COMBINATORIAL CATEGORICAL EQUIVALENCES

STEPHEN LACK AND ROSS STREET

ABSTRACT. In this paper we prove a class of equivalences of additive functor categories that are relevant to enumerative combinatorics, representation theory, and homotopy theory. Let $\mathscr{X}$ denote an additive category with finite direct sums and splitting idempotents. The class includes (a) the Dold-Puppe-Kan theorem that simplicial objects in $\mathscr{X}$ are equivalent to chain complexes in $\mathscr{X}$; (b) the observation of Church-Ellenberg-Farb that $\mathscr{X}$-valued species are equivalent to $\mathscr{X}$-valued functors from the category of finite sets and injective partial functions; (c) a Dold-Kan-type result of Pirashvili concerning Segal's category Γ; and so on. We provide a construction which produces further examples.

Key words: factorization system; partial map; species; simplicial abelian group; chain complex; symmetric groups; pointed set; additive category; semiabelian category.

AMS classification codes 18A25; 18A32; 18B10; 18G30; 18G35; 20C30

CONTENTS

1. INTRODUCTION

The intention of this paper is to prove a class of equivalences of categories that seem of interest in enumerative combinatorics as per [17], representation theory as per [9], and homotopy theory as per [2]. More specifically, for a class of categories $\mathscr{P}$, we construct a

category $\mathscr{D}$ with zero morphisms (that is, $\mathscr{D}$ has homs enriched in the category 1/Set of pointed sets) and an equivalence of categories of the form

$$[\mathscr{P}, \mathscr{X}] \simeq [\mathscr{D}, \mathscr{X}]_{\mathrm{pt}} . \tag{1.1}$$

On the left-hand side we have the usual category of functors from $\mathscr{P}$ into any additive category $\mathscr{X}$ which has finite direct sums and splitting for idempotents. On the right-hand side we have the category of functors which preserve the zero morphisms.

One example has $\mathscr{P} = \Delta_{\perp,\top}$, the category whose objects are finite ordinals with first and last element, and whose morphisms are functions preserving order and first and last elements. Then $\mathscr{D}$ is the category with non-zero and non-identity morphisms

$$0 \xleftarrow{\partial} 1 \xleftarrow{\partial} 2 \xleftarrow{\partial} \dots$$

such that $\partial \circ \partial = 0$. Since there is an isomorphism of categories

$$\Delta_{\perp,\top} \cong \Delta_{+}^{\mathrm{op}} , \tag{1.2}$$

where the right-hand side is the algebraist's simplicial category (finite ordinals and all order-preserving functions), our result (1.1) reproduces the Dold-Puppe-Kan Theorem [11, 12, 20].

Cubical sets also provide an example of our setting; see Example 3.3. We conclude that cubical simplicial abelian groups are equivalent to semi-simplicial abelian groups.

For a category $\mathscr{A}$ equipped with a suitable factorization system $(\mathscr{E}, \mathscr{M})$ [14], write Par$\mathscr{A}$ (strictly the notation should also show the dependence on $\mathscr{M}$) for the category with the same objects as $\mathscr{A}$ and with $\mathscr{M}$-partial maps as morphisms. We identify $\mathscr{E}$ with the subcategory of $\mathscr{A}$ having the same objects but only the morphisms in $\mathscr{E}$. Assume each object of $\mathscr{A}$ has only finitely many $\mathscr{M}$-subobjects. Let $\mathscr{X}$ be any additive category with finite direct sums and splitting idempotents. Our main result Theorem 9.1 includes as a special case an equivalence of categories

$$[\mathrm{Par}\mathscr{A}, \mathscr{X}] \simeq [\mathscr{E}, \mathscr{X}] . \tag{1.3}$$

(Here $\mathscr{D}$ is obtained from $\mathscr{E}$ by freely adjoining zero morphisms.) We give an alternative proof of this particular case in an appendix (Section 13) using the theory of comonads.

Let $\mathfrak{S}$ be the groupoid of finite sets and bijective functions. Let FI$\sharp$ denote the category of finite sets and injective partial functions. Let Mod^{R} denote the category of left modules over the ring R. Our original motivation was to understand and generalize the classification theorem for FI$\sharp$-modules appearing as Theorem 2.24 of [9], which provides an equivalence

$$[\mathrm{FI}\sharp, \mathrm{Mod}^{R}] \simeq [\mathfrak{S}, \mathrm{Mod}^{R}]$$

between the category of functors FI$\sharp \to \mathrm{Mod}^{R}$ and the category of functors $\mathfrak{S} \to \mathrm{Mod}^{R}$. This is the special case of (1.3) above in which $\mathscr{A}$ is the category FI of finite sets and injective functions, and $\mathscr{M}$ consists of all the morphisms. This result has provided a new viewpoint on representations of the symmetric groups, and a new viewpoint on Joyal species [17, 18].

In order to consider stability properties of representations of the symmetric groups, the authors of [9] also consider FI-modules: that is, R-module-valued functors from the category FI. Each FI$\sharp$-module clearly has an underlying FI-module, so their Theorem 2.24 shows how symmetric group representations become FI-modules. One application they give is a structural version of the Murnaghan Theorem [26, 27], a problem which has its combinatorial aspects [31]. We are reminded of the way in which Mackey functors [23, 28] give extra freedom to representation theory.

Another instance of an equivalence of the form (1.3) is when $\mathscr{A}$ is the category of finite sets with its usual (surjective, injective)-factorization system. Then Par$\mathscr{A}$ is equivalent to the category of pointed finite sets, which is equivalent to Graeme Segal's category Γ [30]. After completing this work, we were alerted to Teimuraz Pirashvili's interesting paper [29] which

gives this finite sets example, makes the connection with Dold-Puppe-Kan, and discusses stable homotopy of Γ-spaces.

Consider the basic equivalence (1.1). It is really about Cauchy (or Morita) equivalence of the free additive category on the ordinary category $\mathscr{P}$ and the free additive category on the category with zero morphisms $\mathscr{D}$. By the general theory of Cauchy completeness (see [32] for example), to have (1.1) for all Cauchy complete additive categories $\mathscr{X}$, it suffices to have it when $\mathscr{X}$ is the category of abelian groups. Cauchy completeness amounts to existence of absolute limits (see [33]) and, for additive categories, amounts to the existence of finite direct sums and splittings for idempotents.

Our approach to finding conditions under which (1.1) holds is to consider structure on the category $\mathscr{P}$ satisfying five Assumptions. All this is described in Section 2. As part of the structure we consider that the category $\mathscr{P}$ underlies a locally partially ordered 2-category $\mathbb{P}$. We make use of adjunctions in $\mathbb{P}$ with identity counits. Our construction of $\mathscr{D}$ can be seen as a process of removing, in a systematic way, morphisms in $\mathscr{P}$ which have a one-sided inverse but not a 2-sided inverse. Since one-sided inverses are not unique, we need to choose a particular one-sided inverse with which to work. The 2-category provides a mechanism for making those choices.

In Section 11 we prove that a Grothendieck fibration construction produces new examples of our main result Theorem 9.1. The significance of these constructed examples, let alone the examples obtained by iterating the construction, is not apparent to us.

In Section 12 we prove a monadicity result (rather than an equivalence) when $\mathscr{X}$ is semiabelian [16]. This is due to Bourn [6] in the case $\mathscr{P} = \Delta_{\perp \neq \top}$.

2. The setting

Let $\mathscr{P}$ be a category underlying a 2-category $\mathbb{P}$ whose hom categories are partially ordered sets. That is, for any two objects A and B of $\mathscr{P}$, there is a partially ordered set $\mathbb{P}(A, B)$ whose elements are the morphisms $A \to B$ in $\mathscr{P}$, and this order is preserved by composition on both sides in $\mathscr{P}$.

Suppose $\mathscr{M}$ is a subcategory of $\mathscr{P}$ containing all the objects and all the isomorphisms of $\mathscr{P}$. Assume each $m \in \mathscr{M}$ has a left adjoint $m^* \dashv m$ in $\mathbb{P}$ with identity counit $m^* \circ m = 1$. In particular, the morphisms in $\mathscr{M}$ are split monomorphisms (coretractions) in $\mathscr{P}$.

We write $\mathrm{Sub}A$ for the (partially) ordered set of isomorphism classes of morphisms $m : U \to A$ in $\mathscr{M}$. We will use the term *subobject* rather than "$\mathscr{M}$-subobject" for these elements. The order on $\mathrm{Sub}A$ is the usual one. Abusing notation for this order, we simply write $U \preceq V$ when there exists an $f : U \to V$ such that $m = n \circ f$, where $m : U \to A$ and $n : V \to A$ in $\mathscr{M}$. There can be confusion when $U = V$ as objects, so we assure the reader that we will take care. A subobject of A is *proper* when it is represented by a non-invertible $m : U \to A$ in $\mathscr{M}$; we write $U \prec A$ or $U \prec_m A$.

Define $\mathscr{R}$ to be the class of morphisms $r \in \mathscr{P}$ with the property that, if $r = m \circ x \circ n^*$ with $m, n \in \mathscr{M}$, then m, n are invertible.

Define $\mathscr{D}$ to be the category with zero morphisms (that is, 1/Set-enriched category) obtained from $\mathscr{R}$ by adjoining zero morphisms. Composition in $\mathscr{D}$ of morphisms in $\mathscr{R}$ is as in $\mathscr{P}$ if the result is itself in $\mathscr{R}$, but zero otherwise.

Define $\mathscr{S}$ to be the class of morphisms in $\mathscr{P}$ of the form $r \circ m^*$ with $m \in \mathscr{M}$ and $r \in \mathscr{R}$.

Recall that the limit of a diagram consisting of a family of morphisms into a fixed object A is called a *wide pullback*; the morphisms in the limit cone are called *projections*. The dual is *wide pushout*.

Assumption 2.1. Wide pullbacks of families of morphisms in $\mathscr{M}$ exist, have projections in $\mathscr{M}$, and become wide pushouts under $m \mapsto m^*$.

Assumption 2.2. If $r, r' \in \mathscr{R}$ are composable then $r' \circ r \in \mathscr{S}$.

Assumption 2.3. If $r, m^* \circ r \in \mathscr{R}$ and $m \in \mathscr{M}$ then m is invertible.

Assumption 2.4. The class $\mathscr{M} \circ \mathscr{M}^*$ of morphisms of the form $m \circ n^*$ with $m, n \in \mathscr{M}$ is closed under composition.

Assumption 2.5. For all objects $A \in \mathscr{P}$, the ordered set $\mathrm{Sub}A$ is finite.

Assumption 2.6. The maximal proper elements of $\mathrm{Sub}A$ can be listed $m_1, \ldots, m_n$ such that the idempotents $c_i = m_i \circ m_i^*$ on A satisfy $c_j \circ c_i \circ c_j = c_j \circ c_i$ for all $i < j$.

Remark 2.7. Since $m_i^* \dashv m_i$ in $\mathbb{P}$, the idempotent c_i is in fact an idempotent monad. For idempotent monads c_i and c_j the condition $c_j \circ c_i \circ c_j = c_j \circ c_i$ is equivalent to $c_i \circ c_j \leq c_j \circ c_i$. It then follows also that $c_i \circ c_j \circ c_i = c_j \circ c_i$, and so we are dealing with the relation for Kiselman's semigroup as studied in [21], and for Lawvere's graphic monoids [22].

Proposition 2.8. *Any morphism $f \in \mathscr{P}$ factors as $f = n \circ r \circ m^*$, uniquely up to isomorphism for $m, n \in \mathscr{M}$ and $r \in \mathscr{R}$.*

Proof. Take $f : A \to B$ in $\mathscr{P}$. We use Assumption 2.1 twice. Let $n : Y \to B$ be the wide pullback of all those morphisms $V \to B$ in $\mathscr{M}$ through which f factors. Then $n \in \mathscr{M}$ and there exists a unique f_1 with $f = n \circ f_1$. Let $m : X \to A$ be the wide pullback of those $U \to A$ in $\mathscr{M}$ whose left adjoint in $\mathbb{P}$ the morphism f_1 factors through. Then $m \in \mathscr{M}$ and $f_1 = r \circ m^*$ for a unique r. Clearly $r \in \mathscr{R}$ and we have uniqueness by a familiar argument. $\qquad\square$

Proposition 2.9. *If $t \circ s = m \circ r$ with $s, t \in \mathscr{S}$, $r \in \mathscr{R}$ and $m \in \mathscr{M}$ then both s and t are in $\mathscr{R}$.*

Proof. First we prove the weaker form:

   if $s \circ r = m \circ r'$ with $r, r' \in \mathscr{R}$, $s \in \mathscr{S}$ and $m \in \mathscr{M}$ then $s \in \mathscr{R}$.

In obvious notation, put $s = r_1 \circ m_1^*$, $m_1^* \circ r = m_2 \circ r_2 \circ m_3^*$ and $r_1 \circ m_2 = m_4 \circ r_3 \circ m_5^*$. Then $m \circ r' = s \circ r = r_1 \circ m_1^* \circ r = r_1 \circ m_2 \circ r_2 \circ m_3^* = m_4 \circ r_3 \circ m_5^* \circ r_2 \circ m_3^*$. Let $p, q \in \mathscr{M}$ be the projections in the pullback of m, m_4. Then there exists a unique u into the pullback with $r' = p \circ u$ and $r_3 \circ m_5^* \circ r_2 \circ m_3^* = q \circ u$. Since r' can factor through no proper $\mathscr{M}$, we deduce that p is invertible. Put $n = q \circ p^{-1}$. Then $m = m_4 \circ n$ and $n \circ r' = q \circ u = r_3 \circ m_5^* \circ r_2 \circ m_3^*$. Therefore $r' = n^* \circ r_3 \circ m_5^* \circ r_2 \circ m_3^*$; by definition of $\mathscr{R}$, we obtain that m_3 is invertible. Then $m_1^* \circ (r \circ m_3) = m_2 \circ r_2$ implies $(m_1 \circ m_2)^* \circ (r \circ m_3) = r_2$. Assumption 2.3 applies to yield that $m_1 \circ m_2$ is invertible. So m_1 has a right inverse, as well as its left inverse m_1^*, and so is invertible. So $s = r_1 \circ m_1^*$ is in $\mathscr{R}$ as asserted.

