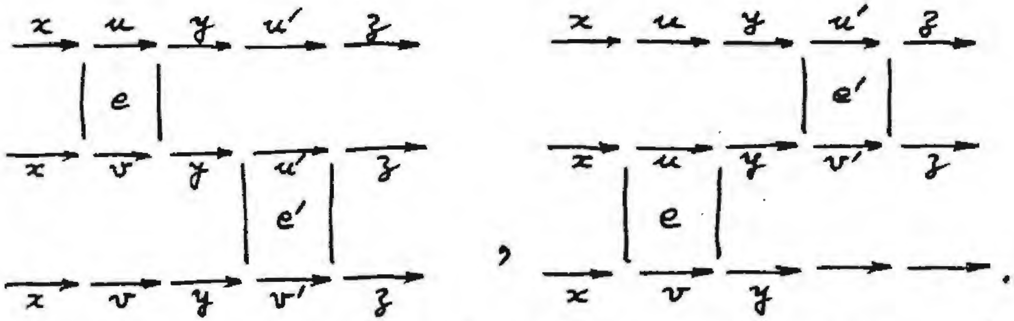
When e, e' are ~~not 1-vertices~~ solid, the lower path around the square gives a derivation

$$(1) \qquad (x, e, y \circ u' \circ z)(x \circ v \circ y, e', z) : x \circ u \circ y \circ u' \circ z \Rightarrow x \circ v \circ y \circ v'$$

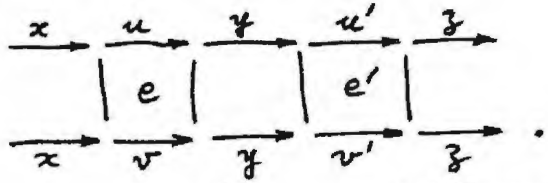
and the upper path around the square gives a derivation

$$(2) \qquad (x \circ u \circ y, e', z)(x, e, y \circ v' \circ z) : x \circ u \circ y \circ u' \circ z \Rightarrow x \circ v \circ y \circ v' \circ z$$

both of length 2, and pictorially represented as follows:



We wish to regard these derivations as the same, and, pictorially representable as follows:



This leads us to the next definition.

Derivations $d = d_1 d_2 \dots d_m$, $d' = d'_1 d'_2 \dots d'_n$ are called <u>equivalent</u> when

(i) $m = n \geqslant 2$,

(ii) $d s_1 = d' s_1$ and $d t_1 = d' t_1$

(iii) d' can be obtained from d by a finite number of steps each of which involves recognizing a consecutive pair of components as of the form (1) or (2) and replacing it by the corresponding (2) or (1).

Equivalence of derivations determines an equivalence relation on $D\partial$ which relates no distinct 1-vertices and is compatible with $s_0, t_0, s_1, t_1, *_0, *_1$. So we obtain a derivation scheme D^* of equivalence classes $[d]$ of elements d of $D\partial$. The composition $*_1$ and, in so far as it has been defined, the composition $*_0$ induce compositions on D^* denoted by the same symbols: that is, for $i = 0, 1$,

$$[d] *_i [d'] = [d *_i d']$$

where, in the case $i = 0$, one of d, d' must be a 1-vertex.

<u>Lemma 1</u>. <u>If</u> $d : u \Rightarrow v$, $d' : u' \Rightarrow v'$ <u>are derivations with</u> $d t_0 = d' s_0$ <u>then the derivations</u>

(3) $\qquad (d *_0 u') *_1 (v *_0 d')$ <u>and</u> $(u *_0 d') *_1 (d *_0 v')$

<u>are equivalent</u>.

<u>Proof</u>. When d, d' are elementary the first derivation in (3) is of the form (1) and the second of the form (2), so we have the result. If $d = d_1 *_1 d_2$ then (3) becomes

$$((d_1 *_0 u') *_1 (v *_0 d')) *_1 ((d_2 *_0 u') *_1 (v *_0 d')) \text{ and } ((u *_0 d') *_1 (d_1 *_0 v')) *_1 ((u *_0 d') *_1 (d_2 *_0 v')).$$

So it suffices to know the result for d_1, d_2. Similarly, if d' is a 1-composite, it suffices to know the result for the components. Since d, d' are 1-composites of elementary derivations, the result follows. $\square$

This leads us to define $*_1$ for D^* by

$$[d] *_1 [d'] = [(d *_0 u') *_1 (v *_0 d')] = [(u *_0 d') *_1 (d *_0 v')]$$

where $d : u \Rightarrow v$, $d' : u' \Rightarrow v'$ in $D\partial$ have $d t_0 = d' s_0$. It now follows quite formally from the associativity conditions (0) that D^* with $*_0, *_1$ forms a 2-category.

A <u>derivation scheme morphism</u> is a 2-graph morphism which induces a functor on 1-vertices.

The inclusion $D \to D^*$ is clearly a derivation scheme morphism.

<u>Proposition 2</u>. <u>Each derivation scheme morphism</u> $\psi : D \to A$ <u>into a 2-category</u> A <u>has a unique extension to a 2-functor</u> $\varphi : D^* \to A$.

<u>Proof</u>. Derivation scheme morphisms $\psi' : D\varepsilon \to A$, $\psi'' : D\partial \to A$ extending ψ are determined by the formulas

$$(x, e, y)\psi' = x\psi *_0 e\psi *_0 y\psi,$$

$$(d_1 d_2 \dots d_n)\psi'' = d_1\psi' *_1 \dots *_1 d_n \psi'.$$

Clearly ψ'' is the unique extension of ψ to $D\partial$ which preserves $*$, and the limited $*_0$. By Proposition 3.1, ψ'' identifies the derivations (1) and (2), and hence, equivalent derivations. So we have φ as desired with $[d]\varphi = d\,\psi''$. $\square$

In other words, D^* *is the free 2-category on the derivation scheme* D.

Suppose A is a category and D is a derivation scheme whose category Ds_1 of 1-vertices is A. Two elements of A are said to be *equivalent modulo* D when they are related by the smallest equivalence relation on A which has $a, b \in A$ related if there exists a derivation $a \Rightarrow b$ of D. Equivalence modulo D is compatible with the category structure on A and so we obtain a category A/D whose elements are equivalence classes modulo D. Call A/D *the quotient of* A *by* D.

§5. Rewrite systems.

A *rewrite system* R is a derivation scheme for which the category Rs_1 is free. Indecomposable elements of Rs_1 are called *letters* while arbitrary elements of Rs_1 are called *words*. Since words can be identified with paths of letters and composition in Rs_1 with concatenation of paths, we use juxtaposition for composition in Rs_1. Solid elements of R are called *rewrite rules*.

A *rewrite system morphism* is a derivation scheme morphism which takes letters to letters.

To give a rewrite system is to give a graph of letters and a set of rewrite rules for words in those letters. The case where the graph of letters has only one vertex and so is just a (pointed) set, or *alphabet*, was considered by Eilenberg[ ]. General rewrite systems are essentially the "computads" of Street[ ].

If R is a rewrite system, so are $R\varepsilon$, $R\partial$ and R^* since these have the same 1-vertices as R. Of course R^* is a 2-category.

We are interested in a class of word problems which can be solved because *unique normal forms* can be derived for the words. The problems are expressible in terms of stability questions for rewrite systems. The considerations of Sections 1 and 2 apply to the graph $(R\varepsilon)_1$, however, a deeper analysis is possible because of the extra structure borne by this graph.

A word u of R is <u>stable</u> (or, <u>in normal form</u>) when u is a stable vertex of the graph $(R\varepsilon)_1$. This means that the only elementary derivation with 1-source u is $(us_0, u, ut_0) = u$. A <u>stabilizer</u> for a word u is a derivation $d : u \Rightarrow v$ with v stable.

In order to apply Lemma 1.2 to obtain the boundedness condition BC for $(R\varepsilon)_1$ the following observation can be used.

<u>Lemma 1.</u> <u>For a rewrite system</u> R, <u>the graph</u> R_1 <u>is LF if and only if</u> $(R\varepsilon)_1$ <u>is LF</u>.

<u>Proof.</u> "if" is obvious. For the converse, note that the number of elementary derivations d with $ds_1 = u$ is no greater than the number of elements of R with 1-source a subword of u. Since u has only finitely many subwords, the result follows. $\square$

Actually it is more usual in practice to invoke Proposition 1.3 to obtain BC for $(R\varepsilon)_1$.

Suppose R is a rewrite system and $\pi : R^* \to A$ is a 2-functor into a 2-category A. The <u>overlap condition</u> (OC) for the pair R, π is as follows:

<u>if</u> $e : u \Rightarrow v$, $e' : u' \Rightarrow v'$ <u>are rewrite rules such that</u> $uz = xu'y$ <u>for words</u> x, y, z <u>with</u> $|x| < |u|$ <u>and either</u> $|y| = 0$ <u>or</u> $|z| = 0$ <u>then there exist derivations</u> $d : vz \Rightarrow w$, $d' : xv'y \Rightarrow w$ <u>with</u>

$$((e *_0 z)\, d)\pi = ((x *_0 e' *_0 y)\, d')\pi.$$

Notice that the length inequality $|x| < |u|$ is precisely what is needed in order that the words u, u' should have a non-trivial overlap. The overlap condition implies the condition obtained from OC by deleting the hypothesis "$|y| = 0$ or $|z| = 0$" (shave the right whisker, apply OC and graft back the right whisker).

