ABSTRACTS OF THE SYDNEY

CATEGORY THEORY SEMINAR 1972

Edited by

G. M. Kelly and Ross Street

SCHOOL OF MATHEMATICS

THE UNIVERSITY OF NEW SOUTH WALES

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INTRODUCTION

With category theorists at each of the three universities in Sydney - Ross Street at Macquarie, R.F.C. (Bob) Walters at Sydney, and Max Kelly and his students (Geoffrey Lewis, Andrew Macfarlane, Robert Blackwell) at New South Wales - we decided in the middle of 1972 to meet, not for an hour a week, but for one whole day a week. This proved a fruitful enterprise, the more so as we were joined for four months by Brian Day, visiting his home town after two years at Chicago before going on to Aarhus for 8 months.

A good deal of mathematics emerged in these sessions; and since this will take a while to write up and publish formally, we conceived the idea of preparing and distributing these abstracts.

The idea seemed all the better, in that we in Australia have less opportunity than Europeans and Americans to disseminate our ideas by personal contacts. Indeed Ross Street is the only one of us who, by an unusual munificence on the part of his university, was able to attend the recent conference at Oberwolfach.

For the same reason, we now appeal to our colleagues oversea to keep us in mind in sending out their preprints and reprints. If we ourselves have sometimes been lax in this regard (as Michael Barr has often chided Kelly), well, we apologise with firm resolutions to sin no more. We are in fact better organised than before, and these present abstracts are an earnest of our repentance.

We had hoped to get these to you by Christmas (or even by Hanukah), to carry our greetings as well as our ideas: but it was beyond the capacity of us and our secretaries. No matter - our belated greetings are still sincere, and are written if not received at the season of goodwill.

Some personal notes: Kelly, after 6 years at the University of N.S.W., is to return at the end of January to the University of Sydney, where he was an undergraduate and, on returning from Cambridge, taught for 10 years. The address is Pure Maths. Dept., University of Sydney N.S.W. 2006, Australia. Lewis is to desert pure mathematics for a Lectureship in Economics at Sydney. We hope to see Day back late in 1973.

Finally as editors we thank all who have so earnestly cooperated, and Pat Roze who has done the typing with her usual excellence.

December 22, 1972.

Max Kelly

Ross Street

CONTENTS

1. Elementary topoi

1.1 The ordered objects in a topos (Street)

As soon as we begin to study elementary topoi, we come up against ordered objects. An ordered object is of course an object A together with a reflexive transitive relation on it. If T is the topos, these form a 2-category Ord-T. The parts of topos theory that have to do with "internal completeness" are assertions about adjunctions and Kan extensions in Ord-T. This part of the theory of topoi can in fact be summed up in the following result:

Theorem *The ordered objects in a topos form a small cosmos.*

For the definition of (elementary) cosmos, see §2 below: the basic results about cosmoi were announced in Street [30], and some more recent insights are described here in §2.

Given $A,B \in$ Ord-T, a profunctor from A to B turns out to be an order-ideal on $A \times B^{op}$. The object ΓA which represents the profunctors out of A is a subobject of $PA = \Omega^A$, corresponding in the case T = **Sets** to the set of order-ideals of A.

1.2 The elements of topoi (Kelly-Street)

In spite of the above theorem of Street, it may still be of some interest to give a simple account of the elements of topos theory, particularly internal completeness, independently of cosmos theory. We have found the following order of development to be simpler than most that we have seen, and we offer it for what it is worth.

(a) <u>Define</u> a topos T as a finitely complete category in which the relations $\mathrm{Rel}(A,B)$ from A to B, made functorial in B by pullback, are representable in the form $T(B,PA)$. The "universal relation" for this representation is called $\in$, from A to PA; the reverse relation from PA to A is $\ni$. The singleton map $\sigma\colon A \to PA$ corresponds to the identity relation.

(b) As Kock has observed, we can prove T to be <u>cartesian closed</u>, defining the internal-hom $[A,B]$ as the pullback of $1 \to PA$ (corresponding to the whole of A) along a suitable map $P(A\times B) \to PA$: the map which in $\underline{\underline{\mathrm{Sets}}}$ takes $R \subset A\times B$ to $\{a \in A \mid (a,b) \in R \text{ for a unique } b \in B\}$.

(c) PA is an ordered object, indeed a lattice-object, in T because $\mathrm{Rel}(A,B)$ is so in $\underline{\underline{\mathrm{Sets}}}$. For <u>any</u> ordered object C and maps $r\colon A \to C$, $s\colon B \to C$ we can define the <u>comma object</u>

$$\begin{tikzcd}
r/s \arrow[r, "d_1"] \arrow[d, "d_o"'] \arrow[dr, phantom, "\leqslant"] & B \arrow[d, "s"] \\
A \arrow[r, "r"'] & C
\end{tikzcd}$$

as the pullback

$$\begin{tikzcd} r/s \arrow[r, tail] \arrow[d, tail] & \Phi C \arrow[d, tail] \\ A\times B \arrow[r, "r\times s"'] & C\times C \end{tikzcd}$$

where ΦC is the order-relation on C. We can then regard r/s as a relation from A to B. We use this construction in particular when $C = PD$ so that $r\colon A \to PD$ and $s\colon B \to PD$ correspond to relations R from D to A and S from D to B. We also then write R/S for r/s.

(d) In particular, given a relation R from A to B, and taking the relation $\in$ from A to PA, we get the relation $R/\in$ from B to PA; we write the corresponding map $PA \to PB$ as $\forall_R$. Three special cases are important: when R is a function f, i.e. the relation $A \xleftarrow[1]{} A \xrightarrow[f]{} B$, we write $\forall_f$; when R is the reverse of a function, i.e. $A \xleftarrow[g]{} B \xrightarrow[1]{} B$, we write Pg for $\forall_R$; and when R is the relation $\ni$ from PB to B, we write $\cap\colon PPB \to PB$ for $\forall_R$. (Added in this reprinting: T. Brook has pointed out that $\cap$ is not as given but is $(\in/\ni)^*$.)

(e) We now construct the <u>image</u> fA of $f\colon A \to B$; in an obvious notation it is $\cap P$, where $P \subset PB$ is the pullback along Pf of the map $1 \to PA$ corresponding to the whole of A. This gives an epi-mono factorization. The idea is due to Mikkelsen.

(f) We shall know that epis are preserved by pullbacks when we know that $T/B \to T/A$ induced by $f\colon A \to B$ has a right adjoint. As Freyd pointed out, it suffices to know that T/B is cartesian closed. In fact we can now prove the <u>fundamental theorem</u> that T/B is a topos; we produce $P(E \xrightarrow[p]{} B)$ as the object $\in/S$, where $\in$ is the relation from E to PE, and S is the relation $E \xleftarrow[1]{} E \xrightarrow[p]{} B$ from E to B; $\in/S$ has an evident map into B.

(g) We now define the <u>composite of relations</u> by pulling back and taking images. In view of (f), this composition is associative, as shown by Freyd; so that we obtain a 2-category Rel. Indeed Rel is a "biclosed 2-category"; we have, for relations R from D to A, S from D to B, and T from A to B,

$$T \circ R \le S \text{ if and only if } T \le R/S,$$

where $\circ$ is composite of relations.

(h) Let R be a relation from A to B. Composing the reverse relation R* from B to A with the relation $\in$ from A to PA gives the relation $\in \circ R^*$ from B to PA; the corresponding map $PA \to PB$ is called $\exists_R$. From (g) it is easily seen that

$\exists_R$ is left adjoint in Ord-$\mathcal{T}$ to $\forall_{R*}$. We get the special cases $\exists_f$ when R is a map f, and $\bigcup$: $\mathcal{P}\mathcal{P}B \to \mathcal{P}B$ when R is $\ni$.

(i) More generally, for any ordered object A in $\mathcal{T}$, the order-relation on A gives an order-preserving map $A \to \mathcal{P}A$, and A can be said to be cocomplete if this has a left adjoint $\bigcup$. Thus $\mathcal{P}B$ may be said to be complete and cocomplete. However we are now saying things that belong in cosmos theory, and further development along these lines is best carried out in terms of Street's theorem in §1.1 above.

1.3 Topological objects in a topos (Macfarlane)

A left topological object in an elementary topos $\mathcal{T}$ is an object X of $\mathcal{T}$ together with a map c: $\mathcal{P}X \to \mathcal{P}X$ satisfying $c \geq 1$, $c^2 = c$, and $\vee\circ(c\times c) = c\circ\vee$. Similarly a right topological object is (X,i) where i: $\mathcal{P}X \to \mathcal{P}X$ is such that $i \leq 1$, $i^2 = i$, and $\wedge\circ(i\times i) = i\circ\wedge$.

The equalizer τ_c of $1_{\mathcal{P}X}$ and c is closed under $\bigcap$ and $\vee$; the equalizer τ_o of $1_{\mathcal{P}X}$ and i is closed under $\bigcup$ and $\wedge$. Moreover any $\tau \subset \mathcal{P}X$ closed under $\bigcap$ and $\vee$ is such a τ_c for some left topology c, and similarly for right topologies. So a left topological object is equally an X with such a τ.

This allows one to define the object of left (or right) topologies on X, and to show that this object is closed under $\bigcap$.