   Now we come to the proof of the Proposition. Since $s \in \mathscr{S}$, $s = r_1 \circ m_1^*$. Then, by definition of $\mathscr{R}$, $m^* \circ t \circ r_1 \circ m_1^* = m^* \circ t \circ s = m^* \circ m \circ r = r$ implies m_1 invertible. So $s \in \mathscr{R}$. Now the weaker form above applies to yield $t \in \mathscr{R}$. $\qquad\square$

3. BASIC EXAMPLES

Example 3.1. Take a category $\mathscr{A}$ with a factorization system $(\mathscr{E}, \mathscr{M})$. Assume that the pullback of any morphism with one in $\mathscr{M}$ exists. Assume every morphism in $\mathscr{M}$ is a monomorphism and every object of $\mathscr{A}$ has only finitely many $\mathscr{M}$-subobjects. A span $f = (X \xleftarrow{f_0} U \xrightarrow{f_1} Y)$ is called a *partial map* $f : X \longrightarrow Y$ when f_0 is in $\mathscr{M}$. If $g = (X \xleftarrow{g_0} V \xrightarrow{g_1} Y)$ is another partial map, we write $f \leq g$ when there exists $h : U \to V$ with $g_0 \circ h = f_0$ and $g_1 \circ h = f_1$. Let $\mathscr{P} = \mathrm{Par}\mathscr{A}$ denote the category whose objects are all those of $\mathscr{A}$ and whose morphisms are isomorphism classes $[f]$ of partial maps. Composition is that of spans: that is, by pullback. We take the **reverse** of the usual order $f \leq g$ on partial maps to give us the required 2-category $\mathbb{P}$; the usual order would give us right adjoints

where we have chosen to work with left adjoints. We identify $f : X \to Y$ in $\mathscr{A}$ with the morphism $[1_X, X, f] : X \longrightarrow Y$ in $\mathscr{P}$. In this way, we have the $\mathscr{M}$ we require for $\mathscr{P}$ as the one in $\mathscr{A}$. For $m : U \to X$ in $\mathscr{M}$, a left adjoint in $\mathbb{P}$ is defined by $m^* = [m, U, 1_U] : X \to U$, and clearly $m^* \circ m = 1$. Every partial map $f = (X \xleftarrow{f_0} U \xrightarrow{f_1} Y)$ has

$$[f] = f_1 \circ f_0^*$$

where $f_0 \in \mathscr{M}$. Furthermore, $f_1 = m \circ e$ uniquely up to isomorphism for $m \in \mathscr{M}$ and $e \in \mathscr{E}$. It follows therefore that $\mathscr{R} = \mathscr{E}$ and that $[f] \in \mathscr{S}$ if and only if $f_1 \in \mathscr{E}$.

Now we look at our Assumptions. To see that Assumption 2.1 holds, first note that since we are assuming the $\mathscr{M}$-subobjects form a finite set, finite wide pullbacks can be obtained from pullbacks. By assumption, pullbacks of $\mathscr{M}$s exist in $\mathscr{A}$ and a pullback of an $\mathscr{M}$ is an $\mathscr{M}$ in a factorization system. It is a pleasant exercise to see that these pullbacks remain pullbacks in $\mathscr{P}$ and become pushouts in $\mathscr{P}$ on taking left adjoints.

We certainly have Assumption 2.2; indeed $\mathscr{R}$ is closed under composition because $\mathscr{E}$ is.

For Assumption 2.3, take $r = [1_A, A, e]$ with $e \in \mathscr{E}$ and $m = [1_V, V, m]$ in $\mathscr{M}$. To have $m^* \circ r = [n, P, u] \in \mathscr{R}$, we must have n invertible and $u \in \mathscr{E}$. Then $e = m \circ u \circ n^{-1}$ implies $m \in \mathscr{E}$; so m is invertible.

The class of partial maps in Assumption 2.4 are those of the form $[m, U, n]$ with $m, n \in \mathscr{M}$; these are closed under composition since pullbacks of $\mathscr{M}$s exist and are in $\mathscr{M}$.

Assumption 2.5 was one of our assumptions on the factorization system on $\mathscr{A}$.

To prove Assumption 2.6 we use the fact that, for every pullback

$$\begin{array}{ccc} P & \xrightarrow{\;g\;} & B \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle m} \\ A & \xrightarrow{\;f\;} & C \end{array} \qquad (3.4)$$

in $\mathscr{A}$ with $m \in \mathscr{M}$, the square

$$\begin{array}{ccc} P & \xrightarrow{\;g\;} & B \\ {\scriptstyle n^*}\uparrow & & \uparrow{\scriptstyle m^*} \\ A & \xrightarrow{\;f\;} & C \end{array} \qquad (3.5)$$

commutes in $\mathrm{Par}\mathscr{A}$. So, for any $m_1, m_2 \in \mathscr{M}$, we see that the idempotents $c_1 = m_1 \circ m_1^*$ and $c_2 = m_2 \circ m_2^*$ commute. To see this, let n_1, n_2 be the projections in the pullback of m_1, m_2. Then

$$c_1 \circ c_2 = m_1 \circ m_1^* \circ m_2 \circ m_2^* = m_1 \circ n_1 \circ n_2^* \circ m_2^*$$
$$= m_2 \circ n_2 \circ n_1^* \circ m_1^* = m_2 \circ m_2^* \circ m_1 \circ m_1^* = c_2 \circ c_1 \ .$$

So $c_2 \circ c_1 \circ c_2 = c_2 \circ c_2 \circ c_1 = c_2 \circ c_1$. Therefore the maximal proper subobjects can be listed in any order and the Assumption is satisfied.

Notice that $\mathscr{S}$ is not closed under composition unless each pullback of an $\mathscr{E}$ along an $\mathscr{M}$ is an $\mathscr{E}$. This is true in many examples.

Example 3.2. Take $\mathscr{P}$ to be the category $\Delta_{\perp,\top}$ of finite non-empty ordinals $n = \{0, 1, \dots, n-1\}$ with morphisms those functions which preserve first element, last element and order. Functors out of this category are augmented simplicial objects because of the isomorphism (1.2). Morphisms $\xi, \zeta : m \to n$ are ordered by taking $\xi \leq \zeta$ iff and only if $\xi(i) \leq \zeta(i)$ for all $i \in m$. This gives our 2-category $\mathbb{P}$; it is a locally full sub-2-category of Cat. Take $\mathscr{M}$ to consist of all the injective functions in $\Delta_{\perp,\top}$; each such injection ∂ has a left adjoint ∂^* (and also a right adjoint for that matter) in $\mathbb{P}$; clearly ∂^* is surjective with $\partial^* \circ \partial = 1$ since ∂ is a

fully faithful functor. A surjection σ in $\mathscr{P}$ is of the form ∂^* if and only if $\sigma(i) = 0$ implies $i = 0$. We write $\sigma_k : m + 1 \to m$ for the order-preserving surjection which takes the value k twice. We write $\partial_k : m \to m + 1$ for the order-preserving injection which does not have k in its image. Note that $\partial_0 \notin \mathscr{P}$, while $\sigma_k \dashv \partial_k \dashv \sigma_{k-1}$ in $\mathbb{P}$ and σ_k is a ∂^* if and only if $k > 0$. Every $\xi \in \mathscr{P}$ factors uniquely as

$$\xi = \partial_{i_s} \circ \cdots \circ \partial_{i_1} \circ \sigma_{j_1} \circ \cdots \circ \sigma_{j_t}$$

for $0 < i_1 < \cdots < i_s < n - 1$ and $0 \le j_1 < \cdots < j_t < m - 1$.

We claim $\mathscr{R}$ consists of the identities and the surjections $\sigma_0 : m + 1 \to m$. The only invertible morphisms in $\mathscr{P}$ are identities. Since members of $\mathscr{R}$ factor through no proper injection, they must be surjective. Every surjection τ is either of the form ∂^* or uniquely of the form $\sigma_0 \circ \partial^*$. Neither of these forms is permissible for $\tau \in \mathscr{R}$ unless the injection ∂ is an identity. This proves our claim. It is also clear then that $\mathscr{S}$ consists of all the surjections in $\mathscr{P}$.

To prove the Assumptions, we make use of the simplicial identities (see page 24 of [13] for example) which, apart from $\sigma_i \circ \partial_i = 1 = \sigma_{i-1} \circ \partial_i$, say that, for all $i < j$,

$$\partial_j \circ \partial_i = \partial_i \circ \partial_{j-1} \;,\; \sigma_{j-1} \circ \sigma_i = \sigma_i \circ \sigma_j \;,\; \sigma_j \circ \partial_i = \partial_i \circ \sigma_{j-1} \;,\; \sigma_i \circ \partial_{j+1} = \partial_j \circ \sigma_i \;.$$

Assumption 2.1 follows from the fact that the pullback of any two monomorphisms in $\mathscr{P}$ exists and is absolute (that is, preserved by all functors); see page 27 of [13] on the Eilenberg-Zilber Theorem. Alternatively, notice that the squares

$$
\begin{array}{ccc}
n-1 & \xrightarrow{\partial_{j-1}} & n \\
{\scriptstyle\partial_i}\downarrow & & \downarrow{\scriptstyle\partial_i} \\
n & \xrightarrow[\partial_j]{} & n+1
\end{array}
\qquad\qquad
\begin{array}{ccc}
n-1 & \xleftarrow{\sigma_{j-1}} & n \\
{\scriptstyle\sigma_i}\uparrow & & \uparrow{\scriptstyle\sigma_i} \\
n & \xleftarrow[\sigma_j]{} & n+1
\end{array}
$$

are respectively a pullback and pushout in $\mathscr{P}$ for $0 < i < j < n$. Then the general result follows by stacking these squares vertically and horizontally.

In the present example, $\mathscr{S}$ is closed under composition, making Assumption 2.2 clear.

For Assumption 2.3, suppose we have $\partial \circ \rho \in \mathscr{R}$ and $\rho \in \mathscr{R}$. Since ρ is surjective, $\partial^* \circ \rho = 1$ implies $\rho = 1$ and hence $\partial = 1$, as required. Otherwise $\partial^* \circ \rho = \sigma_0$. Yet, if $\rho = 1$ this contradicts the lack of right adjoint for σ_0. So $\rho = \sigma_0$. Thus $\partial^* \circ \sigma_0 = \sigma_0$, and we can cancel σ_0 to obtain again $\partial = 1$.

For Assumption 2.4, the class $\mathscr{M} \circ \mathscr{M}^*$ of morphisms consists of those which reflect 0. That is, $\xi = \mu \circ \partial^*$ with $\mu \in \mathscr{M}$ if and only if $\xi(i) = 0$ implies $i = 0$. This class is clearly closed under composition.

Assumption 2.5 is clear.

For Assumption 2.6, notice that the maximal proper subobjects of n are the $\partial_i : n-1 \to n$ which we take in the natural order of the i. We have the idempotents $c_i = \partial_i \circ \sigma_i$ and, using the simplicial identities for $0 < i < j < n - 1$, we have the calculation:

$$
\begin{aligned}
c_j \circ c_i \circ c_j &= \partial_j \circ \sigma_j \circ \partial_i \circ \sigma_i \circ \partial_j \circ \sigma_j \\
&= \partial_j \circ \partial_i \circ \sigma_{j-1} \circ \sigma_i \circ \partial_j \circ \sigma_j \\
&= \partial_j \circ \partial_i \circ \sigma_i \circ \sigma_j \circ \partial_j \circ \sigma_j \\
&= \partial_j \circ \partial_i \circ \sigma_i \circ \sigma_j \\
&= \partial_j \circ \partial_i \circ \sigma_{j-1} \circ \sigma_i \\
&= \partial_j \circ \sigma_j \circ \partial_i \circ \sigma_i \\
&= c_j \circ c_i \;.
\end{aligned}
$$

Notice that the arguments above equally apply to the full subcategory $\Delta_{\perp \neq \top}$ of $\Delta_{\perp, \top}$ obtained by removing the object 1. Functors $\Delta_{\perp \neq \top} \to \mathscr{X}$ are the traditional simplicial objects in $\mathscr{X}$.

Example 3.3. This example is about the cubical category $\mathbb{I}$ as used by Sjoerd Crans [10] and Dominic Verity [34, 35]. Functors with domain $\mathbb{I}$ are cubical objects in the codomain category. Verity constructed $\mathbb{I}$ as the free monoidal category containing a cointerval.

For each natural number k, define a poset $\langle k \rangle = \{-, 1, 2, \ldots, k, +\}$ by adjoining a bottom element $-$ and a top element $+$ to the discrete poset $\{1, 2, \ldots, k\}$. Any function $f : \langle k \rangle \to \langle h \rangle$ which preserves top and bottom is order-preserving. Thus we get a locally partially ordered 2-category with objects the $\langle k \rangle$, with morphisms the top-and-bottom-preserving functions, and with the pointwise order. Take $\mathbb{P}$ to be the locally full sub-2-category consisting of those $f : \langle k \rangle \to \langle h \rangle$ for which, if $f(i), f(j) \notin \{-, +\}$ then $i < j$ if and only if $f(i) < f(j)$.

Let $\mathscr{P} = \mathbb{I}$ be the underlying category of this $\mathbb{P}$. Let $\mathscr{M}$ consist of the morphisms in $\mathscr{P}$ which are injective as functions. Given such an $m : \langle k \rangle \to \langle h \rangle$ in $\mathscr{M}$, define $m^* : \langle h \rangle \to \langle k \rangle$ to send each $m(i)$ in the image of m to i and everything else to $+$. Clearly $m^* \in \mathscr{P}$ and $m^* \circ m = 1$. Furthermore, $mm^*(j)$ is equal to j if $j = m(i)$ for some i, and $+$ otherwise. Therefore $1 \leq m \circ m^*$ showing m^* to be left adjoint to m with identity counit.

We can characterize morphisms of the form m^* as those which are surjective as functions and reflect the bottom element $-$. Consequently $\mathscr{R}$ consists of the morphisms which are surjective as functions and reflect the top element $+$.

Assumption 2.5 and 2.2 are clear.