Notice that a 2-functor $\pi : R^* \to A$ gives a functor $(R\varepsilon)_1^* \to A_1$, also denoted by π, by composing the quotient functor $(R\varepsilon)_1^* \to (R^*)_1$ with $\pi : (R^*)_1 \to A_1$.

<u>Proposition 2.</u> <u>The overlap condition</u> OC <u>is satisfied by</u> $R, \pi : R^* \to A$ <u>if and only if the reuniting condition</u> RC <u>is satisfied by</u> $(R\varepsilon)_1$, $\pi : (R\varepsilon)_1^* \to A_1$.

<u>Proof.</u> Clearly OC is a restricted form of RC. Conversely, suppose OC holds and take elementary derivations (x, e, y), (x', e', y') with the same source $xuy = x'u'y'$ where $e : u \Rightarrow v$, $e' : u' \Rightarrow v'$. We may suppose $|x| \leq |x'|$ so that $x' = xz$ and $uy = zu'y'$. If $|z| < |u|$ then OC applies to give derivations $vy \Rightarrow w'$, $zv'y \Rightarrow w'$; applying $x *_0 -$ to these derivations gives the result with $w = xw'$. If $|z| \geq |u|$ then $z = up$, $y = pu'y'$ so we have a square of derivations

$$\begin{tikzcd}
xuy \arrow[r, Rightarrow, "{(x', e', y')}"] \arrow[d, Rightarrow, "{(x, e, y)}"'] & xupv'y' \arrow[d, Rightarrow, "{(x, e, pv'y')}"] \\
xvpu'y' \arrow[r, Rightarrow, "{(xvp, e', y')}"'] & xvpv'y'
\end{tikzcd}$$

which commutes in R^* and so after application of π. $\square$

The calculation involved in verifying RC for $(R\varepsilon)_1$ is vastly reduced by the above result as we now emphasize.

Proposition 3. Suppose R is a rewrite system and $\pi : R^* \to A$ is a 2-functor. Suppose there is a natural number k such that $|x| \leq k$ for all rewrite rules $e : x \Rightarrow y$. Suppose $(R\varepsilon)_1, \pi$ satisfies RC restricted to those pairs of elementary derivations $u \Rightarrow v$, $u \Rightarrow v'$ with $|u| < 2k$. Then R, π satisfies the overlap condition OC.

Proof. In the notation of OC, if $|z| = 0$ then $|uz| = |u| \leq k < 2k$, while if $|y| = 0$ then $|uz| = |xu'y| = |x| + |u'| < |u| + |u'| = 2k$. $\square$

The above Proposition applies when R has only finitely many rewrite rules. Take k to be the maximum length of their 1-sources. Then we only need to check that all pairs of elementary derivations emanating from words of length less than $2k$ can be reunited.

Call a rewrite system R, together with a 2-functor $\pi : R^* \to A$, perfect when the graph $(R\varepsilon)_1$, together with $\pi : (R\varepsilon)_1^* \to A_1$, is perfect. When we say a rewrite system R is perfect we mean that, together with the unique 2-functor $R^* \to 1$, it is. If (R, π) is perfect for some π then R is perfect.

Combining Propositions 5.2 and 2.6, we obtain:

Proposition 4. Suppose R is a rewrite system and $\pi : R^* \to A$ is a 2-functor. If (R, π) satisfies OC and the graph $(R\varepsilon)_1$ satisfies BC then (R, π) is perfect. $\square$

Suppose R is a perfect rewrite system and A is the category of 1-vertices of R. Recall the definition of the quotient category A/R from the end of Section 4. In this case, the underlying graph of A/R is isomorphic to the graph of stable words in R. The corresponding composition $a * b$ of compatible stable words a, b is defined by taking a stabilizer $ab \Rightarrow a * b$ for the composite ab in A.

A presentation of a category B is a graph G together with a rewrite system R such that

$$G^* = Rs_1 \quad \text{and} \quad B \cong G^*/R.$$

The presentation is called computable when the rewrite system R is perfect. (Notice that, if we regard B as a 2-category with $s_0, t_0, *_0$ the given category structure and $b s_1 = b t_1 = b$, $b *_1 b = b$, we obtain a 2-functor $\pi : R^* \to B$ which is bijective on 0-vertices and has (R, π) perfect.)

Proposition 5. Every category has a computable presentation.

Proof. Let G be the underlying graph of the category $\mathcal{B}$. Let R be the rewrite system with $G^* = Rs_1$ and rewrite rules $r_{a,b} : ab \Rightarrow a * b$ for each non-degenerate compatible pair (a, b) in G where $a * b$ is the composite in $\mathcal{B}$. Length $|\ | : G^* \to \mathbb{N}$ extends to a vertex reflecting graph morphism $(R\varepsilon)_1 \to \mathbb{M}$ (see Example 1.4) since each elementary derivation $d : x \Rightarrow y$ has $|y| < |x|$ unless $d = x = y$. By Proposition 1.3, $(R\varepsilon)_1$ satisfies BC. The overlap condition OC precisely amounts to associativity $(a * b) * c = a * (b * c)$ of composition in $\mathcal{B}$. So R is perfect by Proposition 5.4. The stable words are those of length 1. The stabilizer of $a_1 a_2 \dots a_n$ has target $a_1 * a_2 * \dots * a_n$. (This gives a proof of the general associativity law stated at the beginning of Section 2.) $\square$

Before providing some examples, we shall establish a convenient notation for calculating in a rewrite system. For words w, w', we write

$$d \ \frac{w}{w'}$$

to mean that d is a derivation which, when appropriately whiskered, has source w and target w'. That is, $w = xuy$, $w' = xu'y$ for some words x, y and $d : u \Rightarrow u'$ is a derivation.

Example 6. Let T be a totally ordered set with a distinguished element z. Regard T as a graph with $as = at = z$ for all $a \in T$. Let R be the rewrite system with $Rs_1 = T^*$ and rewrite rules

$$a \wr b : ab \Rightarrow ba$$

for $b < a$ in T. Define a function $\rho : T^* \to \mathbb{N}$ by $z\rho = 0$ and $(a_1 a_2 \dots a_n)\rho$ is the cardinality of the set

$$\{(i,j) \mid 0 < i < j \leq n,\ a_j < a_i\}.$$

Any solid elementary derivation $d : u \Rightarrow v$ has $v\rho < u\rho$. So ρ extends to a graph morphism $\rho : (R\varepsilon)_1 \to \mathbb{M}$ (where $\mathbb{M}$ is as in Example 1.4) which reflects vertices. By Proposition 1.3 $(R\varepsilon)_1$ satisfies BC. In fact, if $d : u \Rightarrow v$ is a derivation and $|u| = n$ then

$$\|d\| \leq u\rho \leq \operatorname{card}\{(i,j) \mid 0 < i < j \leq n\} = \tfrac{1}{2} n(n-1).$$

Verification of the overlap condition is achieved by the following calculations for $c < b < a$.

$$\begin{array}{ll} & abc \\ \hline a\wr b & bac \\ \hline a\wr c & bca \\ \hline b\wr c & cba \end{array} \qquad\qquad \begin{array}{ll} & abc \\ \hline b\wr c & acb \\ \hline a\wr c & cab \\ \hline a\wr b & cba \end{array}$$

By Proposition 5.4, R is a perfect rewrite system. The stable words are z and those $a_1 a_2 \dots a_n$ with $a_1 \leq a_2 \leq \dots \leq a_n$. Each word of length n is equivalent to a unique stable word of the same length n (this follows from perfectness and the invariance of length under the rewrite rules). A given word can have many stabilizers; for example, the two derivations from abc with $c < b < a$ displayed above are stabilizers. The equivalence class of a word $a_1 a_2 \ldots a_n$ with <u>distinct</u> letters is precisely the set of permutations of the n letters, and so, has $n!$ members. Furthermore, (T, R) is a computable presentation of the <u>free commutative monoid</u> on the pointed set T. $\square$

<u>Example 7</u>. Let G be the graph $\{1, a, b\}$ with only vertex 1. Let R be the rewrite system whose category of words is G^* and whose rewrite rules are

$$a \Rightarrow ab, \quad bb \Rightarrow b, \quad ab \Rightarrow b.$$

The only stable word is b and R is perfect. However, if $x \neq b^n$, there are <u>infinitely many</u> derivations $d : x \Rightarrow b$; each such x is unbounded. This shows that R can be perfect without satisfying the hypotheses of Proposition 5.4. In fact, $(R\varepsilon)$, does not satisfy CC since the first rewrite rule can be repeated arbitrarily often. $\square$

<u>Example 8</u>. Let G be the graph $\{1, -, +\}$ with only vertex 1. Let $\square(1)$ be the rewrite system whose category of words is G^* and whose only rewrite rule is

$$0 : - \Rightarrow + .$$

There is functor $\rho : G^* \longrightarrow \mathbb{N}$ which assigns the number of components $-$ in a word. If $d : u \Rightarrow v$ is a solid elementary derivation then $v\rho = u\rho - 1$. So ρ extends to a derivation scheme morphism $\rho : \square(1) \longrightarrow \mathbb{M}$ which reflects 1-vertices. It is easy to see that $\square(1)$ is perfect. The stable words are those with no $-$'s. A word generally has many stabilizers. Any two derivations $d, d' : u \Rightarrow v$ are equivalent. So, in the 2-category $\square(1)^*$, any two elements with the same 1-source and the same 1-target are equal. This means that $\square(1)^*$ amounts to an ordered monoid.