A <u>continuous</u> morphism $f\colon (X,\tau) \to (Y,\sigma)$ is an $f\colon X \to Y$ such that $\sigma \to PY \xrightarrow{Pf} PX$ factors through τ. We thus obtain categories $\mathrm{Top}_l(T)$ and $\mathrm{Top}_r(T)$ of left and right topological objects in T. These are T-categories (hom-objects in T), and are moreover finitely complete and cocomplete.

The contradictions between left and right topologies are summed up in the commuting diagram

$$\begin{tikzcd} \mathrm{Top}_l(T) \arrow[rr, shift right, "L"'] \arrow[dr, "\text{forget}"'] & & \mathrm{Top}_r(T) \arrow[ll, shift right, "R"'] \arrow[dl, "\text{forget}"] \\ & T & \end{tikzcd}$$

However the struggle to resolve these contradictions has only just begun and it is not known what direction it will take.

2. Elementary Cosmoi

2.1 What cosmoi are for. (Street)

Cosmos theory arose from attempts by Walters and myself to characterize that structure on a 2-category which allows the hom-set statement of the Yoneda lemma and the development of its consequences. This Yoneda theory is known to work well in 2-categories of the form V-Cat (including the 2-categories of categories, of ordered sets, of additive categories, ...), but apparently not in Mon (see §4 below). For each object A in V-Cat one has the object $\Gamma A = [A^{op}, V]$ of contravariant V-valued V-functors on A, and one has the representation arrow $A \xrightarrow[y_A]{} \Gamma A$. Our aim was to find an appropriate characterization of the arrow $A \xrightarrow[y_A]{} \Gamma A$ which could be expressed in an arbitrary 2-category, and to deduce theorems in the 2-category corresponding to those theorems of category theory whose statements require hom-sets (for example, the hom-set formulation of adjunction).

An account of such a theory appears in Street [30]. This theory is of the same elementary nature as topos theory. The central point is the right characterization of those spans which correspond to profunctors. In Cat a profunctor is a bifibration with discrete fibres. The notion of bifibration can be expressed in a 2-category (see §2.2 below), we can define fibres (see below), and we can express discreteness (fixity under suspension). However, bifibrations with discrete fibres are not the right notion for a cosmos. One might expect "discreteness" not to be a good notion in V-$\mathbf{Cat}$. In $\mathbf{Ord}$-$\mathbf{Set}$ (= the 2-category of ordered sets) a profunctor from A to B is an order-ideal in $A \times B^{op}$; or better, as a span, a profunctor is a bifibration which is jointly monic (that is, which is a relation). The definition of a profunctor which we take, and which includes these examples, is simply a span which occurs as a comma object (size considerations aside).

We may ask whether elementary cosmoi bear to 2-categories of V-$\mathbf{Cat}$-valued sheaves on a site the same relation as do elementary topoi to categories of $\mathbf{Set}$-valued sheaves on a site. This has not yet been investigated. Other questions may be asked. Further axioms on a topos are known (Tierney [34]) such that the topos should provide a model for the elementary theory of sets (Lawvere [22]). What further axioms on a cosmos are needed in order to capture the elementarily expressibly properties of $\mathbf{Cat}$? What are the relationships with Lawvere [23]?

Progress made on the programme explained above will now be outlined.

2.2 Conceptual view of Yoneda. (Street)

A 2-category K is called stable (or "strongly representable" by Gray [14]) when each object A has a cotensor ΦA, called the suspension of A, with the category $\underline{\underline{2}}$. Thus there is an enriched-natural isomorphism of categories

$$K(X, \Phi A) \cong K(X,A)^{\underline{\underline{2}}},$$

from which we obtain a category object $\Phi A \overset{d_0}{\underset{d_1}{\rightrightarrows}} A$ in K_0. Any limits which may exist in K_0 are then automatically enriched in K.

Suppose K is a stable 2-category with pullbacks. Let Spn denote the 3-category whose objects are those of K, whose hom-2-category Spn(A,B) is the 2-category of isomorphism classes of spans from A to B in K,

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] \arrow[dd, bend right] \arrow[dd, bend left] \arrow[dd, phantom, "\rightarrow"] & \\ A & & B \\ & T \arrow[ul] \arrow[ur] & \end{tikzcd}$$

and whose composition 2-functor

$$\mathrm{Spn}(B,C) \times \mathrm{Spn}(A,B) \xrightarrow{\circ} \mathrm{Spn}(A,C)$$

is given by pulling back (see Benabou [3]). For each A then, Spn(A,A) is a monoidal (2-)category and $A \xleftarrow{d_0} \Phi A \xrightarrow{d_1} A$ is a monoid therein. Composition on one side with ΦA and on the other with ΦB yields a monad

$$\begin{array}{ccc} Spn(A,B) & \longrightarrow & Spn(A,B) \\ S & \longmapsto & S^{\#} = \Phi B \circ S \circ \Phi A \end{array}$$

on Spn(A,B). The (2-) category of algebras $Spn(A,B)^{\#}$ for this monad is denoted by Bim(A,B); spans which are algebras for this monad are called <u>bimodules</u> from A to B. (With coequalizers preserved by pullback we can make Bim into a 3-category using the usual definition of "tensor product of bimodules").

Given arrows $A \xrightarrow{r} C \xleftarrow{s} B$, the composite of the three spans

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, "1"'] \arrow[dr, "r"] & & \Phi C \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & & B \arrow[dl, "s"'] \arrow[dr, "1"] & \\ A & & C & & C & & B \end{tikzcd}$$

is called the <u>comma object</u> of r,s and denoted $A \xleftarrow{d_0} r/s \xrightarrow{d_1} B$.

The further projection $r/s \to \Phi C$ corresponds to a 2-cell

$$\begin{tikzcd} & r/s \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & \\ A \arrow[dr, "r"'] \arrow[rr, phantom, shift right=3ex, "\xrightarrow{\lambda}"] & & B \arrow[dl, "s"] \\ & C & \end{tikzcd}$$

with the obvious universal property.

We will give evidence below for considering the following theorem as expressing the heart of the Yoneda Lemma.

Theorem. For arrows $A \xrightarrow{r} C \xleftarrow{s} B$, the composite 2-cell

$$\begin{tikzcd}
& & rd_1/sd_0 \arrow[dll, "d_0"'] \arrow[drr, "d_1"] & & \\
\Phi A \arrow[d, bend right, "d_0"'] \arrow[d, bend left, "d_1"] \arrow[d, phantom, "\xrightarrow{\lambda}"] & & & & \Phi B \arrow[d, bend right, "d_0"'] \arrow[d, bend left, "d_1"] \arrow[d, phantom, "\xrightarrow{\lambda}"] \\
A \arrow[drr, "r"'] & & {} \arrow[d, phantom, "\xrightarrow{\lambda}"] & & B \arrow[dll, "s"] \\
& & C & &
\end{tikzcd}$$

induces an arrow of spans $(r/s)^{\#} = rd_1/sd_0 \longrightarrow r/s$ which gives r/s a "canonical" structure of bimodule from A to B. Furthermore, any arrow of spans

$$\begin{tikzcd}
& r/s \arrow[dl, "d_0"'] \arrow[dr, "d_1"] \arrow[dd, "f"] & \\
A & & B \\
& u/v \arrow[ul, "d_0"] \arrow[ur, "d_1"'] &
\end{tikzcd}$$

automatically preserves the canonical bimodule structures.

In practice our 2-category K has a distinguished class of objects which are called small (this is what makes category theory more involved than the theory of ordered sets). Given a span $A \xleftarrow{u} S \xrightarrow{v} B$ and arrows $K \xrightarrow{a} A$, $L \xrightarrow{b} B$, the span $K \leftarrow S_{a,b} \rightarrow L$ obtained as the composite

$$\begin{tikzcd}[column sep=small] & K \arrow[dl, "1"'] \arrow[dr, "a"] & & S \arrow[dl, "u"'] \arrow[dr, "v"] & & L \arrow[dl, "b"'] \arrow[dr, "1"] & \\ K & & A & & B & & L \end{tikzcd}$$

is called the fibre of S over a,b. We say S has small fibres when, for all a,b with K,L small, $S_{a,b}$ is small. An arrow $A \xrightarrow{f} B$ is admissible when f/B has small fibres. An object A is legitimate when $A \xrightarrow[1_A]{} A$ is admissible.

Let Bif(A,B) denote the full sub-(2-)category of Bim(A,B) consisting of those bimodules with small fibres; the objects of Bif(A,B) are called bifibrations from A to B. Let Prof(A,B) denote the full sub-(2-)category of Bif(A,B) consisting of those bifibrations which are spans of the form $A \xleftarrow{d_0} r/s \xrightarrow{d_1} B$ with the canonical bimodule structure; the objects of Prof(A,B) are called profunctors from A to B. The last theorem tells us that Prof(A,B) is also a full sub-(2-)category of Spn(A,B).

<u>Corollary</u> <u>If</u> $A \xrightarrow{f} B$ <u>is admissible then the reflection of the object</u> $A \xleftarrow{1} A \xrightarrow{f} B$ <u>of</u> Spn(A,B) <u>in</u> Prof(A,B) <u>is</u> $A \leftarrow f/B \rightarrow B$.

<u>Proof</u>. The free #-algebra (= bimodule) on the span $A \xleftarrow{1} A \xrightarrow{f} B$ is precisely $A \xleftarrow{d_0} f/B \xrightarrow{d_1} B$; and this is a profunctor already.