For Assumption 2.1, the existence of intersections is obvious. However, we must show that taking left adjoints gives cointersections. Take a pullback as in the left-hand diagram of (3.6) and consider the right-hand diagram. Assume $f \circ m^* = g \circ n^*$.

$$
\begin{array}{ccc}
\langle \ell \rangle \xrightarrow{\ q\ } \langle v \rangle & \qquad \langle k \rangle \xrightarrow{\ n^*\ } \langle v \rangle & \\
\ \downarrow{\scriptstyle p} \qquad\quad \downarrow{\scriptstyle n} & \ \downarrow{\scriptstyle m^*} \qquad\quad \downarrow{\scriptstyle q^*} & \\
\langle u \rangle \xrightarrow[\ m\ ]{} \langle k \rangle & \langle u \rangle \xrightarrow[\ p^*\ ]{} \langle \ell \rangle \ \searrow{\scriptstyle g} & \qquad (3.6) \\
& \qquad\qquad \searrow_{f} \quad \langle h \rangle &
\end{array}
$$

It suffices to show $f \circ p \circ p^* = f$. Now $fpp^*(i)$ is equal to $fp(j)$ if $i = p(j)$ for some j, and equal to $+$ otherwise. In the first case, we have $fpp^*(i) = fp(j) = f(i)$, as required. In the second case, if i does not have the form $p(j)$ then $m(i)$ is not in the image of n, and so $gn^*m(i) = +$; thus $f(i) = fm^*m(i) = +$ and we again have $fpp^*(i) = f(i)$.

For Assumption 2.3, suppose $m \in \mathscr{M}$ and $r, m^* \circ r \in \mathscr{R}$. Suppose $m^*(i) = +$. Since r is surjective, we have $i = r(j)$ so that $m^*r(j) = +$. However $m^* \circ r$ reflects $+$. So $j = +$, yielding $i = r(+) = +$. This proves m^* reflects $+$ and therefore must be invertible.

For Assumption 2.4, the composites of the form $m \circ n^*$ are clearly the (not necessarily surjective) morphisms which reflect $-$. Clearly these are closed under composition.

Finally, we come to Assumption 2.5. However, the idempotents of the form $c = m \circ m^*$ are defined by the property that c sends an element i either to itself or to $+$. Such idempotents commute.

There is another possible characterization of $\mathscr{R}$, namely as the category of right adjoints to the morphisms in $\mathscr{M}$. Thus in fact $\mathscr{R}$ is dual to $\mathscr{M}$. Now $\mathscr{M}$ is really just the category Δ_{inj} of finite ordinals and injective order-preserving maps, and $\mathscr{R} \cong \mathscr{M}^{\mathrm{op}}$. Also the category $\mathscr{M}^*$, with morphisms the $m^* \in \mathscr{M}$, is dual to $\mathscr{M}$. The factorization of Proposition 2.8 in

this case shows the category $\mathscr{P}$ is a composite

$$\mathbb{I} = \Delta_{\mathrm{inj}} \circ \Delta_{\mathrm{inj}}^{\mathrm{op}} \circ \Delta_{\mathrm{inj}}^{\mathrm{op}} \ ,$$

relative to suitably defined distributive laws.

An alternative viewpoint is that $\mathbb{I}$ is ParParΔ_{inj}, where in each case partial maps are defined relative to the morphisms in Δ_{inj}.

4. THE TILDE FUNCTOR

Assume we are in the setting of Section 2. Let $\mathscr{X}$ be a category with zero morphisms and finite limits.

There is a functor

$$\widetilde{(-)} : [\mathscr{P}, \mathscr{X}] \longrightarrow [\mathscr{D}, \mathscr{X}]_{\mathrm{pt}} \tag{4.7}$$

where the codomain category consists of the pointed functors. Take any functor $T : \mathscr{P} \to \mathscr{X}$. The definition of $\widetilde{T}$ on objects is:

$$\widetilde{T}A = \bigcap_{U \prec_m A} \ker T(m^* : A \to U) \ . \tag{4.8}$$

This exists because of Assumption 2.5. For $r : A \to B$ in $\mathscr{R}$, the morphism $\widetilde{T}r : \widetilde{T}A \longrightarrow \widetilde{T}B$ is the restriction of $Tr : TA \to TB$. Why does it restrict? Let $i_A : \widetilde{T}A \to TA$ be the inclusion. Take $V \prec_n B$. Proposition 2.8 yields a factorization $n^* \circ r = \ell \circ r' \circ m^*$ for some $\ell, m \in \mathscr{M}$ and $r' \in \mathscr{R}$. If m is invertible then $(n \circ \ell)^* \circ r \circ m = r' \in \mathscr{R}$ and $r \circ m \in \mathscr{R}$. Using Assumption 2.3, we see that $n \circ \ell$ is invertible; so n has a right as well as left inverse, contrary to $n : V \to B$ being proper. So m is proper and we have

$$(Tn^*)(Tr)i_A = T(n^* \circ r)i_A = T(\ell \circ r' \circ m^*)i_A = T(\ell \circ r')(Tm^*)i_A = 0 \ .$$

So there exists $\widetilde{T}r$ such that $i_B(\widetilde{T}r) = (Tr)i_A$, as claimed.

The proof that $\widetilde{T}$ preserves composition is as follows. Take $r : A \to B$ and $r_1 : B \to C$ both in $\mathscr{R}$. Clearly if $r_1 \circ r \in \mathscr{R}$ then we have $\widetilde{T}r_1 \circ \widetilde{T}r = \widetilde{T}r_1 \circ r$ by restriction of functoriality of T. If $r_1 \circ r \notin \mathscr{R}$ then, by Assumption 2.2, $r_1 \circ r \in \mathscr{S}$ and so has the form $r_1 \circ r = r_2 \circ m^*$ with $m \in \mathscr{M}$ non-invertible and $r_2 \in \mathscr{R}$. So

$$i_C \circ \widetilde{T}r_1 \circ \widetilde{T}r = T(r_1 \circ r) \circ i_A = T(r_2 \circ m^*) \circ i_A = Tr_2 \circ Tm^* \circ i_A = 0$$

yielding $\widetilde{T}(r_1 \circ r) = 0 = \widetilde{T}r_1 \circ \widetilde{T}r$.

For a natural transformation $\theta : T \Rightarrow T'$, we define $\widetilde{\theta} : \widetilde{T} \Rightarrow \widetilde{T}'$ to have components $\widetilde{\theta}_A : \widetilde{T}A \to \widetilde{T}'A$ induced by $\theta_A : TA \to T'A$. This works because θ is natural in the morphisms m^*.

5. THE HAT FUNCTOR

We can also construct a functor

$$\widehat{(-)} : [\mathscr{D}, \mathscr{X}]_{\mathrm{pt}} \longrightarrow [\mathscr{P}, \mathscr{X}] \tag{5.9}$$

whose value at the pointed functor $F : \mathscr{D} \longrightarrow \mathscr{X}$ is the functor $\widehat{F} : \mathscr{P} \longrightarrow \mathscr{X}$ defined as follows. On objects, put

$$\widehat{F}A = \sum_{U \preceq A} FU \ .$$

Now we need $\mathscr{X}$ to have finite coproducts (however, if $\mathscr{X}$ is enriched in commutative monoids, or, more specifically, additive, then these follow from finite completeness and are direct sums). We define the morphism $\widehat{F}f : \widehat{F}A \to \widehat{F}B$ by specifying its composite with

each injection $\mathrm{in}_U : FU \to \widehat{F}A$. Let $m : U \to A$ represents $U \preceq A$ and take the factorization $f \circ m = n \circ s$ with $n : V \to B$ in $\mathscr{M}$ and $s \in \mathscr{S}$ as per Proposition 2.8. Define

$$\widehat{F}f \circ \mathrm{in}_U = \begin{cases} \mathrm{in}_V \circ Fs & \text{for } s \in \mathscr{R} \\ 0 & \text{for } s \notin \mathscr{R} \ . \end{cases}$$

Why is $\widehat{F}$ a functor? Preservation of identities is clear since identities are in $\mathscr{R}$. Now take $f : A \to B$ and $g : B \to C$ in $\mathscr{P}$. Take $m : U \to A$ in $\mathscr{M}$. Factorize

$$f \circ m = n \circ s, \ g \circ n = \ell \circ t, \ t \circ s = n_1 \circ s_1$$

with $n : V \to B, \ell : W \to C, n_1 \in \mathscr{M}$ and $s, t, s_1 \in \mathscr{S}$. We first show that $s_1 \in \mathscr{R}$ if and only if $s, t, s \circ t \in \mathscr{R}$. By Assumption 2.2, if $s, t \in \mathscr{R}$ then $t \circ s \in \mathscr{S}$, and so, by uniqueness of the factorisation in Proposition 2.8, n_1 is invertible. If also $t \circ s \in \mathscr{R}$ then $s_1 \in \mathscr{R}$. Conversely, if $s_1 \in \mathscr{R}$, then, by Proposition 2.9, we have $s, t \in \mathscr{R}$; so n_1 is invertible (thus can be taken to be an identity) and so $t \circ s \in \mathscr{R}$. Now we can calculate

$$\widehat{F}g \circ \widehat{F}f \circ \mathrm{in}_U = \widehat{F}g \circ \mathrm{in}_V \circ Fs = \mathrm{in}_W \circ Ft \circ Fs$$

for $s \in \mathscr{R}$ and $t \in \mathscr{R}$, zero otherwise. By definition of $\mathscr{D}$ and functoriality of F, we have $Ft \circ Fs = F(t \circ s)$ for $t \circ s \in \mathscr{R}$, and $Ft \circ Fs = 0$ otherwise. So

$$\widehat{F}g \circ \widehat{F}f \circ \mathrm{in}_U = \mathrm{in}_W \circ F(t \circ s)$$

for $s \in \mathscr{R}$, $t \in \mathscr{R}$ and $t \circ s \in \mathscr{R}$, zero otherwise. From our first observation, this becomes

$$\widehat{F}g \circ \widehat{F}f \circ \mathrm{in}_U = \mathrm{in}_W \circ Fs_1$$

for $s_1 \in \mathscr{R}$, zero otherwise. That is,

$$\widehat{F}g \circ \widehat{F}f \circ \mathrm{in}_U = \widehat{F}(g \circ f) \circ \mathrm{in}_U$$

for all $U \preceq A$.

6. Adjointness

Take $\mathscr{X}$ to have zero morphisms, finite limits, and finite coproducts.

Theorem 6.1.

$$\widehat{(-)} \dashv \widetilde{(-)} : [\mathscr{P}, \mathscr{X}] \longrightarrow [\mathscr{D}, \mathscr{X}]_{\mathrm{pt}}$$

Proof. We must prove that there is a natural isomorphism

$$[\mathscr{P}, \mathscr{X}](\widehat{F}, T) \cong [\mathscr{D}, \mathscr{X}]_{\mathrm{pt}}(F, \widetilde{T}) \ .$$

Take a natural transformation $\theta : \widehat{F} \Rightarrow T : \mathscr{P} \to \mathscr{X}$. As in the definition of $\widehat{F}f$, with $f \circ m = n \circ s$ and $s \in \mathscr{R}$, we have commutativity in the following diagram.

$$
\begin{array}{ccccc}
FU & \xrightarrow{\ \mathrm{in}_U\ } & \widehat{F}A & \xrightarrow{\ \theta_A\ } & TA \\
{\scriptstyle Fs}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{F}f} & & \big\downarrow{\scriptstyle Tf} \\
FV & \xrightarrow[\ \mathrm{in}_V\ ]{} & \widehat{F}B & \xrightarrow[\ \theta_B\ ]{} & TB
\end{array}
\tag{6.10}
$$

Consider a noninvertible $\ell : W \to A$ in $\mathscr{M}$. Then, by the properties of $\mathscr{R}$, $\ell^* \in \mathscr{S}$ and $\ell^* \notin \mathscr{R}$. So $T\ell^* \circ \theta_A \circ \mathrm{in}_A = \theta_W \circ \widehat{F}\ell^* \circ \mathrm{in}_A = \theta_W \circ 0 = 0$. This implies there exists a unique

morphism $\phi_A : FA \longrightarrow \widetilde{T}A$ such that the following square commutes.

$$
\begin{array}{ccc}
FA & \xrightarrow{\phi_A} & \widetilde{T}A \\
{\scriptstyle \mathrm{in}_A}\downarrow & & \downarrow{\scriptstyle i_A} \\
\widehat{F}A & \xrightarrow{\theta_A} & TA
\end{array}
\tag{6.11}
$$

Naturality of ϕ is proved as follows using (6.10) with $f = r \in \mathscr{R}$: $i_B \circ \phi_B \circ Fr = \theta_B \circ \mathrm{in}_B \circ Fr = \theta_B \circ \widehat{F}r \circ \mathrm{in}_A = Tr \circ \theta_A \circ \mathrm{in}_A = Tr \circ i_A \circ \phi_A = i_B \circ \widetilde{T}r \circ \phi_A$.

For the inverse direction, take any natural transformation $\phi : F \Rightarrow \widetilde{T} : \mathscr{D} \to \mathscr{X}$. Define θ by commutativity of the following diagram.

$$
\begin{array}{ccc}
FU & \xrightarrow{\mathrm{in}_U} \widehat{F}A \xrightarrow{\theta_A} & TA \\
{\scriptstyle \phi_U}\downarrow & & \uparrow{\scriptstyle Tm} \\
\widetilde{T}U & \xrightarrow{i_U} & TU
\end{array}
\tag{6.12}
$$

We need to prove the right-hand square of (6.10) commutes when precomposed with any in_U for any $m : U \to A$ in $\mathscr{M}$. Put $f \circ m = n \circ s$ as usual. In the case where $s \in \mathscr{R}$, the desired commutativity is a consequence of the commutativity of the following three squares.

$$
\begin{array}{ccccccc}
FU & \xrightarrow{\phi_U} & \widetilde{T}U & \xrightarrow{i_A} & TU & \xrightarrow{Tm} & TA \\
{\scriptstyle Fs}\downarrow & & {\scriptstyle \widetilde{T}s}\downarrow & & {\scriptstyle Ts}\downarrow & & \downarrow{\scriptstyle Tf} \\
FV & \xrightarrow{\phi_V} & \widetilde{T}V & \xrightarrow{i_B} & TV & \xrightarrow{Tn} & TB
\end{array}
\tag{6.13}
$$

In the case where $s \notin \mathscr{R}$, we can write $s = r \circ \ell^*$ for some noninvertible $\ell \in \mathscr{M}$. Then (using both $U \preceq A$ and $U \preceq U$) we have $Tf \circ \theta_A \circ \mathrm{in}_U = T(f \circ m) \circ i_U \circ \phi_U = T(n \circ r) \circ T\ell^* \circ i_U \circ \phi_U = T(n \circ r) \circ 0 \circ \phi_U = 0 = \theta_B \circ \widehat{F}f \circ \mathrm{in}_U$, as required.