Example 9. Let $\omega = \{0,1,2,3,\dots\}$ be the ordered set of finite ordinals. Each subset p of ω can be written in the form $\{p_0, p_1, p_2, \dots\}$ where $p_0 < p_1 < p_2 < \dots$ in which case we write

$$p = (p_0\, p_1\, p_2 \cdots).$$

Let $\Omega(0)$ denote the set of 1-element subsets of ω; this can be identified with ω, but just as a set. Let $\Omega(1)$ denote the set of subsets of ω of cardinality 1 and 2; we regard this as a graph whose vertices are the 1-element subsets and

$$(ij) : (i) \longrightarrow (j) \quad \text{for } i<j.$$

(The graph $\mathbb{M}$ of Example 1.4 is the opposite graph of $\Omega(1)$.) Let $\Omega(2)$ denote the rewrite system whose category of words is $\Omega(1)^*$ and whose rewrite rules are

$$(ijk) : (ik) \Longrightarrow (ij)(jk) \quad \text{for } i<j<k.$$

Geometrically, we picture the triangle

$$\begin{tikzcd} (i) \arrow[rr, "(ik)"] \arrow[dr, "(ij)"'] & {} \arrow[d, phantom, "\Downarrow(ijk)"] & (k) \\ & (j) \arrow[ur, "(jk)"'] & \end{tikzcd}.$$

There is a graph morphism $\rho : \Omega(1) \longrightarrow \mathbb{N}$ determined by $(ij)\rho = j-i-1$. This uniquely extends to a functor $\rho : \Omega(1)^* \longrightarrow \mathbb{N}$ (Proposition 2.2). Then, for each solid elementary derivation $d : u \Longrightarrow v$ in $\Omega(2)$, we have $v\rho = u\rho - 1$. So ρ extends to a derivation scheme morphism $\rho : \Omega(2) \longrightarrow \mathbb{M}$ which reflects 1-vertices. So $(\Omega(2)\varepsilon)_1$ satisfies BC: in fact, each derivation $d : u \Longrightarrow v$ of $\Omega(2)$ has $\|d\| \leq u\rho$. The 1-sources of the rewrite rules are all of length 1 and so we only need check the rewriting condition emanating from words of length <2 (Proposition 5.3):

$$\begin{array}{rc} (ijl) & \underline{\quad (il) \quad} \\ (jkl) & \underline{\quad (ij)(jl) \quad} \\ & (ij)(jk)(kl) \end{array} \qquad \begin{array}{rc} (ikl) & \underline{\quad (il) \quad} \\ (ijk) & \underline{\quad (ik)(kl) \quad} \\ & (ij)(jk)(kl) \end{array} \qquad \text{for } i<j<k<l.$$

So $\Omega(2)$ satisfies OC. By Proposition 5.4, $\Omega(2)$ is perfect. The stable words are those of the form

$$(i\ i{+}1)(i{+}1\ i{+}2)\cdots(j{-}1\ j) \quad \text{for } i<j$$

and (i). □

Section 6 is dedicated to another example. That example plays a role in the theory of derivation schemes and that theory, in turn, will be developed in Section 7 using rewrite systems.

§6. Braid monoids.

Recall that the braid group A_n on n strings $0, 1, \ldots, n-1$, as considered by Artin [ ], is generated by $n-1$ symbols $\beta_1, \beta_2, \ldots, \beta_{n-1}$ subject to the relations

$$(1) \qquad \beta_i \beta_j = \beta_j \beta_i \quad \text{for} \quad j+1 < i,$$

$$(2) \qquad \beta_i \beta_{i-1} \beta_i = \beta_{i-1} \beta_i \beta_{i-1}.$$

It is well known that the group $\mathfrak{S}_n$ of permutations of n symbols $0, 1, \ldots, n-1$ is obtained by adjoining the relations

$$(3)_1 \qquad \beta_i \beta_i = 1.$$

For the symmetric group $\mathfrak{S}_n$, each β_i is interpreted as the permutation which transposes $i-1$ and i.

Because of their application in the theory of derivations (as well as providing a non-trivial example), the objects of interest to us are not the above groups but closely related monoids with zero. A monoid is a category with precisely one vertex denoted by 1. A monoid with zero is a monoid with a distinguished element 0 such that its composite with any other element is 0.

Let $\mathbb{B}_n$ denote the monoid with zero generated by $\beta_1, \beta_2, \ldots, \beta_{n-1}$ subject to the relations (1), (2) and

$$(3)_0 \qquad \beta_i \beta_i = 0.$$

We call $\mathbb{B}_n$ the braid monoid with zero on n strings.

Since both sides of equations (1) and (2) have the same length, each non-zero element b of $\mathbb{B}_n$ has a well-defined length $|b|$. If we define the length of 0 to be $-\infty$ then, for all $b, b' \in \mathbb{B}_n$, we have the equation

$$|bb'| = |b| + |b'|.$$

It is convenient to define

$$(4) \qquad \beta_{ip} = \beta_i \beta_{i-1} \cdots \beta_{p+1} \beta_p \quad \text{for } p \le i,$$

so that $\beta_{ii} = \beta_i$, and to define

$$(4)' \qquad \beta_{i\,i+1} = 1.$$

The length of β_{ip} is $i-p+1$. As an easy consequence of equations (1) and (2) we have

$$(5) \qquad \beta_{ip} \beta_i = \beta_{i-1} \beta_{ip} \quad \text{for } p < i.$$

In fact, (5) reduces to (2) when $p = i-1$.

Our goal now is to provide a computable presentation for $\mathbb{B}_n$. For this we must replace the equations (1), (2), $(3)_0$ plus the equations for 0 by directed relations, that is, by rewrite rules. If the equations (1), (2) have $=$ replaced by $\Rightarrow$ then it is no longer possible to derive the equivalence of both sides of (5) since the wrong direction of (1) is needed. This problem is overcome by taking a directed version of (5) in place of (2).

Consider the graph $G = \{0, 1, \beta_1, \beta_2, \dots, \beta_{n-1}\}$ whose only vertex is 1 (so that all pairs are compatible). Let $\beta_{ip} \in G^*$ be defined by (4), (4)′. Let R be the rewrite system whose category of 1-vertices is G^* and whose rewrite rules are as follows:

$$\begin{aligned}
t_{ij} &: \beta_i\beta_j \Rightarrow \beta_j\beta_i \quad \text{for } j+1<i,\\
r_{ip} &: \beta_{ip}\beta_i \Rightarrow \beta_{i-1}\beta_{ip} \quad \text{for } p<i,\\
s_i &: \beta_i\beta_i \Rightarrow 0,\\
y_i &: \beta_i 0 \Rightarrow 0,\\
z_i &: 0\beta_i \Rightarrow 0,\\
o &: 00 \Rightarrow 0.
\end{aligned}$$

The <u>rank function</u> $\rho : G^* \to \mathbb{N}$ is defined as follows. Take $w \in G^*$. If $w = 1$, put $w\rho = 0$. Otherwise, let $w = \beta_{i_1}\beta_{i_2}\cdots\beta_{i_m}$, where we allow the subscripts i_p to be 0 and interpret β_0 as 0, and put

$$w\rho = m + \sum_{0<p\le m} i_p + \operatorname{card}\{(i_p, i_q) \mid p<q \text{ and } 0<i_q<i_p\}.$$

If $e : u \Rightarrow v$ is one of the rewrite rules and $d = (x, e, y) : xuy \Rightarrow xvy$ then $(xuy)\rho > (xvy)\rho$. So ρ extends to a vertex reflecting graph morphism $\rho : (R\mathcal{E})_1 \to \mathbb{M}$ where $\mathbb{M}$ is as in Example 1.4. By Proposition 1.3, the graph $(R\mathcal{E})_1$ satisfies the boundedness condition BC (and of course CC). By Proposition 5.4, to see that R is perfect it suffices to prove the overlap condition OC.

<u>Lemma 1</u>. <u>For the above rewrite system</u> R, <u>there are derivations</u>

(6) $\quad \beta_{ip}\beta_{jq} \Longrightarrow \beta_{j-1\,q-1}\beta_{ip}$ <u>for</u> $p<q\le j\le i$,

(7) $\quad \beta_{ip}\beta_{jq} \Longrightarrow 0$ <u>for</u> $q\le p<j\le i$,

<u>whose first components are the rewrite rule</u> r_{jp} <u>with whiskers</u>.