The above corollary is called a Yoneda lemma in Street [30], and the Yoneda lemma of Gray [14] is a consequence. The arrow of spans $A \to f/B$ which gives the reflection in the corollary corresponds to the functor which takes an object a of A to the identity arrow in B of fa. So the reflection property may be stated as a bijection determined by "evaluating at the identity".

There is a similar corollary concerning the reflection of the span $A \xleftarrow{u} B \xrightarrow{1} B$ which also is a form of the Yoneda lemma. The original theorem hence provides the common setting for both corollaries.

<u>2.3</u> <u>Definition of a cosmos</u> (Street)

Note first that pulling back makes Prof (A,B) 2-functorial in B as an object of K in such a way that both

1-cells and 2-cells are reversed.

A <u>pre-cosmos</u> consists of a 2-category K and a distinguished class of <u>small</u> objects satisfying:

<u>Axiom 1</u>. K <u>is stable and has pullbacks, and the small objects are closed under suspension and pullback formation</u>;

<u>Axiom 2</u>. <u>For each legitimate object</u> A, <u>there exists an object</u> ΓA <u>and an enriched-natural equivalence of categories</u>

$$K(B,\Gamma A) \simeq \mathrm{Prof}(A,B);$$

<u>Axiom 3</u>. <u>If</u> A <u>is small then</u> ΓA <u>is legitimate</u>.

For each legitimate object A, the profunctor ΦA from A to A corresponds to arrow $A \xrightarrow[y_A]{} \Gamma A$ called the <u>representation arrow</u> of A. As shown in Street [30], using the corollary of §2.2, the equivalence of categories in Axiom 2 is determined by y_A as follows:

$$B \xrightarrow{h} \Gamma A \quad \longmapsto \quad \begin{tikzcd} & y_A/h \arrow[dl] \arrow[dr] & \\ A & & B. \end{tikzcd}$$

Of course, Prof(A,B) is 2-functorial in A, so that Γ becomes a 2-functor reversing both 1-cells and 2-cells between legitimate objects in K.

Theorem. *If* A *is small and* $A \xrightarrow{f} B$ *is admissible then* $\Gamma B \xrightarrow{\Gamma f} \Gamma A$ *has a right adjoint* $\Gamma A \xrightarrow{\Pi f} \Gamma B$.

The generalized Chevalley (or Beck) condition in the cosmos context is:

the 2-cell

$$\begin{tikzcd} \Gamma(r/s) \arrow[r, "\Pi d_1"] & \Gamma B \\ \Gamma A \arrow[u, "\Gamma d_0"] \arrow[r, "\Pi r"'] \arrow[ur, phantom, "\Leftarrow"] & \Gamma C \arrow[u, "\Gamma s"'] \end{tikzcd}$$

induced by the 2-cell

$$\begin{tikzcd} r/s \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\lambda \Rightarrow"] & B \arrow[d, "s"] \\ A \arrow[r, "r"'] & C \end{tikzcd}$$

of a legitimate comma object, is an isomorphism.

Presumably this condition holds in a pre-cosmos, although we have at present only verified it in the presence of Axioms 4, 5, 6, 7 below. Note that the "pullback" Beck condition of Lawvere [25] does not hold for the hyperdoctrine $\underline{\underline{Cat}}$ with $PX = [X^{op}, \underline{\underline{Set}}]$ (we have a counterexample) but the "comma object" version does (of course this requires the consideration of $\underline{\underline{Cat}}$ as a 2-category, not just a category).

After the last theorem one naturally asks: does Γf have a *left* adjoint? It seems that in a pre-cosmos it need not.

A counter-example should be provided by $\mathcal{V}$-Cat where $\mathcal{V}$ is not cocomplete.

A pre-cosmos is called a cosmos when the following axiom holds, as we henceforth suppose.

Axiom 4. If A is small and $A \xrightarrow{f} B$ is op-admissable then $\Gamma B \xrightarrow[\Gamma f]{} \Gamma A$ has a left adjoint $\Gamma A \xrightarrow[\Sigma f]{} \Gamma B$.

That the notion of cosmos is the "right" one is not yet clear. The fundamental theorem of cosmoi should be: for small A,B, the 2-category Bim(A,B) is a cosmos; I am trying to prove this.

Theorem. For small A,B and op-admissable $B \xrightarrow{t} X$, any arrow $B \xrightarrow{h} \Gamma A$ has a pointwise (see below) left extension along t.

The "pointwise" property referred to here makes sense in any 2-category. A 2-cell

$$\begin{tikzcd} A \arrow[rr, "j"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\kappa}{\Rightarrow}"] & B \arrow[dl, "k"] \\ & X & \end{tikzcd}$$

is said to exhibit k as a pointwise left extension of f along j when, given any arrow $C \xrightarrow{b} B$ for which j/b exists, the composite 2-cell

$$\begin{tikzcd}
j/b \arrow[rr, "d_1"] \arrow[d, "d_0"'] & & C \arrow[d, "b"] \\
A \arrow[rr, "j"] \arrow[dr, "f"'] \arrow[urr, phantom, "\lambda"] & & B \arrow[dl, "k"] \arrow[dll, phantom, "\kappa"] \\
& X &
\end{tikzcd}$$

exhibits kb as a left extension of fd_0 along d_1. The generalized Chevalley condition mentioned above expresses internal pointwiseness of right (and left, on taking adjoints) extensions.

A "biclosed bicategory" Prof_s is then obtained whose objects are the small objects of K, whose hom-categories are given by $\mathrm{Prof}_s(A,B) = \mathrm{Prof}(A,B)$, and whose composition functors correspond under equivalence to

$$K(C,\Gamma B) \times K(B,\Gamma A) \to K(C,\Gamma A)$$
$$(k,h) \longmapsto h'k$$

where h' denotes the pointwise left extension of h along y_B.

In our attempt to capture V-$\underline{\underline{\mathrm{Cat}}}$, cosmoi with further structure have been considered.

<u>Axiom 5</u>. <u>There is a 2-functor</u> $K \times K \xrightarrow{\otimes} K$ <u>such that</u> A⊗B <u>is small or legitimate when</u> A,B <u>are, and a small object</u> I <u>which determine</u>

<u>the structure of an enriched-monoidal 2-category on</u> K.

<u>Axiom 6</u>. <u>There is a 2-functor</u> $(\ )^{op}$ <u>on</u> K <u>which reverses 2-cells, takes small objects into small objects, is involutory, and strictly preserves the monoidal structure</u>.

<u>Axiom 7</u>. <u>For each legitimate object</u> A, <u>there is an evaluation arrow</u> $(\Gamma A)\otimes A^{op} \xrightarrow{e_A} \Gamma I$ <u>which determines an isomorphism of</u> categories

$$K(B,\Gamma A) \cong K(B\otimes A^{op},\Gamma I)$$

<u>for all</u> B.

A pre-cosmos is called <u>small</u> when all the objects are small. (For example, <u>Ord</u>-<u>Set</u> or, more generally, the ordered objects in a topos).

For ordered sets, completeness implies cocompleteness, so one might expect Axiom 4 to be redundant in a small cosmos.

<u>Theorem</u>. <u>A small pre-cosmos satisfying Axioms 5,6,7 and condition (*) below is a cosmos</u>.

The following Chevalley-like conditon is possibly a consequence of Axioms 5,6,7 on a pre-cosmos:

(*) for any arrows $A \xrightarrow{f} B$, $C \xrightarrow{g} D$ between legitimate objects, the canonical 2-cell

$$\begin{tikzcd}
\Gamma(A\otimes C) \arrow[r, "\Pi(1\otimes g)"] & \Gamma(A\otimes D) \\
\Gamma(B\otimes C) \arrow[u, "\Gamma(f\otimes 1)"] \arrow[r, "\Pi(1\otimes g)"'] \arrow[ur, phantom, "\Leftarrow"] & \Gamma(B\otimes D) \arrow[u, "\Gamma(f\otimes 1)"']
\end{tikzcd}$$

is an isomorphism. It is true in V-Cat.

2.4 Cosmoi without comma objects (Walters)

There are many 2-categories which are similar to cosmoi but which do not have comma objects or pullbacks. For example, in a 2-category in which every arrow has a right adjoint the identity arrows are analogous to the representation arrows of a cosmos. (The compact monoidal categories of Kelly [15] are such 2-categories.)

We give a description below of the notion of profunctor in a 2-category without comma objects or pullbacks. (For simplicity, we neglect the problems of legitimacy which occur in Cat.)

The Category Prof(A,B) Let C be a 2-category and let A,B be objects of C. Denote opspans $A \xrightarrow{f} C \xleftarrow{g} B$ by (f,g), and denote diagrams of the form

$$\begin{tikzcd} & C \arrow[dd] & \\ A \arrow[ur, "f"] \arrow[dr, "h"'] \arrow[r, phantom, "\Rightarrow"] & {} \arrow[r, phantom, "\Rightarrow"] & B \arrow[ul, "g"'] \arrow[dl, "k"] \\ & D & \end{tikzcd}$$

by $(f,g) \Rightarrow (h,k)$. If α is such a diagram denote by $\hat{\alpha}$ the function from 2-cells of the form
$$\begin{tikzcd} & \cdot \arrow[dl, "u"'] \arrow[dr, "v"] & \\ \cdot \arrow[dr, "f"'] \arrow[rr, phantom, "\Rightarrow"] & & \cdot \arrow[dl, "g"] \\ & \cdot & \end{tikzcd}$$
to 2-cells of the form
$$\begin{tikzcd} & \cdot \arrow[dl, "u"'] \arrow[dr, "v"] & \\ \cdot \arrow[dr, "h"'] \arrow[rr, phantom, "\Rightarrow"] & & \cdot \arrow[dl, "k"] \\ & \cdot & \end{tikzcd}$$
induced by composition with α.