To show that the assignments are mutually inverse, take θ and define ϕ by (6.11). Let $\bar{\theta}$ be as θ is in (6.12). Then $\bar{\theta}_A \circ \mathrm{in}_U = Tm \circ i_U \circ \phi_U = Tm \circ \theta_U \circ \mathrm{in}_U = \theta_A \circ \widehat{F}m \circ \mathrm{in}_U = \theta_A \circ \mathrm{in}_U$. So $\bar{\theta} = \theta$.

On the other hand, take ϕ and define θ by (6.12). Let ϕ' be as ϕ is in (6.11). Then $i_A \circ \phi'_A = \theta_A \circ \mathrm{in}_A = i_A \circ \phi_A$. So $\phi' = \phi$. $\qquad\square$

Remark 6.2. Here is an alternative way to discover and prove Theorem 6.1. Let $\bar{\mathscr{S}}$ be the subcategory of $\mathscr{P}$ generated by $\mathscr{S}$ under composition. Let $K : \bar{\mathscr{S}} \longrightarrow \mathscr{P}$ be the inclusion. Write $\bar{\mathscr{S}}_{\mathrm{pt}}$ for the free category with zero morphisms on $\bar{\mathscr{S}}$. There is a zero-morphism-preserving functor $H : \bar{\mathscr{S}}_{\mathrm{pt}} \longrightarrow \mathscr{D}$ which is the identity on objects and takes a morphism f to f when $f \in \mathscr{R}$, otherwise it takes f to 0. This gives the two adjunctions

$$
[\mathscr{D}, \mathscr{X}]_{\mathrm{pt}} \underset{\mathrm{Ran}_H}{\overset{[H,1]_{\mathrm{pt}}}{\underset{\perp}{\rightleftarrows}}} [\bar{\mathscr{S}}_{\mathrm{pt}}, \mathscr{X}]_{\mathrm{pt}} \simeq [\bar{\mathscr{S}}, \mathscr{X}] \underset{[K,1]}{\overset{\mathrm{Lan}_K}{\underset{\perp}{\rightleftarrows}}} [\mathscr{P}, \mathscr{X}] \,.
\tag{6.14}
$$

Here Lan_K denotes ordinary left Kan extension along K while Ran_H denotes right Kan extension along H for categories enriched in the category $1/\mathrm{Set}$ of pointed sets. It is quite straightforward using the formula for Kan extension to deduce that the composite $\mathrm{Lan}_K \circ [H, 1]_{\mathrm{pt}}$ is none other than the hat construction of Section 5.9. With somewhat more work one can also deduce that the composite $\mathrm{Ran}_H \circ [K, 1]$ is the tilde construction of Section 4.7. Of course, given Theorem 6.1, only one of these verifications is required.

7. Invertibility of the unit

Now assume $\mathscr{X}$ has homs enriched in commutative monoids and is finitely complete. Take any pointed functor $F : \mathscr{D} \longrightarrow \mathscr{X}$. By Assumption 2.5, we have a direct sum

$$\widehat{F}A = \bigoplus_{V \preceq A} FV$$

over the subobjects $n : V \to A$ of A. For $f : A \to B$ in $\mathscr{P}$, we can represent $\widehat{F}f : \widehat{F}A \longrightarrow \widehat{F}B$ as a matrix with $V \preceq_n A, W \preceq_\ell B$-entry

$$(\widehat{F}f)_{V,W} = \begin{cases} Fs & \text{for } f \circ n = \ell \circ s \text{ and } s \in \mathscr{R} \\ 0 & \text{otherwise .} \end{cases}$$

We have the inclusion $i_A : \widetilde{\widehat{F}}A \longrightarrow \widehat{F}A$. By definition of $\mathscr{R}$, we know m^* is not in $\mathscr{R}$ for a proper subobject $U \prec_m A$. So $\widehat{F}m^* \circ \mathrm{in}_A = 0$. This yields the morphism $\eta_F A : FA \longrightarrow \widetilde{\widehat{F}}A$ satisfying $i_A \circ \eta_F A = \mathrm{in}_A$ which is the component at A of the unit of the component at F of the adjunction in Theorem 6.1.

Consider $\widehat{F}m^* : \widehat{F}A \longrightarrow \widehat{F}U$ for $U \prec_m A$. Notice that $m^* \circ n = \ell \circ s$ with $s \in \mathscr{R}$ implies s invertible. To see this, we have $(m \circ \ell)^* \circ n = s$ and Assumption 2.4 yields $m_1 \circ n_1^* = s$ for some $m_1, n_1 \in \mathscr{M}$, so, by the definition of $\mathscr{R}$, it follows that m_1, n_1 are both invertible, so s is. Hence:

$$(\widehat{F}m^*)_{V,W} = \begin{cases} 1 & \text{for } m^* \circ n = \ell \\ 0 & \text{otherwise .} \end{cases}$$

The inclusion $i_A : \widetilde{\widehat{F}}A \longrightarrow \widehat{F}A$ can be written as a vector

$$i_A = (a_n)_{V \preceq_n A}$$

where $a_n = \mathrm{pr}_V \circ i_A : \widetilde{\widehat{F}}A \longrightarrow FV$. Since $\widehat{F}m^* \circ i_A = 0$ for $U \prec_m A$, we obtain

$$\sum_{m^* \circ n = \ell} a_n = 0 .$$

Lemma 7.1. *If $U \prec_m A$ then $a_m = 0$.*

Proof. We use induction on the number of $n : U \to A$ in $\mathscr{M}$ with $n \leq m$ in the ordered set $\mathbb{P}(U, A)$.

If the number is 1 then m is minimal. Now by adjointness in $\mathbb{P}$, the equation $m^* \circ n = 1$ implies $n \leq m$ and so $n = m$ by minimality. So $\sum_{m^* \circ n = 1} a_n$ has only the one term a_m. So $a_m = 0$.

Assume inductively that $a_p = 0$ for all $p \in \mathbb{P}(U, A)$ with fewer $n \leq p$ than $n \leq m$. We have

$$0 = \sum_{m^* \circ n = 1} a_n = a_m + \sum_{m^* \circ p = 1, \ p \neq m} a_n .$$

By adjointness, $m^* \circ p = 1$ and $p \neq m$ imply $p < m$. Also p is proper since otherwise $m^* \circ p = 1$ would imply m invertible. By the inductive assumption, it follows that each such $a_p = 0$. So $a_m = 0$ as required. $\square$

Proposition 7.2. *The unit $\eta_F : F \Longrightarrow \widetilde{\widehat{F}}$ of the adjunction of Theorem 6.1 is invertible.*

Proof. The component $\eta_F A : FA \to \widehat{F}A$ of the unit is defined by $i_A \circ \eta_F A = \mathrm{in}_A$; it is clearly a monomorphism. Lemma 7.1 can be stated as saying $i_A = \mathrm{in}_A \circ a_A$. So

$$i_A \circ \eta_F A \circ a_A = \mathrm{in}_A \circ a_A = i_A = i_A \circ 1_{\widetilde{\widehat{F}}A} ,$$

yielding $\eta_F A \circ a_A = 1_{\widehat{F}A}$. Hence $\eta_F A$ has inverse $a_A = \mathrm{pr}_A \circ i_A$. $\qquad\square$

8. Remarks on idempotents

Define a relation on any monoid M by $a \sqsubseteq b$ when $ba = a$. This relation is transitive; indeed, we have a stronger property.

Proposition 8.1. *If $ub \sqsubseteq a$ and $vc \sqsubseteq b$ then $uvc \sqsubseteq a$.*
If $u_i a_i \sqsubseteq a_{i-1}$ for $1 \sqsubseteq i \sqsubseteq n$ then $u_1 \dots u_n a_n \sqsubseteq a_0$.

Proof. For the first sentence, the assumptions are $aub = ub$ and $bvc = vc$. So $auvc = aubvc = ubvc = uvc$; that is, $uvc \sqsubseteq a$. The second sentence follows by induction. $\qquad\square$

The relation is only reflexive for idempotents: clearly $a \sqsubseteq a$ is equivalent to $aa = a$.

The unit 1 of the monoid is a largest element in the sense that $a \sqsubseteq 1$ for all $a \in M$. That is, 1 is the empty meet. However, not all meets need exist. A *meet* for $a, b \in M$ is an element $a \wedge b$ with $a \wedge b \sqsubseteq a$ and $a \wedge b \sqsubseteq b$, and, if $x \sqsubseteq a$ and $x \sqsubseteq b$ then $x \sqsubseteq a \wedge b$. In particular, $a \wedge b$ must be an idempotent. Meets of lists of n elements are defined in the obvious way and, for $n \geq 2$, can be constructed from iterated binary meets when they exist.

Proposition 8.2. *If $ab \sqsubseteq b$ and $a \sqsubseteq a$ then $a \wedge b = ab$.*
If $a_1, \dots, a_n$ are idempotents such that $a_i a_j \sqsubseteq a_j$ for $i \sqsubseteq j$ then $a_1 \wedge \cdots \wedge a_n = a_1 \dots a_n$.

Proof. For the first sentence, we are told that $ab \sqsubseteq b$, while $a \sqsubseteq a$ implies $aa = a$, and so $aaab = ab$, yielding $ab \sqsubseteq a$. For the second sentence the result is clear for $n = 1$ since we suppose a_1 idempotent. Assume the result for $n-1$; so $a_1 \wedge \cdots \wedge a_{n-1} = a_1 \dots a_{n-1}$. Apply the second sentence of Proposition 8.1 to the inequalities $a_i a_n \sqsubseteq a_n$ to deduce $a_1 \dots a_{n-1} a_n \sqsubseteq a_n$. So, by the first sentence, $a_1 \dots a_n = a_1 \dots a_{n-1} \wedge a_n = a_1 \wedge \cdots \wedge a_{n-1} \wedge a_n$, as required. $\qquad\square$

Notice that, if $ab = ba$ and b is idempotent, then $bab = abb = ab$, so $ab \sqsubseteq b$. So the proposition applies to commuting idempotents.

Now suppose we have a ring R. We can apply our results to the multiplicative monoid of R. We say idempotents e and f in R are *orthogonal* when $ef = fe = 0$. A list $e_0, e_1, \dots e_n$ of idempotents is *orthogonal* when each pair in the list is orthogonal. The list is *complete* when $e_0 + e_1 + \cdots + e_n = 1$. An easy induction shows that a complete list of idempotents is orthogonal if and only if $e_i e_j = 0$ for $i \leq j$.

For each $a \in R$, put $\bar{a} = 1 - a$. Clearly if a is idempotent, so is $\bar{a}$.

Let R° denote the ring obtained from R by reversing multiplication.

Proposition 8.3.   *(a) $a \sqsubseteq b$ in R if and only if $\bar{b} \sqsubseteq \bar{a}$ in R°.*
  (b) For b an idempotent, $ab \sqsubseteq b$ in R if and only if $\bar{a}\bar{b} \sqsubseteq \bar{b}$ in R°.
  (c) If a and b are idempotents and $ab \sqsubseteq b$ in R then $e_0 = ab, e_1 = \bar{a}b, e_2 = \bar{b}$ is a complete list of orthogonal idempotents.

Proof.   (a) $\bar{b} \sqsubseteq \bar{a}$ in R° means $(1-b)(1-a) = 1-b$ in R; that is, $ba = a$ which means $a \sqsubseteq b$ in R.
  (b) $\bar{a}\bar{b} \sqsubseteq \bar{b}$ in R° means $\bar{b}\bar{a}\bar{b} = \bar{b}\bar{a}$ in R. That is, $(1-b)(1-a)(1-b) = (1-b)(1-a)$. That is, $1 - a - b + ab - b + ba + bb - bab = 1 - b - a + ba$. That is, $bab = ab$, which is $ab \sqsubseteq b$ in R.
  (c) We already know e_0 and e_2 are idempotent. They are also orthogonal: $e_0 e_2 = ab(1-b) = ab - ab = 0$ and $e_2 e_0 = (1-b)ab = ab - bab = 0$. Therefore $e_0 + e_2 = ab + 1 - b = 1 - (1-a)b = \overline{e_1}$ is idempotent. So e_1 is idempotent and $e_0 + e_1 + e_2 = 1$. The calculations $e_0 e_1 = ab(1-a)b = ab - abab = 0$ and $e_1 e_2 = (1-a)b(1-b) = b - ab - b + abb = 0$ complete the proof.

$\qquad\square$

We can extend part (c) inductively to obtain:

Proposition 8.4. *Suppose $a_1, \ldots, a_n$ are idempotents such that $a_i a_j \sqsubseteq a_j$ for $i \leq j$. Then $e_i = \overline{a_i} a_{i+1} a_{i+2} \ldots a_n$ for $0 \leq i \leq n$ (in particular, $e_0 = a_1 a_2 \ldots a_n$ and $e_n = \overline{a_n}$) defines a complete list of orthogonal idempotents.*

Suppose $\mathscr{X}$ is an additive category in which idempotents split. Our results apply to the endomorphism monoid $\mathscr{X}(A, A)$ of each object $A \in \mathscr{X}$. If a is an idempotent on A, we have a splitting:

$$
\begin{array}{ccc}
A & \xrightarrow{\ a\ } & A \\
{\scriptstyle r_a}\downarrow & \nearrow{\scriptstyle i_a} & \downarrow{\scriptstyle r_a} \\
aA & \xrightarrow[\ 1_{aA}\ ]{} & aA \ .
\end{array}
$$

Yet, we also have a splitting for $\bar{a} = 1 - a$ which incidentally provides a kernel $\bar{a}A$ for a and so a direct sum decomposition of A:

$$A \cong \bar{a}A \oplus aA \ .$$

More generally, for any complete list $e_0, e_1, \ldots e_n$ of orthogonal idempotents in $\mathscr{X}(A, A)$, we obtain a direct sum decomposition

$$A \cong e_0 A \oplus e_1 A \oplus \cdots \oplus e_n A \ .$$

9. The equivalence

An adjunction with invertible unit and conservative right adjoint is an equivalence. A right adjoint is conservative if and only if the components of the counit are strong epimorphisms.