<u>Proof</u>. For calculations it is convenient to replace β_{ip}, β_i by (i,p), i, respectively. Let d_j denote the derivation

$$\begin{array}{l|c}
 & (i,p)\,j \\ \hline
r_{jp} & (i,j+1)(j-1)(j,p) \\ \hline
t & (i,j+2)(j-1)(j+1,p) \\ \hline
t & \vdots \\ \hline
t & (j-1)(i,p)
\end{array}$$

for $p<j\le i$. This is the special case of (6) with $q=j$. For (6) in general we have the derivation

$$\begin{array}{l|c}
 & (i,p)(j,q) \\ \hline
d_j & (j-1)(i,p)(j-1,q) \\ \hline
d_{j-1} & (j-1)(j-2)(i,p)(j-2,q) \\ \hline
d_{j-2} & (j-1,j-3)(i,p)(j-3,q) \\ \hline
d_{j-3} & \vdots \\ \hline
d_{q+1} & (j-1,q-1)(i,p)
\end{array}.$$

For (7) notice that we only need prove the case $\beta_{jp}\beta_{jp} \Rightarrow 0$ since the general case will follow by whiskering on the left with β_{ij}, on the right with β_{pq}, then concatenating with whiskered y's and z's. So the following calculation completes the proof.

$$\begin{array}{rc}
 & (j,p)(j,p) \\ \cline{2-2}
r & (j-1)(j,p)(j-1,p) \\ \cline{2-2}
r & (j-1)j(j-2)(j-1,p)(j-2,p) \\ \cline{2-2}
t & (j-1)(j-2)(j,p)(j-2,p) \\ \cline{2-2}
r,t & \vdots \\ \cline{2-2}
 & (j-1,p+1)(j,p)p \\ \cline{2-2}
s & (j-1,p+1)(j,p+1)0 \\ \cline{2-2}
y & \\ \cline{2-2}
y & \vdots \\ \cline{2-2}
y & 0
\end{array} \quad . \quad \square$$

<u>Proposition 2</u>. <u>The above rewrite system</u> R <u>satisfies the overlap condition OC and so is perfect.</u>

<u>Proof</u>. We must consider all overlaps of the rewrite rules pair by pair. The only overlap for the pair (r,r) has source of the form $\beta_{ip}\beta_{iq}\beta_i$ for $p<i$, $q<i$ and we must show that derivations beginning with r_{ip} and r_{iq} can be reunited. If $i=2$, we can easily close with derivations into 0. If $i>2$ and $p<q$ we have the derivations

$$\begin{array}{rc}
 & (i,p)(i,q)i \\ \cline{2-2}
(6) & (i-1,q-1)(i,p)i \\ \cline{2-2}
r & (i-1,q-1)(i-1)(i,p) \\ \cline{2-2}
r & (i-2)(i-1,q-1)(i,p)
\end{array}
\quad \text{and} \quad
\begin{array}{rc}
 & (i,p)(i,q)i \\ \cline{2-2}
r_{iq} & (i,p)(i-1)(i,q) \\ \cline{2-2}
(6) & (i-2)(i,p)(i,q) \\ \cline{2-2}
(6) & (i-2)(i-1,q-1)(i,p)
\end{array}$$

where the left derivation above begins with r_{ip} (whiske[d] by Lemma 6(i)); the bottom lines are the same so we have the desired reunion. If $i>2$ and $q\le p$ we have the derivations

$$\begin{array}{rc}
 & (i,p)(i,q)i \\ \cline{2-2}
(7) & 0\ i \\ \cline{2-2}
z & 0
\end{array}
\quad \text{and} \quad
\begin{array}{rc}
 & (i,p)(i,q)i \\ \cline{2-2}
r_{iq} & (i,p)(i-1)(i,q) \\ \cline{2-2}
(6) & (i-2)(i,p)(i,q) \\ \cline{2-2}
(7) & (i-2)\ 0 \\ \cline{2-2}
y & 0
\end{array}$$

which again give reunion.

For the pair (t,t) we take $j+1<i$ and $k+1<j$ and note the following derivations:

$$\begin{array}{rc}
 & i\,j\,k \\ \cline{2-2}
t_{ij} & j\,i\,k \\ \cline{2-2}
t & j\,k\,i \\ \cline{2-2}
t & k\,j\,i
\end{array}
\qquad
\begin{array}{rc}
 & i\,j\,k \\ \cline{2-2}
t_{jk} & i\,k\,j \\ \cline{2-2}
t & k\,i\,j \\ \cline{2-2}
t & k\,j\,i
\end{array} \quad .$$

For the pair (t,r), take $j+1<i$ and $q<j$ and note:

$$\begin{array}{rc}
 & i\,(j,q)j \\ \cline{2-2}
t & j\,i\,(j-1,q)j \\ \cline{2-2}
t\dots t & (j,q)\,j\,i \\ \cline{2-2}
r & (j-1)(j,q)i
\end{array}
\qquad
\begin{array}{rc}
 & i(j,q)j \\ \cline{2-2}
r & i\,(j-1)(j,q) \\ \cline{2-2}
t & (j-1)\,i\,(j,q) \\ \cline{2-2}
t\dots t & (j-1)(j,q)i
\end{array} \quad .$$

For the pair (r,t), take $p<i$ and $j+1<i$ and note:

$$
\begin{array}{ll}
r & \overline{\quad \cdots \quad} \\
(7) & \dfrac{(i-1)(i,p)j}{} \\
y & \dfrac{(i-1)0}{0}
\end{array}
\qquad\qquad
\begin{array}{ll}
t & \overline{\quad (i,p)\cdots \quad} \\
(7) & \dfrac{(i,p)j\,i}{} \\
z & \dfrac{0i}{0}
\end{array}
\;.
$$

The remaining overlaps clearly all rewrite into 0. $\square$

Theorem 3. *A computable presentation of $\mathbb{B}_n$ is provided by the above rewrite system R. The stable words are 0 and those of the form*

$$(8) \qquad \beta_{1,p_1}\beta_{2,p_2}\cdots\beta_{n-1,p_{n-1}}$$

where $1 \le p_i \le i+1$. Consequently, $\mathbb{B}_n$ is finite with cardinality $n!+1$.

Proof. By Proposition 6.2, R is perfect. Comparison of equations (1), (2), $(3)_0$ and their consequence (5) with the rewrite rules yields $\mathbb{B}_n \cong G^*/R$. This proves the first sentence of the Theorem. No whiskered rewrite rules apply to 0 or (8) so the stated words are stable. We prove that these are the only stable words by induction on n. Let w be a stable word in the symbols $0, \beta_1, \ldots, \beta_n$. If β_n does not occur in w then w has the form (8) by induction, and we are done since $w = w\beta_{n,n+1}$. Otherwise, consider the left-most occurrence of β_n in w. Then $w = u\beta_n v$ where u is a stable 1-vertex of R and so of the form (8). If $v = 1$ we are done since $w = u\beta_{n,n}$. If $v \neq 1$ then $v = \beta_{n-1,q}v'$ for some $q < n$ (or else whiskered versions of $t_{n,j}$ or s_n would apply to w). So $w = u\beta_{n,q}v'$. If $v' = 1$, we are done. If $v' \neq 1$ then $v' = \beta_j v''$ where $j \neq q-1$ (or else we could have taken q smaller). But, by Lemma 6.1, we always have a non-trivial derivation out of $\beta_{n,q}\beta_j$ contradicting the stability of $w = u\beta_{n,q}\beta_j v''$. So this case does not arise.

There are $i+1$ choices of p_i for $i = 1, 2, \ldots, n-1$ so $\mathbb{B}_n$ has cardinality $1 + 2.3\ldots(n-1)n$. $\square$

Corollary 4. (a) *The longest non-zero element of $\mathbb{B}_n$ is*

$$\Delta = \beta_1(\beta_2\beta_1)(\beta_3\beta_2\beta_1)\ldots(\beta_{n-1}\beta_{n-2}\cdots\beta_2\beta_1)$$

of length $\frac{1}{2}n(n-1)$.

(b) *If $k > \frac{1}{2}n(n-1)$ and $(b_1, b_2, \ldots, b_k) \in \mathbb{B}_n^k$ is non-degenerate then $b_1 b_2 \ldots b_k = 0$.*

(c) *If $bc = 1$ in $\mathbb{B}_n$ then $b = c = 1$.*

Proof. (a) The longest word of the form (8) has $p_i = 1$ for $i = 1, \ldots, n-1$ and this gives Δ.

(b) The length of $b_1 b_2 \ldots b_k$ is $> \frac{1}{2}n(n-1)$ and so it not equivalent to a word of the form (8), so it must be 0.

(c) Take $k > \frac{1}{2}n(n-1)$. If $b \neq 1$ then $b^k = 0$ by (b) so $0 = b^k c^k = b^{k-1}bcc^{k-1} = b^{k-1}c^{k-1} = 1$, a contradiction. So $b = 1$. So $c = 1$. $\square$

It is possible to apply the techniques of this Section to the symmetric group $\mathfrak{S}_n$ in place of $\mathbb{B}_n$.

... must be replaced by

$$s_i' : \beta_i \beta_i \Longrightarrow 1.$$

This gives a computable presentation of $\mathfrak{S}_n$ for which the stable words are those of the form (8) (this time there is no 0). There are of course $n!$ such elements corresponding to the permutations of n symbols. The details are left to the reader.

If $m \leq m'$, it is convenient to regard $\mathbb{B}_m$ as a submonoid of $\mathbb{B}_{m'}$ by identifying the $\beta_1, \ldots, \beta_{m-1}$ of $\mathbb{B}_m$ with the $\beta_1, \ldots, \beta_{m-1}$ of $\mathbb{B}_{m'}$. Because of Theorem 6.3 we can be sure that, if two words in $\beta_1, \ldots, \beta_{m-1}$ are equal in $\mathbb{B}_{m'}$, they are equal in $\mathbb{B}_m$.

There is a monoid homomorphism

$$+ \ : \mathbb{B}_m \times \mathbb{B}_n \longrightarrow \mathbb{B}_{m+n}$$

preserving zero and defined by

$$(b, \beta_i)+ \ = \ b + \beta_i \ = \ b\,\beta_{m+i}\,,$$

called <u>braid addition</u>. Notice that β_m is not in the image of this function, and $b\,\beta_{m+i} = \beta_{m+i}\,b$; but $b + \beta_i \neq \beta_i + b$ even when $m = n$ in general.

