Categories in categories, and size matters

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Abstract

We look again at internal categories and internal full subcategories in a category $\mathscr{C}$. We look at the relationship between a generalised Yoneda lemma and the descent construction. Application to $\mathscr{C} = \mathrm{Cat}$ gives results on double categories.

Introduction

We revisit the theory of categories in a category. We look in particular at categories in the category of categories; that is, at double categories.

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1 Categories in categories

A *category C in* a category $\mathscr{C}$ is a diagram

$$\begin{tikzcd}[column sep=large]
C: & C_3 \arrow[r, shift left=4.5, "d_0"] \arrow[r, shift left=1.5, "d_1"] \arrow[r, shift right=1.5, "d_2"] \arrow[r, shift right=4.5, "d_3"] & C_2 \arrow[r, shift left=6, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=6, "d_2"] & C_1 \arrow[l, shift right=3, "i_0"'] \arrow[l, shift left=3, "i_1"] \arrow[r, shift left=3, "d_0"] \arrow[r, shift right=3, "d_1"'] & C_0 \arrow[l, "i"]
\end{tikzcd} \tag{1.1}$$

in $\mathscr{C}$ such that

C1. $d_pd_q = d_{q-1}d_p$ for $p < q$

C2. $d_0i = d_1i = 1_{C_0}, d_0i_0 = d_1i_0 = d_1i_1 = d_2i_1 = 1_{C_1}$

C3. $i_0i = i_1i, d_2i_0 = id_1, d_0i_1 = id_0$

C4. the following two squares are pullbacks.

$$\begin{tikzcd}
C_2 \arrow[r, "d_0"] \arrow[d, "d_2"'] & C_1 \arrow[d, "d_1"] & C_3 \arrow[r, "d_0"] \arrow[d, "d_3"'] & C_2 \arrow[d, "d_2"] \\
C_1 \arrow[r, "d_0"'] & C_0 & C_2 \arrow[r, "d_0"'] & C_1
\end{tikzcd} \tag{1.2}$$

The definition goes back to Ehresmann [11].

For any object U of $\mathscr{C}$ and any category C in $\mathscr{C}$, we obtain a category $\mathscr{C}(U, C)$ as follows. An object is a morphism $u : U \longrightarrow C_0$ in $\mathscr{C}$. A morphism $\gamma : u \longrightarrow v$ is a morphism $\gamma : U \longrightarrow C_1$ in $\mathscr{C}$ such that $d_0\gamma = u$ and $d_1\gamma = v$. The identity $1_u : u \longrightarrow u$ is $iu : U \longrightarrow C_1$. The composite of

$\gamma : u \longrightarrow v$ and $\delta : v \longrightarrow w$ is $d_1(\gamma, \delta) : U \longrightarrow C_1$ where $(\gamma, \delta) : U \longrightarrow C_2$ is defined by $d_0(\gamma, \delta) = \gamma$ and $d_2(\gamma, \delta) = \delta$.

For each morphism $h : V \longrightarrow U$ in $\mathscr{C}$, we have a functor

$$\mathscr{C}(h, C) : \mathscr{C}(U, C) \longrightarrow \mathscr{C}(V, C)$$

taking u to uh and γ to γh. Thus we obtain a functor

$$\mathscr{C}(-, C) : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathrm{Cat}\ .$$

A *functor* $f : C \longrightarrow D$ between categories C and D in $\mathscr{C}$ is a morphism of diagrams

$$\begin{tikzcd}
C_2 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_2"'] \arrow[d, "f_2"'] & C_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[d, "f_1"'] & C_0 \arrow[l, "i" description] \arrow[d, "f_0"'] \\
D_2 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_2"'] & D_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & D_0 \arrow[l, "i" description]
\end{tikzcd}$$

Write $\mathrm{Cat}\mathscr{C}$ for the category of categories in $\mathscr{C}$ and functors between them.

Each functor $f : C \longrightarrow D$ defines a functor

$$\mathscr{C}(U, f) : \mathscr{C}(U, C) \longrightarrow \mathscr{C}(U, D)$$

taking u to $f_0 u$ and γ to $f_1 \gamma$. Thus we obtain a functor

$$\mathscr{C}(-, -) : \mathscr{C}^{\mathrm{op}} \times \mathrm{Cat}\mathscr{C} \longrightarrow \mathrm{Cat}$$

which can be recast as a functor

$$\mathrm{Yon} : \mathrm{Cat}\mathscr{C} \longrightarrow [\mathscr{C}^{\mathrm{op}}, \mathrm{Cat}] \tag{1.3}$$

extending the usual Yoneda embedding $\mathscr{C} \longrightarrow [\mathscr{C}^{\mathrm{op}}, \mathrm{Set}]$.

A *natural transformation* $\theta : f \Longrightarrow g : C \longrightarrow D$ between functors f and g is a morphism

$$\theta : C_0 \longrightarrow D_1$$

in $\mathscr{C}$ such that

N1. $d_0 \theta = f_0$ and $d_1 \theta = g_0$

N2. the following diagram commutes.

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "{(f_1,\theta d_1)}"] \arrow[r, shift right, "{(\theta d_0,g_1)}"'] & D_2 \arrow[r, "d_1"] & D_1
\end{tikzcd} \tag{1.4}$$

Given a diagram

$$\begin{tikzcd}
B \arrow[r, "r"] & C \arrow[r, bend left, "f", ""{name=U, below}] \arrow[r, bend right, "g"', ""{name=D, above}] & D \arrow[r, "s"] & E
\arrow[Rightarrow, from=U, to=D, "\theta"]
\end{tikzcd} \tag{1.5}$$

of functors r, f, g, s and natural transformation θ, we obtain a 'whiskered' natural transformation $s\theta r : sfr \Longrightarrow sgr$ defined as the composite

$$\begin{tikzcd}
B_0 \arrow[r, "r_0"] & C_0 \arrow[r, "\theta"] & D_1 \arrow[r, "s_1"] & E_1
\end{tikzcd} \quad . \tag{1.6}$$

Given a diagram

$$\begin{tikzcd}
C \arrow[r, bend left=50, "f", ""{name=F, below}] \arrow[r, "g" description, ""{name=G1, above}, ""{name=G2, below}] \arrow[r, bend right=50, "h"', ""{name=H, above}] & D
\arrow[Rightarrow, from=F, to=G1, "\theta"]
\arrow[Rightarrow, from=G2, to=H, "\phi"]
\end{tikzcd} \tag{1.7}$$

of functors f, g, h and natural transformations θ, ϕ, we obtain a natural transformation $\phi \circ \theta : f \Longrightarrow h$ defined as the composite

$$\begin{tikzcd}
C_0 \arrow[r, "{(\theta,\phi)}"] & D_2 \arrow[r, "d_1"] & D_1
\end{tikzcd} \quad . \tag{1.8}$$

In this way, Cat$\mathscr{C}$ becomes a 2-category.

Each object U of $\mathscr{C}$ and natural transformation $\theta : f \Longrightarrow g : C \longrightarrow D$ define a natural transformation

$$\begin{tikzcd}
\mathscr{C}(U,C) \arrow[r, bend left, "{\mathscr{C}(U,f)}", ""{name=A, below}] \arrow[r, bend right, "{\mathscr{C}(U,g)}"', ""{name=B, above}] & \mathscr{C}(U,D)
\arrow[Rightarrow, from=A, to=B, "{\mathscr{C}(U,\theta)}"]
\end{tikzcd}$$

whose component at $u \in \mathscr{C}(U,C)$ is $\theta u : fu \longrightarrow gu$ in $\mathscr{C}(U,D)$. In this way, we see that the extended Yoneda embedding (1.3) is a 2-functor.

For each category C in $\mathscr{C}$, there is a category $\int C$ defined as follows. The objects are pairs (U,u) where U is an object of $\mathscr{C}$ and u is an object of $\mathscr{C}(U,C)$. A morphism $(h,\theta) : (U,u) \longrightarrow (V,v)$ consists of a morphism $h : U \longrightarrow V$ in $\mathscr{C}$ and a morphism $\theta : u \longrightarrow vh$

in $\mathscr{C}(U,C)$. Regarding each object U of $\mathscr{C}$ as a 'discrete category' in $\mathscr{C}$ (that is, all $U_n = U$ and $d_p = 1_U$), we can say that morphisms of $\int C$ are diagrams

$$\begin{tikzcd}
U \arrow[rr, "h"] \arrow[dr, "u"'] \arrow[rr, phantom, shift right=3, "\overset{\theta}{\Longrightarrow}"] & & V \arrow[dl, "v"] \\
& C &
\end{tikzcd} \tag{1.9}$$

in Cat$\mathscr{C}$; composition in $\int C$ is by pasting these triangles. There is a functor

$$P : \int C \longrightarrow \mathscr{C} \tag{1.10}$$

defined by $P(U,u) = U$ and $P(h,\theta) = h$.

2 Fibrations and the generalized Yoneda Lemma

Consider a functor $P : \mathscr{X} \longrightarrow \mathscr{C}$. A morphism $x : Y \longrightarrow X$ in $\mathscr{X}$ is *cartesian* (with respect to P) when the following square is a pullback for all objects K of $\mathscr{X}$.

$$\begin{tikzcd}
\mathscr{X}(K,Y) \arrow[r, "\mathscr{X}(K{,}x)"] \arrow[d, "P"'] & \mathscr{X}(K,X) \arrow[d, "P"] \\
\mathscr{X}(PK,PY) \arrow[r, "\mathscr{X}(PK{,}Px)"'] & \mathscr{X}(PK,PX)
\end{tikzcd} \tag{2.1}$$

Lemma 2.1 *Consider a commutative square*

$$\begin{tikzcd}
Y' \arrow[r, "x'"] \arrow[d, "k"'] & X' \arrow[d, "h"] \\
Y \arrow[r, "x"'] & X
\end{tikzcd}$$

in $\mathscr{X}$, which is taken to a pullback in $\mathscr{C}$ by the functor $P : \mathscr{X} \longrightarrow \mathscr{C}$, and where x is cartesian. The square is a pullback if and only if x' is cartesian.

Proof Contemplate the two diagrams below.

$$\begin{tikzcd}
\mathscr{X}(K,Y') \arrow[r, "\mathscr{X}(1{,}x')"] \arrow[d, "\mathscr{X}(1{,}k)"'] & \mathscr{X}(K,X') \arrow[d, "\mathscr{X}(1{,}h)"] \\
\mathscr{X}(K,Y) \arrow[r, "\mathscr{X}(1{,}x)"'] \arrow[d, "P"'] & \mathscr{X}(K,X) \arrow[d, "P"] \\
\mathscr{C}(PK,PY) \arrow[r, "\mathscr{C}(1{,}Px)"'] & \mathscr{C}(PK,PX)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{X}(K,Y') \arrow[r, "\mathscr{X}(1{,}x')"] \arrow[d, "P"'] & \mathscr{X}(K,X') \arrow[d, "P"] \\
\mathscr{C}(PK,PY') \arrow[r, "\mathscr{C}(1{,}Px')"'] \arrow[d, "\mathscr{C}(1{,}Pk)"'] & \mathscr{C}(PK,PX') \arrow[d, "\mathscr{C}(1{,}Ph)"] \\
\mathscr{C}(PK,PY) \arrow[r, "\mathscr{C}(1{,}Px)"'] & \mathscr{C}(PK,PX
\end{tikzcd}$$

The two squares obtained by composing both diagrams vertically are equal. By hypothesis, the bottom square in each diagram is a pullback. It follows that the top square in one diagram is a pullback if and only if the top square in the other is. □

For any functor $P : \mathscr{X} \longrightarrow \mathscr{C}$ and any category C in $\mathscr{C}$, we define a category $\mathscr{X}^C$. The objects are categories X in $\mathscr{X}$ such that $PX = C$ and $d_0 : X_1 \longrightarrow X_0$ is cartesian. A morphism $r : X \longrightarrow Y$ is a functor in $\mathscr{X}$ such that $Pr : PX \longrightarrow PY$ is the identity functor of C.

Remark 2.2 *By Lemma 2.1, the pullback requirement C4* (1.2) *on a category X in $\mathscr{X}$ belonging to $\mathscr{X}^C$ can be replaced by the requirement that*

$$d_0 : X_2 \longrightarrow X_1 \quad \textit{and} \quad d_0 : X_3 \longrightarrow X_2$$

should be cartesian.

For functors $P : \mathscr{X} \longrightarrow \mathscr{C}$ and $Q : \mathscr{Y} \longrightarrow \mathscr{C}$, we write

$$\mathrm{Cat}_{/\mathscr{C}}(\mathscr{X}, \mathscr{Y})$$

for the category of functors $F : \mathscr{X} \longrightarrow \mathscr{Y}$ with $QF = P$. The morphisms are natural transformations taken to identities by Q. We write

$$\mathrm{Cart}_{/\mathscr{C}}(\mathscr{X}, \mathscr{Y})$$

for the full subcategory of $\mathrm{Cat}_{/\mathscr{C}}(\mathscr{X}, \mathscr{Y})$ consisting of those functors F which preserve cartesian morphisms. Such an F defines a functor

$$F^C : \mathscr{X}^C \longrightarrow \mathscr{Y}^C$$

given by $F^C X = FX$; this uses Remark 2.2. We obtain a functor

$$(-)^C : \mathrm{Cart}_{/\mathscr{C}}(\mathscr{X}, \mathscr{Y}) \longrightarrow [\mathscr{X}^C, \mathscr{Y}^C] \tag{2.2}$$

for each category C in $\mathscr{C}$.

A functor $P : \mathscr{X} \longrightarrow \mathscr{C}$ is called a *fibration* (or "category fibred over $\mathscr{C}$" in the sense of Grothendieck [17], not [16]) when, for all $h : V \longrightarrow U$ in $\mathscr{C}$ and X in $\mathscr{X}$ with $PX = U$, there exists a cartesian morphism $x : Y \longrightarrow X$ with $Px = h$. Such an $x : Y \longrightarrow X$ is called an *inverse* *image* of X along h. A choice of inverse image for all such X and h is called a *cleavage* for P; we write

$$h^X : h^\star X \longrightarrow X$$

for the chosen cartesian morphisms.

Convention Given a cleavage, by $\mathscr{X}^C$ we will then mean the restriction of $\mathscr{X}^C$ to those objects X with all the morphisms $d_0 : X_{n+1} \longrightarrow X_n$ taken to be chosen cartesian.

Let us now return to the functor $P : \int C \longrightarrow \mathscr{C}$ of (1.10). A morphism $(h, \theta) : (U, u) \longrightarrow (V, v)$ is cartesian if and only if $\theta : u \longrightarrow vh$ is invertible in the category $\mathscr{C}(U, C)$. In fact, $P : \int C \longrightarrow \mathscr{C}$ is a fibration and there is a cleavage for which the chosen cartesian (h, θ) have θ an identity.

The following diagram defines a category $\hat{C}$ in $\int C$:

$$\begin{tikzcd}
(C_2, d_0 d_0) \arrow[r, shift left=2, "{(d_0, id_0 d_0)}"] \arrow[r, "{(d_1, id_0 d_0)}" description] \arrow[r, shift right=2, "{(d_2, d_0)}"'] & (C_1, d_0) \arrow[r, shift left=2, "{(d_0, id_0)}"] \arrow[r, shift right=2, "{(d_1, 1_{C_1})}"'] & (C_0, 1_{C_0}) \arrow[l, "{(i,i)}" description]
\end{tikzcd} \quad . \tag{2.3}$$

For each object (U, u) of $\int C$, we can identify the category

$$\int C((U, u), \hat{C}) = u / \mathscr{C}(U, C) \tag{2.4}$$

as the category of objects of $\mathscr{C}(U, C)$ under the object u.

Note moreover that $\hat{C}$ is actually an object of $(\int C)^C$. The generalized Yoneda Lemma of [39] Theorem 5.15 expresses the 'generic' property of $\hat{C} \in (\int C)^C$.

Theorem 2.3 *For any fibration* $P : \mathscr{X} \longrightarrow \mathscr{C}$ *and category* C *in* $\mathscr{C}$*, the composite functor*

$$\mathrm{Cart}_{/\mathscr{C}}(\int C, \mathscr{X}) \xrightarrow{(-)^C} [(\int C)^C, \mathscr{X}^C] \xrightarrow{\mathrm{eval}_{\hat{C}}} \mathscr{X}^C$$

is an equivalence of categories.