So it remains to prove, for any functor $T : \mathscr{P} \to \mathscr{X}$ and any object A in $\mathscr{P}$, the component $\varepsilon_T A : \widehat{\widetilde{T}} A \to T A$ of the counit of the adjunction of Theorem 6.1 is a strong epimorphism. We have $\widehat{\widetilde{T}} A = \bigoplus_{U \preceq A} \widetilde{T} U$ and the restriction of $\varepsilon_T A$ to the $U \preceq_m A$ term is

$$\alpha_m : \widetilde{T} U \xrightarrow{\ i_U\ } T U \xrightarrow{\ Tm\ } T A \ . \tag{9.15}$$

It follows from Assumption 2.6 that the idempotents $a_i = \overline{T c_i}$ on $T A$ in $\mathscr{X}$ satisfy the conditions of Proposition 8.4.

Theorem 9.1. *For any additive category $\mathscr{X}$ which has finite products and splittings for idempotents, the adjunction of Theorem 6.1 is an equivalence*

$$[\mathscr{P}, \mathscr{X}] \simeq [\mathscr{D}, \mathscr{X}]_{\mathrm{pt}} \ .$$

Proof. As remarked in the Introduction, it suffices to take $\mathscr{X}$ to be the category of abelian groups. We say this to assure the reader that the limits involved in the tilde construction do exist in $\mathscr{X}$ as given in the Theorem.

We already know that it suffices to show that the morphisms α_m of (9.15) are jointly strongly epimorphic. Put $\alpha_0 = \alpha_{1_A}$ and $\alpha_i = \alpha_{m_i}$ for $1 \leq i \leq n$ and we shall show that these α_i for $0 \leq i \leq n$ are already jointly strongly epimorphic. The proof is by induction on the number of subobjects of A (using Assumption 2.5). If A has no proper subobjects, $\widetilde{T} A = T A$ and α_0 is invertible. For $1 \leq i \leq n$, we have a commuting square

$$
\begin{array}{ccc}
\widehat{\widetilde{T}} U_i & \xrightarrow{\ \varepsilon_T U_i\ } & T U_i \\
{\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle T m_i} \\
\widetilde{T} U_i & \xrightarrow[\ \alpha_i\ ]{} & T A \ .
\end{array}
$$

Each U_i has fewer subobjects than A so the inductive hypothesis yields invertibility of each $\varepsilon_T U_i$. So it suffices to prove that α_0 together with the Tm_i are jointly strongly epimorphic. Every proper subobject of A factors through one of the m_i; so

$$\widetilde{T}A = \bigcap_{i=1}^{n} \ker Tm_i^* = e_0 TA$$

in the notation of Proposition 8.4. Also, in that notation, $e_i = (Tm_i)(Tm_i^*)a_{i+1}\ldots a_n$, so we put $s_i = (Tm_i^*)a_{i+1}\ldots a_n : TA \to TU_i$. Take s_0 to be the splitting of $\alpha_0 : e_0 TA \to TA$. Then

$$[\alpha_0, Tm_1, \ldots, Tm_n] : \bigcap_{i=0}^{n} TU_i \longrightarrow TA$$

has a right inverse s with entries s_i since

$$\alpha_0 s_0 + Tm_1 s_1 + \cdots + Tm_n s_n = e_0 + e_1 + \cdots + e_n = 1 .$$

Thus $\alpha_0, Tm_1 \ldots, Tm_n$ are jointly strongly epimorphic, as required. $\qquad\square$

10. Examples of Theorem 9.1

Example 10.1. We begin with a baby version of the Dold-Puppe-Kan Theorem. Let $\mathrm{Pt}\mathscr{X}$ denote the category whose objects are split epimorphisms in $\mathscr{X}$, the morphisms are morphisms of the epimorphisms which commute with the splittings; this is what Bourn [5] calls the category of points in $\mathscr{X}$. Take $\mathbb{P}$ to be the free-living adjunction $\mu^* \dashv \mu$ with identity counit $\mu^* \circ \mu = 1$. So $[\mathscr{P}, \mathscr{X}] \cong \mathrm{Pt}\mathscr{X}$. Let $\mathscr{M}$ consist of all the monomorphisms. Then $\mathscr{R}$ contains only the identities. Theorem 9.1 yields

$$\mathrm{Pt}\mathscr{X} \simeq \mathscr{X} \times \mathscr{X} .$$

This example also shows the necessity of $\mathscr{X}$ having homs enriched in abelian groups (not merely commutative monoids). We need $\mathscr{X}$ to have kernels of split epimorphisms and coproducts already. If we also ask that it have finite products then considering the split epimorphism $X \times Y \to Y$ given by the projection, the counit is the canonical map $X + Y \to X \times Y$, so if this is invertible we have hom enrichment in commutative monoids. Now considering the codiagonal $X + X \to X$, split by one of the injections, it is not hard to show that 1_X has an additive inverse.

Example 10.2. ([11, 12, 20]) Applying Theorem 9.1 to Example 3.2 yields that $[\Delta_{\perp,\top}, \mathscr{X}]$ is equivalent to the category of chain complexes in $\mathscr{X}$.

Example 10.3. Applying Theorem 9.1 to Example 3.3 yields $[\mathbb{I}, \mathscr{X}] \simeq [\Delta_{\mathrm{inj}}, \mathscr{X}]$, the category of semi-simplicial objects in $\mathscr{X}$.

Here are some examples of the general type described in Example 3.1.

Example 10.4. A *(set) species* in the sense of Joyal [17] is a functor $F : \mathfrak{S} \longrightarrow \mathrm{Set}$ where $\mathfrak{S}$ is the category of sets and bijective functions. A *pointed-set species* is a functor $F : \mathfrak{S} \longrightarrow 1/\mathrm{Set}$. An *R-module species* is a functor $F : \mathfrak{S} \longrightarrow \mathrm{Mod}^R$; the case where R is a field is the basic situation of [18].

Following [9], we write FI for the category of finite sets and injective functions. We then see that $\mathscr{M} = \mathrm{FI}$ and $\mathscr{E} = \mathfrak{S}$, while $\mathscr{P} = \mathrm{FI}\sharp$, the category of injective partial functions.

Corollary 10.5. [9] *The functor*

$$\widehat{(-)} : [\mathfrak{S}, \mathrm{Mod}^R] \longrightarrow [\mathrm{FI}\sharp, \mathrm{Mod}^R]$$

is an equivalence of categories.

Example 10.6. Another example relevant to [19] is the category $\mathscr{A} = \mathrm{FIVect}_{\mathbb{F}_q}$ of finite vector spaces over the field $\mathbb{F}_q$ of cardinality q (a prime power) and injective linear functions. Let $\mathscr{E} = \mathfrak{Gl}_q$ be the category of finite $\mathbb{F}_q$-vector spaces and linear isomorphisms. Then $\mathscr{P} = \mathrm{FI}\sharp_q$ is the category of finite $\mathbb{F}_q$-vector spaces and injective partial linear functions.

Corollary 10.7. *The functor*

$$\widehat{(-)} : [\mathfrak{Gl}_q, \mathrm{Mod}^R] \longrightarrow [\mathrm{FI}\sharp_q, \mathrm{Mod}^R]$$

is an equivalence of categories.

Example 10.8. Here are a few examples of categories $\mathscr{A}$ as in Example 3.1 to which Theorem 9.1 applies with $\mathscr{E}$ the epimorphisms and $\mathscr{M}$ the monomorphisms:

(a) the category of finite abelian groups and group morphisms;
(b) the category of finite abelian p-groups and group morphisms;
(c) the category of finite sets and all functions.

Theorem 9.1 also applies to $\mathscr{M}$ in place of $\mathscr{A}$ in these examples. Then $\mathscr{E}$ is replaced by the groupoid of invertible morphisms in $\mathscr{A}$. In case of example (a), the paper [15] describes the groupoid being represented in $\mathscr{X}$.

Example 10.9. Consider the "algebraic" simplicial category Δ_+ whose objects are all the natural numbers and whose morphisms $\xi : m \longrightarrow n$ are order-preserving functions

$$\xi : \{0, 1, \dots, m-1\} \longrightarrow \{0, 1, \dots, n-1\} .$$

Put $\mathscr{A} = \Delta_+^{\mathrm{op}}$. Take $\mathscr{M}$ in $\mathscr{A}$ to consist of the surjections in Δ_+. Pushouts of surjections along arbitrary morphisms exist in Δ_+. Then $\mathscr{E} = \Delta_{+\mathrm{inj}}^{\mathrm{op}}$ and $\mathscr{P}$ is the opposite of the category whose morphisms $m \longrightarrow n$ are cospans

$$m \xrightarrow{\ \xi\ } r \xleftarrow{\ \sigma\ } n$$

in Δ_+ with σ surjective. We could also take the "topological" simplicial category Δ (omit the object 0) to obtain a reinterpretation of the preoperads in $\mathscr{X}$ in the sense of [2].

Example 10.10. Here is a rather trivial example involving Δ. Take $\mathscr{A}$ to be the category of non-empty ordinals and morphisms the order-preserving functions which preserve first element. Let $\mathscr{M}$ be the class of morphisms which are inclusions of initial segments. This is part of a factorization system where $\mathscr{E} = \Delta_{\perp \neq \top}$ is the category of ordinals with distinct first and last element and morphisms the order-preserving functions which preserve first and last element. Sometimes $\mathscr{E}$ is called the category of *intervals*; there is a duality isomorphism

$$\mathscr{E} \cong \Delta^{\mathrm{op}} .$$

In this case, not only do we have the equivalence

$$[\mathscr{E}, \mathscr{X}] \simeq [\mathscr{P}, \mathscr{X}]$$

of Theorem 9.1, we actually also have an isomorphism

$$\mathscr{P} \cong \mathscr{E} .$$

Example 10.11. Take $\mathscr{A}$ to be a (partially) ordered set with finite infima and the descending chain condition. Then every morphism is a monomorphism and the strong epimorphisms are equalities. So $\mathscr{E}$ is the discrete category $\mathrm{ob}\mathscr{A}$ on the set of elements of the ordered set. The reader may like to contemplate the case where $\mathscr{A}$ is the set of strictly positive integers ordered by division.

11. A construction for new examples

Let $\Delta_{\perp \neq \top}$ be the category of intervals; that is, the full subcategory of $\Delta_{\perp,\top}$ obtained by deleting the object 1. This provides an example of the setting in Section 2; see Example 3.2. The 2-category "$\mathbb{P}$" for this example will be denoted by $\mathbb{D}$; it is the 2-category whose underlying category is $\Delta_{\perp \neq \top}$ and whose 2-cells are pointwise order. $\mathbb{D}$ is a locally full sub-2-category of Cat.

Let $\mathscr{P}$ be any category with the structure and assumptions laid out in Section 2; in particular, we have the locally ordered 2-category $\mathbb{P}$.

There is a kind of wreath product $\mathbb{Q}$ of $\mathbb{D}$ with $\mathbb{P}$. It is another locally ordered 2-category. The objects are functors $A : a \to \mathscr{P}$ (not just families) with $a \in \mathbb{D}$ and for which each $A(i \leq j) : A_i \to A_j$ is in $\mathscr{M} \circ \mathscr{M}^*$. A morphism $(\xi, u) : (a, A) \to (b, B)$ is a diagram

$$
\begin{array}{ccc}
a & \xrightarrow{\quad \xi \quad} & b \\
& \underset{A}{\searrow} \overset{u}{\underset{\Longrightarrow}{}} \swarrow{B} & \\
& \mathscr{P} &
\end{array}
\tag{11.16}
$$

in Cat with $\xi : a \to b$ in $\mathbb{D}$ and $u_0 : A_0 \to B_0$ invertible (or, if the class $\mathscr{S}$ for $\mathscr{P}$ is closed under composition, we can merely ask that $u_0 \in \mathscr{S}$). Define $(\xi, u) \leq (\zeta, v) : (a, A) \to (b, B)$ when $\xi \leq \zeta$ and, for each $i \in a$, we have $B(\xi i \leq \zeta i) \circ u_i \leq v_i$.

This $\mathbb{Q}$ is a locally full sub-2-category of the Grothendieck fibration construction applied to the 2-functor

$$
\mathbb{D}^{\mathrm{op}} \longrightarrow \mathrm{Cat}^{\mathrm{op}} \xrightarrow{[-, \mathbb{P}]} 2\text{-Cat} .
$$

Let $\mathscr{Q}$ be the underlying category of $\mathbb{Q}$. Define $\mathscr{M}$ for $\mathscr{Q}$ to consist of the morphisms $(\partial, m) : (a, A) \to (b, B)$ where ∂ is injective and $m_i \in \mathscr{M}$ for all $i \in a$. The following result is routine.

Proposition 11.1. *If σ is a left adjoint with identity counit for ∂ in $\mathbb{D}$ and m_i^* is a left adjoint with identity counit in $\mathbb{P}$ for $m_i \in \mathscr{M}$, $i \in a$, then (σ, z) is a left adjoint with identity counit for $(\partial, m) : (a, A) \to (b, B)$, where $z_k = m_{\sigma k}^* \circ B(k \leq \partial \sigma k)$ for all $k \in b$.*

Proposition 11.2. *Assumptions 2.1 and 2.5 are satisfied by this $\mathscr{M}$ for $\mathscr{Q}$.*

Proof. Assumption 2.5 is immediate since it is true of $\Delta_{\perp \neq \top}$ and assumed for $\mathscr{P}$. Thus we only need to check Assumption 2.1 for binary pullbacks. The pullback of two subobjects $(\partial, m) : (a, A) \longrightarrow (c, C)$ and $(\partial', m') : (b, B) \longrightarrow (c, C)$ is obtained by forming the pullback

$$
\begin{array}{ccc}
p & \xrightarrow{\quad \pi' \quad} & b \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \partial'} \\
a & \xrightarrow{\quad \partial \quad} & c
\end{array}
$$

in $\Delta_{\perp \neq \top}$, and, for each $k \in p$, forming the pullback

$$
\begin{array}{ccc}
P_k & \xrightarrow{\quad \ell'_k \quad} & B_{\pi' k} \\
{\scriptstyle \ell_k}\downarrow & & \downarrow{\scriptstyle m'_{\pi' k}} \\
A_{\pi k} & \xrightarrow[\quad m_{\pi k} \quad]{} & C_{\partial \pi k}
\end{array}
$$

in $\mathscr{P}$. Note that, for $k_1 \leq k_2$, the induced $P(k_1 \leq k_2)$ satisfies $\ell_{k_2} \circ P(k_1 \leq k_2) = A(\pi k_1 \leq \pi k_2) \circ \ell_{k_2}$ so that $P(k_1 \leq k_2) = \ell_{k_2}^* \circ A(\pi k_1 \leq \pi k_2) \circ \ell_{k_2} \in \mathscr{M} \circ \mathscr{M}^*$ by Assumption 2.4 for

$\mathscr{P}$. This gives the pullback

$$
\begin{array}{ccc}
(p, P) & \xrightarrow{\ (\pi', \ell')\ } & (b, B) \\
{\scriptstyle (\pi, \ell)} \downarrow & & \downarrow {\scriptstyle (\partial', m')} \\
(a, A) & \xrightarrow[\ (\partial, m)\ ]{} & (c, C)
\end{array}
$$

in $\mathscr{Q}$.