<u>The 2-category</u> $\mathbb{B}$ <u>of (positive) braids</u> <u>(with zero)</u> consists of the set

$$\mathbb{B} = \{(m, b) \mid m \in \mathbb{N},\ b \in \mathbb{B}_m\},$$

with 2-graph structure given by the equations

$$(m,b)s_0 = (m,b)t_0 = (0,1)$$
$$(m,b)s_1 = (m,b)t_1 = (m,1),$$

and, with the following compositions

$$(m,b) *_0 (n,c) \ = \ (m+n,\ b+c)$$
$$(m,b) *_1 (m,c) \ = \ (m,\ bc).$$

There is only one 0-vertex $(0,1)$. The 1-vertices have the form $(m,1)$ and so can be identified with the natural numbers. We sometimes denote (m,b) by

$$b : m \Longrightarrow m \quad \text{in} \quad \mathbb{B}.$$

The <u>fundamental element</u> of $\mathbb{B}$ is $\beta_1 : 2 \Longrightarrow 2$ since each $\beta_i : m \Longrightarrow m$ can be obtained from it by whiskering; in fact,

$$(m, \beta_i) = (i-1) *_0 (2, \beta_1) *_0 (m-i-1)$$

$$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\beta_i"] & \cdot \end{tikzcd} \quad = \quad \begin{tikzcd} \cdot \arrow[r, "(i-1)"] & \cdot \arrow[r, bend left, "2"] \arrow[r, bend right, "2"'] \arrow[r, phantom, "\Downarrow\beta_1"] & \cdot \arrow[r, "(m-i-1)"] & \cdot \end{tikzcd} .$$

Every element of $\mathbb{B}$ is a 1-composite of elements (m, β_i).

Equivalence of derivations was described in Section 4. In this Section we shall examine the computability of this equivalence.

An element e of a derivation scheme D is called _penetrating_ when it is solid and $es_1 = es_0$. Call D _virgin_ when it has no penetrating elements.

Penetrating elements cause a problem for equivalence of derivations using rewrites. This stems from Propositions 3.1(2) and 3.2. By putting a restriction on the existence of penetrating elements we have an elegant solution to the problem in terms of the braid monoids with zero. The restriction is not so severe as to exclude any examples we wish to consider. In fact many important examples are virgin, but this is stronger than we need.

A derivation scheme D is called _pure_ when, for all $e, e' \in D$, if $et_1 = e's_1 = es_0$ then e and e′ are central. This means that the presence of an element

$$\begin{tikzcd} p \arrow[r, bend left, "u"] \arrow[r, bend right, "p"'] \arrow[r, phantom, "\Downarrow e"] & p \end{tikzcd}$$

with $u \neq p$ disallows the presence of elements of the form

$$\begin{tikzcd} p \arrow[r, bend left, "p"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow e'"] & p \end{tikzcd},$$

and similarly with the directions of e, e' reversed. Notice that a derivation scheme in which _all_ elements are central is pure while a virgin derivation scheme has no solid central elements.

However, to deal with solid central elements we do need to assume that they have been _totally ordered_. The axiom of choice even ensures the existence of a well ordering. In specific examples there may be natural choices for the order; certainly when D has only finitely many solid central elements existence is no problem.

Let $D^{\uparrow}$ be the rewrite system which has the free category $(D\varepsilon)_1^* = (D\partial)_1$ of derivations in D as its category $D^{\uparrow}s_1$ of 1-vertices, and, has rewrite rules

$$\uparrow(x,e,y,e',z) : (xuy,e',z)(x,e,yv'z) \Rightarrow (x,e,yu'z)(xvy,e',z)$$

for each 0-compatible list (x,e,y,e',z) in D where x, y, z are 1-vertices, where $e : u \Rightarrow v$, $e' : u' \Rightarrow v'$ are solid, and where, if y is a 0-vertex and e, e' are central, then $e < e'$.

There is a 2-functor $\pi : D^{\uparrow *} \to \mathbb{B}$ which is determined (Proposition 4.2) by putting

... for each derivation d of D;

$$\rho\pi = (2, \beta_1) \text{ for each rewrite rule } \rho \text{ of } D^{\uparrow}.$$

From the comments at the end of Section 6 we see that, for any rewrite rule ρ of $D^{\uparrow}$ and elementary derivation (d, ρ, d'), there is an i such that

$$(d, \rho, d')\pi = (m, \beta_i) \text{ where } m = \|d\| + 2 + \|d'\|.$$

<u>Proposition 1</u>. *Suppose D is a pure derivation scheme in which 0-vertices are 0-indecomposable. Then the overlap condition OC is satisfied by $D^{\uparrow}, \pi$.*

<u>Proof</u>. We must take a derivation $d = d_1 d_2 d_3$ of D and rewrite rules $\rho : d_1 d_2 \Rightarrow d_1' d_2'$, $\tau : d_2 d_3 \Rightarrow d_2'' d_3'$ of $D^{\uparrow}$ and show that $\rho *_0 d_3$, $d_1 *_0 \tau$ can be reunited compatibly with π. We have $d_1 = (x_2 u_2 y, e_1, y_1)$, $d_2 = (x_2, e_2, y v_1 y_1) = (x_3 u_3 y', e_2, y_2)$, $d_3 = (x_3, e_3, y' v_2 y_2)$, $\rho = \uparrow(x_2, e_2, y, e_1, y_1)$, $\tau = \uparrow(x_3, e_3, y', e_2, y_2)$ and hence the two composite derivations

$$\begin{array}{r l}
\rho & (x_3 u_3 y' u_2 y, e_1, y_1)(x_3 u_3 y', e_2, y v_1 y_1)(x_3, e_3, y' v_2 y v_1 y_1) \\ \cline{2-2}
 & (x_3 u_3 y', e_2, y u_1 y_1)(x_3 u_3 y' v_2 y, e_1, y_1)(x_3, e_3, y' v_2 y v_1 y_1) \\ \cline{2-2}
\uparrow(x_3, e_3, y' v_2 y, e_1, y_1) & \\
 & (x_3 u_3 y', e_2, y u_1 y_1)(x_3, e_3, y' v_2 y u_1 y_1)(x_3 v_3 y' v_2 y, e_1, y_1) \\ \cline{2-2}
\uparrow(x_3, e_3, y', e_2, y u_1 y_1) & \\
 & (x_3, e_3, y' u_2 y u_1 y_1)(x_3 v_3 y', e_2, y u_1 y_1)(x_3 v_3 y' v_2 y, e_1, y_1)
\end{array}$$

and

$$\begin{array}{r l}
\tau & (x_3 u_3 y' u_2 y, e_1, y_1)(x_3 u_3 y', e_2, y v_1 y_1)(x_3, e_3, y' v_2 y v_1 y_1) \\ \cline{2-2}
 & (x_3 u_3 y' u_2 y, e_1, y_1)(x_3, e_3, y' u_2 y v_1 y_1)(x_3 v_3 y', e_2, y v_1 y_1) \\ \cline{2-2}
\uparrow(x_3, e_3, y' u_2 y, e_1, y_1) & \\
 & (x_3, e_3, y' u_2 y u_1 y_1)(x_3 v_3 y' u_2 y, e_1, y_1)(x_3 v_3 y', e_2, y v_1 y_1) \\ \cline{2-2}
\uparrow(x_3 v_3 y', e_2, y, e_1, y_1) & \\
 & (x_3, e_3, y' u_2 y u_1 y_1)(x_3 v_3 y', e_2, y u_1 y_1)(x_3 v_3 y' v_2 y, e_1, y_1)
\end{array}$$

in $D^{\uparrow}$ which reunite $\rho *_0 d_3$, $d_1 *_0 \tau$. The legitimacy of the rewrite rules used above must be verified. For $\uparrow(x_3, e_3, y' v_2 y, e_1, y_1)$ to be illegitimate would require $y' v_2 y$ to be a 0-vertex, e_3, e_1 to be central and $e_1 \leq e_3$. But 0-vertices are 0-indecomposable, so y, v_2, y' would then be 0-vertices. By purity, $e_2 t_1 = v_2 = y = e_1 s_1$ implies e_2 central. But the legitimacy of ρ, τ gives $e_3 < e_2 < e_1$ contrary to $e_1 \leq e_3$. A similar argument applies to $\uparrow(x_3, e_3, y' u_2 y, e_1, y_1)$, while the legitimacy of the other two rewrite rules is immediate from that of ρ, τ.

Finally notice that π takes the first composite derivation above to $(3, \beta_1\beta_2\beta_1)$ while it takes the second to $(3, \beta_2\beta_1\beta_2)$. Since $\beta_1\beta_2\beta_1 = \beta_2\beta_1\beta_2$ in $\mathbb{B}_3$, we have the result. $\square$

The full force of purity was not used in the above proof. All we needed was: $e, e' \in D$, e central and either $e s_0 = e' s_1$ or $e s_0 = e' t_1$ imply e' central.

We now show what can happen without purity. Suppose p is a 0-vertex and $e: p \Rightarrow v$, $e': u' \Rightarrow p$ are solid but not both central. Then we have the following derivation of length 2 in $D^{\uparrow}$:

$$\begin{array}{l} \uparrow(x,e',p,e,z) \\ \uparrow(x,e,p,e',z) \end{array} \begin{array}{c} (xu',e,z)(x,e',vz) \\ \hline (x,e',z)(x,e,z) \\ \hline (x,e,u'z)(xv,e',z). \end{array}$$

Under π this derivation goes to $(2,\beta_1\beta_1) = (2,0)$. When D is a pure rewrite system, we shall show that π never takes a value of the form $(n,0)$ in $\mathbb{B}$.