Proof Take $X \in \mathscr{X}^C$. We must define a cartesian-morphism-preserving functor

$$\bar{X} : \int C \longrightarrow \mathscr{X}$$

over $\mathscr{C}$. For $(U, u) \in \int C$, put

$$\bar{X}(U, u) = u^\star X_0 \quad .$$

For $(h, \theta) : (U, u) \longrightarrow (V, v)$ in $\int C$, notice that the composite of cartesian morphisms

$$\theta^\star X_1 \xrightarrow{\theta^{X_1}} X_1 \xrightarrow{d_0} X_0$$

is cartesian over $u : U \longrightarrow C_0$ (since $d_0\theta = u$), so we have an isomorphism $u^\star X_0 \cong \theta^\star X_1$ over U and commuting with u^{X_0} and $d_0\theta^{X_1}$. Define $\bar{X}(h,\theta) : \bar{X}(U,u) \longrightarrow \bar{X}(V,v)$ such that

$$\begin{tikzcd} u^\star X_0 \arrow[r, "\cong"] \arrow[d, "\bar{X}(h,\theta)"'] & \theta^\star X_1 \arrow[r, "\theta^{X_1}"] & X_1 \arrow[d, "d_1"] \\ V \arrow[rr, "v^{X_0}"'] & & X_0 \end{tikzcd}$$

commutes and $P\bar{X}(h,\theta) = h$. Please see [39] Theorem 5.15 for further details. $\square$

A *split* fibration is a fibration equipped with a cleavage satisfying the identities

$$1_U{}^\star X = X \quad , \quad 1_U{}^X = 1_X \quad ,$$

and

$$(uv)^\star X = v^\star u^\star X \quad , \quad (uv)^X = u^X v^{u^\star X} \quad .$$

Write $\mathrm{Spl}_{/\mathscr{C}}(\mathscr{X},\mathscr{Y})$ for the full subcategory of $\mathrm{Cart}_{/\mathscr{C}}(\mathscr{X},\mathscr{Y})$ consisting of those $F : \mathscr{X} \longrightarrow \mathscr{Y}$ which preserve the cleavages given in $\mathscr{X}$ and $\mathscr{Y}$. Recall our convention about $\mathscr{X}^C$ in the cloven case.

Notice that $P : \int C \longrightarrow \mathscr{C}$ is a split fibration. In fact, [39] Theorem 5.15 actually included the following result. Here we make implicit use of our convention.

Proposition 2.4 *For any split fibration $P : \mathscr{X} \longrightarrow \mathscr{C}$ and category C in $\mathscr{C}$, the equivalence of Theorem 2.3 restricts to an isomorphism of categories*

$$\mathrm{Spl}_{/\mathscr{C}}(\textstyle\int C, \mathscr{X}) \cong \mathscr{X}^C \quad .$$

Corollary 2.5 *The 2-functor* (1.3)

$$\mathrm{Yon} : \mathrm{Cat}\mathscr{C} \longrightarrow [\mathscr{C}^{\mathrm{op}}, \mathrm{Cat}]$$

is fully faithful. That is, the inclusion 2-functor

$$\mathscr{C} \longrightarrow \mathrm{Cat}\mathscr{C}$$

is dense.

Proof For categories C and D in $\mathscr{C}$, we have a commutative square

$$\begin{tikzcd}
(\int D)^C \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathrm{Spl}_{/\mathscr{C}}(\int C, \int D) \arrow[d, "\cong"] \\
\mathrm{Cat}\mathscr{C}(C,D) \arrow[r, "\mathrm{Yon}"'] & {[\mathscr{C}^{\mathrm{op}}, \mathrm{Cat}](\mathscr{C}(-,C), \mathscr{C}(-,D))}
\end{tikzcd}$$

where the vertical isomorphisms are fairly straightforward and the top isomorphism comes from Proposition 2.4. □

Corollary 2.6 *Limits in the 2-category* $\mathrm{Cat}\mathscr{C}$ *are detected by discrete categories in* $\mathscr{C}$. *That is, a (weighted) diagram in* $\mathrm{Cat}\mathscr{C}$ *is a limit if and only if the 2-functor* (1.3) *takes it to a limit in* $[\mathscr{C}^{\mathrm{op}}, \mathrm{Cat}]$.

A morphism $f : A \to B$ in $\mathrm{Cat}\mathscr{C}$ is called *fully faithful* when it is taken to a pointwise fully faithful morphism by the 2-functor (1.3). In other words, when the following square is a pullback.

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "{(d_0,d_1)}"'] & B_1 \arrow[d, "{(d_0,d_1)}"] \\
A_0 \times A_0 \arrow[r, "f_0 \times f_0"'] & B_0 \times B_0
\end{tikzcd} \tag{2.5}$$

Perhaps the reader noticed that $d_1 : \hat{C}_2 \longrightarrow \hat{C}_1$ in the generic example is actually cartesian; in fact, it is part of the cleavage. So the following is actually a consequence of Theorem 2.3. We give a direct proof.

Proposition 2.7 *For any object* X *of* $\mathscr{X}^C$, *the morphism* $d_1 : X_2 \longrightarrow X_1$ *is cartesian.*

Proof We have $d_0 d_1 = d_0 d_0$ and commutative diagrams.

$$\begin{tikzcd}
\mathscr{X}(K,X_2) \arrow[r, "P"] \arrow[d, "{\mathscr{X}(1,d_1)}"'] & \mathscr{C}(PK,C_2) \arrow[d, "{\mathscr{C}(1,d_1)}"] \\
\mathscr{X}(K,X_1) \arrow[r, "P"'] \arrow[d, "{\mathscr{X}(1,d_0)}"'] & \mathscr{C}(PK,C_1) \arrow[d, "{\mathscr{C}(1,d_0)}"] \\
\mathscr{X}(K,X_0) \arrow[r, "P"'] & \mathscr{C}(PK,C_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{X}(K,X_2) \arrow[r, "P"] \arrow[d, "{\mathscr{X}(1,d_0)}"'] & \mathscr{C}(PK,C_2) \arrow[d, "{\mathscr{C}(1,d_0)}"] \\
\mathscr{X}(K,X_1) \arrow[r, "P"'] \arrow[d, "{\mathscr{X}(1,d_0)}"'] & \mathscr{C}(PK,C_1) \arrow[d, "{\mathscr{C}(1,d_0)}"] \\
\mathscr{X}(K,X_0) \arrow[r, "P"'] & \mathscr{C}(PK,C_0)
\end{tikzcd}$$

So the top left square is a pullback, implying $d_1 : X_2 \longrightarrow X_1$ is cartesian. □

We conclude this section with some facts all due to Grothendieck [17].

A cleavage for a fibration $P : \mathscr{X} \longrightarrow \mathscr{C}$ allows the definition of a pseudofunctor

$$\mathscr{X}^{-} : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathrm{Cat} \quad .$$

The value at $U \in \mathscr{X}$ is the fibre $\mathscr{X}^{U}$ of P over U: it is the subcategory of $\mathscr{X}$ consisting of the objects $X \in \mathscr{X}$ with $PU = U$ and morphisms $x : Y \longrightarrow X$ with $Px = 1_U$. (This agrees with the previous notation $\mathscr{X}^{C}$ when $C = U$ is discrete.) The value of the pseudofunctor at $h : V \longrightarrow U$ in $\mathscr{C}$ is the functor

$$h^{\star} : \mathscr{X}^{U} \longrightarrow \mathscr{X}^{V}$$

taking X to $h^{\star}X$ and using the universal property of cartesian morphisms to define $h^{\star}$ on morphisms. If $P : \mathscr{X} \longrightarrow \mathscr{C}$ is a split fibration, $\mathscr{X}^{-} : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$ is a functor. If $\mathrm{Cat}^{\mathscr{C}^{\mathrm{op}}}$ is the 2-category of split fibrations over $\mathscr{C}$ and distinguished-cartesian-morphism-preserving functors over $\mathscr{C}$, we obtain an equivalence of 2-categories

$$\mathrm{Cat}^{\mathscr{C}^{\mathrm{op}}} \simeq [\mathscr{C}^{\mathrm{op}}, \mathrm{Cat}] \tag{2.6}$$

taking $P : \mathscr{X} \longrightarrow \mathscr{C}$ to $\mathscr{X}^{-}$.

3 Internal full subcategories

Suppose $\mathscr{C}$ is a category admitting pullbacks. Let $\mathscr{C}^{\mathbf{2}} = [\mathbf{2}, \mathscr{C}]$ be the category of morphisms in $\mathscr{C}$. The functor

$$\mathrm{cod} : \mathscr{C}^{\mathbf{2}} \longrightarrow \mathscr{C} \quad ,$$

taking each morphism to its codomain, is a fibration. The cartesian morphisms in $\mathscr{C}^{\mathbf{2}}$ are the pullback squares in $\mathscr{C}$.

For any category C in $\mathscr{C}$, the category $(\mathscr{C}^{\mathbf{2}})^{C}$ has objects those functors $p : E \longrightarrow C$ in $\mathscr{C}$ such that the square

$$\begin{tikzcd} E_1 \arrow[r, "d_0"] \arrow[d, "p_1"'] & E_0 \arrow[d, "p_0"] \\ C_1 \arrow[r, "d_0"'] & C_0 \end{tikzcd} \tag{3.1}$$

is a pullback. Such functors p are called *discrete opfibrations over C*. We think of them as internalized functors $C \longrightarrow \mathscr{C}$. It is more usual to write $\mathscr{C}^{C}$ rather than $(\mathscr{C}^{\mathbf{2}})^{C}$. By

Theorem 2.3, each discrete opfibration $p : E \longrightarrow C$ corresponds to a cartesian-morphism-preserving functor

$$\bar{p} : \textstyle\int C \longrightarrow \mathscr{C}^{\mathbf{2}} \tag{3.2}$$

taking (U, u) to $p_u : E_u \longrightarrow U$ where the square

$$\begin{tikzcd} E_u \arrow[r, "i_u"] \arrow[d, "p_u"'] & E \arrow[d, "p"] \\ U \arrow[r, "u"'] & C \end{tikzcd} \tag{3.3}$$

is a pullback. For a morphism $(h, \theta) : (U, u) \longrightarrow (V, v)$, we obtain the morphism $\bar{p}(h, \theta)$:

$$\begin{tikzcd} E_u \arrow[r, "\bar{\theta}"] \arrow[d, "p_u"'] & E_v \arrow[d, "p_v"] \\ U \arrow[r, "h"'] & V \end{tikzcd} \tag{3.4}$$

in $\mathscr{C}^{\mathbf{2}}$ using the pullback (3.3) for v and the diagram

$$\begin{tikzcd} E_u \arrow[rr, "d_1(\theta p_u, p_0 i_u)"] \arrow[d, "p_u"'] & & E \arrow[d, "p"] \\ U \arrow[r, "h"'] & V \arrow[r, "v"'] & C \end{tikzcd}$$

for which the pullback (3.1) is used to define $(\theta p_u, p_0 i_u) : E_u \longrightarrow E_1$ satisfying

$$d_0(\theta p_u, p_0 i_u) = i_u \quad \text{and} \quad p_1(\theta p_u, p_0 i_u) = \theta p_u \quad .$$

The split fibration $P : \int C \longrightarrow \mathscr{C}$ corresponds to the functor

$$\mathscr{C}(-, C) : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathrm{Cat} \quad .$$

A choice of pullbacks in $\mathscr{C}$ means that the fibration $\mathrm{cod} : \mathscr{C}^{\mathbf{2}} \longrightarrow \mathscr{C}$ leads to a pseudofunctor

$$\mathscr{C}/- : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$$

taking each object U to the slice category $\mathscr{C}/U$ and defined on morphisms by pullback along them. It follows that $\bar{p}$ induces a pseudonatural transformation with component at $U \in \mathscr{C}$ denoted by

$$\bar{p}^U : \mathscr{C}(U, C) \longrightarrow \mathscr{C}/U \quad . \tag{3.5}$$

Proposition 3.1 *A functor*

$$\begin{tikzcd}
\mathscr{X} \arrow[rr, "F"] \arrow[dr, swap, "P"] & & \mathscr{Y} \arrow[dl, "Q"] \\
& \mathscr{C} &
\end{tikzcd} \tag{3.6}$$

over $\mathscr{C}$ induces a family of functors

$$F^U : \mathscr{X}^U \longrightarrow \mathscr{Y}^U \quad , \quad U \in \mathscr{C} \quad , \tag{3.7}$$

on the fibres. If F is fully faithful then so are all the functors F^U, $U \in \mathscr{C}$. The converse holds if P is a fibration and F preserves cartesian morphisms.

Proof If $y : FX \longrightarrow FX'$ in $\mathscr{Y}$ is in $\mathscr{Y}^U$ then any $x : X \longrightarrow X'$ in $\mathscr{X}$ with $Fx = y$ is in $\mathscr{X}^U$. This proves the second sentence of the Proposition. For the converse, take any $h : U \longrightarrow U'$ in $\mathscr{C}$ and $x : Z \longrightarrow X'$ cartesian with $Px = h$. We have a diagram of four pullbacks as below.

$$\begin{tikzcd}
\mathscr{X}^U(X,Z) \arrow[r, "\mathrm{incl}"] \arrow[d, swap, "F^U"] & \mathscr{X}(X,Z) \arrow[r, "\mathscr{X}(1,x)"] \arrow[d, "F"] & \mathscr{X}(X,X') \arrow[d, "F"] \\
\mathscr{Y}^U(FX,FZ) \arrow[r, swap, "\mathrm{incl}"] \arrow[d, swap, "!"] & \mathscr{Y}(FX,FZ) \arrow[r, swap, "\mathscr{Y}(1,Fx)"] \arrow[d, "Q"] & \mathscr{Y}(FX,FX') \arrow[d, "Q"] \\
1 \arrow[r, swap, "\ulcorner 1_U \urcorner"] & \mathscr{C}(U,U) \arrow[r, swap, "\mathscr{C}(1,h)"] & \mathscr{C}(U,U')
\end{tikzcd}$$

By assumption, the top left vertical function is invertible. So F is fully faithful on those morphisms over h; but all morphisms are over some h. So F is fully faithful. □

An *internal full subcategory* of the category $\mathscr{C}$ is a discrete opfibration $p : E \longrightarrow C$ for which each of the functors $\bar{p}^U$ (3.5) is fully faithful. By Proposition 3.1, this is the same as the requirement that the functor $\bar{p} : \int C \longrightarrow \mathscr{C}^{\mathbf{2}}$ (3.2) be fully faithful. Obviously:

Proposition 3.2 *If $j : S \to C$ is a fully faithful morphism of* $\mathrm{Cat}\mathscr{C}$ *(see* (4.2)*) and C with the discrete opfibration $p : E \to C$ is an internal full subcategory of $\mathscr{C}$ then S with the pullback of p along j is also an internal full subcategory of $\mathscr{C}$.*

For any discrete opfibration $p : E \longrightarrow C$, we have a factorization

$$\begin{tikzcd}
\mathscr{C}/C_0 \arrow[rr, "\bar{p}_0"] \arrow[dr, swap, "\mathrm{incl}"] & & \mathscr{C}^{\mathbf{2}} \\
& \int C \arrow[ur, swap, "\bar{p}"] &
\end{tikzcd} \tag{3.8}$$

of $\bar{p}_0$ over $\mathscr{C}$ in which the first functor is bijective on objects. All three functors preserve cartesian morphisms; indeed, $\mathscr{C}/C_0 \longrightarrow \int C$ preserves the cleavages.

For any cartesian-morphism-preserving functor F (3.6) over $\mathscr{C}$, we can take the 'full image'

$$\begin{tikzcd} \mathscr{X} \arrow[r, "H"] \arrow[dr, "P"'] & \mathscr{Z} \arrow[r, "G"] \arrow[d, "R"] & \mathscr{Y} \arrow[dl, "Q"] \\ & \mathscr{C} & \end{tikzcd} \tag{3.9}$$

where $\mathscr{Z}$ is the category with the same objects as $\mathscr{X}$ and with homsets $\mathscr{Z}(X', X) = \mathscr{Y}(FX', FX)$. The functor H is the identity on objects and defined by

$$F : \mathscr{X}(X', X) \longrightarrow \mathscr{Y}(FX', FX) = \mathscr{Z}(X', X)$$

on morphisms. The functor G is defined to be F on objects and the identity on morphisms. The functor R is unique making (3.9) commutative. The cartesian morphisms in $\mathscr{Z}$ are those which are cartesian in $\mathscr{Y}$. If P is a fibration then so is R; given $h : V \longrightarrow U$ in $\mathscr{C}$ and $X \in \mathscr{Z}$ with $RX = PX = U$, there is a cartesian $x : X' \longrightarrow X$ over h for P, and $Fx : X' \longrightarrow X$ in $\mathscr{Z}$ is cartesian over h for R. Both H and G preserve cartesian morphisms.

Let us write $P_F : \mathscr{X}[F] \longrightarrow \mathscr{C}$ for the functor $R : \mathscr{Z} \longrightarrow \mathscr{C}$ obtained using the above full image construction.

A morphism $k : M \longrightarrow N$ in $\mathscr{C}$ (can be regarded as a discrete opfibration between discrete categories in $\mathscr{C}$ and so) gives rise to a cartesian-morphism-preserving functor

$$\begin{tikzcd} \mathscr{C}/N \arrow[rr, "\bar{k}"] \arrow[dr, "\mathrm{dom}"'] & & \mathscr{C}^{\mathbf{2}} \arrow[dl, "\mathrm{cod}"] \\ & \mathscr{C} & \end{tikzcd} \tag{3.10}$$

over $\mathscr{C}$. We can factor $\bar{k}$ as

$$\begin{tikzcd} \mathscr{C}/N \arrow[r, "H"] \arrow[dr, "\mathrm{dom}"'] & (\mathscr{C}/N)[\bar{k}] \arrow[r, "G"] \arrow[d, "P_{\bar{k}}"] & \mathscr{C}^{\mathbf{2}} \arrow[dl, "\mathrm{cod}"] \\ & \mathscr{C} & \end{tikzcd} \tag{3.11}$$

where H is the identity on objects and G is fully faithful. The objects of $(\mathscr{C}/N)[\bar{k}]$ are pairs (U, u) for $u : U \longrightarrow N$ in $\mathscr{C}$ and the morphisms are commutative squares

$$\begin{tikzcd} M_u \arrow[r] \arrow[d, "k_u"'] & M_v \arrow[d, "k_v"] \\ U \arrow[r, "h"'] & V \end{tikzcd} \tag{3.12}$$

where k_u is the pullback of k along u.