Consider finite sequences of diagrams

$$(*) \qquad (f,g) \overset{\alpha_1}{\Longleftarrow} \cdot \overset{\beta_1}{\Longrightarrow} \cdot \overset{\alpha_2}{\Longleftarrow} \cdots \overset{\beta_n}{\Longrightarrow} (h,k)$$

in which each $\hat{\alpha}_i$ $(i = 1,2, \ldots, n)$ is a bijection. Two such sequences $(*)$ and $(f,g) \overset{\gamma_1}{\Longleftarrow} \cdot \overset{\delta_1}{\Longrightarrow} \cdots \overset{\delta_m}{\Longrightarrow} (h,k)$ are called <u>equivalent</u> if

$$\hat{\beta}_n \ldots \hat{\alpha}_2^{-1}\, \hat{\beta}_1\, \hat{\alpha}_1^{-1} = \hat{\delta}_m \ldots \hat{\delta}_1\, \hat{\gamma}_1^{-1}.$$

Now the category Prof(A,B) is defined as follows. The objects of Prof(A,B) are the opspans from A to B. The arrows from (f,g) to (h,k) are the equivalence classes of sequences from (f,g) to (h,k).

It can be proved that if either (i) $\mathcal{C}$ is a cosmos, or (ii) $\mathcal{C}$ has comma and opcomma objects, then Prof(A,B) is equivalent to the category of profunctors described in §2.2 above.

The Universal Property Prof(A,B) is made 2-functorial in B (reversing both arrows and 2-cells) by composition. Then $\mathcal{C}$ is a *precosmos* if it satisfies the following

Axiom For each A in $\mathcal{C}$ there is an object ΓA and there are equivalences $\mathrm{Prof}(A,B) \to \mathcal{C}(B,\Gamma A)$ natural in B. Further if $yA\colon A \to \Gamma A$ corresponds to $(1_A, 1_A)$ then the universal profunctor is $(y_A, 1_{\Gamma A})$. Further if (f,g) corresponds to $k\colon B \to \Gamma A$ then the isomorphism between (f,g) and (yA,k) is achieved by a diagram of the form

$$\begin{tikzcd}
 & & C \arrow[dd] & & \\
A \arrow[urr, "f"] \arrow[drr, "yA"'] & \Rightarrow & & = & B \arrow[ull, "g"'] \arrow[dll, "k"] \\
 & & \Gamma A & &
\end{tikzcd}$$

It can be shown that this axiom is an elementary condition on $\mathcal{C}$. A precosmos as defined in §2.3 satisfies this axiom. Conversely, if $\mathcal{C}$ has comma and opcomma objects and satisfies this axiom then $\mathcal{C}$ is a precosmos in the sense of 2.3.

Many properties of the representation functor in *Cat* may be deduced from this axiom; for example, the lifting and extension properties described in Street [30] and Walters [35]. With the further assumption that each Γf has a left adjoint Prof becomes a biclosed bicategory.

2.5 A factorization system for functors. (Street-Walters)

Given a factorization system (E,M) on a category A in the sense of Freyd-Kelly [11], let $M(A)$ denote the full subcategory of $A/\ulcorner A \urcorner$ consisting of the arrows in M with target A. Then the inclusion of $M(A)$ in $A/\ulcorner A \urcorner$ has a left adjoint (namely, "take the (E,M)-image")

$$M(A) \rightleftarrows A/\ulcorner A \urcorner .$$

An example of this occurs when A is the category of sets and M is the class of injective functions; then E is the class of surjective functions and the above adjunction is an example of a comprehension scheme (Lawvere [25]).

One naturally asks whether the comprehension scheme for the hyperdoctrine $\underline{\underline{\mathrm{Cat}}}$ with $PX = [X^{op},\underline{\underline{\mathrm{Set}}}]$ arises from a factorization system on $\underline{\underline{\mathrm{Cat}}}$, and if so, what are the elements of E? These questions are answered in Street-Walters [31], the idea being suggested by the method of constructing the factorization of an arrow in a topos, and the analogue of this in a suitable cosmos. This comprehension scheme does indeed arise in such a way, and :

M = the class of discrete 0-fibrations (Gray [13]);

E = the class of initial functors (Mac Lane [27]).

3. Categories of functors.

3.1 Note on monoidal localisations (Day) (text by editors).

{Editors' comment: Day provided the abstracts given in §3.2 and §3.3 below, but not one of the paper of the above title, which is shortly to appear [6]. To aid in the appreciation of the succeeding sections, the editors have taken the liberty of inserting the following remarks on the above paper.}

The author shows that the category of fractions $A[Z^{-1}]$ is monoidal when A is, and solves the universal problem for monoidal functors that invert Z, provided that Z contains, along with a morphism s, every $1_A \otimes s$ and $s \otimes 1_B$. (In future we take for simplicity $\otimes$ to be symmetric.) In the author's applications, the functor $A \to A[Z^{-1}]$ has a right adjoint, so that $A[Z^{-1}]$ is a reflective subcategory of A, and we are in the situation considered in the author's earlier paper [5].

This is the case in particular when A is a complete symmetric monoidal closed category with a set of strong generators, if we have some left-adjoint functor $S: A \to B$ and take Z to consist of those s such that every $s \otimes 1_A$ is inverted by S. So if B is a full reflective subcategory of A and S the reflection, the category $\bar{B} = A[Z^{-1}]$ is the smallest reflective subcategory containing B that is monoidal (and hence closed).

Taking A to be $[C^{op}, \underline{\underline{Sets}}]$ for a small monoidal C, and B to be a Lambek completion of C, the author obtains a completion theorem for monoidal categories. For $\overline{B}$ is a complete and cocomplete monoidal closed category, and the inclusion $C \to \overline{B}$ is dense, is continuous, is $\otimes$-preserving, and preserves those colimits in C preserved by each $-\otimes C$. Moreover if C is already closed, $C \to \overline{B}$ preserves all colimits and preserves internal-homs.

For a closed category V, the functor category $[A,V]$ is not available unless the V-category A is small, and many a theorem has to be proved by hard work because of this lack. The author applies his results to "legitimize" the shorter functor-category arguments, by passing to a new universe of sets and suitably completing V.

3.2 On adjoint-functor factorisation. (Day)

This article gives an account of the factorisation of an adjunction $S \dashv T\colon \mathcal{C} \to \mathcal{B}$ through the associated idempotent monad on $\mathcal{B}$, where $\mathcal{B}$ is a complete category. The result was first obtained by Applegate and Tierney [2] and Fakir [10] and, for the relative $\mathcal{V}$-based case, where $\mathcal{V}$ is a complete symmetric monoidal closed category, the result implies the existence of the second relative completion process discussed by Dubuc [7], Theorem III.3.2.

We do not use the cotriple-tower construction of Applegate-Tierney. The approach used is simply to first factor the given adjunction unit into an epimorphism followed by a monomorphism. This process determines the full reflective subcategory $\mathcal{B}' \subset \mathcal{B}$ comprising all subobjects of objects in the image of $T\colon \mathcal{C} \to \mathcal{B}$. An object $B \in \mathcal{B}$ is called orthogonal to a morphism $s \in \mathcal{B}$ if $\mathcal{B}(s,B)$ is an isomorphism. The category $\mathcal{B}'$ is then seen to have the property that the class of objects in $\mathcal{B}'$ which are orthogonal to <u>any</u> given class of morphisms inverted by the restriction of S to $\mathcal{B}' \subset \mathcal{B}$ forms a reflective subcategory of $\mathcal{B}'$.

To summarise these remarks, the class of reflective subcategories of $\mathcal{B}'$ which reflectively contain the full subcategory of $\mathcal{B}'$ determined by the objects orthogonal to all the morphisms inverted by S: $\mathcal{B}' \to \mathcal{C}$ forms a complete lattice. Thus, for example, if $\mathcal{B}'$ has a monoidal closed structure, each such subcategory of $\mathcal{B}'$ can be embedded in a "monoidal closure". This generalises a result of P. Antoine [1] on the embedding of the category *Top* of all topological spaces and continuous maps into a "minimal" cartesian closed extension.

3.3 On closed categories of functors II. (Day)

This paper discusses conditions under which a monoidal or, more generally, a promonoidal structure on a category A generates, along a given dense functor M: $A^{op} \to \mathcal{C}$, a monoidal biclosed structure on the codomain category $\mathcal{C}$.

There are two cases of special interest. The first is the Yoneda embedding M: $A^{op} \to [A,\mathcal{S}]$ of the category A^{op} into the category $[A,\mathcal{S}]$ of all set-valued functors on A; this embedding generates the "convolution" structure on $[A,\mathcal{S}]$ (as treated in [4]). The second case is that in which M: $A^{op} \to \mathcal{C}$ is the left adjoint part of a monoidal reflective embedding, A being the monoidal-dual structure of a monoidal biclosed category A^{op} (as treated in [5]).