Now take left adjoints of the morphisms in this pullback in $\mathbb{Q}$. We must prove the result is a pushout in $\mathscr{Q}$. Take $(\xi, u) : (a, A) \to (x, X)$ and $(\zeta, v) : (b, B) \to (x, X)$ such that $(\xi, u) \circ (\partial, m)^* = (\zeta, v) \circ (\partial', m')^*$. By Assumption 2.1 for $\Delta_{\perp \neq \top}$, there exists a unique $\theta : p \to x$ such that $\theta \circ \pi^* = \xi$ and $\theta \circ \pi'^* = \zeta$. By Assumption 2.1 for $\mathscr{P}$, for each $k \in p$, there exists a unique $w_k : P_k \to X_{\theta k}$ such that $w_k \circ \ell_k^* = u_{\pi k}$ and $w_k \circ \ell_k'^* = v_{\pi' k}$. (Note that this uses $\pi^* \circ \pi = 1$ and $\pi'^* \circ \pi' = 1$.) This gives $(\theta, w) : (p, P) \to (x, X)$ as required. $\qquad \square$

Defining $\mathscr{R}$ and $\mathscr{S}$ for $\mathscr{Q}$ as we must, we conclude from Proposition 11.2 that we have the unique factorization property $(\xi, u) = (\mu, n) \circ (\sigma, s)$ with $(\mu, n) \in \mathscr{M}$ and $(\sigma, s) \in \mathscr{S}$; furthermore, we have the unique factorization $(\sigma, s) = (\rho, r) \circ (\partial, m)^*$ with $(\partial, m) \in \mathscr{M}$ and $(\rho, r) \in \mathscr{R}$.

Before continuing, we require more explicit descriptions of these classes $\mathscr{S}$ and $\mathscr{R}$. To do this, we require some restrictions on $\mathscr{P}$.

Suppose $\mathscr{N}$ is a subcategory of a category $\mathscr{C}$ and $\mathscr{H}, \mathscr{K}, \mathscr{L}$ are three classes of morphisms in $\mathscr{C}$. We say $(\mathscr{H}, \mathscr{K})$-*factorization in* $\mathscr{L}$ *is* $\mathscr{N}$-*functorial* when, for all $f_1 = k_1 \circ h_1 \in \mathscr{L}$ and $f_2 = k_2 \circ h_2 \in \mathscr{L}$ with $h_1, h_2 \in \mathscr{H}$ and $k_1, k_2 \in \mathscr{K}$, if $b \circ f_1 = f_2 \circ a$ with $a, b \in \mathscr{N}$ then there exists a unique $c \in \mathscr{N}$ for which the following diagram commutes.

$$
\begin{array}{ccccc}
A_1 & \xrightarrow{\ h_1\ } & C_1 & \xrightarrow{\ k_1\ } & B_1 \\
{\scriptstyle a} \downarrow & & \vdots\,{\scriptstyle c} \downarrow & & \downarrow {\scriptstyle b} \\
A_2 & \xrightarrow[\ h_2\ ]{} & C_2 & \xrightarrow[\ k_2\ ]{} & B_2
\end{array}
$$

Example 11.3. In Example 3.1, where $\mathscr{P} = \mathrm{Par}\mathscr{A}$, we see that $(\mathscr{M}^*, \mathscr{R})$-factorization in $\mathscr{S}$ is $\mathscr{P}$-functorial. Moreover, $(\mathscr{S}, \mathscr{M})$-factorization in $\mathscr{P}$ is $\mathscr{P}$-functorial *if each pullback in* $\mathscr{A}$ *of a morphism in* $\mathscr{E}$ *along a morphism in* $\mathscr{M}$ *is in* $\mathscr{E}$. Under this condition, $\mathscr{S}$ is closed under composition.

Example 11.4. In Example 3.2, $(\mathscr{S}, \mathscr{M})$-factorization in $\mathscr{P} = \Delta_{\perp, \top}$ (or $\Delta_{\perp \neq \top}$) is $\mathscr{P}$-functorial. Moreover, $(\mathscr{M}^*, \mathscr{R})$-factorization in $\mathscr{S}$ is $\mathscr{M} \circ \mathscr{M}^*$-functorial. To see this last sentence, suppose $\rho \circ \xi = \zeta \circ \sigma$ where $\sigma \dashv \partial$, $\rho = 1$ or σ_0, and ξ reflects 0. We claim $\theta = \xi \circ \partial$, which clearly reflects 0, satisfies $\rho \circ \theta = \zeta$ and $\theta \circ \sigma = \xi$. The first of these is easy: $\rho \circ \theta = \rho \circ \xi \circ \partial = \zeta \circ \sigma \circ \partial = \zeta$. The second is clear when $\rho = 1$. For the second with $\rho = \sigma_0$, we use that σ_0 is injective except that $\sigma_0 1 = \sigma_0 0$. Now $1 \leq \partial \circ \sigma$ so $\xi i \neq \xi \partial \sigma i$ implies $\xi i < \xi \partial \sigma i$, and the only possibility for this is $\xi i = 0$ and $\xi \partial \sigma i = 1$. Since $\xi \in \mathscr{M} \circ \mathscr{M}^*$, $\xi i = 0$ implies $i = 0$. Then $\xi \partial \sigma i = \xi \partial \sigma 0 = 0 \neq 1$, a contradiction. This proves $\theta \circ \sigma = \xi$. Clearly θ is unique since σ is epimorphic.

Proposition 11.5. *Assume* $(\mathscr{S}, \mathscr{M})$-*factorization in* $\mathscr{P}$ *is* $\mathscr{M} \circ \mathscr{M}^*$-*functorial. A morphism* $(\xi, u) : (a, A) \longrightarrow (b, B)$ *in* $\mathscr{Q}$ *is in* $\mathscr{S}$ *if and only if for each* $j \in b$, *there exists* $i \in a$ *with* $\sigma i = j$ *and* $u_i \in \mathscr{S}$.

Proof. To prove "if", assume $(\xi, u) = (\partial, m) \circ (\zeta, v)$ with $(\partial, m) \in \mathscr{M}$. Since $\xi = \partial \circ \zeta$ is surjective, ∂ is an identity and $\xi = \zeta$. So we have $u_i = m_{\xi i} \circ v_i$ for all $i \in a$. For any given

$j \in b$, choose i in the fibre of ξ over j with $u_i \in \mathscr{S}$. Then $m_j = m_{\xi i}$ is invertible. This proves (∂, m) invertible. So $(\xi, u) \in \mathscr{S}$.

For "only if", first note that, writing $\xi = \mu \circ \sigma$ with μ injective and σ surjective, we have $(\xi, u) = (\mu, 1_{B_\mu}) \circ (\sigma, u)$ with $(\mu, 1_{B_\mu}) \in \mathscr{M}$. So μ is an identity and ξ is surjective. Now notice that, by naturality of u, if $i_1 \leq i_2$ in the fibre of ξ over $j \in b$, then $u_{i_1} = u_{i_2} \circ A(i_1 \leq i_2)$. So $u_{i_1} \in \mathscr{S}$ implies $u_{i_2} \in \mathscr{S}$. Let ∂j denote the largest element of the fibre of ξ over j. So we are required to prove that each $u_{\partial j}$ is in $\mathscr{S}$. (In fact, $\xi \dashv \partial$ in Cat although $\partial : b \to a$ may not be in $\Delta_{\perp \neq \top}$). We have $(\xi, w) : (a, A) \longrightarrow (b, A_\partial)$ defined by $w_i = A(i \leq \partial \xi i) : A_i \longrightarrow A_{\partial \xi}$. However, $(\xi, u) = (1_b, u_\partial) \circ (\xi, w)$ and $(\xi, u) \in \mathscr{S}$ imply $(1_b, u_\partial) \in \mathscr{S}$. It remains to show that any morphism in $\mathscr{S}$ of the form $(1_b, y) : (b, C) \longrightarrow (b, B)$ has each y_j in $\mathscr{S}$. Factor $y_j = m_j \circ s_j$ with $m_j \in \mathscr{M}$ and $s_j \in \mathscr{S}$. Using functoriality of the factorization in $\mathscr{P}$, we obtain a factorization $(1_b, y) = (1_b, m) \circ (1_b, s)$ in $\mathscr{Q}$. So $(1_b, m)$ is invertible. Hence each m_j is invertible yielding $y_j \in \mathscr{S}$. □

Proposition 11.6. *Assume* $(\mathscr{S}, \mathscr{M})$-*factorization in* $\mathscr{P}$ *and* $(\mathscr{M}^*, \mathscr{R})$-*factorization in* $\mathscr{S}$ *are* $\mathscr{M} \circ \mathscr{M}^*$-*functorial. The morphisms in the* $\mathscr{R}$ *for* $\mathscr{Q}$ *consist of those of the form* $(1_a, r) :$ $(a, A) \to (a, B)$ *or* $(\sigma_0, r) : (b+1, A) \to (b, B)$ *where* $r_i \in \mathscr{R}$ *for all* $i \in a$.

Proof. We have the characterization of $\mathscr{S}$ as in Proposition 11.5.

The morphisms (ρ, r) of the given form are clearly in $\mathscr{S}$ so it suffices the show that $(\rho, r) = (\xi, u) \circ (\partial, m)^*$, for $(\partial, m) \in \mathscr{M}$, implies (∂, m) invertible. We have $\rho = \xi \circ \sigma$ and $r_k = u_{\sigma k} \circ z_k$ where $(\sigma, z) = (\partial, m)^*$. It follows that σ is an identity, that $\rho = \xi$ and $r_k = u_k \circ m_k^*$. Since $r_k \in \mathscr{R}$ and $m_k \in \mathscr{M}$, it follows that each m_k is invertible. So (∂, m) is invertible.

Conversely, suppose $(\xi, u) \in \mathscr{R}$. Then $(\xi, u) \in \mathscr{S}$ so ξ is surjective and, for each j, there is an i with $\xi i = j$ and $u_i \in \mathscr{S}$. By definition of $\mathscr{Q}$, we also have $u_0 \in \mathscr{S}$. For each i with $i \in \mathscr{S}$, write $u_i = r_i \circ m_i^*$ functorially with $r_i \in \mathscr{R}$ and $m_i \in \mathscr{M}$. If ξ is an identity then, by definition of $\mathscr{R}$ in $\mathscr{P}$, all m_i are invertible and so all $u_i \in \mathscr{R}$; so (ξ, u) has the desired form. Suppose ξ is not an identity. Then $\xi = \sigma \circ \sigma_k$ where σ is a (possibly empty) composite of morphisms σ_i with $i < k$. So $(\xi, u) = (\sigma, 1) \circ (\sigma_k, u)$. Since $(\xi, u) \in \mathscr{R}$, it follows that $(\sigma_k, u) \in \mathscr{R}$. Assume $k > 0$. Then we have $(\partial_k, m) \in \mathscr{M}$ and $(\sigma_k, z) = (\partial_k, m)^*$ such that $(\sigma_k, u) = (1, r) \circ (\sigma_k, z)$. It follows from the definition of $\mathscr{R}$ that (∂_k, m) is invertible, a contradiction. So $k = 0$, and thus $\sigma = 1$ and $\xi = \sigma_0$. Consequently we have $u_i \in \mathscr{S}$ for all i. So $(\xi, u) = (\sigma_0, r_i) \circ (1, m_i^*)$ and the definition of $\mathscr{R}$ yields the m_i invertible. So $u_i \in \mathscr{R}$ for all i, and (ξ, u) has the desired form. □

Proposition 11.7. *Assume* $(\mathscr{S}, \mathscr{M})$-*factorization in* $\mathscr{P}$ *and* $(\mathscr{M}^*, \mathscr{R})$-*factorization in* $\mathscr{S}$ *are* $\mathscr{M} \circ \mathscr{M}^*$-*functorial. The morphisms in the the class* $\mathscr{M} \circ \mathscr{M}^*$ *for* $\mathscr{Q}$ *consist of those of the form* $(\xi, u) : (a, A) \to (b, B)$ *such that* ξ *reflects 0 and* $u_i \in \mathscr{M} \circ \mathscr{M}^*$ *for all* $i \in a$.

Proof. Take $(\xi, u) = (\nu, n) \circ (\sigma, z) \in \mathscr{M} \circ \mathscr{M}^*$ where $(\sigma, z) = (\partial, m)^*$ as usual. Then $\xi = \nu \circ \sigma$ reflects 0. Also $u_i = n_{\sigma i} \circ z_i = n_{\sigma i} \circ m_{\sigma i}^* \circ A(i \leq \partial \sigma i)$ is a composite of morphisms in $\mathscr{M} \circ \mathscr{M}^*$ for $\mathscr{P}$ and so is in $\mathscr{M} \circ \mathscr{M}^*$ by Assumption 2.4 for $\mathscr{P}$.