The following result is due to Bénabou [2]. It is Theorem 6.2 of [39] and Example 2.38 of [18]. Also see [3], [45] and [19].

Proposition 3.3 *For a morphism $k : M \longrightarrow N$ in a finitely complete category $\mathscr{C}$, the following three conditions are equivalent:*

(i) there exists an internal full subcategory $p : E \longrightarrow C$ of $\mathscr{C}$ with $p_0 = k$;

(ii) there exists a category C in $\mathscr{C}$ with $C_0 = N$ and an isomorphism

$$\textstyle\int C \cong (\mathscr{C}/N)[\bar{k}]$$

of categories over $\mathscr{C}$;

(iii) there exists a cartesian internal hom for the two objects

$$k \times 1 : M \times N \longrightarrow N \times N \quad \textit{and} \quad 1 \times k : N \times M \longrightarrow N \times N$$

of the category $\mathscr{C}/N \times N$.

Proof (i) ⇒ (ii) Given (i), compare the factorizations (3.8) and (3.11). Since bijective-on-objects and fully faithful functors form a factorization system on Cat, the two factorizations are isomorphic, yielding (ii).

(ii) ⇒ (iii) The product of $(u, v) : U \longrightarrow N \times N$ and $k \times 1 : M \times N \longrightarrow N \times N$ in $\mathscr{C}/N \times N$ is the main diagonal of the pullback

$$\begin{tikzcd}
M_u \arrow[r] \arrow[d, "k_u"'] & M \times N \arrow[d, "k\times 1"] \\
U \arrow[r, "(u,v)"'] & N \times N
\end{tikzcd}$$

in $\mathscr{C}$. So

$$(\mathscr{C}/N \times N)(U \underset{N\times N}{\times} (M \times N) \to N \times N, N \times M \xrightarrow{1\times k} N \times N) \cong (\mathscr{C}/U)(M_u \xrightarrow{k_u} U, M_v \xrightarrow{k_v} U) \ . \tag{3.13}$$

On the other hand, for any category C with $C_0 = N$, we have

$$(\mathscr{C}/N \times N)(U \xrightarrow{(u,v)} N \times N, C_1 \xrightarrow{(d_0,d_1)} N \times N) \cong \mathscr{C}(U, C)(u, v) \ . \tag{3.14}$$

The isomorphism of (ii) yields an isomorphism between the right hand side of (3.13) and the right hand side of (3.14). This shows that

$$(d_0, d_1) : C_1 \longrightarrow N \times N$$

is an internal hom as required for (iii).

(iii) ⇒ (i) Let $(d_0, d_1) : C_1 \longrightarrow N \times N$ be an internal hom as in (iii) and let $C_0 = N$. Despite C not yet being a category, only a graph, the left hand side of (3.13) and that of (3.14) are isomorphic. Therefore, looking at the right hand sides, we have a fully faithful graph morphism

$$\mathscr{C}(U, C) \longrightarrow \mathscr{C}/U \ , \quad u \mapsto k_u \ .$$

Since the codomain is a category, a category structure is induced on $\mathscr{C}(U, C)$. The family of functors is pseudonatural in U and so yields a fully faithful cartesian-morphism-preserving functor $\int C \longrightarrow \mathscr{C}^{\mathbf{2}}$ over $\mathscr{C}$. By Theorem 2.3, (i) follows. □

4 Slices and left extensions

For functors $F : \mathscr{A} \longrightarrow \mathscr{C}$ and $G : \mathscr{B} \longrightarrow \mathscr{C}$, the so-called *comma category* F/G (for example, see [30]) has objects $(A, h : FA \longrightarrow GB, B)$ where A is an object of $\mathscr{A}$, B is an object of $\mathscr{B}$, and $h : FA \longrightarrow GB$ is a morphism of $\mathscr{C}$. A morphism $(f, g) : (A, h, B) \longrightarrow (A', h', B')$ in F/G consists of a morphism $f : A \longrightarrow A'$ in $\mathscr{A}$ and a morphism $f : B \longrightarrow B'$ in $\mathscr{B}$ such that $h' \circ Ff = Gg \circ h$. We write $P : F/G \longrightarrow \mathscr{A}$ and $Q : F/G \longrightarrow \mathscr{B}$ for the projection functors. The special case where F is the identity functor of $\mathscr{C}$ and $\mathscr{B}$ is the terminal category, so that G can be identified with an object U of $\mathscr{C}$, is called traditionally called the *slice* of $\mathscr{C}$ over U.

We propose to use the term *slice of F over G* for the general comma category F/G. This terminology seems to work better than my traditional term ‘comma object’ [35] for the construction internalized to a pair of morphisms in a 2-category (recalled below). Weber [44] uses the term ‘lax pullback’.

Before internalizing definitions to a 2-category, we remind the reader (see [30]) that the left Kan extension $K : \mathscr{B} \longrightarrow \mathscr{C}$ of a functor $F : \mathscr{A} \longrightarrow \mathscr{C}$ along a functor $H : \mathscr{A} \longrightarrow \mathscr{B}$ can be calculated by Lawvere’s pointwise formula

$$KB = \operatorname{colim}(H/B \xrightarrow{P} \mathscr{A} \xrightarrow{F} \mathscr{C})$$

when the colimits exist. Furthermore, the colimit of a functor is none other than a left Kan extension along a functor into the terminal category **1**.

Let $\mathscr{K}$ be a 2-category. In $\mathscr{K}$, a *slice* of a morphism $f : A \longrightarrow C$ over a morphism $g : B \longrightarrow C$ is a square

$$\begin{tikzcd} f/g \arrow[r, "q"] \arrow[d, "p"'] \arrow[rd, phantom, "\overset{\lambda}{\Longrightarrow}"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \tag{4.1}$$

such that, for all objects X, the functor

$$\mathscr{K}(X, f/g) \longrightarrow \mathscr{K}(X, f)/\mathscr{K}(X, g) \ , \tag{4.2}$$

taking $r : X \longrightarrow f/g$ to $(pr, \lambda r, qr)$, is an isomorphism. A slice of the identity morphism $1_C : C \longrightarrow C$ over itself is the *cotensor* $C^{\mathbf{2}}$ *of* C *with the arrow category* **2**.

A 2-category $\mathscr{K}$ is *(finitely) complete* [37] when it has (finite) products and equalizers (in the Cat-enriched sense), and has cotensors with **2**. It follows that all slices exist. Also all cotensors with (finite) categories exist.

If $\mathscr{C}$ is a (finitely) complete category then $\mathscr{K} = \mathrm{Cat}\mathscr{C}$ is a (finitely) complete 2-category; see (7.1) for the construction of cotensors with **2**. By Corollary 2.6, the square (4.1) is a slice if and only if the functor (4.2) is invertible for all discrete X.

The 2-category $\mathrm{Cat}/\mathscr{C}$ is complete. In particular, we are interested in the slice $F/_{\mathscr{C}}F'$ in $\mathrm{Cat}/\mathscr{C}$ of two functors over $\mathscr{C}$ as in the following diagram.

$$\begin{tikzcd} \mathscr{X} \arrow[r, "F"] \arrow[rd, "P"'] & \mathscr{Y} \arrow[d, "Q"] & \mathscr{X}' \arrow[l, "F'"'] \arrow[ld, "P'"] \\ & \mathscr{C} & \end{tikzcd} \tag{4.3}$$

Indeed, $F/_{\mathscr{C}}F'$ is the full subcategory of the slice category F/F' consisting of those objects (X, y, X'), where $X \in \mathscr{X}$, $X' \in \mathscr{X}'$ and $y \in \mathscr{Y}(FX, F'X')$, for which Qy is an identity morphism in $\mathscr{C}$. We have a universal square containing a natural transformation, as in the following square, taken to an identity by $Q : \mathscr{Y} \longrightarrow \mathscr{C}$.

$$\begin{tikzcd} F/_{\mathscr{C}}F' \arrow[r, "\mathrm{cod}"] \arrow[d, "\mathrm{dom}"'] \arrow[rd, phantom, "\overset{\lambda}{\Longrightarrow}"] & \mathscr{X}' \arrow[d, "F'"] \\ \mathscr{X} \arrow[r, "F"'] & \mathscr{Y} \end{tikzcd} \tag{4.4}$$

If F and F' preserve cartesian morphisms then so do dom and cod, and the cartesian morphisms of $F/_{\mathscr{C}}F'$ are precisely those taken to cartesian morphisms by dom and cod.

Proposition 4.1 *The 2-functor*

$$\int : \mathrm{Cat}\mathscr{C} \longrightarrow \mathrm{Cat}/\mathscr{C}$$

preserves slices.

Proof The universal property of a slice

$$\begin{tikzcd}
f/f' \arrow[r, "d'"] \arrow[d, "d"'] \arrow[dr, phantom, "\overset{\lambda}{\Longrightarrow}"] & C' \arrow[d, "f'"] \\
C \arrow[r, "f"'] & D
\end{tikzcd} \tag{4.5}$$

in $\mathrm{Cat}\mathscr{C}$ immediately yields that the functor

$$\int(f/f') \longrightarrow \int f/_{\mathscr{C}} \int f' \quad ,$$

taking the morphism

$$\begin{tikzcd}
U \arrow[rr, "h"] \arrow[dr, "u"'] & \arrow[d, phantom, "\overset{\theta}{\Longrightarrow}"] & V \arrow[dl, "v"] \\
& f/f' &
\end{tikzcd}$$

in $\int f/_{\mathscr{C}} \int f'$ to the morphism

$$((h, d\theta), (h, d'\theta)) : ((U, du), \lambda u, (U, d'u)) \longrightarrow ((V, dv), \lambda v, (V, d'v))$$

in $\int f/_{\mathscr{C}} \int f'$, is an isomorphism. □

In a 2-category $\mathscr{K}$, a diagram

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\kappa}{\Longrightarrow}"] & B \arrow[dl, "k"] \\
& C &
\end{tikzcd} \tag{4.6}$$

is said [34] to exhibit k as a *left extension* of f along h when, for all $g : B \longrightarrow C$, pasting with the triangle yields a bijection between 2-cells $k \Longrightarrow g$ and 2-cells $f \Longrightarrow gh$. To say such a left extension exists for each f, is to say the functor

$$\mathscr{K}(h, C) : \mathscr{K}(B, C) \longrightarrow \mathscr{K}(A, C)$$

has a left adjoint. We write $k = \mathrm{lan}_h(f)$.

The diagram (4.6) is said [35] to exhibit k as a *pointwise left extension* of f along h when, for all morphisms $b : X \longrightarrow B$ for which the slice of h over b exists, the diagram

$$\begin{tikzcd}
h/b \arrow[rr, "q"] \arrow[d, "p"'] \arrow[drr, phantom, "\overset{\lambda}{\Longrightarrow}"] & & X \arrow[d, "b"] \\
A \arrow[rr, "h"] \arrow[dr, "f"'] \arrow[rr, phantom, shift right=3ex, "\overset{\kappa}{\Longrightarrow}"] & & B \arrow[dl, "k"] \\
& C &
\end{tikzcd} \tag{4.7}$$

exhibits kb as a left extension of fp along q. It is shown in [35] that every pointwise left extension is a left extension, provided the slice $h/1_B$ exists, and that (4.7) furthermore exhibits kb as a pointwise left extension of fp along q.

Definition 4.2 [35] A morphism $f : A \longrightarrow B$ in $\mathscr{K}$ is *(representably) fully faithful* when the functor $\mathscr{K}(X, f) : \mathscr{K}(X, A) \longrightarrow \mathscr{K}(X, B)$ is fully faithful for all objects X of $\mathscr{K}$. This is equivalent, when the cotensor with $\mathbf{2}$ exists, to saying that the square

$$\begin{tikzcd}
A^{\mathbf{2}} \arrow[rr, "d_1"] \arrow[d, "d_0"'] \arrow[drr, phantom, "\overset{f\lambda}{\Longrightarrow}"] & & A \arrow[d, "f"] \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}$$

exhibits $A^{\mathbf{2}}$ as the slice f/f.

Definition 4.3 [36] A morphism $f : A \longrightarrow B$ in $\mathscr{K}$ is *dense* when the triangle

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "f"'] \arrow[rr, phantom, shift right=3ex, "\overset{1_f}{\Longrightarrow}"] & & B \arrow[dl, "1_B"] \\
& B &
\end{tikzcd}$$

exhibits 1_B as a pointwise left extension of f along f.

5 Descent categories

As usual we write $\mathbf{\Delta}$ for the (topologists') simplicial category. The objects are the non-empty linearly ordered sets

$$[n] = \{0, 1, \ldots, n\}$$

for $n \geq 0$, and the morphisms are order-preserving functions. We write

$$\partial_p : [n-1] \longrightarrow [n] \quad \text{and} \quad \sigma_q : [n+1] \longrightarrow [n]$$

for the injective order-preserving function which omits the element p in its image and the surjective order-preserving function which identifies only the elements q and $q+1$. The squares

$$\begin{tikzcd}
{[n-1]} \arrow[r, "\partial_0"] \arrow[d, "\partial_n"'] & {[n]} \arrow[d, "\partial_{n+1}"] \\
{[n]} \arrow[r, "\partial_0"'] & {[n+1]}
\end{tikzcd}$$

are pushouts in $\mathbf{\Delta}$.

Convention When thinking of ∂_p and σ_q as morphisms of the category $\mathbf{\Delta}^{\mathrm{op}}$, we write them respectively as

$$d_p : [n] \longrightarrow [n-1] \quad \text{and} \quad i_q : [n] \longrightarrow [n+1] \ .$$

Let $\mathbf{D}$ denote the category

$$\begin{tikzcd}[column sep=large]
{[3]} \arrow[r, shift left=6, "d_0" description] \arrow[r, shift left=2, "d_1" description] \arrow[r, shift right=2, "d_2" description] \arrow[r, shift right=6, "d_3" description] & {[2]} \arrow[r, shift left=8, "d_0" description] \arrow[r, shift right=0, "d_1" description] \arrow[r, shift right=8, "d_2" description] & {[1]} \arrow[l, shift right=4, "i_0" description] \arrow[l, shift left=4, "i_1" description] \arrow[r, shift left=4, "d_0" description] \arrow[r, shift right=4, "d_1" description] & {[0]} \arrow[l, "i" description]
\end{tikzcd} \tag{5.1}$$

in $\mathbf{\Delta}^{\mathrm{op}}$.

The origins of the following construction go back to Grothendieck [16].

Definition 5.1 The *descent category* for a functor $P : \mathscr{E} \longrightarrow \mathbf{\Delta}^{\mathrm{op}}$ is

$$\mathrm{Desc}\mathscr{E} = \mathscr{E}^{\mathbf{D}} \ .$$

Objects E of $\mathrm{Desc}\mathscr{E}$ are called *descent data* for P.

If, in Definition 5.1, the functor P is a fibration, a cleavage determines a pseudofunctor

$$\mathscr{E}^{-} : \mathbf{\Delta} \longrightarrow \mathrm{Cat} \ ;$$

that is, a pseudocosimplicial category. We write $\mathscr{E}_n$ for the fibre $\mathscr{E}^{[n]}$ of P over $[n]$, yielding a diagram

$$\begin{tikzcd}[column sep=large]
\mathscr{E}_3 & \mathscr{E}_2 \arrow[l, shift right=6, "\partial_0" description] \arrow[l, shift right=2, "\partial_1" description] \arrow[l, shift left=2, "\partial_2" description] \arrow[l, shift left=6, "\partial_3" description] \arrow[r, shift left=4, "\sigma_0" description] \arrow[r, shift right=4, "\sigma_1" description] & \mathscr{E}_1 \arrow[l, shift right=8, "\partial_0" description] \arrow[l, "\partial_1" description] \arrow[l, shift left=8, "\partial_2" description] \arrow[r, "\sigma" description] & \mathscr{E}_0 \arrow[l, shift right=4, "\partial_0" description] \arrow[l, shift left=4, "\partial_1" description]
\end{tikzcd} \tag{5.2}$$

in Cat in which the cosimplicial identities hold up to coherent isomorphism: for example,

$$\partial_q \partial_p \cong \partial_p \partial_{q-1} \quad \text{for} \quad p < q \ .$$

For descent data E, define

$$e : \partial_0 E_0 \longrightarrow \partial_1 E_0 \tag{5.3}$$

to be the unique morphism in $\mathscr{E}_1$ such that the triangle

$$\begin{tikzcd}
E_1 = \partial_0 E_0 \arrow[rr, "e"] \arrow[dr, "d_1"'] & & \partial_1 E_0 \arrow[dl, "{d_1}^{E_0}"] \\
& E_0 &
\end{tikzcd}$$

commutes. We readily see that the hexagon

$$\begin{tikzcd}
& \partial_1 \partial_0 E_0 \arrow[r, "\partial_1 e"] & \partial_1 \partial_1 E_0 \arrow[dr, "\cong"] & \\
\partial_0 \partial_0 E_0 \arrow[ur, "\cong"] \arrow[dr, "\partial_0 e"'] & & & \partial_2 \partial_1 E_0 \\
& \partial_0 \partial_1 E_0 \arrow[r, "\cong"'] & \partial_2 \partial_0 E_0 \arrow[ur, "\partial_2 e"'] &
\end{tikzcd} \tag{5.4}$$

commutes in $\mathscr{E}_2$ and

$$\begin{tikzcd}
\sigma_0 \partial_0 E_0 \arrow[rr, "\sigma_0 e"] \arrow[dr, "\cong"'] & & \sigma_0 \partial_1 E_0 \arrow[dl, "\cong"] \\
& E_0 &
\end{tikzcd} \tag{5.5}$$

commutes in $\mathscr{E}_0$. In fact, we can reconstruct E from E_0 and e (5.3) satisfying (5.4) and (5.5).