The main features of the proof can be summarised as follows. A dense functor from a small category A^{op} to a cocomplete category $\mathcal{C}$ can be written as the composite $A^{op} \to [A,\mathcal{S}] \to \mathcal{C}$ of a

Yoneda embedding and a reflection; the general theorem would thus be established by combining the two special cases except that $[A,S]$ is not a small category. However, a more refined version of the theorem with A a large category and C admitting only certain necessary limits and colimits may be obtained by embedding C in a suitable completion $\hat{C}$ with respect to a larger universe $\hat{S}$ which contains A, C, and S as elements; that is, as category objects. The biclosed monoidal structure which occurs on C is simply the restriction of the structure obtained on $\hat{C}$ as a reflective subcategory of the closed functor category $[A,\hat{S}]$.

The monoidal version of the theorem is as follows:

<u>Theorem</u>: <u>Let</u> A <u>be a monoidal category and let</u> $M: A^{op} \to C$ <u>be a dense functor into a category</u> C <u>with suitable copowers</u>. <u>Then there exists a monoidal biclosed structure on</u> C <u>and a tensor-product--preserving monoidal enrichment of</u> M <u>if and only if the colimits and limits</u>

$$X \otimes Y = \int^{AB} (C(MA,X) \times C(MB,Y)) \cdot M(A \otimes B)$$

$$H(BX) = \int^{A} C(M(A \otimes B),X) \cdot MA$$

$$K(BX) = \int^{A} C(M(B \otimes A),X) \cdot MA$$

$$Y/X = \int_{A} [C(MA,X), H(AY)]$$

$$X \backslash Y = \int_{A} [C(MA,X), K(AY)]$$

exist in $\mathcal{C}$ and the canonical maps

$$\mathcal{C}(M(A\otimes B),X) \to \mathcal{C}(MA,\ H(BX))$$

$$\mathcal{C}(M(B\otimes A),X) \to \mathcal{C}(MA,\ K(BX))$$

are isomorphisms for all $A,B \in \mathcal{A}$ and $X,Y \in \mathcal{C}$.

To obtain the refined version of this theorem in the case where the category theory is relative to a suitably complete symmetric monoidal closed category $\mathcal{V}$, we need a workable notion of "change of $\mathcal{V}$-universe" (as treated in [6]).

The concepts of convolution and reflection (or, more generally, "localisation") are available in the setting of bicategories and biclosed bicategories (Benabou [3]) and analogous results are available here.

3.4 Categories of continuous functors (Kelly) (with P.J. Freyd)

Let $\mathcal{C}$ and $\mathcal{A}$ be categories with $\mathcal{C}$ small, and let Γ be a family (perhaps large) of (projective) cones in $\mathcal{C}$. The functors $\mathcal{C} \to \mathcal{A}$ which turn each cone in Γ into a limiting cone in $\mathcal{A}$ form a full subcategory $[\mathcal{C},\mathcal{A}]_\Gamma$ of the functor category $[\mathcal{C},\mathcal{A}]$. This subcategory is closed under limits, and one seeks conditions on $\mathcal{A}$ in order that it should be, for all $\mathcal{C}$ and all Γ, a reflective subcategory.

Gabriel-Ulmer [12] showed that it is so if A is locally presentable. Peter Freyd and I showed independently [11] that it is so if A belongs to the larger class of boundable categories, which includes non-locally-presentable ones like Top. The condition of boundability, however, like that of local presentability, being a generalization of AB5, is not except in trivial cases possessed by A^{op} if it is possessed by A. Freyd and I promised a sequel in which we would show that $[C,A]_{\Gamma}$ is reflective for certain A's where A^{op} was boundable.

This has never been written up, because we have so far failed to prove it in the kind of generality we should like, e.g. for A^{op} boundable, or for A^{op} locally presentable. Since there have been some enquiries, however, it may be worth recording what we can do in this direction.

We have a proof when A^{op} is the category B of algebras for a monad-with-rank on Sets. Here we could surely replace Sets by a power of Sets. If we could do it for a reflective-subcategory-with-a-rank of such a B, we should then have it by [12], for A^{op} locally presentable: but this generalization eludes us.

We also have a proof for A^{op} = Top, and we can combine these to get a proof for cases like A^{op} = T-algebras in Top, at least for a finitary theory T. Similarly with Top replaced by k-spaces and so on. Indeed these Top -like extensions are covered by the recent results of Wischnewsky [36], who shows roughly that if it works for A then it works for A', if there is a faithful functor $A' \to A$ with adjoints on both sides.

The proof is totally different from that in [11], and uses the special adjoint functor theorem.

4. Categories with structure, and coherence

4.1 Doctrinal adjunction (Kelly)

We use the prefix "2-" to mean "enriched over $\underline{\underline{Cat}}$", not in the more general sense of Gray [13]. Following Lawvere [24] we call a 2-monad D on $\underline{\underline{Cat}}$ a doctrine. A D-category is an algebra for D; a D-functor is a lax morphism of such algebras; a D-natural transformation is suitably defined; and these elements constitute a 2-category D-$\underline{\underline{Cat}}$.

More precisely, if A and B are D-categories with actions $\theta: DA \to A$ and $\theta': DB \to B$, a D-functor $\Phi: A \to B$ consists of a functor $\phi: A \to B$ together with a natural transformation $\overline{\phi}: \theta'.D\phi \Rightarrow \phi\theta$, satisfying the evident axioms. A D-natural transformation $\eta: \Phi \Rightarrow \Psi: A \to B$ is a natural transformation $\eta: \phi \Rightarrow \psi$ satisfying the evident axiom relating it to $\overline{\phi}$ and $\overline{\psi}$. Reversing the sense of $\overline{\phi}$ gives the notion of op-D-functor. A D-functor Φ is called strong if $\overline{\phi}$ is an isomorphism, and strict if $\overline{\phi} = 1$.

An example of D-$\underline{\underline{Cat}}$ is the 2-category $\underline{\underline{Mon}}$ of monoidal categories, monoidal functors, and monoidal natural transformations. Another is $\underline{\underline{SMon}}$, of symmetric monoidal categories etc. Yet another has categories-with-a-monad as its objects, and the monad-functors of Street [29] as its 1-cells. More generally, we can take D to be $K\circ-$ where K is any covariant club (see §4.2 below); and there are still other examples where D is not derived from a club. Moreover we could just as well replace D by a 2-monad not on $\underline{\underline{\mathrm{Cat}}}$ but on <u>any</u> 2-category at all.

<u>Proposition 4.1</u> <u>Let</u> $\varepsilon,\eta\colon \psi \dashv \phi\colon A \to B$ <u>be an adjunction in</u> $\underline{\underline{\mathrm{Cat}}}$, <u>where</u> A <u>and</u> B are D-<u>categories. There is a bijection between enrichments</u> $\overline{\phi}$ <u>of</u> ϕ <u>to a</u> D-<u>functor</u> $\Phi = (\phi,\overline{\phi})$ <u>and enrichments</u> $\overline{\psi}'$ <u>of</u> ψ <u>to an op</u>-D-<u>functor</u> $\Psi' = (\psi,\overline{\psi}')$.

(Loosely, "the left adjoint of a D-functor is an op-D-functor".)

<u>Proposition 4.2</u> <u>A</u> D-<u>functor</u> $\Phi = (\phi,\overline{\phi})\colon A \to B$ <u>has a left adjoint in</u> D-$\underline{\underline{\mathrm{Cat}}}$ <u>if and only if</u> ϕ <u>has a left adjoint</u> ψ <u>in</u> $\underline{\underline{\mathrm{Cat}}}$ <u>as in Proposition</u> 4.1 <u>and the corresponding</u> $\overline{\psi}'$ <u>is an isomorphism. Then the left adjoint</u> Ψ <u>of</u> Φ <u>in</u> D-$\underline{\underline{\mathrm{Cat}}}$ <u>is</u> $(\psi,\overline{\psi})$ <u>where</u> $\overline{\psi} = \overline{\psi}'^{-1}$. <u>In particular</u> Ψ <u>is necessarily strong. Similarly a</u> D-<u>functor</u> $\Psi\colon B \to A$ <u>with a right adjoint in</u> $\underline{\underline{\mathrm{Cat}}}$ <u>has one in</u> D-$\underline{\underline{\mathrm{Cat}}}$ <u>if and only if it is strong</u>.

In the special case where D-$\underline{\underline{\mathrm{Cat}}}$ = $\underline{\underline{\mathrm{Mon}}}$, a strong monoidal functor is just a $\otimes$-preserving one, and an op-monoidal-functor $A \to B$ is just a monoidal functor $A^{op} \to B^{op}$. The condition that $\overline{\psi}'$ be an isomorphism can be written in terms of $\overline{\phi}$; in the present case it is that ϕ° be normal plus a condition on $\tilde{\phi}\colon \phi A \otimes \phi B \to \phi(A\otimes B)$. If A and B are closed as well as monoidal, this latter condition can be written in terms of

$\hat{\phi}: \phi[A,B] \to [\phi A, \phi B]$. In this form the condition still makes sense for non-monoidal closed categories A and B, and a closed functor $\Phi: A \to B$; and these conditions are still necessary and sufficient for a left adjoint ψ of ϕ in Cat to admit enrichment to a left adjoint Ψ of Φ in Closed_Cat. This should simplify greatly the forthcoming paper of Wolff [37]. Presumably the generalization of this is to pro-D-structures for a general D, in the spirit of Day [4].