Recall that an element a of a category A is called <u>epic</u> (or <u>left cancellable</u>) when $a*b = a*c$ implies $b=c$. It is called <u>monic</u> (or <u>right cancellable</u>) when $b*a = c*a$ implies $b=c$. Call a <u>cancellable</u> when it is epic and monic. Call A a <u>cancellation category</u> when each element is cancellable. Call A <u>retractless</u> when its vertices are indecomposable. If A is free, it is a retractless cancellation category.

<u>Proposition 2</u>. <u>Suppose D is a pure derivation scheme for which Ds_1 is a retractless cancellation category.</u>

(i) <u>If d is a derivation of D of length 2 then any derivation $\delta: d \Rightarrow d'$ of $D^{\uparrow}$ has length</u> < 2.

(ii) <u>If $\delta: d \Rightarrow d'$ is a derivation of $D^{\uparrow}$ with $\delta\pi = (n, bc) \in \mathbb{B}$ then there exist derivations $\delta_1: d \Rightarrow d''$, $\delta_2: d'' \Rightarrow d'$ such that</u> $\delta_1\pi = (n,b)$, $\delta_2\pi = (n,c)$.

(iii) <u>If $\delta: d \Rightarrow d'$ is a derivation of $D^{\uparrow}$ with $\delta\pi = (n,b)$ then</u> $b \neq 0$.

<u>Proof</u>. (i) If δ has length ≥ 2 then its first two components must be rewrite rules of $D^{\uparrow}$. So we would have compatible rewrites $\uparrow(x,e,y,e',z)$, $\uparrow(x_1,e_1,y_1,e_1',z_1)$. This would mean $(x,e,yu'z)(xvy,e',z) = (x_1u_1y_1,e_1',z_1)(x_1, e_1, y_1v_1'z_1)$. So $x = x_1u_1y_1$, $e = e'$, $yu'z = z_1$, $xvy = x_1$, $e' = e_1$, $z = y_1v_1'z_1$. So $x = xvyu_1y_1$, $z = y_1v_1'yu'z$. Since x, z are cancellable, vyu_1y_1 and $y_1v_1'yu'$ are 0-vertices of D. Since 0-vertices are 0-indecomposable, $v = y = u_1 = y_1$ and $y_1 = v_1' = y = u'$ and these are both 0-vertices. So $et_1 = v = u_1 = e_1s_1$, $e_1't_1 = v_1' = u' = e's_1$ are 0-vertices. By purity e, e_1, e_1', e' are central. By the restriction on rewrite rules, $e < e'$ and $e_1 < e_1'$. But $e < e' = e_1 < e_1' = e$ is a contradiction.

(ii) The derivation $\delta: d \Rightarrow d'$ is a path of elementary derivations each of which is taken by π to some (n,β_i). Taking these β_i's in order, we obtain a word α in the β_i's. Let λ, μ be words in $0, 1, \beta_1, \dots, \beta_{n-1}$ which represent $b, c \in \mathbb{B}_n$, respectively.

[illegible], the words α and $\lambda\mu$ represent the same element of $\mathbb{B}_n$. Thus it is possible to pass from α to $\lambda\mu$ by a finite sequence of replacements using equations (1), (2), $(3)_0$ of Section 6. What we must see is that, when one side of these equations comes from a derivation in $D^\uparrow$, the other side does too. Now 0 does not occur in the word α nor in any word coming from a derivation in $D^\uparrow$ (we are not saying yet that the word does not represent 0 in $\mathbb{B}_n$). By (i) above, $\beta_i\beta_i$ cannot occur in a word obtained from a derivation in $D^\uparrow$ via π. So $(3)_0$ is not used in obtaining $\lambda\mu$ from α. Now consider equation (1). If the left-hand side of this equation comes from a derivation in $D^\uparrow$, the derivation must have the form $(d_1d_2d_3, \rho_2, d_5)(d_1, \rho_1, d_3d_4d_5)$ in which case the other side of (1) comes from the derivation $(d_1, \rho_1, d_3d_4d_5)(d_1d_2d_3, \rho_2, d_5)$. And vice versa. This brings us to equation (2) of Section 6. By suitable whiskering we only need consider the equation $\beta_1\beta_2\beta_1 = \beta_2\beta_1\beta_2$. But if the left-hand side comes from a derivation in $D^\uparrow$ that derivation has precisely the form of the one displayed in the proof of Proposition 1 beginning with ρ. Then the right-hand side comes from the derivation beginning with τ displayed in that same proof. And vice versa. So (ii) is proved.

(iii) If $\delta\pi = (n, 0)$ then $\delta\pi = (n, \beta_1\beta_1)$. By (ii) we have δ_1, δ_2 with $\delta_1\pi = (n, \beta_1)$, $\delta_2\pi = (n, \beta_1)$ and δ_1, δ_2 compatible. This contradicts (i). $\square$

<u>Corollary 3</u>. <u>For D as in Proposition 7.2, if</u> $\delta : d \Rightarrow d'$, $\delta' : d \Rightarrow d''$ <u>are derivations in</u> $D^\uparrow$ <u>with</u> $\delta\pi = \delta'\pi$, <u>then</u> $d' = d''$.

<u>Proof</u>. If δ, δ' are elementary derivations the result is clear. In general, put $\delta\pi = \delta'\pi = \beta_{i_1}\beta_{i_2}\cdots\beta_{i_p}$ and apply Proposition 7.2 (ii) repeatedly to obtain paths $\delta_1\delta_2\ldots\delta_p : d \Rightarrow d'$, $\delta_1'\delta_2'\ldots\delta_p' : d \Rightarrow d''$ of elementary derivations δ_j, δ_j' with $\delta_j\pi = (n, \beta_{i_j}) = \delta_j'\pi$. Since δ_1, δ_1' have the same source d, we must have $\delta_1 = \delta_1'$. Continuing we obtain $\delta_j = \delta_j'$. So $d' = d''$. $\square$

Corollary 7.3 shows that each element b of $\mathbb{B}_n$ determines a partial function, also denoted by b, from the set of derivations d in D with $\|d\| = n$ to itself; namely,

$db = d'$ when there exists a derivation $\delta : d \Rightarrow d'$ in $D^\uparrow$ with $\delta\pi = (n, b)$.

<u>Proposition 4</u>. <u>For D as in Proposition 7.2, the pair</u> $(D^\uparrow, \pi)$ <u>is perfect</u>.

<u>Proof</u>. After Propositions 5.4 and 7.1, it suffices to show that $(D^\uparrow\varepsilon)_1$ satisfies BC. We claim that paths $\delta : d \Rightarrow d'$, with $\|d\| = n$, in $(D^\uparrow\varepsilon)_1$ can have

length at most $\frac{1}{2}n(n-1)$. This is an immediate consequence of Corollary 6.4(b) and Proposition 7.2(iii). $\square$

A derivation d of D is called <u>stable</u> when it is a stable 1-vertex of the rewrite system $D^{\uparrow}$. This means that no consecutive pair of components of the word d is the source of a rewrite rule $\uparrow(x,e,y,e',z)$ of $D^{\uparrow}$.

For D as in Proposition 7.2, to say a derivation d of length n is stable is to say that $d\beta_i$ is undefined for $0<i<n$. Also, a stabilizer for a derivation d of length n can be identified with an element b of $\mathbb{B}_n$ such that the derivation db is stable. The work above also provides an algorithm for obtaining this stabilizer b in normal form (Theorem 6.3). Consider $d = d_1 d_2 \dots d_n$ as a path of elementary derivations $d_1, d_2, \dots, d_n$. Find the first i for which $d_i d_{i+1}$ is unstable. Then β_i is the first component of the normal form for b. Replace $d_i d_{i+1}$ by the target of the rewrite rule of $D^{\uparrow}$ which applies to $d_i d_{i+1}$ to obtain a new derivation d' from d. Proceed now with d' in place of d. The process stops when we reach a stable derivation. The sequence of β's is the normal form for b.

When D is a rewrite system R, there is a criterion involving word length to detect stability for derivations. Since stability is checked by looking at consecutive pairs of elementary derivations, it suffices to look at stability for derivations of length 2.

<u>Proposition 5</u>. <u>Suppose</u> $d_1 = (x_1, e_1, y_1) : x_1 u_1 y \Longrightarrow x_1 v_1 y_1$, $d_2 = (x_2, e_2, y_2) : x_2 u_2 y_2 \Longrightarrow x_2 v_2 y_2$ <u>are elementary derivations with</u> $x_1 v_1 y_1 = x_2 u_2 y_2$ <u>in a rewrite system</u> R. <u>The derivation</u> $d = d_1 d_2$ <u>is stable if and only if either</u>

(a) $|x_1| < |x_2 u_2|$, <u>or</u>

(b) $|x_1| = |x_2|$, e_1, e_2 <u>are central and</u> $e_1 \leq e_2$.

<u>Proof</u>. If d is unstable then some $\uparrow(x,e,y,e',z)$ applies. So $|x_1| = |xuy| \geq |xu| = |x_2 u_2|$ while, if $|x_1| = |x_2|$ and e_1, e_2 are central, y must be a 0-vertex and $e_2 = e < e' = e_1$. So (a), (b) are both false. Conversely, suppose (a), (b) both false. Then $|x_1| \geq |x_2 u_2|$, so $x_1 = x_2 u_2 w$ for some w. Using $x_1 v_1 y_1 = x_2 u_2 y_2$, we obtain $y_2 = w v_1 y_1$. So $\uparrow(x_2, e_2, w, e_1, y_1)$ applies to d unless e_1, e_2 are central, w is a 0-vertex and $e_1 \leq e_2$. The last proviso is impossible since (b) is false. So d is unstable. $\square$

Although the above Proposition 7.5 does not require R to be pure, we only know that each derivation is equivalent to a unique stable derivation when R is pure.