While we have defined the descent category, for a category 'parametrized' by $\mathbf{\Delta}^{\text{op}}$, in terms of the $\mathscr{X}^C$ construction, we can go the other way too. Suppose $P : \mathscr{X} \longrightarrow \mathscr{C}$ is a functor and C is a category in $\mathscr{C}$. For simplicity we suppose $\mathscr{C}$ has pullbacks chosen so that we can complete the diagram for C to a simplicial object (functor)

$$C : \mathbf{\Delta}^{\text{op}} \longrightarrow \mathscr{C} \ .$$

Form the pullback

$$\begin{tikzcd}
C^{\star} \mathscr{X} \arrow[r] \arrow[d, "P_C"'] & \mathscr{X} \arrow[d, "P"] \\
\mathbf{\Delta}^{\text{op}} \arrow[r, "C"'] & \mathscr{C}
\end{tikzcd} \tag{5.6}$$

in Cat.

Proposition 5.2 *There is a canonical equivalence of categories*

$$\mathscr{X}^C \simeq \mathrm{Desc}C^\star \mathscr{X}$$

$$E \longleftrightarrow (\mathbf{D}, E)\ .$$

For cloven P, this is an isomorphism.

For any functor $P : \mathscr{X} \longrightarrow \mathscr{C}$ and category $\mathscr{H}$, the cotensor of $\mathscr{H}$ with P in the 2-category $\mathrm{Cat}/\mathscr{C}$ is defined by the pullback

$$\begin{tikzcd}
{[\mathscr{H},\mathscr{X}]_{\mathscr{C}}} \arrow[r] \arrow[d, "P^{\mathscr{H}}"'] & {[\mathscr{H},\mathscr{X}]} \arrow[d, "{[1,P]}"] \\
\mathscr{C} \arrow[r, "\mathrm{diag}"'] & {[\mathscr{H},\mathscr{C}]}
\end{tikzcd}$$

in Cat. For a cloven fibration P, the pseudofunctor corresponding to the cloven fibration $P^{\mathscr{H}}$ is the composite

$$\mathscr{C}^{\mathrm{op}} \xrightarrow{\mathscr{X}^-} \mathrm{Cat} \xrightarrow{[\mathscr{H},-]} \mathrm{Cat}\ .$$

Proposition 5.3 *For any functor $P : \mathscr{E} \longrightarrow \mathbf{\Delta}^{\mathbf{op}}$ and category $\mathscr{H}$, there is a canonical isomorphism of categories*

$$[\mathscr{H}, \mathrm{Desc}\mathscr{E}] \cong \mathrm{Desc}[\mathscr{H}, \mathscr{E}]_{\mathbf{\Delta}^{\mathbf{op}}}\ .$$

Loosely speaking, this says that the descent construction is preserved by representable 2-functors out of Cat. It therefore makes sense to define the construction in a 2-category $\mathscr{K}$.

6 Descent and codescent objects

For a pseudofunctor $T : \mathbf{\Delta} \longrightarrow \mathrm{Cat}$, we write $\mathrm{Desc}T$ for the category $\mathrm{Desc}\mathscr{E}$ where $P : \mathscr{E} \longrightarrow \mathbf{\Delta}^{\mathbf{op}}$ is the cloven fibration obtained from T by the Grothendieck construction described at the end of Section 2.

Definition 6.1 *For a pseudofunctor*

$$T : \mathbf{\Delta} \longrightarrow \mathscr{K}\ ,$$

a descent object is an object $\mathrm{Desc}T$ *of* $\mathscr{K}$ *equipped with a 2-natural isomorphism*

$$\mathscr{K}(K, \mathrm{Desc}T) \cong \mathrm{Desc}\mathscr{K}(K, T) \ .$$

A codescent object in $\mathscr{K}$ *for a pseudofunctor* $S : \mathbf{\Delta}^{\mathbf{op}} \longrightarrow \mathscr{K}$ *is a descent object for* $S^{\mathrm{op}} : \mathbf{\Delta} \longrightarrow \mathscr{K}^{\mathrm{op}}$.

Proposition 6.2 *A descent object for a functor* $T : \mathbf{\Delta} \longrightarrow \mathscr{K}$ *is a limit for* T *weighted by the inclusion* $\mathbf{\Delta} \longrightarrow \mathrm{Cat}$.

Bénabou-Roubaud [4] and Jon Beck (unpublished) expressed descent data in terms of Eilenberg-Moore algebras for monads. We shall describe a version of this.

A functor $P : \mathscr{X} \longrightarrow \mathscr{C}$ is called an *opfibration* (originally Grothendieck used the term 'cofibration' and some authors persist with this) when $P^{\mathbf{op}} : \mathscr{X}^{\mathbf{op}} \longrightarrow \mathscr{C}^{\mathbf{op}}$ is a fibration. Cartesian morphisms and inverse images for $P^{\mathbf{op}}$ are called *opcartesian morphisms* and *direct images* for P.

Proposition 6.3 *If a cloven fibration* $P : \mathscr{X} \longrightarrow \mathscr{C}$ *is also an opfibration then the inverse image functor*

$$h^{\star} : \mathscr{X}^{U} \longrightarrow \mathscr{X}^{V} \ ,$$

for each $h : V \longrightarrow U$ *in* $\mathscr{C}$, *has a left adjoint defined by direct image* $h_{\star}$ *along* h.

Proof $\mathscr{X}^{V}(Y, h^{\star}X) \cong \{x : Y \longrightarrow X \mid Px = h\} \cong \mathscr{X}^{U}(h_{\star}Y, X) \ .$ □

For any functor $P : \mathscr{X} \longrightarrow \mathscr{C}$, consider a commutative square

$$\begin{tikzcd}
R \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "y"] \\
X \arrow[r, "x"'] & Z
\end{tikzcd} \tag{6.1}$$

in $\mathscr{X}$. The following condition is attributed to Chevalley in [4] and to Jon Beck in other places.

Chevalley-Beck condition *If* (6.1) *is a pullback in* $\mathscr{X}$ *preserved by* P *with* x *cartesian and* y *opcartesian then* p *is opcartesian.*

By Lemma 2.1, the hypothesis here that (6.1) be a pullback in $\mathscr{X}$ is equivalent to the assumption that q be cartesian.

Example 6.4 The Chevalley-Beck condition holds for $\mathrm{cod} : \mathscr{C}^{\mathbf{2}} \longrightarrow \mathscr{C}$ for any category $\mathscr{C}$. To see this, notice that a square (6.1) in $\mathscr{C}^{\mathbf{2}}$ is a cube (6.2).

$$\begin{tikzcd}
R \arrow[rrr, "q_0"] \arrow[dr] \arrow[ddd, "p_0"'] & & & Y \arrow[dl] \arrow[ddd, "y_0"] \\
& M \arrow[r, "q_1"] \arrow[d, "p_1"'] & V \arrow[d, "y_1"] & \\
& U \arrow[r, "x_1"'] & W & \\
X \arrow[rrr, "x_0"'] \arrow[ur] & & & Z \arrow[ul]
\end{tikzcd} \tag{6.2}$$

The assumptions of Chevalley-Beck are that the front, back, top and bottom faces of (6.2) are pullbacks, and that y_0 is invertible. Since the top face is a pullback, p_0 is also invertible. So p is opcartesian.

If P is a cloven fibration and opfibration, the Chevalley-Beck condition is the requirement that, for all pullback squares

$$\begin{tikzcd}
M \arrow[r, "q"] \arrow[d, "p"'] & V \arrow[d, "k"] \\
U \arrow[r, "h"'] & W
\end{tikzcd} \tag{6.3}$$

in $\mathscr{C}$, the canonical natural transformation

$$\begin{tikzcd}
\mathscr{X}^U \arrow[rr, "h_\star"] \arrow[d, "p^\star"'] & & \mathscr{X}^W \arrow[d, "k^\star"] \\
\mathscr{X}^M \arrow[rr, "q_\star"'] \arrow[from=1-1, to=2-3, phantom, "\overset{\gamma}{\Longrightarrow}" description] & & \mathscr{X}^V
\end{tikzcd} \tag{6.4}$$

(mate to $k_\star q_\star = h_\star p_\star$) is invertible. Under these circumstances, each category C in $\mathscr{C}$ determines a monad T_C on the category $\mathscr{X}^{C_0}$ as follows. The endofunctor for the monad is the composite

$$\mathscr{X}^{C_0} \xrightarrow{d_0^\star} \mathscr{X}^{C_1} \xrightarrow{d_{1\star}} \mathscr{X}^{C_0} \ .$$

The unit is the composite natural transformation

$$1 \cong (d_1 i)_\star (d_0 i)^\star \cong d_{1\star} i_\star i^\star d_0^\star \xrightarrow{d_{1\star}\varepsilon d_0^\star} d_{1\star} d_0^\star$$

where $\varepsilon : i_\star i^\star \longrightarrow 1$ is the counit for $i_\star \dashv i^\star$. The multiplication is the composite natural transformation

$$d_{1\star} d_0^\star d_{1\star} d_0^\star \xrightarrow{d_{1\star}\gamma^{-1} d_0^\star} d_{1\star} d_{2\star} d_0^\star d_0^\star \cong d_{1\star} d_{1\star} d_1^\star d_0^\star \xrightarrow{d_{1\star}\varepsilon d_0^\star} d_{1\star} d_0^\star$$

using the γ^{-1} of 6.3 for a pullback in condition (C4) of Section 1 and the counit ε for $d_{1\star} \dashv d_0^\star$.

The following result appears as Proposition (9.10) of [39].

Proposition 6.5 *If $P : \mathscr{X} \longrightarrow \mathscr{C}$ is a cloven fibration and opfibration satisfying the Chevalley-Beck condition then, for each category C in $\mathscr{C}$ there is an equivalence of categories*

$$\mathscr{X}^C \simeq (\mathscr{X}^{C_0})^{T_C}$$

where the right-hand side is the category of Eilenberg-Moore algebras for the monad T_C.

Proof Just as for descent data with (5.3), for each $X \in \mathscr{X}^C$, we obtain a unique morphism

$$e : d_0^\star X_0 \longrightarrow d_1^\star X_0 \tag{6.5}$$

in $\mathscr{X}_1^C$ defined by the equation

$$(d_0^\star X_0 \xrightarrow{e} d_1^\star X_0 \xrightarrow{d_1^{X_0}} X_0) = (d_0^\star X_0 = X_1 \xrightarrow{d_1} X_0) \ .$$

Diagrams as in (5.4) and (5.5) hold, yielding that the mate

$$\hat{e} : T_C X_0 = d_{1\star} d_0^\star X_0 \longrightarrow X_0 \ ,$$

of e under the adjunction $d_{1\star} \dashv d_1^\star$, is a T_C-algebra structure on X_0. The remainder is left to the reader. $\square$

Example 6.6 For the moment, consider Cat as a category and let us look at the fibration

$$\mathrm{cod} : \mathrm{Cat}^{\mathbf{2}} \longrightarrow \mathrm{Cat} \ .$$

The cartesian morphisms for a cleavage are the chosen pullbacks in Cat. The chosen opcartesian morphisms over $H : \mathscr{C} \longrightarrow \mathscr{D}$ are commutative squares of the following form.

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "1_{\mathscr{X}}"] \arrow[d] & \mathscr{X} \arrow[d] \\
\mathscr{C} \arrow[r, "H"'] & \mathscr{D}
\end{tikzcd}$$

As we saw using (6.2), the Chevalley-Beck condition holds. For any category $\mathscr{C}$, we have a category

$$\begin{tikzcd}
\mathrm{sq}\mathscr{C} : \mathscr{C}^{\mathbf{3}} \arrow[r, shift left=3] \arrow[r, "\mathrm{comp}" description] \arrow[r, shift right=3] & \mathscr{C}^{\mathbf{2}} \arrow[r, shift left=3, "\mathrm{dom}" description] \arrow[r, shift right=3, "\mathrm{cod}" description] & \mathscr{C} \arrow[l, "\mathrm{id}" description]
\end{tikzcd} \tag{6.6}$$

in Cat obtained as

$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{incl}} \mathrm{Cat}^{\mathrm{op}} \xrightarrow{[-,\mathscr{C}]} \mathrm{Cat}\ .$$

The reason for the name $\mathrm{sq}\mathscr{C}$ is because it is the 'double category of squares' in the category $\mathscr{C}$. Proposition 6.5 yields an equivalence of categories

$$(\mathrm{Cat})^{\mathrm{sq}\mathscr{C}} \simeq (\mathrm{Cat}/\mathscr{C})^{T_{\mathrm{sq}\mathscr{C}}}\ . \tag{6.7}$$

The monad $T_{\mathrm{sq}\mathscr{C}}$ is easily described. The endofunctor of $\mathrm{Cat}/\mathscr{C}$ takes a functor $P : \mathscr{X} \longrightarrow \mathscr{C}$ to the split opfibration

$$\mathrm{cod} : P/\mathscr{C} \longrightarrow \mathscr{C}$$

with domain the slice $P/\mathscr{C}$ of the functor $P : \mathscr{X} \longrightarrow \mathscr{C}$ over $1_{\mathscr{C}} : \mathscr{C} \longrightarrow \mathscr{C}$. The objects of $P/\mathscr{C}$ are (X, h, U) where $X \in \mathscr{X}$ and $h : PX \longrightarrow U$ in $\mathscr{C}$. The multiplication of the monad at P is the functor

$$\begin{tikzcd}
\mathrm{cod}/\mathscr{C} \arrow[rr, "\mu"] \arrow[dr, "\mathrm{cod}"'] & & P/\mathscr{C} \arrow[dl, "\mathrm{cod}"] \\
& \mathscr{C} &
\end{tikzcd}$$

taking (X, h, U, r, V) to (X, rh, V) and the unit of the monad at P is the functor

$$\begin{tikzcd}
\mathscr{X} \arrow[rr, "\eta"] \arrow[dr, "P"'] & & P/\mathscr{C} \arrow[dl, "\mathrm{cod}"] \\
& \mathscr{C} &
\end{tikzcd}$$

taking X to $(X, 1_{PX}, PX)$. In fact, the categories in (6.7) are 2-categories in an obvious way ($T_{\mathrm{sq}\mathscr{C}}$ is a 2-monad on $\mathrm{Cat}/\mathscr{C}$) and the equivalence is an equivalence of 2-categories. The $T_{\mathrm{sq}\mathscr{C}}$-algebras are easily identified as split opfibrations $P : \mathscr{X} \longrightarrow \mathscr{C}$; the $T_{\mathrm{sq}\mathscr{C}}$-action is defined by the functor

$$\alpha : P/\mathscr{C} \longrightarrow \mathscr{X}$$

taking (X, h, U) to $h_\star X \in \mathscr{X}^U$. If we write $\mathrm{Spl}_{\mathrm{op}\mathscr{C}}$ for the 2-category of split opfibrations over $\mathscr{C}$, cleavage preserving functors over $\mathscr{C}$, and natural transformations over $\mathscr{C}$, we see that (6.7) can be prolonged to equivalences of 2-categories

$$(\mathrm{Cat})^{\mathrm{sq}\mathscr{C}} \simeq (\mathrm{Cat}/\mathscr{C})^{T_{\mathrm{sq}\mathscr{C}}} \cong \mathrm{Cat}^{\mathscr{C}} \simeq [\mathscr{C}, \mathrm{Cat}]\ . \tag{6.8}$$

Example 6.7 For any category B in $\mathscr{C}$, the codescent object of the simplicial object

$$\begin{tikzcd} B_2 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_2"'] & B_1 \arrow[r, shift left=2, "d_0"] \arrow[r, leftarrow, "i" description] \arrow[r, shift right=2, "d_1"'] & B_0 \end{tikzcd} \tag{6.9}$$

in Cat$\mathscr{C}$ is easily seen to be B itself. For, the definitions at the beginning of Section 1 make it clear that

$$(\mathrm{Cat}\mathscr{C})(B,C)$$

is the descent category for the simplicial category

$$\begin{tikzcd} \ldots(\mathrm{Cat}\mathscr{C})(B_2,C) \arrow[r, leftarrow, shift left=4, "\partial_0"] \arrow[r, shift left=2, "\sigma_0"] \arrow[r, leftarrow, "\partial_1" description] \arrow[r, shift right=2, "\sigma_1"'] \arrow[r, leftarrow, shift right=4, "\partial_2"'] & (\mathrm{Cat}\mathscr{C})(B_1,C) \arrow[r, leftarrow, shift left=2, "\partial_0"] \arrow[r, "\sigma" description] \arrow[r, leftarrow, shift right=2, "\partial_1"'] & (\mathrm{Cat}\mathscr{C})(B_0,C) \end{tikzcd} \tag{6.10}$$

where $\partial_p = (\mathrm{Cat}\mathscr{C})(d_p, 1_C)$. Observe too that, for each U in $\mathscr{C}$, the nerve of the category $\mathscr{C}(U,B)$ is

$$\begin{tikzcd} \ldots\mathscr{C}(U,B_2) \arrow[r, shift left=2, "\mathscr{C}(1{,}d_0)"] \arrow[r, "\mathscr{C}(1{,}d_1)" description] \arrow[r, shift right=2, "\mathscr{C}(1{,}d_2)"'] & \mathscr{C}(U,B_1) \arrow[r, shift left=2, "\mathscr{C}(1{,}d_0)"] \arrow[r, leftarrow, "\mathscr{C}(1{,}i)" description] \arrow[r, shift right=2, "\mathscr{C}(1{,}d_1)"'] & \mathscr{C}(U,B_0)\ ; \end{tikzcd}$$

so the codescent object of (6.9) is preserved by the 2-functors $\mathscr{C}(U,-) : \mathrm{Cat}\mathscr{C} \longrightarrow \mathrm{Cat}$. This provides another proof of Corollary 2.5 that $\mathscr{C} \longrightarrow \mathrm{Cat}\mathscr{C}$ is dense, by showing that there is a density presentation via a codescent construction.