If $\varepsilon = 1$, so that A is a full reflective subcategory of B with inclusion ϕ and reflexion ψ, and if we suppose that only B, not A, is given with a D-category structure, we can ask whether A can be enriched to a D-category and the reflexion to one in D-Cat; a generalization from Mon to D-Cat of the problem studied by Day in [5]. A necessary condition is that $\psi.\theta'.D\eta$ be an isomorphism, and this is sufficient if the doctrine D is <u>flexible</u> enough: roughly, if a category equivalent to a D-category is also a D-category. The doctrine for Mon is flexible, and the above condition becomes Day's condition that $\psi(\eta \otimes \eta)$ be an isomorphism; the doctrine for <u>strict</u> monoidal categories is an example of an inflexible one.

4.2 Extension of the notion of covariant club (Kelly)

The idea of a (covariant) club was introduced in [15] and [16] as a setting for assertions about "general diagrams commuting" and about "coherence"; the idea being that the generic category K bearing an extra structure of a given kind comes, in favourable cases, with an augmentation functor $\Gamma: K \to G$ sending each "natural transformation" to its "graph"; so that a necessary condition for commutativity of a diagram in K is commutativity of its image under Γ, and to say that "all diagrams commute" is to say that Γ is faithful. The graph was to be the information as to which variables in the domain and in the codomain of the natural transformation were to be set equal. The above papers dealt primarily with the case where the graphs lay in the category $G = \underline{\underline{P}}$ of finite sets and permutations; but it was suggested in [15] that everything doubtless carries over to the cases $G = \underline{\underline{Sets}}$ and $G = \underline{\underline{Cat}}$, as well as to $G = \underline{\underline{Sets}}^{op}$ and $G = \underline{\underline{Cat}}^{op}$.

The extension to the case $G = \underline{\underline{Cat}}$ has now been checked out; I am indebted to Street for helpful suggestions. The situation is the following.

We said earlier that for us the prefix "2-" means "enriched over $\underline{\underline{Cat}}$". For the more general sense in which Gray in [13] uses "2-natural transformation", "2-comma-object", "2-colimit", we use instead the prefix "lax". We ignore size considerations in this sketch.

For categories $\mathcal{B}$ and $\mathcal{C}$ we define a "multi-functor category" $\{\mathcal{B},\mathcal{C}\}$. An object is a small category n together with a functor $T\colon \mathcal{B}^n \to \mathcal{C}$, where $\mathcal{B}^n$ is the usual functor category. A morphism from (n,T) to (m,S) is a functor $\phi\colon n \to m$ together with an ordinary natural transformation f:

$$\begin{tikzcd} & \mathcal{B}^n \arrow[dr, "T"] \arrow[d, phantom, "\Downarrow f"] & \\ \mathcal{B}^m \arrow[ur, "\mathcal{B}^\phi"] \arrow[rr, "S"'] & & \mathcal{C}. \end{tikzcd}$$

Composition is evident. We have an augmentation functor $\Gamma\colon \{\mathcal{B},\mathcal{C}\} \to \underline{\underline{Cat}}$ sending (n,T) to n and (ϕ, f) to ϕ. We can call (n,T), or T for short, a functor from $\mathcal{B}$ to $\mathcal{C}$ of type n; and call (ϕ, f), or f for short, a natural transformation from T to S of graph ϕ. As a category over $\underline{\underline{Cat}}$, $\{\mathcal{B},\mathcal{C}\}$ may be said to be an object of $\underline{\underline{Cat}}/\underline{\underline{Cat}}$, where the second $\underline{\underline{Cat}}$ is regarded as an object of the first one. Then $\{-,-\}$ is a 2-functor $\underline{\underline{Cat}}^{op} \times \underline{\underline{Cat}} \to \underline{\underline{Cat}}/\underline{\underline{Cat}}$.

It has a left 2-adjoint $\circ\colon \underline{\underline{Cat}}/\underline{\underline{Cat}} \times \underline{\underline{Cat}} \to \underline{\underline{Cat}}$, so that $\underline{\underline{Cat}}(A\circ\mathcal{B},\mathcal{C}) \cong \underline{\underline{Cat}}/\underline{\underline{Cat}}(A,\{\mathcal{B},\mathcal{C}\})$. It turns out that $A\circ\mathcal{B}$ is the lax comma category formed from the functors

$$A \xrightarrow{\Gamma} \underline{\underline{Cat}} \xleftarrow{\ulcorner\mathcal{B}\urcorner} 1,$$

where Γ is the augmentation of $\mathcal{A}$, $\mathcal{I}$ is the unit category, and $\ulcorner \mathcal{B} \urcorner$ is the name of $\mathcal{B}$. An object of $\mathcal{A} \circ \mathcal{B}$ is a pair $A[X]$ where $A \in \mathcal{A}$ and $X \colon \Gamma A \to \mathcal{B}$; a morphism is $f[g]$ where $f \colon A \to A'$ and

$$\begin{tikzcd}
\Gamma A \arrow[rr, "\Gamma f"] \arrow[dr, "X"'] & \arrow[d, phantom, "\overset{\Rightarrow}{g}"] & \Gamma A' \arrow[dl, "X'"] \\
& \mathcal{B} &
\end{tikzcd}$$

Now regard $\mathbf{Cat}$ as a full subcategory of $\mathbf{Cat}/\mathbf{Cat}$ by giving to $\mathcal{B}$ in $\mathbf{Cat}$ the trivial augmentation $\mathcal{B} \to \mathbf{Cat}$ which is the constant functor at the empty category. We extend $\circ$ to a 2-functor $\mathbf{Cat}/\mathbf{Cat} \times \mathbf{Cat}/\mathbf{Cat} \to \mathbf{Cat}/\mathbf{Cat}$. We define $\mathcal{A} \circ \mathcal{B}$ as above, but have now to augment it given an augmentation of $\mathcal{B}$. We define $\Gamma(A[X])$ to be

$$\Gamma(A[X]) = \text{lax colim } (\Gamma A \xrightarrow[X]{} \mathcal{B} \xrightarrow[\Gamma]{} \mathbf{Cat}).$$

Then $\circ$ is a coherently associative tensor product on $\mathbf{Cat}/\mathbf{Cat}$, with coherent identity $\mathcal{J}$, where $\mathcal{J}$ is the unit category $\mathcal{I}$ with augmentation $\Gamma = \ulcorner \mathcal{I} \urcorner \colon \mathcal{I} \to \mathbf{Cat}$. Moreover the monoidal category $\mathbf{Cat}/\mathbf{Cat}$ is closed, $-\circ\mathcal{B}$ having the right adjoint $\{\mathcal{B},-\}$, where for augmented $\mathcal{B}$ and $\mathcal{C}$ the category $\{\mathcal{B},\mathcal{C}\}$ only contains those (n,T) rendering commutative

$$\begin{tikzcd}
\mathcal{B}^n \arrow[r, "T"] \arrow[d, "\Gamma^n"'] & \mathcal{C} \arrow[d, "\Gamma"] \\
\mathbf{Cat}^n \arrow[r, "\text{lax colim}"'] & \mathbf{Cat},
\end{tikzcd}$$

with a similar restriction on the (ϕ,f): it reduces to the earlier $\{B,C\}$ when B and C have trivial augmentations.

A covariant club (of the first kind) is now a $\circ$-monoid K in $\underline{\underline{Cat}}/\underline{\underline{Cat}}$. It determines a doctrine $D = K\circ -$ on $\underline{\underline{Cat}}$. A D-category is a category A together with an action $K\circ A \to A$, or equally together with a club-map $K \to \{A,A\}$. Each object T of K of type n gives therefore a functor $|T|: A^n \to A$, and each morphism f of graph ϕ gives a natural transformation $|f|: |T|\ A^{\phi} \Rightarrow |S|$.

We can similarly speak of a club of the second kind: a $\circ$- monoid for a suitable tensor product $\circ$ in $\underline{\underline{Cat}}/\underline{\underline{Cat}}^{op}$. But such a club L is of the form K^{op} for a club K of the first kind; and $L\circ A$ is $(K\circ A^{op})^{op}$, so that $L\circ -$ is what Lawvere in [24] calls the opposite doctrine to $K\circ -$.

Of course not every doctrine comes in this way from a club of one kind or the other. Clubs of the second kind involve natural transformations of the form $|f|: |T| \Rightarrow |S|A^{\psi}$; a general doctrine will have natural transformations irreducibly of the more general form $|f|: |T|A^{\phi} \Rightarrow |S|A^{\psi}$.

Just as a theory may have a rank, so a club may not need all of $\underline{\underline{Cat}}$ to receive its augmentation. If G is a subcategory of $\underline{\underline{Cat}}$, we can embed $\underline{\underline{Cat}}/G$ as a full subcategory of $\underline{\underline{Cat}}/\underline{\underline{Cat}}$; and for some suitable G's this is closed under $\circ$, e.g. for

G = $\underline{\underline{\text{Sets}}}$ or $G = \underline{\underline{P}}$. A club in $\underline{\underline{\text{Cat}}}/G$ may be called a G-club. The $\circ$ in $\underline{\underline{\text{Cat}}}/G$ is that in $\underline{\underline{\text{Cat}}}/\underline{\underline{\text{Cat}}}$; the $\{\ ,\ \}'$ in $\underline{\underline{\text{Cat}}}/G$ is smaller than that in $\underline{\underline{\text{Cat}}}/\underline{\underline{\text{Cat}}}$, having restricted types and graphs.