For the converse, we use the characterizations of $\mathscr{S}$ and $\mathscr{R}$ in Propositions 11.5 and 11.6. Suppose (ξ, u) has the form given in the Proposition. Factorize as $(\xi, u) = (\nu, n) \circ (\rho, r) \circ (\sigma, z)$ with $(\nu, n) \in \mathscr{M}$, $(\rho, r) \in \mathscr{R}$, and $(\sigma, z) \in \mathscr{M}^*$ as usual. Then $\xi = \nu \circ \rho \circ \sigma$ reflects 0 implies $\rho = 1$ by the uniqueness of factorization in $\Delta_{\perp \neq \top}$. However, this means we have $u_{\partial j} = n_{\nu j} \circ r_j \circ m_j^*$ which implies $r_j = 1$ by uniqueness of factorization in $\mathscr{P}$. So $(\xi, u) = (\nu, n) \circ (\sigma, z) \in \mathscr{M} \circ \mathscr{M}^*$. □

Theorem 11.8. *Assume* $(\mathscr{S}, \mathscr{M})$-*factorization in* $\mathscr{P}$ *and* $(\mathscr{M}^*, \mathscr{R})$-*factorization in* $\mathscr{S}$ *are* $\mathscr{M} \circ \mathscr{M}^*$-*functorial. Then Assumptions 2.1 to 2.6 hold for* $\mathscr{Q}$.

Proof. Assumptions 2.1 and 2.5 were dealt with in Proposition 11.2. Assumption 2.2 is straightforward. Assumption 2.4 is immediate given Proposition 11.7. So turn to Assumption 2.3. By Assumption 2.3 for $\Delta_{\perp \neq \top}$ and $\mathscr{P}$, we only need consider the case of a composite

$$(b+1, A) \xrightarrow{(\sigma_0, u)} (b, B) \xrightarrow{(1, z)} (b, C)$$

in $\mathscr{R}$, where $u_i \in \mathscr{R}$ for $0 < i \leq b$, $(1, m) \in \mathscr{M}$ and $(1, z) = (1, m)^*$. Then $z_{\sigma_0 i} \circ u_i$ is in $\mathscr{R}$ for $0 < i \leq b$. However, $z_j = m_j^*$. So, by Assumption 2.3 for $\mathscr{P}$ and the fact that σ_0 is surjective, each m_j is invertible. So $(1, m)$ is invertible as required.

For Assumption 2.6, let (a, A) be an object of $\mathscr{Q}$. Subobjects of (a, A) are isomorphism classes of morphisms $(\partial, m) : (x, X) \to (a, A)$ where ∂ is injective and each $m_i : X_i \to A_{\partial i}$ represents a subobject of an A_j in $\mathscr{P}$. The maximal subobjects are represented by those (∂, m) of two types. The first type have $\partial = \partial_k$ for some $0 < k < a-1$ and all m_i an identity. The second type have ∂ an identity and have m_i an identity for all but one component $i = i_0$ for which $m_{i_0} : X_0 \to A_{i_0}$ represents a maximal subobject of A_{i_0}. Any endomorphism of the form $(1, u) : (a, A) \to (a, A)$ commutes with any of the form $(\xi, v) : (a, A) \to (a, A)$ where $1 \leq \xi$ and $v(i) = A(i \leq \xi i)$ by naturality of u. So the idempotents (as needed in Assumption 2.6) obtained from the maximal subobjects of the first type commute with those of the second type. Clearly the idempotents obtained from maximal subobjects of the second type for different i_0 also commute. So we can take any listing of our idempotents in $\mathscr{Q}$ which keeps the order of those of the first type consistent with the order used in $\Delta_{\perp \neq \top}$, and, those of the second type, consistent for each i_0 with the order used in $\mathscr{P}$. □

12. WHEN $\mathscr{X}$ IS SEMIABELIAN

Semiabelian categories include the category Grp of (not necessarily abelian) groups and group morphisms. In [6] Dominique Bourn gave a version of the Dold-Puppe-Kan Theorem (Example 10.2) for the case where the codomain category $\mathscr{X}$ was semiabelian. In that case it asserted monadicity of the right adjoint in Theorem 6.1. In this section, we provide a version of this for $\mathscr{P}$ as in Section 2 in place of Δ^{op}.

Throughout we assume our category $\mathscr{X}$ has zero morphisms (that is, has homs enriched in pointed sets).

We begin by providing a non-additive version of the material at the end of Section 8 on idempotents.

Proposition 12.1. *Suppose the category $\mathscr{X}$ has kernels of idempotents. Let e, f be idempotents on an object A of $\mathscr{X}$. If $e \circ f \circ e = e \circ f$ then the intersection of the kernels of e and f exists.*

Proof. Let $k : K \to A$ be the kernel of e. Since $e \circ f \circ k = e \circ f \circ e \circ f = 0$, there exists a unique g with $f \circ k = k \circ g$. Then $k \circ g \circ g = f \circ k \circ g = f \circ f \circ k = f \circ k = k \circ g$ and k is a monomorphism. So g is idempotent. Then the kernel $\ell : L \to K$ of g is easily verified to be the intersection of the kernels of e and f. □

Protomodular categories were defined by Bourn [5]: a category $\mathscr{X}$ (with zero morphisms) is *protomodular* when it is finitely complete and, for each object A, the functor ker : $\mathrm{Pt}A \to \mathscr{X}$ is conservative. Here $\mathrm{Pt}A$ is the category whose objects (p, X, s) consist of morphisms $p : X \to A, s : A \to X$ with $p \circ s = 1_A$, and whose morphisms $f : (p, X, s) \to (q, Y, t)$ are morphisms $f : X \to Y$ such that $q \circ f = p$ and $f \circ s = t$. Also the functor ker takes (p, X, s) to the kernel of p.

The following property is sometimes [4] taken as the definition of protomodular.

Lemma 12.2. *In a protomodular category, if (p, X, s) is an object of $\mathrm{Pt}A$ and $k : K \to X$ is the kernel of p then $s : A \to X, k : K \to X$ are jointly strongly epimorphic.*

Proof. Suppose $m : Y \to X$ is a monomorphism and $m \circ u = s, m \circ v = k$ for some u, v. Then $m : (p \circ m, Y, u) \to (p, X, s)$ is a morphism of PtA. Using v, we see that m induces an isomorphism between the kernel of $p \circ m$ and K. Since $\ker : \mathrm{Pt}A \to \mathcal{X}$ is conservative, $m : (p \circ m, Y, u) \to (p, X, s)$ is invertible. So m is invertible. $\qquad\square$

Proposition 12.3. *Let $a_1, \ldots, a_n$ be a list of idempotents on an object A of a protomodular category $\mathcal{X}$. Suppose $a_i \circ a_j \circ a_i = a_i \circ a_j$ for $i < j$. Suppose $a_i = m_i \circ m_i^*$ is a splitting of a_i via a subobject $m_i : A_i \to A$ and retraction m_i^*. Let $k_i : K_i \to A$ be the kernel of m_i^* (or equally of a_i). Then the morphisms $m_1, \ldots, m_n$ along with the inclusion $\cap_i \ker m_i^* \to A$ are jointly strongly epimorphic.*

Proof. By Lemma 12.2, for each i, the morphisms $m_i : A_i \to A$ and $k_i : K_i \to A$ are jointly strongly epimorphic; we will loosely say "A_i and K_i cover A".

If $i > 1$ then $a_1 \circ a_i \circ k_1 = a_1 \circ a_i \circ a_1 \circ k_1 = 0$, and so $a_i \circ k_1$ lands in K_1, providing a factorization $a_i \circ k_1 = k_1 \circ a_i^1$. Now a_i^1 is also an idempotent, and, for $1 < i < j$, $k_1 \circ a_i^1 \circ a_j^1 \circ a_i^1 = a_i \circ a_j \circ a_i \circ k_1 = a_i \circ a_j \circ k_1 = k_1 \circ a_i^1 \circ a_j^1$, and so $a_i^1 \circ a_j^1 \circ a_i^1 = a_i^1 \circ a_j^1$. Clearly the splitting A_i^1 of a_i^1 is contained in A_i.

The kernel K_i^1 of a_i^1 is $K_i^1 = K_1 \cap K_i$ since $a_1 \circ x = a_i \circ x = 0$ is equivalent to $x = k_1 \circ y$ and $k_1 \circ a_i^1 \circ y = a_i \circ k_1 \circ y = s_i \circ x = 0$; so in fact $a_i^1 \circ y = 0$.

We know that A may be covered by A_1 and K_1. By Lemma 12.2 again, we know, for each $i > 1$, that K_1 may be covered by the splitting of A_i^1 and the kernel $K_i^1 = K_1 \cap K_i$ of a_i^1. Since $A_i^1 \leq A_i$, we see that A may be covered by A_1, A_i, and $K_1 \cap K_i$.

Now continue inductively. $\qquad\square$

Theorem 12.4. *Suppose $\mathscr{P}$ is as in Section 2 and $\mathcal{X}$ is protomodular (with zero morphisms) with finite coproducts. Then the components of the adjunction of Theorem 6.1 are strong epimorphisms.*

Proof. The counit has components

$$\sum_{B \preceq_m A} \bigcap_{C \prec_n B} \ker T n^* \longrightarrow TA \ .$$

We prove this is a strong epimorphism by induction on the number k of maximal proper subobjects $A_1, \ldots, A_k$ of A, with $m_i : A_i \to A$. The result is clear for $k = 1$. For $k > 1$, consider the following diagram.

$$
\begin{array}{ccc}
\sum_i \sum_{B \preceq_m A} \bigcap_{C \prec_n B} \ker T n^* & \longrightarrow & \sum_i T A_i \\
\Big\downarrow {\scriptstyle \delta} & & \Big\downarrow {\scriptstyle \gamma} \\
\sum_{B \prec_m A} \bigcap_{C \prec_n B} \ker T n^* \xrightarrow{\ \alpha\ } & FA & \xleftarrow{\ \beta\ } \bigcap_{C \prec_n A} \ker T n^*
\end{array}
$$

The component of the counit is strongly epimorphic if and only if α and β are jointly strongly epimorphic. The top row is a coproduct of components of the counit already known to be strongly epimorphic by induction. So it suffices to show that γ and β are jointly strongly epimorphic. Rewriting the domain of β as

$$\bigcap_{C \prec_n A} \ker T n^* = \bigcap_{i=1}^{k} \ker T m_i^* \ ,$$

we see that Proposition 12.3 applies to yield what we want. $\qquad\square$

A category $\mathcal{X}$ is *semiabelian* [16] when it has zero morphisms, is protomodular, is Barr exact, and has finite coproducts. A category is *regular* [1] when it is finitely complete, and has the (strong epimorphism, monomorphism)-factorization system existing and stable under

pullbacks. It follows that every strong epimorphism is regular (that is, a coequalizer); see [8] for a proof. A category is *Barr exact* when it is regular and every equivalence relation is a kernel pair.

We mentioned Bourn's category $\text{Pt}\mathscr{X}$ in Example 10.1. We will use the following routine fact.

Lemma 12.5. *If $\mathscr{X}$ is a semiabelian category then the functor $\text{Pt}\mathscr{X} \to \mathscr{X}$, sending each split epimorphism to its kernel, preserves strong epimorphisms.*

Proof. A strong epimorphism

$$
\begin{array}{ccc}
X & \xrightarrow{\ \ g\ \ } & Y \\
s\uparrow\downarrow p & & t\uparrow\downarrow q \\
A & \xrightarrow{\ \ f\ \ } & B
\end{array}
$$

in $\text{Pt}\mathscr{X}$ has f and g strong epimorphisms in $\mathscr{X}$. From this it is easily verified that the square involving the downward-pointing arrows is a pushout. Factor the morphism in $\text{Pt}\mathscr{X}$ as

$$
\begin{array}{ccccc}
X & \xrightarrow{\ \ g_1\ \ } & Z & \xrightarrow{\ \ f_1\ \ } & Y \\
s\uparrow\downarrow p & & t_1\uparrow\downarrow q_1 & & t\uparrow\downarrow q \\
A & \xrightarrow{\ 1_A\ } & A & \xrightarrow{\ \ f\ \ } & B
\end{array}
$$

in which the right-hand square involving the downward-pointing arrows is a pullback. The induced morphism $\ker q_1 \to \ker q$ is invertible. Semiabelian categories are Maltsev [16]. Therefore, as a comparison morphism to the pullback in a pushout square in a Maltsev exact category, g_1 is a strong epimorphism (see Theorem 5.7 of [7]). The induced morphism $\ker p \to \ker q_1$ is the pullback of g_1 along $\ker q_1 \to Z$ and so is a strong epimorphism by regularity. $\qquad\square$

Theorem 12.6. *If $\mathscr{P}$ is as in Section 2 and $\mathscr{X}$ is semiabelian then the adjunction of Theorem 6.1 is (crudely) monadic.*

Proof. By Theorem 12.4, the right adjoint (tilde) is conservative (since this is logically equivalent to the counit being a strong epimorphism). Since $\mathscr{X}$ is semiabelian, it has coequalizers. Therefore $[\mathscr{P}, \mathscr{X}]$ has coequalizers, so, for crude monadicity [25], it suffices to show that tilde preserves coequalizers of reflexive pairs.

Both $[\mathscr{P}, \mathscr{X}]$ and $[\mathscr{D}, \mathscr{X}]_{\text{pt}}$ are semiabelian. A limit-preserving functor between semiabelian categories preserves coequalizers of reflexive pairs provided it preserves strong (= regular) epimorphisms (see Lemma 5.1.12 of [3]).

Let $q : S \to T$ be a strong epimorphism in $[\mathscr{P}, \mathscr{X}]$. Each $q_A : SA \to TA$ is a strong epimorphism. We must show each $\widetilde{q}_A : \widetilde{S}A \to \widetilde{T}A$ is a strong epimorphism.

Recall that $\widetilde{T}A$ is calculated by a sequence of kernels of split epimorphisms. This sequence depends only on the object A and the category $\mathscr{P}$, not on the particular functor T. The desired result follows on repeated application of Lemma 12.5. $\qquad\square$

13. Appendix: Partial maps using two comonads

In this Appendix we provide a more categorical proof of the equivalence (1.3) in the situation of Example 3.1. Let $\mathscr{A}$, $\mathscr{E}$, $\mathscr{M}$ and $\mathscr{P}$ be as in Example 3.1.

Let $J : \mathscr{E} \longrightarrow \mathscr{A}$ be the inclusion functor and let $I = (-)_* : \mathscr{A} \longrightarrow \mathscr{P}$. Both functors are the identity on objects.