7 Split fibrations in a category

Let $\mathscr{C}$ be a category with pullbacks. For each category B in $\mathscr{C}$, we can construct a category $B^{\mathbf{2}}$ in $\mathscr{C}$ as follows. Form the diagram

$$\begin{tikzcd} & & P \arrow[dl, "p"'] \arrow[dr, "q"] & & \\ & B_2 \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & & B_2 \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & \\ B_1 & & B_1 & & B_1\ , \end{tikzcd}$$

in which the diamond is a pullback, to obtain the graph

$$\begin{tikzcd} P \arrow[r, shift left=1, "d_0 p"] \arrow[r, shift right=1, "d_2 q"'] & B_1\ , \end{tikzcd}$$

which underlies our category $B^{\mathbf{2}}$. There is a graph isomorphism

$$\mathscr{C}(U,B^{\mathbf{2}}) \cong \mathscr{C}(U,B)^{\mathbf{2}} \tag{7.1}$$

natural in U. The right-hand side of (7.1) has a category structure: the arrow category of $\mathscr{C}(U,B)$. Since pullbacks exist in $\mathscr{C}$, we can construct the objects required to complete $B^{\mathbf{2}}$ to a simplicial object in $\mathscr{C}$ and, by Yoneda, we transport the morphisms across (7.1) then internalize them to $\mathscr{C}$. Using Corollary 2.5 or (6.9), we extend the isomorphism (7.1) to an isomorphism

$$(\mathrm{Cat}\mathscr{C})(A,B^{\mathbf{2}}) \cong (\mathrm{Cat}\mathscr{C})(U,B)^{\mathbf{2}} \ , \tag{7.2}$$

2-natural in A. This says that $B^{\mathbf{2}}$ is the cotensor of $\mathbf{2}$ and B in the 2-category $\mathrm{Cat}\mathscr{C}$.

The 2-category $\mathscr{K} = \mathrm{Cat}\mathscr{C}$ has pullbacks formed pointwise with the simplicial objects. We can therefore compose the span

$$\begin{tikzcd}B & B^{\mathbf{2}} \arrow[l, "d_0"'] \arrow[r, "d_1"] & B\end{tikzcd}$$

with itself; for example, we obtain

$$\begin{tikzcd}
 & & B^{\mathbf{3}} \arrow[dl, "p"'] \arrow[dr, "q"] & & \\
 & B^{\mathbf{2}} \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & & B^{\mathbf{2}} \arrow[dl, "d_1"'] \arrow[dr, "d_2"] & \\
B & & B & & B
\end{tikzcd}$$

and then a functor $d_1 : B^{\mathbf{3}} \longrightarrow B^{\mathbf{2}}$ lifting $d_1 : B_2 \longrightarrow B_1$. This gives a category

$$\mathrm{sq}B \ : \ \begin{tikzcd}B^{\mathbf{3}} \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_2"'] & B^{\mathbf{2}} \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & B \arrow[l, "i" description]\end{tikzcd}$$

in $\mathscr{K} = \mathrm{Cat}\mathscr{C}$ such that there is a natural isomorphism

$$\mathscr{C}(U,\mathrm{sq}B) \cong \mathrm{sq}\mathscr{C}(U,B)$$

where the right-hand side is explained at (6.6). Using (6.9), we extend the isomorphism to

$$\mathscr{K}(A,\mathrm{sq}B) \cong \mathrm{sq}\mathscr{K}(U,B) \ . \tag{7.3}$$

Looking at the fibration

$$\mathrm{cod} : \mathscr{K}^{\mathbf{2}} \longrightarrow \mathscr{K}$$

and the category $\mathrm{sq}B$ in $\mathrm{Cat}\mathscr{C}$, we obtain a category

$$\mathscr{K}^B := (\mathscr{K}^{\mathbf{2}})^{\mathrm{sq}B} \ .$$

By Proposition 6.5, we have an equivalence

$$\mathscr{K}^B \simeq (\mathscr{K}/B)^{T_{\mathrm{sq}B}} \ . \tag{7.4}$$

The monad $T_{\mathrm{sq}B}$ is easily identified. For any functor $f : X \longrightarrow B$ in $\mathscr{C}$, form the pullback as in the square

$$\begin{tikzcd}
f/B \arrow[r, "q"] \arrow[d, "d_0"'] & B^{\mathbf{2}} \arrow[r, "d_1"] \arrow[d, "d_0"] & B \\
X \arrow[r, "f"'] & B \ .
\end{tikzcd}$$

The endofunctor of $T_{\mathrm{sq}B}$ takes $f : X \longrightarrow B$ to $d_1 = d_1 q : f/B \longrightarrow B$. We see that $T_{\mathrm{sq}B}$ is actually a 2-monad on $\mathscr{K}/B$ yielding a 2-category $\mathscr{K}^B$.

The objects of $\mathscr{K}^B$ are called *split opfibrations over* B in $\mathscr{C}$. Such an object is a functor $p : E \longrightarrow B$ in $\mathscr{C}$ equipped with a functor $d_1 : p/B \longrightarrow E$ for which there exists a category structure in $\mathscr{K}^{\mathbf{2}}$ extending the following diagram.

$$\begin{tikzcd}
p/B \arrow[r, shift left=2, "d_0"] \arrow[r, leftarrow, "i" description] \arrow[r, shift right=2, "d_1"'] \arrow[d] & E \arrow[d, "p"] \\
B^{\mathbf{2}} \arrow[r, shift left=2, "d_0"] \arrow[r, leftarrow, "i" description] \arrow[r, shift right=2, "d_1"'] & B
\end{tikzcd} \tag{7.5}$$

The functor i in the top line of (7.5) is defined by the requirement that the pasted composite 2-cell

$$\begin{tikzcd}
E \arrow[dr, dashed, "i"] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "1"'] & & \\
& p/B \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\lambda}{\Longrightarrow}"] & B \arrow[d, "1"] \\
& E \arrow[r, "p"'] & B
\end{tikzcd} \tag{7.6}$$

should be the identity 2-cell of p. In fact, the action $d_1 : p/B \longrightarrow E$ of the monad $T_{\mathrm{sq}B}$ is unique up to isomorphism since one can show that it is left adjoint $d_1 \dashv i$ to i.

We already defined discrete opfibration in Section 3. Replacing C by B in the pullback (3.1), we can deduce the pullback

$$\begin{tikzcd}
E^{\mathbf{2}} \arrow[r, "d_0"] \arrow[d, "p^{\mathbf{2}}"'] & E \arrow[d, "p"] \\
B^{\mathbf{2}} \arrow[r, "d_0"'] & B
\end{tikzcd} \tag{7.7}$$

in $\mathscr{K}$. Define $j : E^{\mathbf{2}} \longrightarrow p/B$ by the following equation.

$$\begin{tikzcd}
E^{\mathbf{2}} \arrow[dr, dashed, "j"] \arrow[drr, bend left, "pd_1"] \arrow[ddr, bend right, "d_0"'] & & \\
& p/B \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\xRightarrow{\lambda}"] & B \arrow[d, "1"] \\
& E \arrow[r, "p"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
E^{\mathbf{2}} \arrow[r, bend left, "d_0"] \arrow[r, bend right, "d_1"'] \arrow[r, phantom, "\Downarrow \lambda"] & E \arrow[r, "p"] & B
\end{tikzcd}
\tag{7.8}$$

It follows that a functor $p : E \longrightarrow B$ in $\mathscr{C}$ is a discrete opfibration if and only if j is invertible. We can see then that p is indeed an opfibration by taking the d_1 of (7.5) to be the composite

$$p/B \xrightarrow{j^{-1}} E^{\mathbf{2}} \xrightarrow{d_1} E\ . \tag{7.9}$$

We can also then see that every commutative triangle

$$\begin{tikzcd}
E \arrow[rr, "f"] \arrow[dr, "p"'] & & F \arrow[dl, "q"] \\
& B &
\end{tikzcd}
\tag{7.10}$$

in $\mathscr{K}$, with q a discrete opfibration and p any split opfibration, is a morphism in $\mathscr{K}^B$. Write $\mathscr{C}^B$ for the full subcategory of $\mathscr{K}^B$ whose objects are the discrete opfibrations. Our observations show that the 2-functor

$$\mathscr{C}^B \xrightarrow{\text{incl.}} \mathscr{K}^B \xrightarrow{\text{und.}} \mathscr{K}/B \tag{7.11}$$

is fully faithful.

Let $p : E \longrightarrow C$ be a discrete opfibration in $\mathscr{C}$. Recall the definition of $\bar{p}^U : \mathscr{C}(U, C) \longrightarrow \mathscr{C}/U$ in Section 3 (3.5). For each category B in $\mathscr{C}$, we have a morphism of diagrams

$$\begin{tikzcd}[column sep=large]
\mathscr{C}(B_0, C) \arrow[r, shift left=3, "\mathscr{C}(d_0{,}1)"] \arrow[r, leftarrow, "\mathscr{C}(i{,}1)" description] \arrow[r, shift right=3, "\mathscr{C}(d_2{,}1)"'] \arrow[d, "\bar{p}^{B_0}"'] & \mathscr{C}(B_1, C) \arrow[r, shift left=3, "\mathscr{C}(d_0{,}1)"] \arrow[r, "\mathscr{C}(d_1{,}1)" description] \arrow[r, shift right=3, "\mathscr{C}(d_2{,}1)"'] \arrow[d, "\bar{p}^{B_1}"'] & \mathscr{C}(B_2, C) \arrow[d, "\bar{p}^{B_2}"'] \\
\mathscr{C}/B_0 \arrow[r, shift left=3, "d_0^\star"] \arrow[r, leftarrow, "i^\star" description] \arrow[r, shift right=3, "d_1^\star"'] & \mathscr{C}/B_1 \arrow[r, shift left=3, "d_0^\star"] \arrow[r, "d_1^\star" description] \arrow[r, shift right=3, "d_2^\star"'] & \mathscr{C}/B_2
\end{tikzcd}
\tag{7.12}$$

in the ‘pseudo’ sense. This induces a functor

$$\bar{p}^B : \mathscr{K}(B, C) \longrightarrow \mathscr{C}^B \tag{7.13}$$

on the descent categories. Clearly then:

Proposition 7.1 *If $p : E \longrightarrow C$ is an internal full subcategory of $\mathscr{C}$ then, for all categories B in $\mathscr{C}$, the functors $\bar{p}^B$ of (7.13) are fully faithful.*

Corollary 7.2 *If $\mathscr{C}$ is a finitely complete, cartesian closed category and $p : E \longrightarrow C$ is an internal full subcategory then the 2-category $\mathscr{K} = \mathrm{Cat}\mathscr{C}$ equipped with the usual duality involution interchanging d_0 and d_1, and the classifying discrete opfibration p, is a 2-topos in the sense of Weber [44], who shows that this leads to a* good Yoneda structure*, and so a Yoneda structure [43] on $\mathscr{K}$.*

8 Two-sided discrete fibrations

George Janelidze recently pointed out to the author that Nobuo Yoneda had the notion of two-sided fibration in his 1960 paper [46] under the name "regular span". Jean Bénabou [1] cites that paper for the term "span".

This Section is a slight reworking of some material from [35, 39]. Some standard results in those papers will be quoted without reproving them here.

Let $\mathscr{K}$ be a 2-category. The *identee* of a morphism $f : A \to B$ is a 2-universal 2-cell

$$\begin{tikzcd} C \arrow[r, bend left, "u"{name=U}] \arrow[r, bend right, "v"'{name=V}] \arrow[from=U, to=V, phantom, "\Downarrow \theta"] & A \end{tikzcd}$$

with the property that the 2-cell $f \circ \theta$ is an identity. If 1 is terminal in $\mathscr{K}$, the identee of the unique $A \to 1$ is $A^{\mathbf{2}}$ with its universal 2-cell.

If $\mathscr{K}$ is finitely complete, as we henceforth suppose, we can construct an identee of f as a pullback

$$\begin{tikzcd} C \arrow[r, "\hat{\theta}"] \arrow[d] & A^{\mathbf{2}} \arrow[d, "f^{\mathbf{2}}"] \\ B \arrow[r, "i"'] & B^{\mathbf{2}} \end{tikzcd}$$

where $\hat{\theta}$ composes with the universal 2-cell out of $A^{\mathbf{2}}$ to give θ.

A *discrete fibration* (p, E, q) from B to A in $\mathscr{K}$ is a diagram

$$\begin{tikzcd} A & E \arrow[l, "p"'] \arrow[r, "q"] & B \end{tikzcd} \tag{8.1}$$

with an identee of q of the form

$$\begin{tikzcd} A/p \arrow[r, bend left, "m^*"{name=U}] \arrow[r, bend right, "d_1"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow \lambda"] & E \end{tikzcd} \tag{8.2}$$

and an identee of p of the form

$$\begin{tikzcd} q/B \arrow[r, bend left, "d_0"{name=U}] \arrow[r, bend right, "m_*"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow \rho"] & E \end{tikzcd} \tag{8.3}$$

such that the square (8.4) is a pullback square (in which the left side is induced by p and the top side is induced by q).

$$\begin{tikzcd} E^{\mathbf{2}} \arrow[r] \arrow[d] & q/B \arrow[d, "m_*"] \\ A/p \arrow[r, "m^*"'] & E \end{tikzcd} \tag{8.4}$$

The structure m^*, m_*, λ, ρ is unique if it exists; that is, being a discrete fibration is a property of a span.

A *morphism* $f : (p, E, q) \to (r, F, s)$ *of discrete fibrations* from B to A is simply a morphism of spans; that is, a commutative diagram (8.5).

$$\begin{tikzcd} & E \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "f"] & \\ A & & B \\ & F \arrow[ul, "r"] \arrow[ur, "s"'] & \end{tikzcd} \tag{8.5}$$

It turns out that morphisms do automatically commute with the m^*, m_*, λ, ρ structures. We write $\mathrm{DFib}(\mathscr{K})(B, A)$ for the category so obtained.

We call $p : E \to A$ a discrete fibration in $\mathscr{K}$ when $(p, E, !)$ is a discrete fibration from 1 to A. We call $q : E \to B$ a discrete opfibration in $\mathscr{K}$ when $(!, E, q)$ is a discrete fibration from B to 1.

Given a discrete fibration (p, E, q) from B to A and morphisms $a : C \to A$ and $b : D \to B$

in $\mathscr{K}$, the limit diagram (8.6)

$$\begin{tikzcd}
 & & E(a,b) \arrow[dll, "\bar{p}"'] \arrow[d] \arrow[drr, "\bar{q}"] & & \\
C \arrow[dr, "a"'] & & E \arrow[dl, "p"'] \arrow[dr, "q"] & & D \arrow[dl, "b"] \\
 & A & & B &
\end{tikzcd} \tag{8.6}$$

yields a discrete fibration $(\bar{p}, E(a,b), \bar{q})$ from D to C. With limits chosen in $\mathscr{K}$, this defines the object function of a functor

$$-(a,b) : \mathrm{DFib}(\mathscr{K})(B,A) \longrightarrow \mathrm{DFib}(\mathscr{K})(D,C)$$

the definition on morphisms uses the universal property of limit. This defines a pseudofunctor

$$\mathrm{DFib}(\mathscr{K})(-,-) : \mathscr{K}^{\mathrm{op}} \times \mathscr{K}^{\mathrm{coop}} \longrightarrow \mathrm{CAT}$$

on morphisms; the definition on 2-cells uses the discrete fibration property.

Here we are interested in $\mathscr{K} = \mathrm{Cat}(\mathscr{C})$. A discrete fibration from B to 1 is precisely a discrete opfibration over B in $\mathscr{C}$, as defined in Section 3. As an exercise the reader might like to show that, for a discrete fibration (p, E, q) from B to A, the morphism $q : E \to B$ is a split opfibration in $\mathscr{K}$ in the sense of Section 7.