In practice, a club K is defined by generators and relations; the coherence problem for the structure in question is that of finding K, with its augmentation and monoid structure, from these generators and relations; a diagram commutes in K if and only if its image commutes in $\{A,A\}$ for each K-category A.

4.3 Operads (Kelly)

A paper "on the operads of P.J. May", originally submitted in January 1972, and returned in July 1972 with a request from the referee to clarify a few connexions (especially that with clubs), is still I fear awaiting revision. The attempt to see more clearly the connexion with clubs led me first to §4.1 above, and then I got side-tracked generally. Perhaps a short report is in order.

Let V be a complete and cocomplete symmetric monoidal closed category, and $\underline{\underline{P}}$ the skeletal category of finite sets and permutations. The functor category $[\underline{\underline{P}},V]$ has itself a symmetric monoidal closed structure, the convolution in the sense of Day [4] of the monoidal structure on $\underline{\underline{P}}$ and the closed structure on V; the tensor product is given by

$$T \otimes S = \int^{m,n} \underline{\underline{P}}(m+n, -) \otimes Tm \otimes Sn.$$

However $[\underline{\underline{P}}, V]$ has another monoidal closed structure, this time not symmetric (and not biclosed), whose tensor product is given by

$$T \circ S = \int^{m} Tm \otimes S^m,$$

where S^m is the m-fold $\otimes$-power of S, regarded as a contravariant functor of m. The corresponding internal hom $\{S,R\}$ is given by

$$\{S,R\}m = \int_n [(S^m)n, Rn].$$

We call a $\circ$-monoid K in $[\underline{\underline{P}}, V]$ an <u>operad</u>; it determines a monad $K\circ-$ on $[\underline{\underline{P}}, V]$, which restricts to a monad $K\circ-$ on V, suitably embedded in $[\underline{\underline{P}}, V]$. If we demand of K that $K(0)$ be the identity object I of V, the algebras for the monad $K\circ-$ on V are also the algebras for a suitable monad on the category I/V of pointed V-objects. If V = hausdorff k-spaces, an operad satisfying this latter condition coincides with the notion of operad introduced by May in [28]. The present treatment therefore extends May's concept to any symmetric monoidal closed V; it behaves well under change of V, so that an operad on $\underline{\underline{Top}}$ gives one on chain complexes and so on; it would be pleasant if this enriched the value of operads in algebraic topology, but I haven't pursued this. It further turns out that V-operads can be identified with V-props of a special kind.

One can always replace the domain category $\underline{\underline{P}}$ by $\underline{\underline{N}}$, the discrete skeletal category of finite sets. This gives the "non-Σ operads" of May [28]. If V is cartesian closed, one can replace the domain category $\underline{\underline{P}}$ by $\underline{\underline{S}}$, the skeletal category of finite sets. When this is done, V-prop is replaced by V-theory, and V-operads may be identified with all the finitary V-theories.

When we take $V = \underline{\underline{Cat}}$, there is a strong formal resemblance between operads and clubs, for the domain category $\underline{\underline{P}}$; a club is a $\circ$-monoid in $\underline{\underline{Cat}}/\underline{\underline{P}}$, and an operad is a $\circ$-monoid in $[\underline{\underline{P}},\underline{\underline{Cat}}]$. I am beginning to understand this better: it seems that the canonical left-adjoint functor $\underline{\underline{Cat}}/\underline{\underline{P}} \to [\underline{\underline{P}},\underline{\underline{Cat}}]$ preserves $\circ$, and sends a club to an operad, with the same algebras. I hoped at one time that a general doctrine could be defined by a club in $\underline{\underline{Cat}}/G$, where G was something containing both $\underline{\underline{Cat}}$ and $\underline{\underline{Cat}}^{op}$ - presumably the category of categories and op-spans; but now I doubt that this makes sense. A general doctrine is an operad in $[\underline{\underline{Cat}}, \underline{\underline{Cat}}]$, for there $\circ$ reduces to composition; I am still sorting this out.

4.4 Mixed-variance clubs (Kelly)

Structures such as a monoidal closed one, involving functors such as $[\ ,\ ] : A^{op} \times A \to A$ that are not covariant in every variable, may as was shown in [16] in some cases be regarded as algebras for an extended kind of club, called a <u>mixed-variance club</u>. We restrict ourselves here to the simplest case where the natural transformations are of the generalized kind introduced by Eilenberg-Kelly in [8], whose graphs are fixed-point-free involutions on the totality of variables in the domain and the codomain. Let $\underline{\underline{P}}'$ be the category of such graphs, but with added zero-morphisms * to act as the composite of incompatible graphs. Then we can define $A \circ B$ for $A, B \in \underline{\underline{Cat}}/\underline{\underline{P}}'$ provided that the augmentations of A and of B do not contain * in their images: i.e., if no incompatibles occur. With this restriction, $\circ$ is coherently associative with identity (but has no right adjoint $\{-,-\}$). A $\circ$-monoid L in this sense is a <u>mixed-variance club</u>, and determines a monad $L\circ-$ on $\underline{\underline{Cat}}$, whose algebras we call L-categories.

This gives a convenient setting for discussing the coherence of L-categories in terms of $\Gamma\colon L \to \underline{\underline{P}}'$, and gives an explicit construction of the free L-category on the category A in the form $L \circ A$. It further shows that L-categories are monadic over $\underline{\underline{Cat}}$. Thus (see below) monoidal closed categories are monadic, being algebras for such a club L. It must be noted however that in the mixed-variance case the functor $\circ$ is <u>not</u> a 2-functor, so that the monad $L\circ-$ on $\underline{\underline{Cat}}$ is <u>not</u> a doctrine. So monoidal closed categories, while monadic over $\underline{\underline{Cat}}$, are <u>not</u> doctrinal. It also follows that in general there is no concept of "L-functor" or "lax morphism of L-algebras" in the sense of §4.1 above.

Unfortunately it is not possible to tell at once from generators and relations for L that no incompatibles will arise, i.e. that we do get a club. It is shown however in [17], by a cut-elimination argument, that if we start from a covariant $\underline{\underline{P}}$-club K, and consider those K-categories in which some or all of the structure-functors have right adjoints, then these form the algebras for a mixed-variance club L. Thus monoidal closed categories, monoidal biclosed categories, symmetric monoidal closed categories, etc., are categories of algebras for mixed-variance clubs.

It was conjectured in [17] that the canonical club-map $K \to L$ in this situation was faithful. This has now been verified; it turns out that the Yoneda embedding $K \to [K^{op}, \underline{\underline{Sets}}]$ behaves ideally with respect to $\circ$ and $\{-,-\}$, so that $[K^{op}, \underline{\underline{Sets}}]$ admits a K-structure in which <u>every</u> functor has a right adjoint (use left Kan extension to extend the K-structure on K). Thus $[K^{op}, \underline{\underline{Sets}}]$ is an L-category, so that the Yoneda embedding factorizes as $K \to L \to [K^{op}, \underline{\underline{Sets}}]$, whence $K \to L$ is faithful.

More generally, I should take to make the following

<u>Fidelity Conjecture</u>. <u>Let M be a mixed-variance club, and consider those M-categories in which some specified structure-</u>

functor has, in some specified variable, a right adjoint. Then these are the algebras for a mixed-variance club L, and $M \to L$ is faithful.

It is a matter of showing that, if say $-\otimes B$ were to have a right adjoint $[B,-]$, with unit and counit d and e, then any diagram not involving [,], d, or e, which commutes when these are allowed, i.e. which can be "filled in" by little diagrams using these, can be filled in without using these. One can imagine a purely combinatorial proof, and I have a few preliminary ideas for one. However the Yoneda-argument above gives a "transcendental" proof when M is in fact covariant. Perhaps analysis of this will suggest a combinatorial proof.

The fidelity conjecture, if true, would also apply to adding a left adjoint to some functor: just replace M by M^{op}. It would then solve at one stroke a host of coherence problems. For suppose we start with a covariant $\underline{\underline{P}}$-club K, add some right adjoints to get a mixed-variance club L, and then drop some of the original left adjoints to get a new club M. (It is then clear that M is in fact a club). We have faithful functors $K \to L \leftarrow M$. It is often easy to prove a coherence result for K; we know in a general way how to use cut-elimination techniques to get at least a partial coherence-result for L, although each case at the moment needs separate treatment; if $M \to L$ is indeed faithful we then get a coherence result for

M. Of course to say that $M \to L$ is faithful when L and M are got in this rather special way is less than the full fidelity conjecture, and for suitable K the fidelity of $M \to L$ can be established by a functor-category argument imitating §4.5 below. (The "specialness" here is that an M-category is a "pro-K-category").

The simplest example is the following. Let K_1 be the club for monoidal categories, and let L_1 be the club for monoidal closed categories, i.e. those monoidal categories in which $-\otimes B$ has a right adjoint $[B,-]$.

Then the data for L_1 are the functors $\otimes, I$, and $[\ ,\ ]$, together with the natural transformations $a\colon (A\otimes B)\otimes C \to A\otimes(B\otimes C)$, $l\colon I\otimes A \to A$, $r\colon A\otimes I \to A$, $e\colon [A,B]\otimes A \to B$, $d\colon A \to [B,A\otimes B]$, of which a,l,r are to be isomorphisms satisfying the coherence conditions and d,e are to satisfy the triangular equations making $[B,-]$ right adjoint to $-\otimes B$. To say that a is an isomorphism is really to give an extra datum $\bar{a}\colon A\otimes(B\otimes C) \to (A\otimes B)\otimes C$ and two extra axioms $a\bar{a} = 1$, $\bar{a}a = 1$, and similarly for l and r.