We shall describe a technique of blowing up any rewrite system R to obtain a closely related virgin rewrite system $\bar{R}$.

Let G be the graph of letters of the rewrite system R. Let P be the set of 0-vertices p of R of the form $p = e s_1$ for some solid (necessarily penetrating) element e of R. Introduce a symbol $\bar{p}$ for each $p \in P$. Let $\bar{G}$ be the graph obtained from G by adjoining elements $\bar{p} : p \to p$ for all $p \in P$. That is, $\bar{G}$ is the disjoint union of G and P where p in P is represented by $\bar{p}$ in $\bar{G}$; also G is a subgraph of $\bar{G}$ while $\bar{p}s = \bar{p}t = p$. There is a graph morphism $\tau : \bar{G} \to G$ which is the identity on G and takes $\bar{p}$ to p. This extends to a functor $\tau : \bar{G}^* \to G^*$ taking letters to letters. The height of $u \in \bar{G}^*$ is $|u| - |u\tau|$; this is the number of components of u which are of the form $\bar{p}$ with $p \in P$.

The blow up $\bar{R}$ of R is the rewrite system with $\bar{R}s_1 = \bar{G}^*$ and rewrite rules as follows:

for each rewrite rule $e : u \Rightarrow v$ in R and each $w \in \bar{G}^*$ with $w\tau = v$, there is a rewrite rule $\langle e, w\rangle$ in $\bar{R}$ with $\langle e,w\rangle t_1 = w$ and

$$\langle e,w\rangle s_1 = \begin{cases} \bar{u} & \text{when } e \text{ is penetrating} \\ u & \text{otherwise.} \end{cases}$$

Clearly $\bar{R}$ is virgin. If R is virgin then $\bar{R} = R$. In general, τ extends to a rewrite system morphism $\tau : \bar{R} \to R$ taking $\langle e,w\rangle$ to e. This extends to a rewrite system morphism $\tau\partial : \bar{R}\partial \to R\partial$ which preserves equivalence of derivations. Since $\bar{R}$ is virgin (a fortiori, pure), each derivation is equivalent to a unique stable derivation (Proposition 7.4) and we have a criterion for stability (Proposition 7.5(a)).

An elementary derivation d in $\bar{R}$ is of height m when m is the maximum of the heights of ds_1 and dt_1. A derivation d in $\bar{R}$ of length >1 has height m when m is the maximum of the heights of its component elementary derivations.

Proposition 6. *Each derivation d in R is the value of $\tau\partial : \bar{R}\partial \to R\partial$ at a derivation in $\bar{R}$ whose height is at most equal to the number of components (x, e, y) of d with e penetrating.*

Proof. For an elementary derivation (x,e,y) in R, notice that $(x, \langle e, et_1\rangle, y)$ is taken to (x,e,y) under $\tau\partial$ and has height ≤ 1 with equality when e is penetrating. Suppose $d = (x,e,y)d'$ is a derivation in R where $e : u \Rightarrow v$. By induction there is a derivation d_1 in $\bar{R}$ with $d_1(\tau\partial) = d'$ and height m at most equal to the number of components of d' involving penetrating elements. By inserting at most m elements $\bar{p}$, for $p \in P$,

... we obtain a word $x_1 w y_1$ equal to $d_1 s_1$. Then $\tau\partial$ takes $(x_1, \langle e, w\rangle, y_1) d_1$ to d while the height of this derivation in $\overline{R}$ is m unless e is penetrating and $w = v$, in which case its height is at most $m+1$. □

<u>Proposition 7.</u> <u>In any rewrite system R, each solid derivation is equivalent to one of the form</u>

(1) $\qquad (x_1, e_1, y_1)(x_2, e_2, y_2) \cdots (x_n, e_n, y_n)$

<u>where</u> $|x_i| \leqslant |x_{i+1}| + |e_{i+1} s_1|$ <u>with equality only if</u> e_{i+1} <u>is penetrating</u>.

<u>Proof.</u> Take any solid derivation in R. It is the value of $\tau\partial$ at some derivation in $\overline{R}$ by Proposition 7.6. By Propositions 7.4 and 7.5, this derivation in $\overline{R}$ is equivalent to one of the form

$$(a_1, f_1, b_1)(a_2, f_2, b_2) \cdots (a_n, f_n, b_n)$$

with $|a_i| < |a_{i+1}| + |f_{i+1} s_1|$. The value of this under $\tau\partial$ is a derivation of the form (1) in R equivalent to the original derivation in R since $\tau\partial$ preserves equivalence. It remains to prove the inequalities. We have $a_i c_i = a_{i+1}(f_{i+1} s_1)$ with $|c_i| > 0$. Applying τ we obtain $x_i(c_i\tau) = x_{i+1}(e_{i+1} s_1)$. So $|x_i| \leqslant |x_{i+1}| + |e_{i+1} s_1|$ with equality iff $c_i\tau = 0$. But $c_i\tau = 0$ and $|c_i| > 0$ imply that c_i has all components of the form $\overline{p}$ and there is at least one component. The equation $a_i c_i = a_{i+1}(f_{i+1} s_1)$ then gives that $f_{i+1} s_1$ has the form $\overline{p}$. Looking at the rewrite rules for $\overline{R}$ we see that this implies $f_{i+1} = \langle e_{i+1}, w\rangle$ with e_{i+1} penetrating. □

In general the form in Proposition 7.7 is not unique although it does simplify the word problem for derivations quite considerably. If the given derivation only involves at most one penetrating element in its components then the form in Proposition 7.7 is unique (for then the rewrite system, with the same words but only the rewrite rules occurring in the derivation, is pure).

An element of a 2-category A is called *i-indecomposable* when it is indecomposable in the category $A_i = (A, s_i, t_i, *_i)$. The element is called *indecomposable* when it is both 0- and 1-indecomposable.

Each 2-category A has a sub-derivation-scheme $A\gamma$ whose category of 1-vertices is As_1 and whose solid elements are the solid indecomposables of A. The inclusion $A\gamma \longrightarrow A$ extends uniquely to a 2-functor $(A\gamma)^* \longrightarrow A$ by Proposition 4.2.

For any derivation scheme D, the 1-indecomposables in D^* are the elementary derivations of D. Moreover, $D^*\gamma = D$.

A 2-category A is called *free on a derivation scheme* when the 2-functor $(A\gamma)^* \longrightarrow A$ is an isomorphism. From the last paragraph, if $A \cong D^*$ for some derivation scheme D then A is free on a derivation scheme.

A 2-category A is called *free* when it is free on a derivation scheme and As_1 is a free category. That is to say, there is a rewrite system R and an isomorphism of 2-categories $R^* \cong A$.

A 2-category A is called *pure* when the derivation scheme $A\gamma$ is pure. Where necessary we shall suppose that a *total order* has been chosen on the set of central solid indecomposable elements of A.

If A is free on a derivation scheme then each solid 1-indecomposable c in A can be written uniquely in the form

$$c = x *_0 g *_0 y$$

where g is solid indecomposable and x, y are 1-vertices. Here (x, g, y) is the elementary derivation corresponding to c under the isomorphism $(A\gamma)^* \cong A$.

When A is free and $c = x *_0 g *_0 y$, $c' = x' *_0 g' *_0 y'$ are the unique decompositions of c, c' as in the last paragraph, we write

$$c \prec c'$$

when either (a) $|x| < |x'| + |g's_1|$,

or (b) $|x| = |x'|$, g, g' are central, and $g \leq g'$.

Theorem 1. *A pure 2-category A is free if and only if the following four conditions hold:*

(i) *the category* As_1 *is free;*

(ii) *each 1-vertex is 1-indecomposable and every 0-decomposition of a 1-vertex has all components 1-vertices;*

(iii) *each solid 1-indecomposable c can be written uniquely as*

$$c = x *_0 g *_0 y$$

where g is indecomposable and x, y are 1-vertices;

(iv) *each solid a can be written uniquely as*

$$a = c_1 *_1 c_2 *_1 \dots *_1 c_k$$

where each c_i *is solid 1-indecomposable and*

$$c_1 \prec c_2 \prec c_3 \prec \dots \prec c_k .$$

Proof. By Propositions 7.4 and 7.5, the 2-category R^* satisfies the conditions for any pure rewrite system R. Consequently any pure free 2-category satisfies them. Conversely, if A is a pure 2-category satisfying the conditions then $A\gamma$ is a rewrite system and it is routine to check that $(A\gamma)^* \longrightarrow A$ is an isomorphism using the conditions together with Propositions 7.4 and 7.5. □

If A is free on a derivation scheme, we write $\|a\|$ for the length of the derivation in $A\gamma$ which corresponds to $a \in A$. In the situation of the above Theorem 8.1(iv), $\|a\| = k$. If a is a 1-vertex then $\|a\| = 0$.