Let $\mathscr{X}$ be any category admitting coproducts and products indexed by sets of $\mathscr{M}$-subobjects of any given object of $\mathscr{A}$.

Lemma 13.1. *Each functor $F : \mathscr{E} \longrightarrow \mathscr{X}$ has a pointwise left Kan extension*

$$
\begin{array}{ccc}
\mathscr{E} & \xrightarrow{\quad J \quad} & \mathscr{A} \\
& \overset{\kappa}{\Longrightarrow} & \\
{}_{F}\searrow & & \swarrow{}_{\mathrm{Lan}_J F} \\
& \mathscr{X} &
\end{array}
\tag{13.17}
$$

along $J : \mathscr{E} \longrightarrow \mathscr{A}$ defined on objects by:

$$
(\mathrm{Lan}_J F)X = \sum_{U \leq X} FU \ .
$$

For morphisms $f : X \longrightarrow Y$ in $\mathscr{A}$, the following square commutes.

$$
\begin{array}{ccc}
FU & \xrightarrow{\quad \mathrm{in}_U \quad} & (\mathrm{Lan}_J F)X \\
{}_{Fe}\downarrow & & \downarrow{}_{(\mathrm{Lan}_J F)f} \\
FfU & \xrightarrow{\quad \mathrm{in}_{fU} \quad} & (\mathrm{Lan}_J F)Y
\end{array}
\tag{13.18}
$$

Proof. The inclusion $U \leq X \mapsto (U, i_U : JU \longrightarrow X)$ of the discrete category on the set $\{U : U \leq X\}$ into the comma category J/X has a left adjoint, taking $(V, f : JV \longrightarrow X)$ to $fV \leq X$, and so is final. It follows that the colimit of $J/X \xrightarrow{\mathrm{dom}} \mathscr{E} \xrightarrow{F} \mathscr{X}$ can be calculated by restricting along the inclusion and so is the coproduct displayed. $\qquad\square$

Lemma 13.2. *Each functor $T : \mathscr{A} \longrightarrow \mathscr{X}$ has a pointwise right Kan extension*

$$
\begin{array}{ccc}
\mathscr{A} & \xrightarrow{\quad I \quad} & \mathscr{P} \\
& \overset{\rho}{\Longleftarrow} & \\
{}_{T}\searrow & & \swarrow{}_{\mathrm{Ran}_I T} \\
& \mathscr{X} &
\end{array}
\tag{13.19}
$$

along $I : \mathscr{A} \longrightarrow \mathscr{P}$ defined on objects by:

$$
(\mathrm{Ran}_I T)X = \prod_{U \leq X} TU \ .
$$

For morphisms $(W, h) : X \longrightarrow Y$ in $\mathscr{P}$, the square

$$
\begin{array}{ccc}
(\mathrm{Ran}_I T)X & \xrightarrow{\quad \mathrm{pr}_{h^{-1}V} \quad} & Th^{-1}V \\
{}_{(\mathrm{Ran}_I T)(W,h)}\downarrow & & \downarrow{}_{T\bar{h}} \\
(\mathrm{Ran}_I T)Y & \xrightarrow{\quad \mathrm{pr}_V \quad} & TV
\end{array}
\tag{13.20}
$$

commutes, where $\bar{h}$ is the pullback of h along i_V.

Proof. The functor $U \leq X \mapsto (i_U^* : X \longrightarrow IU, U)$ from the discrete category on the set $\{U : U \leq X\}$ into the comma category X/I has a right adjoint, taking $((W, h) : X \longrightarrow IA, A)$ to $W \leq X$, and so is initial. It follows that the limit of $X/I \xrightarrow{\mathrm{cod}} \mathscr{A} \xrightarrow{T} \mathscr{X}$ can be calculated as the product displayed. $\qquad\square$

This gives the two adjunctions

$$[\mathscr{E},\mathscr{X}] \xrightarrow[\underset{[J,1]}{\longleftarrow}]{\overset{\mathrm{Lan}_J}{\longrightarrow}} [\mathscr{A},\mathscr{X}] \xrightarrow[\underset{[I,1]}{\longleftarrow}]{\overset{\mathrm{Ran}_I}{\longrightarrow}} [\mathscr{P},\mathscr{X}] . \tag{13.21}$$

Each adjunction generates a comonad on $[\mathscr{A},\mathscr{X}]$:

$$G = \mathrm{Lan}_J \circ [J,1] \quad \text{and} \quad H = [I,1] \circ \mathrm{Ran}_I . \tag{13.22}$$

To be more explicit, the functors GF and HF are defined on objects by

$$(GF)A = \sum_{U \leq A} FU \quad \text{and} \quad (HF)A = \prod_{U \leq A} FU . $$

On morphisms $f : A \longrightarrow B$ in $\mathscr{A}$, they are defined by commutativity in the squares

$$
\begin{array}{ccc}
FU \xrightarrow{\ Fe\ } FfU & \qquad & (HF)A \xrightarrow{\ (HF)f\ } (HF)B \\
{\scriptstyle \mathrm{in}_U}\downarrow \qquad \downarrow{\scriptstyle \mathrm{in}_{fU}} & & {\scriptstyle \mathrm{pr}_{f^{-1}V}}\downarrow \qquad \downarrow{\scriptstyle \mathrm{pr}_V} \\
(GF)A \xrightarrow[(GF)f]{} (GF)B & & Ff^{-1}V \xrightarrow[F\bar{f}]{} FV
\end{array}
\tag{13.23}
$$

The counits $\varepsilon_F : GF \longrightarrow F$ and $\varepsilon_F : HF \longrightarrow F$ have components respectively defined by $\varepsilon_F A \circ \mathrm{in}_U = F\mathrm{in}_U$ and $\varepsilon_F A = \mathrm{pr}_A$. The comultiplications $\delta_F : GF \longrightarrow G^2 F$ and $\delta_F : HF \longrightarrow H^2 F$ have components respectively defined as follows.

$$
\begin{array}{ccc}
FU \xrightarrow{\ \mathrm{in}_U\ } \sum_{U \leq A} FU & \qquad & \prod_{U \leq A} FU \xrightarrow{\ \mathrm{pr}_V\ } FV \\
{\scriptstyle \mathrm{in}_{U \leq U}}\searrow \quad \swarrow{\scriptstyle \delta_F A} & & {\scriptstyle \delta_F A}\searrow \quad \nearrow{\scriptstyle \mathrm{pr}_{V \leq U}} \\
\sum_{V \leq U \leq A} FV & & \prod_{V \leq U \leq A} FV
\end{array}
$$

We shall construct a comonad morphism $\Theta : G \Longrightarrow H$ when $\mathscr{X}$ is a pointed category. Take $F : \mathscr{A} \longrightarrow \mathscr{X}$ and $A \in \mathscr{A}$. We define

$$\Theta_F A : \sum_{U \leq A} FU \longrightarrow \prod_{V \leq A} FV \tag{13.24}$$

by taking the composite

$$FU \xrightarrow{\ \mathrm{in}_U\ } \sum_{U \leq A} FU \xrightarrow{\ \Theta_F A\ } \prod_{V \leq A} FV \xrightarrow{\ \mathrm{pr}_V\ } FV$$

to be zero unless $U \leq V$, in which case the composite is $F(U \leq V) : FU \longrightarrow FV$.

Proposition 13.3. *For the two comonads G and H of* (13.22) *on* $[\mathscr{A},\mathscr{X}]$, *a comonad morphism* $\Theta : G \Longrightarrow H$ *is defined by the components* (13.24).

Proof. Naturality in A is proved by contemplating the following diagram.

$$
\begin{array}{ccccccc}
FU & \xrightarrow{\ \mathrm{in}_U\ } & \sum_{U \leq A} FU & \xrightarrow{\ \Theta_F A\ } & \prod_{U_1 \leq A} FU_1 & \xrightarrow{\ \mathrm{pr}_{f^{-1}V}\ } & Ff^{-1}V \\
{\scriptstyle Fe}\downarrow & & \downarrow{\scriptstyle (GF)f} & & \downarrow{\scriptstyle (HF)f} & & \downarrow{\scriptstyle F\bar{f}} \\
FfU & \xrightarrow[\mathrm{in}_{fU}]{} & \sum_{V_1 \leq B} FV_1 & \xrightarrow[\Theta_F B]{} & \prod_{V \leq B} FV & \xrightarrow[\mathrm{pr}_V]{} & FV
\end{array}
$$

The top composite is zero unless $U \leq f^{-1}V$ (say with inclusion i). The bottom composite is zero unless $fU \leq V$ (say with inclusion j). These conditions are the same. When they hold, the diagram commutes since $\bar{f} \circ i = j \circ e$.

Naturality in F is obvious.

Preservation of the counits is proved by the calculation

$$\varepsilon_F A \circ \Theta_F A \circ \text{in}_U = \text{pr}_A \circ \Theta_F A \circ \text{in}_U = F(U \leq A) = \varepsilon_F A \circ \text{in}_U \ .$$

Preservation of the comultiplications is proved by observing that

$$\text{pr}_{V \leq U} \circ \delta_F A \circ \Theta_F A \circ \text{in}_W = \text{pr}_V \circ \Theta_F A \circ \text{in}_W$$

while

$$\text{pr}_{V \leq U} \circ \Theta_F^2 A \delta_F A \circ \text{in}_W = \text{pr}_{V \leq U} \circ \Theta_F^2 A \circ \text{in}_{W \leq W} \ .$$

Both of these right-hand sides are zero unless $W \leq V$, in which case they are both equal to $F(W \leq V)$. $\qquad \square$

Proposition 13.4. *The functor* $[I, 1] : [\mathscr{P}, \mathscr{X}] \longrightarrow [\mathscr{A}, \mathscr{X}]$ *is comonadic.*

Proof. The functor $[I, 1]$ is conservative since I is bijective on objects. It also preserves all limits, including all equalizers. The result follows (for example) by the Beck comonadicity theorem [24]. $\qquad \square$

It follows that $\Theta : G \Longrightarrow H$ induces a functor $\bar{\Theta} : [\mathscr{A}, \mathscr{X}]^G \longrightarrow [\mathscr{P}, \mathscr{X}]$ over $[\mathscr{A}, \mathscr{X}]$, where $[\mathscr{A}, \mathscr{X}]^G$ is the category of Eilenberg-Moore G-coalgebras. The composite of $\bar{\Theta}$ with the comparison functor $[\mathscr{E}, \mathscr{X}] \longrightarrow [\mathscr{A}, \mathscr{X}]^G$ is isomorphic to

$$\widehat{(-)} : [\mathscr{E}, \mathscr{X}] \longrightarrow [\mathscr{P}, \mathscr{X}] \tag{13.25}$$

as defined in (5.9).

With a finite well-poweredness assumption on the factorization system of $\mathscr{A}$, we shall show that the adjunction of Theorem 6.1 is an equivalence. In other words, we shall show that the free additive categories on $\mathscr{E}$ and $\mathscr{P}$ are Morita equivalent (= additively Cauchy equivalent).

Our goal now is to prove:

Theorem 13.5. *Let* $(\mathscr{E}, \mathscr{M})$ *be a factorisation system on a category* $\mathscr{A}$*. Assume that* $\mathscr{M}$ *is contained in the class of monomorphisms and that all pairs* $(m : U \to X, f : A \to X)$ *of morphisms with* $m \in \mathscr{M}$ *have a pullback in* $\mathscr{A}$*. Assume that the* $\mathscr{M}$*-subobjects of each object form a finite set. Regard* $\mathscr{E}$ *as a subcategory of* $\mathscr{A}$ *with the same objects and put* $\mathscr{P} = \text{Par}\mathscr{A}$*. Let* $\mathscr{X}$ *be any finitely complete additive category. Then the functor* (13.25) *is an equivalence of categories*

$$[\mathscr{E}, \mathscr{X}] \simeq [\mathscr{P}, \mathscr{X}] \ .$$

We first prove that the comonads G and H of (13.22) are isomorphic.

Proposition 13.6. *Under the assumptions of Theorem 13.5, the comonad morphism* $\Theta : G \Longrightarrow H$*, with components* (13.24)*, is invertible.*

Proof. Since $\mathscr{X}$ is additive, every finite product is a coproduct. The ordered set of $\mathscr{M}$-subobjects of $A \in \mathscr{A}$ is assumed finite and so has a linear refinement. So we can list all the subobjects as $0 = U_0, U_1, \ldots, U_n = A$ such that $U_i \leq U_j$ implies $i \leq j$. Then the morphisms (13.24) are represented by an upper-triangular matrix with identity morphisms in the main diagonal. Since $\mathscr{X}$ is additive (including existence of additive inverses), such matrices are invertible. $\qquad \square$

Proof of Theorem 13.5. By Proposition 13.6, the functor (13.25) becomes the comparison from $[\mathscr{E}, \mathscr{X}]$ into the category of G-coalgebras. So the Theorem is now equivalent to comonadicity of the functor $\text{Lan}_J : [\mathscr{E}, \mathscr{X}] \longrightarrow [\mathscr{A}, \mathscr{X}]$. Since the unit of the adjunction $\text{Lan}_J \dashv [J, 1]$ is a pointwise coretraction $FA \longrightarrow FA \oplus \bigoplus_{U < A} FU$, and hence a strong monomorphism, the functor Lan_J is conservative. So it remains to prove that Lan_J preserves

certain equalizers. In fact, it preserves all finite limits. Since limits in $[\mathscr{A}, \mathscr{X}]$ are formed pointwise, it suffices to see that each

$$\mathrm{ev}_A \circ \mathrm{Lan}_J : [\mathscr{E}, \mathscr{X}] \longrightarrow \mathscr{X}$$

preserves finite limits where $\mathrm{ev}_A : [\mathscr{A}, \mathscr{X}] \longrightarrow \mathscr{X}$ is evaluation at $A \in \mathscr{A}$. By Lemma 13.1,

$$\mathrm{ev}_A \circ \mathrm{Lan}_J \cong \bigoplus_{U \leq A} \mathrm{ev}_U \ .$$

Each evaluation ev_U preserves limits so their direct sum does. $\qquad \square$

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Department of Mathematics, Macquarie University, NSW 2109, Australia
E-mail address: steve.lack@mq.edu.au

Department of Mathematics, Macquarie University, NSW 2109, Australia
E-mail address: ross.street@mq.edu.au