Proposition 8.1 *For* $\mathscr{K} = \mathrm{Cat}(\mathscr{C})$*, there is a pseudonatural equivalence of categories*

$$\mathrm{DFib}(\mathscr{K})(B,A) \simeq \mathrm{DFib}(\mathscr{K})(A^{\mathrm{op}} \times B, 1)(= \mathscr{C}^{A^{\mathrm{op}} \times B}) .$$

Proof Let (p, E, q) be a discrete fibration from B to A. Observe that, by taking objects of objects in the pullback (8.4), we see that E_1 can be reconstructed from A, B, E_0, p_0, q_0 and $(m^*)_0, (m_*)_0$. We wish to define a discrete opfibration $(r,s) : \widetilde{E} \to A^{\mathrm{op}} \times B$. Put

$$\widetilde{E_0} = E_0, r_0 = p_0 : E_0 \to A_0, s_0 = q_0 : E_0 \to B_0$$

and define, as we must, $\widetilde{E_1}$ via the pullback (8.7).

$$\begin{tikzcd}
\widetilde{E_1} \arrow[r, "d_0"] \arrow[d, "{(r_1,s_1)}"'] & E_0 \arrow[d, "{(p_0,q_0)}"] \\
A_1 \times B_1 \arrow[r, "d_1 \times d_0"'] & A_0 \times B_0
\end{tikzcd} \tag{8.7}$$

This gives the pullback (8.8).

$$\begin{tikzcd}
\widetilde{E_1} \arrow[r, "d_2"] \arrow[d, "d_0"'] & (q/B)_0 \arrow[d, "d_0"] \\
(A/p)_0 \arrow[r, "d_1"'] & E_0
\end{tikzcd} \tag{8.8}$$

The morphism $r_1 : \widetilde{E_1} \to A_1$ is the composite of the left side d_0 of (8.8) with the canonical $(A/p)_0 \to A_1$. The morphism $s_1 : \widetilde{E_1} \to B_1$ is the composite of the top side d_2 of (8.8) with the canonical $(q/B)_0 \to B_1$. Onto the square (8.8) paste the λ of (8.9) on the bottom side at d_1 and paste the ρ of (8.3) on the right side at d_0 to obtain a 2-cell

$$\begin{tikzcd}
\widetilde{E_1} \arrow[r, bend left, "m^* d_0"{name=U}] \arrow[r, bend right, "m_* d_1"'{name=D}] & E
\arrow[from=U, to=D, phantom, "\Downarrow (\rho d_2)(\lambda d_0)"]
\end{tikzcd} \tag{8.9}$$

which yields a morphism $\widetilde{E_1} \to E_1$. The morphism $d_1 : \widetilde{E_1} \to E_0$ is then defined as the composite of $\widetilde{E_1} \to E_1$ with the diagonal of the square (8.4). So we have the underlying graph in $\mathscr{C}$ of the desired category $\widetilde{E}$ in $\mathscr{C}$. To complete the proof, since we are dealing purely with limits in $\mathscr{C}$, it suffices to check that this construction gives the composite of the well-known equivalences

$$\mathrm{DFib}(\mathrm{Cat})(B, A) \simeq \mathrm{CAT}(A^{\mathrm{op}} \times B, \mathrm{Set}) \simeq \mathrm{DFib}(\mathrm{Cat})(A^{\mathrm{op}} \times B, 1)$$

in the case $\mathscr{C} = \mathrm{Set}$. □

The image of the discrete fibration $(d_0, A^{\mathbf{2}}, d_1)$ from A to A under the equivalence of Proposition 8.1 is the discrete opfibration

$$(d, c) : \widetilde{A^{\mathbf{2}}} \longrightarrow A^{\mathrm{op}} \times A$$

where $\widetilde{A^{\mathbf{2}}}$ is the *twisted arrow category* of A whose underlying graph is

$$\begin{tikzcd}
A_3 \arrow[r, shift left, "d_3 d_0"] \arrow[r, shift right, "d_1 d_2"'] & A_1
\end{tikzcd} .$$

9 Size and cocompleteness

In Section 5 of Weber [44], given a 2-topos, it is shown how to construct a Yoneda structure in the sense of Street-Walters [43] which is 'good' in Weber's sense (compare Theorem 7 of [36]). After Corollary 7.2, we are in a position to do this. However, it is of particular interest when the Yoneda morphisms provide some kind of cocompletion of their domain object. This section addresses that question by looking at cocompleteness concepts for an internal full subcategory.

If $p : E \longrightarrow C$ is an internal full subcategory of $\mathscr{C}$, we say a morphism $q : F \longrightarrow U$ in $\mathscr{C}$ is *C-fibred* when there exist a morphism $f : U \longrightarrow C_0$ and a pullback square (9.1).

$$\begin{tikzcd} F \arrow[r] \arrow[d, "q"'] & E_0 \arrow[d, "p_0"] \\ U \arrow[r, "f"'] & C_0 \end{tikzcd} \tag{9.1}$$

Clearly C-fibred morphisms are stable under pullback. A discrete opfibration $q : F \longrightarrow B$ will be called *C-fibred* when $q_0 : F_0 \longrightarrow B_0$ is C-fibred. This is reasonable in light of:

Proposition 9.1 *If $p : E \to C$ is an internal full subcategory of $\mathscr{C}$ and $q : F \to B$ is a discrete opfibration in $\mathscr{C}$ such that $q_0 : F_0 \to B_0$ is C-fibred then there exist a functor $f : B \to C$ in $\mathscr{C}$ and a pullback square in the 2-category* Cat$\mathscr{C}$ *as below.*

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ B \arrow[r, "f"'] & C \end{tikzcd}$$

Consequently, $q : F \longrightarrow B$ is in the essential image of (7.13) *if and only if $q_0 : F_0 \longrightarrow B_0$ is C-fibred.*

Proof We will begin by giving an explicit construction. Then we will give a simple conceptual proof of the result.

By assumption we have a pullback (9.2).

$$\begin{tikzcd} F_0 \arrow[r, "g_0"] \arrow[d, "q_0"'] & E_0 \arrow[d, "p_0"] \\ B_0 \arrow[r, "f_0"'] & C_0 \end{tikzcd} \tag{9.2}$$

We must define a graph morphism $f : B \to C$ with f_0 as in (9.2). This is to give a morphism $\phi : f_0 d_0 \to f_0 d_1$ in the category $\mathscr{C}(B_1, C)$. Using the fully faithful functor $\mathscr{C}(B_1, C) \longrightarrow \mathscr{C}/B_1$

determined by the internal full subcategory, we see that such morphisms are in bijection with morphisms $(q/B)_0 \to (B/q)_0$ over B_1. However, the commutative square

$$\begin{tikzcd}
F_1 \arrow[r, "d_1"] \arrow[d, "q_1"'] & F_0 \arrow[d, "q_0"] \\
B_1 \arrow[r, "d_1"'] & B_0
\end{tikzcd}$$

induces an arrow $\widehat{d_1} : F_1 \to (B/q)_0$ whose composite with the isomorphism $(q/B)_0 \cong F_1$, gives what we want to obtain $\phi : f_0 d_0 \to f_0 d_1$. As part of this construction we have the morphism $\rho : g_0 d_0 \to d_1 \widehat{d_1}$ in $\mathscr{C}(F_1, E)$, opcartesian for $\mathscr{C}(F_1, p)$, over the morphism $\phi q_1 : f_0 d_0 q_1 \to f_0 d_1 q_1$ in $\mathscr{C}(F_1, C)$. This ρ is a morphism $g_1 : F_1 \to E_1$ which, together with g_0 as in (9.2), gives a graph morphism $g : F \to E$. It remains to check that f and g are functors in $\mathscr{C}$ and form the pullback of the proposition. Instead of doing this we will give a Yoneda-lemma-style proof.

The pullback (9.2) implies that the following solid square commutes up to isomorphism.

$$\begin{tikzcd}
\mathscr{C}(U, B_0) \arrow[r] \arrow[d, "\mathscr{C}(U,f_0)"'] & \mathscr{C}(U, B) \arrow[d, "\bar{q}_U"] \arrow[dl, dashed, "f_U"'] \\
\mathscr{C}(U, C) \arrow[r, "\bar{p}_U"'] & \mathscr{C}/U
\end{tikzcd}$$

The top functor of the square is bijective on objects. The bottom functor is fully faithful. It follows (see Proposition 23 of [43]) that there is a unique functor $f_U : \mathscr{C}(U, B) \to \mathscr{C}(U, C)$, as shown by the dotted arrow, such that the left triangle commutes and the right triangle commutes up to an isomorphism which gives back the isomorphism in the square on pasting the two triangles. By the generalized Yoneda lemma (Theorem 2.3), f_U is isomorphic to $\mathscr{C}(U, f)$ for some functor $f : B \to C$. The right triangle then gives the pullback of the Proposition. □

We say the internal full subcategory C *has coproducts* when the composite of any two composable C-fibred morphisms is C-fibred. We say the internal full subcategory C *has a terminator* when every identity morphism $1_U : U \to U$ in $\mathscr{C}$ is C-fibred. If $\mathscr{C}$ has a terminal object 1 then, in the last sentence, it suffices for the identity morphism 1_1 of 1 to be C-fibred; we have a terminal object $t : 1 \to C$ in the category $\mathscr{C}(1, C)$ and a terminal object $U \xrightarrow{!} 1 \xrightarrow{t} C$ in $\mathscr{C}(U, C)$.

An object A of any 2-category $\mathscr{K}$ is said to *admit coequalizers* when, for all objects X, the category $\mathscr{K}(X,A)$ admits coequalizers and these are preserved by all functors of the form $\mathscr{K}(h,A) : \mathscr{K}(X,A) \longrightarrow \mathscr{K}(Y,A)$ where $h : Y \longrightarrow X$. Let $\mathbf{Pp}$ be the free category on the 'parallel pair' directed graph

$$\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd} .$$

A functor from $\mathbf{Pp}$ to a category $\mathscr{A}$ amounts to a pair of morphisms in $\mathscr{A}$ with the same domain and the same codomain. Suppose the cotensor $A^{\mathbf{Pp}}$ of $\mathbf{Pp}$ with A exists in the 2-category $\mathscr{K}$. There is a 'diagonal morphism' $\delta : A \longrightarrow A^{\mathbf{Pp}}$ corresponding to the parallel pair $(1_A, 1_A)$ of morphisms in the category $\mathscr{K}(A,A)$. It is easy to see that the object A admits coequalizers if and only if the morphism δ has a left adjoint.

Proposition 9.2 *Suppose C is an internal full subcategory of the finitely complete category $\mathscr{C}$. The following properties pertain to the 2-category $\mathscr{K} = \mathrm{Cat}\mathscr{C}$.*

- *(i) If C has a terminator then each category $\mathscr{C}(U,C)$ has a terminal object preserved by each functor $\mathscr{C}(r,C) : \mathscr{C}(U,C) \longrightarrow \mathscr{C}(V,C)$ for $r : V \longrightarrow U$. Indeed, there exists a right adjoint t to the unique functor $C \longrightarrow 1$.*
- *(ii) If C has coproducts and $h : U \longrightarrow V$ in $\mathscr{C}$ is C-fibred then every morphism $f : U \longrightarrow C$ has a pointwise left extension along h.*
- *(iii) If C has coproducts and admits coequalizers, and $h : A \longrightarrow B$ is a functor for which $h_0 : A_0 \longrightarrow B_0$, $d_1 : A_1 \longrightarrow A_0$ and $d_1 : B_1 \longrightarrow B_0$ are C-fibred, then every functor $f : A \longrightarrow C$ has a left extension along h.*

Proof (i) By hypothesis, there is an object u in $\mathscr{C}(U,C)$ taken by the fully faithful $\bar{p}^U$ of (3.5) to the terminal object 1_U of $\mathscr{C}/U$. It follows that u is a terminal object of $\mathscr{C}(U,C)$. By pseudonaturality of the $\bar{p}^U$, the functor $\mathscr{C}(r,C)$ preserves terminal objects since pullback $r^\star : \mathscr{C}/U \longrightarrow \mathscr{C}/V$ along r does. A right adjoint for $C \longrightarrow 1$ is any $t : 1 \longrightarrow C$ with $\bar{p}^1(t) = 1_1$.

(ii) Since $h^\star : \mathscr{C}/V \longrightarrow \mathscr{C}/U$ has a left adjoint Σ_h defined by composition with h, and since each $\bar{p}^U(f)$ is C-fibred, C having coproducts implies $\Sigma_h(\bar{p}^U(f))$ is in the image of $\bar{p}^V$.

So $\mathscr{C}(h,1) : \mathscr{C}(V,C) \longrightarrow \mathscr{C}(U,C)$ has a left adjoint obtained by restricting Σ_h along the components of $\bar{p}$. Next notice that each slice of the form h/b is actually a pullback

$$\begin{tikzcd}
P \arrow[r, "s"] \arrow[d, "r"'] & X \arrow[d, "b"] \\
U \arrow[r, "h"'] & V
\end{tikzcd} .$$

in Cat$\mathscr{C}$ since V is in $\mathscr{C}$. Form the cube

$$\begin{tikzcd}
\mathscr{C}(V,C) \arrow[rrr, "\mathscr{C}(h,1)"] \arrow[dddd, "\mathscr{C}(b,1)"'] \arrow[dr] & & & \mathscr{C}(U,C) \arrow[dddd, "\mathscr{C}(r,1)"] \arrow[dl] \\
& \mathscr{C}/V \arrow[r, "h^\star"] \arrow[d, "b^\star"'] & \mathscr{C}/U \arrow[d, "r^\star"] & \\
& \mathscr{C}^X \arrow[r, "s^\star"'] & \mathscr{C}^P & \\
& & & \\
\mathscr{C}(X,C) \arrow[rrr, "\mathscr{C}(s,1)"'] \arrow[uur] & & & \mathscr{C}(P,C) \arrow[uul]
\end{tikzcd}$$

in which all face squares commute up to isomorphism (indeed, the big front face commutes on the nose). The sloping inward edges are all fully faithful since they are components of $\bar{p}$. We need to see that the big front face commutes when the top and bottom edges are replaced by their left adjoints. Since U and V are discrete objects of Cat$\mathscr{C}$, the morphism h is a discrete opfibration. So the pullback s of h along b is also a discrete opfibration. So the left adjoint of $s^\star$ is the functor Σ_s defined by composition with s. Since a horizontally pasted composite of pullbacks is a pullback, the square

$$\begin{tikzcd}
\mathscr{C}/U \arrow[r, "\Sigma_s"] \arrow[d, "r^\star"'] & \mathscr{C}/V \arrow[d, "b^\star"] \\
\mathscr{C}^P \arrow[r, "\Sigma_h"'] & \mathscr{C}^X
\end{tikzcd}$$

commutes up to isomorphism. Evaluating this square at a C-fibred object $t : F \longrightarrow U$ of $\mathscr{C}$ and using that ht is C-fibred, we obtain an object of $\mathscr{C}^X$ in the replete image of $\bar{p}^X$, as required.

(iii) Consider commutative the square

$$\begin{tikzcd}
\mathscr{K}(B,C) \arrow[r, "\mathscr{K}(h{,}1)"] \arrow[d] & \mathscr{K}(A,C) \arrow[d] \\
\mathscr{C}(B_0,C) \arrow[r, "\mathscr{C}(h_0{,}1)"'] & \mathscr{C}(A_0,C).
\end{tikzcd}$$

It follows from Example 6.7 and Proposition 6.5 that the left and right sides of the square are monadic since $\mathscr{C}(d_1, 1_C)$ has a left adjoint for both $d_1 : A_1 \longrightarrow A_0$ and $d_1 : B_1 \longrightarrow B_0$. Since $\mathscr{K}(B, C)$ has coequalizers and $\mathscr{C}(h_0, 1)$ has a left adjoint, the adjoint triangle theorem of Dubuc [10] implies that $\mathscr{K}(h, 1)$ has a left adjoint, as required. □

Using ideas of Theorems 3 and 28 of [36], I strongly suspect that the left extensions in (iii) of Proposition 9.2 are pointwise. At this point, a proof eludes me.

Proposition 9.3 *Suppose C is an internal full subcategory of the finitely complete category $\mathscr{C}$. Suppose C has a terminator with $t : 1 \longrightarrow C$ right adjoint to the unique $C \longrightarrow 1$. Then there exists a 2-cell*

$$\begin{tikzcd}
E \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\overset{\lambda}{\Longrightarrow}"] & C \arrow[d, "1_C"] \\
1 \arrow[r, "t"'] & C
\end{tikzcd} \tag{9.3}$$

in Cat$\mathscr{C}$ *exhibiting E as the slice t/C. Moreover, the morphism $t : 1 \longrightarrow C$ of* Cat$\mathscr{C}$ *is dense* (4.3).

Proof First note that the fully faithful functor $\bar{p}^X$ takes $X \to 1 \overset{t}{\to} C$ to $1_X : X \to X$. So $\bar{p}^X$ on morphisms determines a bijection between 2-cells

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\overset{\theta}{\Longrightarrow}"] & C \arrow[d, "1_C"] \\
1 \arrow[r, "t"'] & C
\end{tikzcd}$$

and morphisms $X \to \bar{p}^X(f)$ over X. Since $\bar{p}^X(f)$ is a pullback of p along f, these are in bijection with morphisms $g : X \to E$ such that $pg = f$. Taking λ to correspond to $g = 1_E$ and a bit more work with 2-cells, we obtain the slice property required.