Example 2. The set N^3 of triples of natural numbers becomes a 2-category with

$$(n,k,n')s_0 = (n,k,n')t_0 = (0,0,0),$$

$$(n,k,n')s_1 = (n,0,n), \qquad (n,k,n')t_1 = (n',0,n'),$$

$$(m,h,m') *_0 (n,k,n') = (m+n,\ h+k,\ m'+n'),$$

$$(m,h,n) *_1 (n,k,p) = (m,\ h+k,\ p).$$

The only 0-vertex is $(0,0,0)$ and the 1-vertices are of the form $(n,0,n)$. Thus $N^3 s_1$ is isomorphic to the free category N of Example 2.3. Each element has 1-decompositions

$$(n,k,n') = (n,k,m) *_1 (m,0,n')$$

where m is not n or n'; so there are no

N^3 is not a free 2-category. Yet any free 2-category A has a length 2-functor $|\ |: A \to N^3$ given by

$$|a| = (|as_1|, \|a\|, |at_1|). \quad \square$$

A derivation scheme D will be called *spotless* when, for all $e \in D$, if es_1 or et_1 is a 0-vertex then e is central. Spotless implies pure.

Lemma 3. *Suppose D is a spotless derivation scheme for which Ds_1 is a retractless cancellation category. Suppose $d_1 d_2$, $d_1' d_2'$ are equivalent derivations of length 2. If either $d_1 = d_1'$ or $d_2 = d_2'$ then $d_1 = d_1'$ and $d_2 = d_2'$.*

Proof. By Proposition 7.4, we have unique stable derivations in each equivalence class. Suppose $d_2 = d_2'$ and $d_1' d_2$ is stable. If $d_1 d_2$ is stable then $d_1 = d_1'$ by uniqueness. If $d_1 d_2$ is unstable then $d_1 = (x_2 u_2 w, e_1, y_1)$, $d_2 = (x_2, e_2, w v_1 y_1)$ where $u_i = e_i s_1$, $v_i = e_i t_1$, $d_i = (x_i, e_i, y_i)$. By Proposition 7.2, $d_1 d_2$ is equivalent to the stable derivation

$$(x_2, e_2, w u_1 y_1)(x_2 v_2 w, e_1, y_1).$$

By uniqueness of the stable representative $d_1' d_2$, we have $d_2 = (x_2, e_2, y_2) = (x_2 v_2 w, e_1, y_1)$. So $x_2 = x_2 v_2 w$, $e_2 = e_1$, $y_2 = y_1$. By cancellation in Ds_1, we have that $v_2 w$ is a 0-vertex. Since Ds_1 is retractless, $v_2 = w = ws_0$. Since D is spotless, $u_2 = ws_0$. So

$$d_1 = (x_2 u_2 w, e_1, y_1) = (x_2, e_2, y_2) = d_2,$$

contrary to $d_1 d_2$ unstable. So $d_2 = d_2'$ implies $d_1 = d_1'$. The other case is similar. $\square$

Recall that, for a 2-category $A = (A, s_0, t_0, *_0, s_1, t_1, *_1)$, we write A_0, A_1 for the categories $(A, s_0, t_0, *_0)$, $(A, s_1, t_1, *_1)$, respectively.

Theorem 4. *Suppose the 2-category A is free on a spotless derivation scheme. If As_1 is a retractless cancellation category then so are A_0 and A_1.*

Proof. We assume $A = D^*$ where D is a spotless derivation scheme. We look at A_1 first. Since $\|d\| = 0$ implies d is a 1-vertex, A_1 is retractless. So suppose $b *_1 a = b' *_1 a$ in A_1. If a is a 1-vertex then $b = b'$. By induction on $\|a\|$, we may assume $\|a\| = 1$. If $\|b\| = k$ then $b *_1 a = b' *_1 a$ implies $\|b'\| = k$. If $k = 0$, both b, b' are 1-vertices, so $b = as_1 = b'$. We may suppose $k > 0$. Let dc, $d'c'$ be the stable derivations representing b, b' where c, c' are elementary derivations. If ca and $c'a$ are stable then dca, $d'c'a$ are equivalent and stable so $d = d'$, $c = c'$ and $b = b'$. If $c'a$ is stable and ca is unstable then ca is equivalent to a stable $\bar{a}\bar{c}$. By equation (5) of Section 6 and Proposition 7.2(ii) we see that $\bar{c}$ is the last component in the stable derivation equivalent to dca. But that stable derivation is $d'c'a$, so by uniqueness, $\bar{c} = a$. So ca is equivalent to $\bar{a}\bar{c} = \bar{a}a$. So, by Lemma 8.3 above, $c = \bar{a}$. So $ca (= \bar{a}\bar{c})$ is stable, a contradiction. If ca and $c'a$ are both unstable then they are respectively equivalent to stable derivations $\bar{a}\bar{c}$ and $\bar{a}'\bar{c}'$. As before $\bar{c}$, $\bar{c}'$ are the last components in the ~~stable~~ derivations representing the equivalent dca, $d'c'a$. By uniqueness, $\bar{c} = \bar{c}'$. Looking at the derivations $ca \Rightarrow \bar{a}\bar{c}$ and $c'a \Rightarrow \bar{a}'\bar{c}$ in $D^{\uparrow}$ we see that $c = c'$. So we have dca and $d'ca$ equivalent. By induction on k, we have $d = d'$. So $b = b'$.

Similarly, $a *_1 b = a *_1 b'$ implies $b = b'$.

Now we look at A_0. Suppose $a *_0 b$ is a 0-vertex. Then $\|a\| + \|b\| = \|a *_0 b\| = 0$. So $\|a\| = \|b\| = 0$. So a, b are 1-vertices. Since As_1 is retractless, a, b are 0-vertices. So A_0 is retractless.

Suppose $a *_0 b = a *_0 b'$ in A_0. Then $as_1 *_0 bs_1 = as_1 *_0 b's_1$. Using cancellation in As_1, we obtain $bs_1 = b's_1$. By Proposition 3.1 (i),

$$(a *_0 bs_1) *_1 (at_1 *_0 b) = (a *_0 b's_1) *_1 (at_1 *_0 b').$$

Using cancellation in A_1, we obtain $at_1 *_0 b = at_1 *_0 b'$. So we may assume a is a 1-vertex. Then $\|b\| = \|a *_0 b\| = \|a *_0 b'\| = \|b'\| = k$, say. If $k = 0$ then $b = bs_1 = b's_1 = b'$. Suppose $k > 0$ and let $d_1 \dots d_k$, $d'_1 \dots d'_k$ be stable derivations representing b, b' with d_i, d'_i elementary. Then $(a *_0 d_1) \dots (a *_0 d_k)$, $(a *_0 d'_1) \dots (a *_0 d'_k)$ are stable derivations representing $a *_0 b = a *_0 b'$. By uniqueness $a *_0 d_i = a *_0 d'_i$. Looking at the definition of this $*_0$ (see Section 4) and since As_1 is a cancellation category, we obtain $d_i = d'_i$. So $b = b'$. $\square$

Example 5. As in Example 5.6, let T be a totally ordered set with distinguished element z. This time we regard T as a 2-graph with $as_0 = at_0 = as_1 = at_1 = z$ for all $a \in T$. In fact, T is then a rewrite system since $Ts_1 = \{z\}$ supports a unique category structure which is moreover free. The elements of T distinct from z become the rewrite rules

$$a : z \Longrightarrow z.$$

Since z is the only 1-vertex in T and this is also a 0-vertex, the only elementary derivations are of the form (z, a, z) for $a \in T$ and this is identified with a itself. So $T\mathcal{E} = T$. Also, all elements of T are central. Since T is totally ordered, we do have a total order on the central elements. So we can form the rewrite system $T^{\uparrow}$ as in Section 7. This $T^{\uparrow}$ is precisely the rewrite system R of Example 5.6. Hence we have the 2-functor $\pi : T^{\uparrow *} \longrightarrow \mathbb{B}$ between 2-categories which both have only one 0-vertex. The other 1-vertices of $T^{\uparrow *}$ are words in the letters of T excluding z. The elements of $T^{\uparrow *}$ can be identified with stable derivations $d : u \longrightarrow v$ and these are determined by the word u and an element b of $\mathbb{B}_n$ where $n = |u|$ and $d\pi = (n, b)$. □

Example 6. Let $\square(1)$ be the rewrite system of Example 5.8 and put $\square(2) = \square(1)^{\uparrow}$. The 0-vertices of $\square(2)$, apart from 1, are strings of $-$'s and $+$'s (such as $-++-+---$). The other 1-vertices have the form

$$(x_1 0 y_1)(x_2 0 y_2) \cdots (x_k 0 y_k)$$

where each x_i, y_i is a 0-vertex. The rewrite rules of $\square(2)$ have the form

$$(x0y0z) : (x-y0z)(x0y+z) \Longrightarrow (x0y-z)(x+y0z),$$

which should be viewed geometrically as a square

$$\begin{tikzcd}
x-y-z \arrow[r, "x+y0z"] \arrow[d, "x-y0z"'] \arrow[dr, phantom, "\overset{x0y0z}{\Longrightarrow}"] & x+y-z \arrow[d, "x+y0z"] \\
x-y+z \arrow[r, "x0y+z"'] & x+y+z,
\end{tikzcd}$$

where x, y, z are 0-vertices of $\square(2)$. The 2-functor $\pi : \square(2)^* \longrightarrow \mathbb{B}$ explains the appearance of braids in the work of Gray [ ]. The notation using $-, 0, +$ compares with Aitchison [ ]. The pair $(\square(2), \pi)$ is perfect by Proposition 7.4. The 2-category $\square(2)^*$ plays a role in the cubical version of non-abelian cohomology. A global description of $\square(2)^*$ appears in the work of Aitchison-Street [ ]; the connection can be established using Theorem 8.1. □