For the last sentence of the Proposition, we need to see that the diagram

$$\begin{tikzcd}
t/b \arrow[r] \arrow[d, "!"'] \arrow[dr, phantom, "\overset{\lambda}{\Longrightarrow}"] & X \arrow[d, "b"] \\
1 \arrow[r, "t"'] & C
\end{tikzcd}$$

exhibits b as a left extension of $t!$ along the top horizontal morphism. By the universal property of the slice t/f, 2-cells

$$\begin{tikzcd}
t/b \arrow[r] \arrow[d, "!"'] \arrow[dr, phantom, "\overset{\theta}{\Longrightarrow}"] & X \arrow[d, "f"] \\
1 \arrow[r, "t"'] & C
\end{tikzcd}$$

are in bijection with morphisms $t/b \longrightarrow t/f$ over X. Since the fully faithful $\bar{p}^X$ is defined by slicing $t/-$, these morphisms $t/b \longrightarrow t/f$ over X are in bijection with 2-cells $\phi : b \Longrightarrow f : X \longrightarrow C$, as required. □

Proposition 9.3 suggests a simplification of the description of an internal full subcategory with terminator. Recall Definitions 4.2 and 4.3.

Proposition 9.4 *Suppose $t : 1 \longrightarrow C$ is a fully faithful dense morphism in* Cat$\mathscr{C}$ *where $\mathscr{C}$ is a finitely complete category. Then C together with the discrete opfibration $d_1 : t/C \longrightarrow C$ is an internal full subcategory of C with terminator.*

Proof Since t is dense, the square

$$\begin{tikzcd}
t/f \arrow[r, "d_1"] \arrow[d, "!"'] \arrow[dr, phantom, "\overset{\lambda}{\Longrightarrow}"] & U \arrow[d, "f"] \\
1 \arrow[r, "t"'] & C
\end{tikzcd}$$

exhibits f as a left extension of $t!$ along d_1. It follows that morphisms $\theta : f \longrightarrow g$ in the category $\mathscr{C}(U, C)$ are in bijection with 2-cells $\phi : t! \longrightarrow gd_1$. But such ϕ are in bijection with morphisms $t/f \longrightarrow t/g$ over U by the universal property of the slice t/g. So the functor $\mathscr{C}(U, C) \longrightarrow \mathscr{C}/U$, taking f to $d_1 : t/f \longrightarrow U$, is fully faithful. That functor is pullback along $d_1 : t/C \longrightarrow C$. It follows that C is an internal full subcategory as asserted.

To say $t : 1 \longrightarrow C$ is fully faithful is to say the square

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] \arrow[dr, phantom, "="] & 1 \arrow[d, "t"] \\
1 \arrow[r, "t"'] & C
\end{tikzcd}$$

has the slice property for t/t. So the square

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "1_U"'] \arrow[dr, phantom, "="] & 1 \arrow[d, "t"] \\
U \arrow[r, "t!"'] & C
\end{tikzcd}$$

has the slice property for $t!/t$. It follows that C has a terminator. □

In the situation of Proposition 9.4, we have an explicit inverse equivalence to

$$\bar{p}^B : \mathscr{K}(B, C) \simeq C\text{-}\mathscr{C}^B \tag{9.4}$$

where $C\text{-}\mathscr{C}^B$ is the full subcategory of $\mathscr{C}^B$ consisting of the C-fibred discrete opfibrations. It takes the C-fibred discrete opfibration $q : F \to B$ to the pointwise left extension (9.5) of $t!$ along it.

$$\begin{tikzcd}
F \arrow[r, "q"] \arrow[d, "!"'] \arrow[dr, phantom, "\overset{\kappa}{\Longrightarrow}"] & B \arrow[d, "\mathrm{lan}_q(t!)"] \\
1 \arrow[r, "t"'] & C
\end{tikzcd} \tag{9.5}$$

Proposition 9.5 *In the situation of Proposition 9.4, C has coproducts if and only if, for all C-fibred $h : U \longrightarrow V$ in $\mathscr{C}$, every morphism $f : U \longrightarrow C$ has a pointwise left extension along h.*

Proof "Only if" is part (ii) of Proposition 9.2. So suppose C has the pointwise left extensions. Assume $f : U \to V$ and $g : V \to W$ are C-fibred. Then we have $h = \mathrm{lan}_f(t!)$, $U \cong t/h$ and $k = \mathrm{lan}_g(h)$.

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\overset{\lambda}{\Longrightarrow}"] & V \arrow[r, "g"] \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\kappa}{\Longrightarrow}"] & W \arrow[dl, "k"] \\
1 \arrow[r, "t"'] & C &
\end{tikzcd}$$

So $k = \mathrm{lan}_{g \circ f}(t!)$, yielding $\bar{p}^U(k) = g \circ f$ (see (9.4)). □

Using Yoneda structure terminology (see Corollary 7.2), we say a functor $f : A \to B$ is *C-admissible* when the pullback $(d_0, d_1) : (f/B)_0 \to A_0 \times B_0$ of $(d_0, d_1) : B_1 \to B_0 \times B_0$ along $f_0 \times 1_{B_0} : A_0 \times B_0 \to B_0 \times B_0$ is C-fibred. It follows from Proposition 9.1 that there exists a pullback

$$\begin{tikzcd}
\widetilde{f/B} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
A^{\mathrm{op}} \times B \arrow[r, "{B[f,1]}"'] & C
\end{tikzcd} \tag{9.6}$$

where the left side corresponds under Proposition 8.1 to the discrete fibration f/B from B to A. When $\mathscr{C}$ is cartesian closed, we obtain a morphism $B(f,1) : B \to [A^{\mathrm{op}}, C] = \widehat{A}$ corresponding to $B[f,1]$ in (9.6). Indeed, when A (that is, $1 : A \to A$) is also C-admissible, we have the diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "y_A"'] & \arrow[d, phantom, "\overset{\chi^f}{\Longrightarrow}", near start] & B \arrow[dl, "{B(f,1)}"] \\
& \widehat{A} &
\end{tikzcd} \tag{9.7}$$

as required for a Yoneda structure on $\mathscr{K} = \mathrm{Cat}$ (see Section 2 of [43]), where $y_A = A(1,1)$ and χ^f arises (using Propositions 7.1 and 8.1) from the canonical morphism $A^{\mathbf{2}} = A/A \to f/f$ of discrete fibrations from A to A.

An admissible object X of $\mathscr{K}$ will be called *C-total* (or "totally C-cocomplete") (see Section 6 of [43]) when $y_X : X \to \widehat{X}$ has a left adjoint. An object A of $\mathscr{K}$ will be called *C-small* (see Section 5 of [43]) when A and $\widehat{A}$ are admissible. The main result we wish to stress here, holding in a Yoneda structure, is the following.

Proposition 9.6 [43 Corollary 14] *If A is C-small then $\widehat{A}$ is C-total.*

When C has a terminator, the terminal object 1 of $\mathscr{C}$ is C-admissible and we have $\widehat{1} = C$ and $y_1 = t : 1 \to C$ (see Proposition 9.3).

It is not necessarily the case that C is C-admissible. For example, take $\mathscr{C} = \mathrm{Set}$ and C to be a full subcategory containing the sets of all finite cardinalities except 4. This C is not C-admissible since $C(2,2) \notin C$.

To say C is C-admissible is to say $(d_0, d_1) : C_1 \to C_0 \times C_0$ is C-fibred. Then we have a slice square (9.8).

$$\begin{tikzcd}
\widetilde{C^{\mathbf{2}}} \arrow[r, "!"] \arrow[d, "{(r,s)}"'] \arrow[dr, phantom, "\overset{\lambda}{\Longleftarrow}"] & 1 \arrow[d, "t"] \\
C^{\mathrm{op}} \times C \arrow[r, "{C[1,1]}"'] & C
\end{tikzcd} \tag{9.8}$$

Corollary 9.7 *If C is C-admissible then C is C-total.*

10 Internal full subcategories of Cat

An algorithm for finding internal full subcategories of a locally presentable category $\mathscr{C}$ was provided in [39]. This was applied to $\mathscr{C} = \mathrm{Cat}$ in Section 8.10 of that paper. However, here we shall give an internal full subcategory of Cat without going through the discovery process.

For a (small) category A, write $\hat{A}$ for the category $[A^{\mathrm{op}}, \mathrm{Set}]$ of contravariant set-valued functors on A. It is the small-colimit completion of A in the sense that restriction along the Yoneda embedding $y_A : A \longrightarrow \hat{A}$ yields an equivalence of categories

$$\mathrm{Cocts}(\hat{A}, X) \simeq [A, X] \ , \tag{10.1}$$

where the left side is the full subcategory of $[\hat{A}, X]$ consisting of the small-colimit-preserving functors into the small cocomplete category X.

Let mod_0 denote the category whose objects are small categories A and whose morphisms $m : A \longrightarrow B$ are colimit-preserving functors $m : \hat{A} \longrightarrow \hat{B}$. Each morphism $m : A \longrightarrow B$ determines a functor

$$\overline{m} : B^{\mathrm{op}} \times A \longrightarrow \mathrm{Set}$$

defined by

$$\overline{m}(b, a) = m(A(-, a))(b) \ . \tag{10.2}$$

By (10.1), m is uniquely determined up to isomorphism by $\overline{m}$. For each $\phi \in A$, we have a natural family

$$\kappa : \phi(a) \times \overline{m}(b, a) \longrightarrow m(\phi)(b)$$

defined by

$$\kappa(x, y) = m(\hat{x})_b(y) \tag{10.3}$$

where $\hat{x} : A(-, a) \longrightarrow \phi$ is the unique natural transformation with

$$\hat{x}_a(1, a) = x \in \phi(a) \ .$$

In particular, for $m : A \longrightarrow B$ and $n : B \longrightarrow C$ in mod_0, we have

$$\kappa : \overline{m}(b, a) \times \overline{n}(c, b) \longrightarrow \overline{nm}(c, a) \ ,$$

which is a universal extraordinary natural family in the variable b; in other words, κ induces an isomorphism

$$\int^b \overline{m}(b, a) \times \overline{n}(c, b) \cong \overline{nm}(c, a) \ . \tag{10.4}$$

Let $T : X \longrightarrow \mathrm{mod}_0$ be a functor and define a category El(T) as follows. The objects are pairs (x, a) where $x \in X$ and $a \in Tx$. Morphisms $(\xi, \tau) : (x, a) \longrightarrow (y, b)$ consist of $\xi : x \longrightarrow y$ in X and $\tau \in \overline{T\xi}(b, a)$. Composition is defined by

$$\begin{tikzcd}
(x, a) \arrow[rr, "(\zeta\xi, \upsilon \star \tau)"] \arrow[dr, "(\xi, \tau)"'] & & (z, c) \\
& (y, b) \arrow[ur, "(\zeta, \upsilon)"'] &
\end{tikzcd} \tag{10.5}$$

where $\upsilon \star \tau$ is the value of the function

$$\kappa : \overline{T\xi}(b, a) \times \overline{T\zeta}(c, b) \longrightarrow \overline{T(\zeta\xi)}(c, a)$$

at (τ, υ). The identity morphism of (x, a) is $(1_x, 1_a)$ where we note that

$$\overline{T1_x}(a, a) = (Tx)(a, a) \ .$$

There is a projection functor

$$p_T : \mathrm{El(T)} \longrightarrow X \tag{10.6}$$

defined by $p_T(x, a) = x$ and $p_T(\xi, \tau) = \xi$.

Let mod_1 denote the category whose objects are functors $f : A \longrightarrow B$ between small categories. A morphism is a square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "m"'] \arrow[dr, phantom, "\overset{\theta}{\Longrightarrow}"] & B \arrow[d, "n"] \\
C \arrow[r, "g"'] & D
\end{tikzcd} \tag{10.7}$$

where the functors f and g are the domain and codomain of the morphism, where m and n are morphisms of mod_0, and where θ is a natural family of functions

$$\theta_{b,a} : \overline{m}(c,a) \longrightarrow \overline{n}(gc, fa) \ .$$

There is a bijection between such θ and natural transformations θ as in the diagram

$$\begin{tikzcd}
\hat{A} \arrow[d, "m"'] \arrow[r, phantom, "\overset{\theta}{\Longrightarrow}" near end] & & \hat{B} \arrow[ll, "\hat{f}"'] \arrow[d, "n"] \\
\hat{C} & & \hat{D} \arrow[ll, "\hat{g}"]
\end{tikzcd} \qquad . \tag{10.8}$$

Composition is achieved by vertically pasting the squares of the form (10.8).

We have functors

$$d_0, d_1 : \mathrm{mod}_1 \longrightarrow \mathrm{mod}_0 \tag{10.9}$$

defined by

$$d_0(f) = A, \quad d_0(m, \theta, n) = m,$$

$$d_1(f) = B, \quad d_1(m, \theta, n) = n,$$

referring to (10.7). This defines a graph in Cat. There is also a canonical structure of category in Cat having (10.9) as underlying graph. Composition for this category mod in Cat is derived from horizontal pasting of squares of the form (10.8).

At present, we are regarding Cat as a category, not a 2-category. When we write $\mathrm{Cat}(X, \mathrm{mod})$ we mean in the sense of $\mathscr{C}(U, C)$ as described at the beginning of Section 1.

Now we shall extend the construction of (10.6) to a functor

$$\mathrm{El} : \mathrm{Cat}(X, \mathrm{mod}) \longrightarrow \mathrm{Cat}/X \ . \tag{10.10}$$

Let T and S be objects of $\mathrm{Cat}(X, \mathrm{mod})$; that is, they are functors from X to mod_0. A morphism $\Theta : T \longrightarrow S$ in $\mathrm{Cat}(X, \mathrm{mod})$ is a functor $\Theta : X \longrightarrow \mathrm{mod}$ with $d_0 \Theta = T$ and $d_1 \Theta = S$. Each morphism $\xi : x \longrightarrow y$ in X yields a diagram

$$\begin{tikzcd}
Tx \arrow[rr, "\Theta_x"] \arrow[d, "T\xi"'] \arrow[r, phantom, "\overset{\theta_\xi}{\Longrightarrow}" near end] & & Sx \arrow[d, "S\xi"] \\
Ty \arrow[rr, "\Theta_y"'] & & Sy
\end{tikzcd}$$

in which Θ_x and $\Theta - y$ are functors. Define a functor

$$\begin{tikzcd}
\mathrm{El}(T) \arrow[rr, "\mathrm{El}(\Theta)=F"] \arrow[dr, "p_T"'] & & \mathrm{El}(S) \arrow[dl, "p_T"] \\
& X &
\end{tikzcd} \tag{10.11}$$

over X as follows:

$$F(x,a) = (x, \Theta_x(a)), \quad F(\xi,\tau) = (\xi, \theta_\xi(\tau))$$

using $\theta_\xi : \overline{T\xi}(b,a) \longrightarrow \overline{S\xi}(\Theta_y(b), \Theta_x(a))$. By looking at (10.5) and using the fact that Θ is a functor, we see that F is a functor. Clearly (10.11) commutes. Composition in $\mathrm{Cat}(X, \mathrm{mod})$ involves horizontal pasting of squares (10.8), from which we see that we do have a functor (10.10).

The following result is related to Gray's Yoneda-like lemma on page 290 of [15]; also see page 210 of Kelly [22].

Theorem 10.1 *The functors* El *of* (10.10) *are fully faithful for all* X. *The family of these functors is pseudonatural in* X.

Proof The proof of the first sentence is a mere re-tracing of the steps in the definition of El on morphisms. The second sentence follows from the observation that we have a pullback square

$$\begin{tikzcd}
\mathrm{El}(Tr) \arrow[r] \arrow[d, "p_{Tr}"'] & \mathrm{El}(T) \arrow[d, "p_T"] \\
Y \arrow[r, "r"'] & X
\end{tikzcd}$$

for all functors $r : Y \longrightarrow X$. □

Corollary 10.2 *The family of functors* (10.10) *exhibits* mod *as an internal full subcategory of* Cat.

By the generalized Yoneda lemma of Section 2 (see Theorem 2.3 and (3.1)), the pseudonatural family (10.10) is determined by a discrete opfibration

$$p : \mathrm{obj} \longrightarrow \mathrm{mod} \tag{10.12}$$

between categories in Cat. We shall describe obj explicitly.

The category obj_0 has objects pairs (A, a) where A is a small category and a is an object of A. A morphism $(m, \mu) : (A, a) \longrightarrow (B, b)$ in obj_0 consists of a morphism $m : A \longrightarrow B$ in mod_0 together with $\mu \in \overline{m}(b, a)$. Composition is a special case of (10.5) and uses the κ of (10.4). We have the projection functor

$$p_0 : \mathrm{obj}_0 \longrightarrow \mathrm{mod}_0 \tag{10.13}$$

taking $(m, \mu) : (A, a) \longrightarrow (B, b)$ to $m : A \longrightarrow B$.

Proposition 10.3 *The functor* p_0 *of* (10.13) *is powerful in the category* Cat. *That is, the functor*

$$p_0^\star : \mathrm{Cat}/\mathrm{mod}_0 \longrightarrow \mathrm{Cat}/\mathrm{obj}_0$$

has a right adjoint.

Proof We must show that the functor p_0 has the factorization lifting property of Giraud-Conduché (see [13], [8] and [42]). Take a composable pair

$$A \xrightarrow{m} B \xrightarrow{n} C$$

in mod_0 and a lifting

$$(A, a) \overset{(nm,\lambda)}{\longrightarrow} (C, c)$$

to obj_0 of the composite nm. By (10.4), there exists $b \in B$ and $(\mu, \nu) \in \overline{m}(b, a) \times \overline{n}(c, b)$ such that $\kappa(\mu, \nu) = \lambda$. This gives a factorization

$$(A, a) \overset{(m,\mu)}{\longrightarrow} (B, b) \overset{(n,\nu)}{\longrightarrow} (C, c)$$

of (nm, λ) which, using (10.4) again, determines a unique path component in the category of such liftings, as required. □

Now we shall define the category obj_1. The objects are pairs (f, a) where $f : A \longrightarrow B$ is a functor and $a \in A$. A morphism

$$(m, \mu, n, \theta) : (f, a) \longrightarrow (g, c) \tag{10.14}$$

consists of a morphism $(m, n, \theta) : f \longrightarrow g$ as in (10.7) and $\mu \in \overline{m}(c, a)$. Composition is such that we have the functor

$$p_1 : \mathrm{obj}_1 \longrightarrow \mathrm{mod}_1 \tag{10.15}$$

taking (10.14) to (m, n, θ). We also have the functor

$$d_0 : \mathrm{obj}_1 \longrightarrow \mathrm{obj}_0 \tag{10.16}$$

taking (10.14) to $(m, \mu) : (A, a) \longrightarrow (C, c)$, and the functor

$$d_1 : \mathrm{obj}_1 \longrightarrow \mathrm{obj}_0 \tag{10.17}$$

taking (10.14) to $(n, \theta_{c,a}(\mu)) : (B, fa) \longrightarrow (D, gc)$. By the general theory in Section 3, the functors d_0 and d_1 of (10.16) and (10.17) form the underlying graph in Cat of a category obj in Cat, and the functors p_0 and p_1 of (10.13) and (10.15) form a functor

$$p : \mathrm{obj} \longrightarrow \mathrm{mod}$$

in Cat. Indeed, we also know that p is a discrete opfibration, pullback along which gives the functor El of (10.10).

After Corollary 10.2, it is of interest to know whether mod is mod-admissible (see (9.8)). The strict answer is "No". However, the answer is "Essentially yes", as we now explain.

There is a natural choice of 2-cells to make the category mod_0 into a 2-category; just take the natural transformations between the colimit-preserving functors $m : \hat{A} \to \hat{B}$. There is a pseudofunctor

$$\mathbb{F} : \mathrm{mod}_0 \times \mathrm{mod}_0 \longrightarrow \mathrm{mod}_0 \tag{10.18}$$

defined on objects by $\mathbb{F}(A, B) = [A, B]$, the functor category. For morphisms $m : A \to C$ and $n : B \to D$ in mod_0, define

$$\mathbb{F}(m, n) : [A, B] \longrightarrow [C, D]$$

(up to isomorphism) to be a morphism in mod_0 with

$$\overline{\mathbb{F}(m, n)}(g, f) = \mathrm{mod}_0(A, D)(g_* m, n f_*) \ .$$

For a functor $f : A \to B$, the meaning of $f_* : A \to B$ is the morphism of mod_0 amounting to the functor $f_* : \widehat{A} \to \widehat{B}$ which is a left Kan extension along Yoneda exhibited by an equality:

$$\begin{tikzcd} A \arrow[r, "\mathrm{y}_A"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & \widehat{A} \arrow[d, "f_*"] \\ B \arrow[r, "\mathrm{y}_B"'] & \widehat{B} \end{tikzcd} \tag{10.19}$$

A little book keeping shows:

Proposition 10.4 *The following diagram is a pullback in the category of 2-categories and pseudofunctors.*

$$\begin{tikzcd}
\mathrm{mod}_1 \arrow[r] \arrow[d, "(d_0{,}d_1)"'] & \mathrm{obj}_0 \arrow[d, "p_0"] \\
\mathrm{mod}_0 \times \mathrm{mod}_0 \arrow[r, "\mathbb{F}"'] & \mathrm{mod}_0
\end{tikzcd}$$

If $\mathbb{F}$ were a (2-)functor rather than a pseudofunctor, we would have the admissibility of mod. By cutting mod down a bit, we can obtain an internal full subcategory of Cat which is admissible with respect to itself.

Let fun_0 be the category cat of small categories and functors between them. Let fun_1 be the category whose objects are functors $f : A \to B$ between small categories and whose morphisms from f to g are squares

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] \arrow[rd, phantom, "\overset{\theta}{\Longrightarrow}"] & B \arrow[d, "v"] \\
C \arrow[r, "g"'] & D
\end{tikzcd} \tag{10.20}$$

where u and v are functors and $\theta : gu \Rightarrow vf$ is a natural transformation. Composition is vertical pasting. We obtain a category fun with a functor $\mathrm{j} : \mathrm{fun} \longrightarrow \mathrm{mod}$ in Cat such that the following square is a pullback.

$$\begin{tikzcd}
\mathrm{fun}_1 \arrow[r, "\mathrm{j}_1"] \arrow[d, "(d_0{,}d_1)"'] & \mathrm{mod}_1 \arrow[d, "(d_0{,}d_1)"] \\
\mathrm{fun}_0 \times \mathrm{fun}_0 \arrow[r, "\mathrm{j}_0 \times \mathrm{j}_0"'] & \mathrm{mod}_0 \times \mathrm{mod}_0
\end{tikzcd} \tag{10.21}$$

Here the functor $\mathrm{j}_0 : \mathrm{fun}_0 \longrightarrow \mathrm{mod}_0$ is the identity on objects and takes $u : A \to C$ in fun_0 to $u_* : A \to C$ in mod_0. From Proposition 3.2 we now have:

Corollary 10.5 *With the obvious discrete opfibration,* fun *is an internal full subcategory of* Cat*.*

From the definition (9.1) of C-fibred, we have:

Corollary 10.6 *A functor $q : F \to U$ is* fun*-fibred if and only if it is a split opfibration with small fibres.*

The pseudofunctor (10.18) restricts to an actual functor

$$\mathbb{F} : \mathrm{fun}_0 \times \mathrm{fun}_0 \to \mathrm{fun}_0 \ ,$$

yielding:

Corollary 10.7 fun *is* fun*-admissible in* Cat(Cat)*. Consequently,* $\widehat{\mathrm{fun}}$ *is total.*

Categories in Cat are called *double categories.* They were defined by Ehresmann in [11]. Another reference is [25]. We write Dbl for the 2-category Cat(Cat) of double categories. Actually, looking only at this 2-category structure on Dbl loses quite a bit of the symmetry of the situation. Since Cat is cartesian closed, so too is Dbl and that is important from the viewpoint of the 2-topos structure. It also means that Dbl is hom-enriched in itself. There are two underlying functors

$$\mathrm{Dbl} \longrightarrow \mathrm{Cat}$$

which induce two 2-functors

$$\mathrm{Dbl}\text{-}\mathrm{Cat} \longrightarrow \mathrm{Cat}\text{-}\mathrm{Cat} = 2\text{-}\mathrm{Cat}$$

whose values at Dbl itself give two ways to regard Dbl as a 2-category.

We need to use some fairly standard terminology to express this distinction. For an object $A \in \mathrm{Cat}(\mathrm{Cat})$ with underlying graph $d_0, d_1 : A_1 \to A_0$, we call the morphisms of A_0 *horizontal morphisms* and the objects of A_1 *vertical morphisms.* So A_{00} is the set of objects, A_{01} is the set of horizontal morphisms, A_{10} is the set of vertical morphisms, and A_{11} is the set of *squares* of A. Notation for horizontal ◫ and vertical ⊟ composition is explained by the following diagrams.

$$\begin{tikzcd}
a \arrow[r, "\alpha"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\theta"] & a' \arrow[r, "\alpha'"] \arrow[d, "\gamma'"] \arrow[dr, phantom, "\theta'"] & a'' \arrow[d, "\gamma''"] & = & a \arrow[r, "\alpha \boxbar \alpha'"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\theta \boxbar \theta'"] & a'' \arrow[d, "\gamma''"] \\
a_1 \arrow[r, "\alpha_1"'] & a'_1 \arrow[r, "\alpha'_1"'] & a''_2 & & a_1 \arrow[r, "\alpha_1 \boxbar \alpha'_1"'] & a''_1
\end{tikzcd} \tag{10.22}$$

$$\begin{tikzcd}
a \arrow[r, "\alpha"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\theta"] & a' \arrow[d, "\gamma'"] & & & \\
a_1 \arrow[r, "\alpha_1"'] \arrow[d, "\gamma_1"'] \arrow[dr, phantom, "\theta_1"] & a'_1 \arrow[d, "\gamma'_1"] & = & a \arrow[r, "\alpha"] \arrow[d, "\gamma \boxminus \gamma_1"'] \arrow[dr, phantom, "\theta \boxminus \theta_1"] & a' \arrow[d, "\gamma' \boxminus \gamma'_1"] \\
a_2 \arrow[r, "\alpha_2"'] & a'_2 & & a_2 \arrow[r, "\alpha_2"'] & a'_2
\end{tikzcd} \tag{10.23}$$

Notation that can be helpful comes from the fact that a double category A can be identified with a double simplicial set A_{**}. The category A_0 might also be denoted A_{0*} and the category A_1 as A_{1*} where the $*$ runs over the simplicial category. Symmetrically, we have categories A_{*0} and A_{*1}; objects of A_{*0} are objects and morphisms are vertical morphisms; objects of A_{*1} are horizontal morphisms and morphisms are squares.

So now back to Dbl as the 2-category Cat(Cat) of Corollary 10.7. The morphisms are quite symmetric with respect to horizontal and vertical morphisms: they are *double functors*. A double functor $f : A \to B$ assigns objects in A to objects in B, horizontal morphisms to horizontal morphisms, vertical morphisms to vertical morphisms, and squares to squares, in such a way as to preserves domains, codomains, compositions, and identities. A 2-cell $\sigma : f \Rightarrow g : A \to B$ in this 2-category Dbl, called a *vertical transformation*, assigns to each horizontal morphism $\alpha : a \to a'$ in A a square

$$\begin{tikzcd}
f(a) \arrow[r, "f(\alpha)"] \arrow[d, "\sigma_a"'] \arrow[dr, phantom, "\sigma_\alpha"] & f(a') \arrow[d, "\sigma_{a'}"] \\
g(a) \arrow[r, "g(\alpha)"'] & g(a')
\end{tikzcd} \tag{10.24}$$

in B such that

$$\sigma_\alpha \boxbar \sigma_{\alpha'} = \sigma_{\alpha \boxbar \alpha'} \quad , \quad \sigma_{1_a} = 1_{\sigma_a} \quad \text{and} \quad f(\theta) \boxminus \sigma_{\alpha_1} = \sigma_\alpha \boxminus g(\theta) \ .$$

Proposition 10.8 *A double functor $q : F \to B$ is a discrete opfibration between categories in* Cat *if and only if the functors $q_{*0} : F_{*0} \to B_{*0}$ and $q_{*1} : F_{*1} \to B_{*1}$ are discrete opfibrations (between categories in* Set*).*

Proof Contemplate the discrete opfibration requirement that the following should be a pullback in Cat.

$$\begin{tikzcd}
F_1 \arrow[r, "d_0"] \arrow[d, "q_1"'] & F_0 \arrow[d, "q_0"] \\
B_1 \arrow[r, "d_0"'] & B_0
\end{tikzcd}$$

Looking at what it means to be a pullback on objects gives $q_{*0} : F_{*0} \to B_{*0}$ a discrete opfibration and what it means on morphisms gives $q_{*1} : F_{*1} \to B_{*1}$ a discrete opfibration. □

Corollary 10.9 *The replete image of the fully faithful functor* $\mathrm{Dbl}(B, \mathrm{fun}) \to \mathrm{Dbl}/B$ *consists of those double functors* $q : F \to B$ *with* $q_{*0} : F_{*0} \to B_{*0}$ *and* $q_{*1} : F_{*1} \to B_{*1}$ *discrete opfibrations, and* $q_0 : F_0 \to B_0$ *a split opfibration with small fibres.*

We already mentioned that Dbl is cartesian closed as a 2-category. This is because it is the 2-category of categories in a finitely complete, cartesian closed category. Symmetry is restored by taking into account this cartesian internal hom $[A, B]$. The double category $[A, B]$ has double functors as objects and vertical transformations as vertical morphisms. *Horizontal transformations* are defined by switching horizontal and vertical in the definition of vertical transformation; and these are the horizontal transformations. The squares of $[A, B]$ are *double squares*:

$$\begin{tikzcd}
f \arrow[r, "\lambda"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "s"] & f' \arrow[d, "\sigma'"] \\
g \arrow[r, "\kappa"'] & g' .
\end{tikzcd} \tag{10.25}$$

Here f, f', g, g' are double functors, σ, σ' are vertical transformations, λ, κ are horizontal transformations, and s assigns to each object $a \in A$, a square

$$\begin{tikzcd}
f(a) \arrow[r, "\lambda_a"] \arrow[d, "\sigma_a"'] \arrow[dr, phantom, "s_a"] & f'(a) \arrow[d, "\sigma'_a"] \\
g(a) \arrow[r, "\kappa_a"'] & g'(a)
\end{tikzcd} \tag{10.26}$$

in B such that $s_a \boxbar \sigma'_\alpha = \sigma_\alpha \boxbar s_{a'}$ and $s_a \boxminus \kappa_\gamma = \lambda_\gamma \boxminus s_{a_1}$. Compositions are pointwise in B.

Let A be a 2-category. There is a double category A_{h} with the same objects as A, with horizontal morphisms the morphisms of A, with only identity vertical morphisms, and with squares the 2-cells of A oriented as in (10.27). Indeed, Ehresmann defined 2-categories as double categories with all vertical morphisms identities.

$$\begin{tikzcd}
a \arrow[r, "\alpha"] \arrow[d, "1_a"'] \arrow[dr, phantom, "\overset{\theta}{\Longleftarrow}"] & a' \arrow[d, "1_{a'}"] \\
a \arrow[r, "\alpha_1"'] & a'
\end{tikzcd} \tag{10.27}$$

There is also a double category A_{v} with the same objects as A, with only identity horizontal morphisms of A, with vertical morphisms the morphisms of A, and with squares the 2-cells of A oriented as in (10.28). Of course, this is just an appropriate double categorical dual of A_{h}.

$$\begin{tikzcd}
a \arrow[r, "1_a"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\overset{\theta}{\Longrightarrow}"] & a \arrow[d, "\gamma'"] \\
a_1 \arrow[r, "1_{a_1}"'] & a_1
\end{tikzcd} \tag{10.28}$$

There is a third double category A_{sq} associated with a 2-category A. The objects are those of A, both the horizontal and vertical morphisms are the morphisms of A, and the squares are the squares containing a 2-cell in A oriented as in (10.29).

$$\begin{tikzcd}
a \arrow[r, "\alpha"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\overset{\theta}{\Longrightarrow}"] & a' \arrow[d, "\gamma'"] \\
a_1 \arrow[r, "\alpha_1"'] & a'_1
\end{tikzcd} \tag{10.29}$$

If cat denotes the 2-category of small categories, notice that our internal full subcategory fun of Cat is none other than $\mathrm{cat}_{\mathrm{sq}}$.

For the terminology in our next statement see [25, 22, 27].

Proposition 10.10 *Each 2-functor $f : A \to B$ between 2-categories A and B induces double functors $f_{\mathrm{h}} : A_{\mathrm{h}} \to B_{\mathrm{h}}$, $f_{\mathrm{v}} : A_{\mathrm{v}} \to B_{\mathrm{v}}$, $f_{\mathrm{sq}} : A_{\mathrm{sq}} \to B_{\mathrm{sq}}$, and $f_{\mathrm{m}} : A_{\mathrm{h}} \to B_{\mathrm{sq}}$. For 2-functors $f, g : A \to B$,*

(a) vertical transformations $f_{\mathrm{h}} \Rightarrow g_{\mathrm{h}}$ are icons $f \Rightarrow g$,

(b) vertical transformations $f_{\mathrm{v}} \Rightarrow g_{\mathrm{v}}$ are 2-natural transformations $f \Rightarrow g$,

(c) vertical transformations $f_{\mathrm{sq}} \Rightarrow g_{\mathrm{sq}}$ are 2-natural transformations $f \Rightarrow g$, and

(d) vertical transformations $f_{\mathrm{m}} \Rightarrow g_{\mathrm{m}}$ are lax natural transformations $f \Rightarrow g$.

Recall from [42] that a category B in a category $\mathscr{C}$ is *amenable* when the morphism $d_1 : B_1 \to B_0$ is powerful (that is, pullback along it has a right adjoint). All fibrations and opfibrations are powerful in Cat; see [13, 8, 42].

Proposition 10.11 *For any 2-category A, the functors $d_1 : A_{\mathrm{sq}1} \to A_{\mathrm{sq}0}$ and $d_1 : A_{\mathrm{v}1} \to A_{\mathrm{v}0}$ are opfibrations while $d_1 : A_{\mathrm{h}1} \to A_{\mathrm{h}0}$ is a discrete opfibration. Consequently, A_{sq}, A_{v} and A_{h} are all amenable categories in* Cat*.*

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