Now it is known (cf. [9] §II.3) that in the presence of d and e the giving of a,l,r is equivalent to the giving of natural transformations

$$L\colon [A,B] \to [[C,A],[C,B]],$$
$$j\colon I \to [A,A],$$
$$i\colon A \to [I,A].$$

Moreover the coherence conditions MC1 - MC5 of [9] p. 472 are then equivalent to the coherence conditions CC1 - CC4 of [9] p. 429 together with $i_I = j_I \colon I \to [I,I]$. To say that r is an isomorphism is just to say that i is an isomorphism; to say that l is an isomorphism is to say that (in terms of a model V)

(*) $f \mapsto [f,1]j \ : \ V(A,B) \to V(I,[A,B])$ is an isomorphism.

There is no simple way of asserting in terms of L,j,i that a is an isomorphism.

Let then K be the covariant club got from K_1 by discarding the inverses to a and l, but keeping that to r; let L be the mixed-variance club got by adding a right adjoint [B,-] to $-\otimes B$ with d and e. Then L can be given alternatively in terms of $\otimes$, I, [,] and the natural transformations L,j,i,d,e, with i an isomorphism, satisfying CC1 - CC4, $i_I = j_I$, and the triangular equations for d and e. Now let M be the club got from L by dropping $\otimes$,d,e; it is given by I, [,] and L,j,i with the above axioms (and i an isomorphism).

An M-category V satisfying the extra axiom (*) above is essentially what was called in [9] §I.2 a closed category; we have merely dropped the requirement of a chosen isomorph to $V(I,-)$ turning the isomorphism (*) into an equality.

We have seen that monoidal closed categories, while not doctrinal over **Cat**, are at any rate monadic. It would seem that closed-categories-without-⊗ are not even monadic. For I am pretty sure that (*) is not a consequence of the other axioms in M, and at the same time I am pretty sure that (*) is in fact satisfied by M itself, and hence by any free M-category $M \circ A$. If so, Closed_Cat contains the Kleisli category of $M\circ-$ and is strictly contained in the Eilenberg-Moore category of $M\circ-$, so it is not monadic. Of course (*) is quite different from club-type axioms, asserting an isomorphism in Sets and not an equality in V.

Moreover, if I am right in believing that M satisfies (*), it follows that the only coherence problem for closed-categories--without-⊗ is the coherence problem for M. The fidelity of $M \to L$ in the present case is proved by the embedding result of Day-Laplaza in §4.5 below, so a coherence result for M will follow from one for L. We know coherence results for K_1 and L_1 (at least I take it that the result of Kelly-MacLane in [18] will go over without symmetry to give one for L_1); Laplaza [20] has a coherence result for K (at least for the a part; l and r will surely give no trouble); it is an intriguing but not trivial problem to get a coherence result for L, by cut-elimination techniques, from one for K. I have only had a preliminary look at it; one cannot imitate [18] because a is not an isomorphism; but there seem to be possibilities starting from the cut-elimination theorem of [17]. Even without j and i, the coherence-problem for L is an old one that it would be pleasant to settle. For symmetric closed-categories-without-$\otimes$, a turns out to be an isomorphism, and the Day-Laplaza result below gives a coherence result by the use of [18].

4.5 An embedding theorem for closed categories (Day)(with Laplaza)

The initial aim of this paper is to establish a coherence result for diagrams constructed from the data of a closed category. A closed category is a category A equipped with an internal-hom functor $[-,-]: A^{op} \times A \to A$ and an identity object $I \in A$ and related natural transformations $i: A \cong [IA]$, $j: I \to [AA]$, and $L: [BC] \to [[AB][AC]]$, all satisfying the basic "coherence" axioms CC1 to CC4 of Eilenberg-Kelly [9], together with (*) of §4.4 above; the axiom $i_I = j_I$ is then a consequence.

On defining functors $P: A^{op} \times A^{op} \times A \to S$ and $J: A \to S$ by $P(ABC) = A(A[BC])$ and $JA = A(IA)$ one obtains a "non-associative" promonoidal structure on the category A. In general such a structure is taken to consist of functors P and J together with natural transformations

$$\lambda: J \circ P(-AB) \cong A(AB)$$

$$\rho: J \circ P(A-B) \cong A(AB)$$

$$\alpha: P(A-D) \circ P(BC-) \to P(AB-) \circ P(-CD),$$

where the symbol "$\circ$" denotes profunctor composition. The data are to satisfy axioms analogous to MC1 to MC5 of Eilenberg-Kelly [9] (cf. Day [4]). When the functor J is representable one obtains a non-associative monoidal category precisely when P is representable in the last variable (i.e., when $P(AB-) \cong A(A\otimes B,-)$), and a closed category precisely when P is representable in the first variable (i.e., when $P(-AB) \cong A(-, [AB])$.)

Each non-associative promonoidal structure on a small category A generates a non-associative monoidal structure on the opposite of the category $[A,S]$ of all set-valued functors on A. The tensor product on $[A,S]$ is given by the convolution formula:

$$F\otimes G = \int^{AB} FA \times GB \times P(AB-).$$

With respect to this structure the image of the Yoneda embedding $A \to [A,S]^{op}$ generates a small non-associative monoidal category $\bar{A}$ containing A. When the convolution construction is repeated using $\bar{A}^{op}$ we obtain a composite embedding

$$A \to \bar{A} \to [\bar{A}^{op}, S]$$

of A into a non-associative monoidal category whose tensor product has a right adjoint on each side.

In particular, if A is a closed category then the above composite is an internal-hom-preserving closed embedding of A into a non-associative monoidal closed structure on $[\overline{A}^{op}, S]$. Thus the study of coherence of the structure on A can be reduced to the study of coherence in $[\overline{A}^{op}, S]$ where a tensor-product functor is available for the internal hom.

A study of coherence for "non-associative" monoidal categories has been made by Laplaza [20]. Furthermore, it seems likely that a cut-elimination process suggested by Kelly (of the type described in [17]) provides a method for establishing a coherence result involving both the tensor product and the internal hom and hence a coherence result for the internal hom alone.

It turns out that if the closed category A is symmetric as a promonoidal category (see [4] §3.4) then the monoidal structure generated on $[\overline{A}^{op}, S]$ is associative as well as being symmetric. Thus the study of coherence for such a closed structure on A can be reduced to the study of coherence in a symmetric monoidal closed category (as discussed by Kelly and Mac Lane [18]).

4.6 Coherence for V-natural transformations (Blackwell)

Kelly-Mac Lane in [19] proved a coherence result for the three-category structure consisting of a symmetric monoidal closed category V, two V-categories A and B, two V-functors $T,S: A \to B$, and a V-natural transformation $k: T \to S: A \to B$.

Further coherence problems arise if either or both of A and B is tensored, or cotensored, or both. There should be a single coherence result covering all these cases.

Let K be the club for the three-category purely covariant structure given as follows: we have categories V,A,B; functors $I: V^0 \to V$, $\otimes: V \times V \to V$, $\bar{\otimes}: V \times A \to A$, $\bar{\bar{\otimes}}: V \times B \to B$, $T: A \to B$, $S: A \to B$. We have natural isomorphisms a,l,r,c making V symmetric monoidal; natural isomorphisms $\bar{l}: I\bar{\otimes}A \to A$ and $\bar{a}: (U\otimes V) \bar{\otimes} A \to U\bar{\otimes}(V\bar{\otimes}A)$ satisfying the obvious coherence requirements; and similarly $\bar{\bar{l}}$ and $\bar{\bar{a}}$. We have natural transformations $\tau: V\bar{\bar{\otimes}}TA \to T(V\bar{\otimes}A)$, $\sigma: V\bar{\bar{\otimes}}SA \to S(V\bar{\otimes}A)$, satisfying evident axioms involving $\bar{a}$, $\bar{l}$, $\bar{\bar{a}}$, $\bar{\bar{l}}$; and we have a natural transformation $k: T \to S$ satisfying the evident axiom involving τ and σ. For K we prove the coherence result: all diagrams commute.

Now let L be the mixed-variance club got by adding right adjoints of $-\otimes V$ (and hence of $V\otimes-$ because of the symmetry), of $-\bar{\otimes}A$, of $-\bar{\bar{\otimes}}B$, of $V\bar{\otimes}-$, and of $V\bar{\bar{\otimes}}-$. An L-algebra is then a symmetric monoidal closed category V, two tensored and cotensored V-categories A and B, two V-functors $T,S: A \to B$, and a V-natural $k: T \to S$. (The right adjoint of $-\bar{\otimes}A$ is the V-valued hom $A^{op} \times A \to V$; the right adjoint of $V\bar{\otimes}-$ is the cotensor $V^{op} \times A \to A$, and so on; τ gives the "V-enrichment" $T_{AB}: A(A,B) \to B(TA,TB)$ of T, etc.)

By imitating Kelly-Mac Lane [18] we prove for L the coherence result that $f,g: P \to Q$ are equal if $\Gamma f = \Gamma g$ and if P,Q are proper (cf. [18]).

Now let M be the club in which V is still symmetric monoidal closed, but A drops its tensor or cotensor or both, and similarly for B. The case in which all four are dropped is that considered by Kelly-Mac Lane in [19]. We intend to adapt §4.5 above to show that $M \to L$ is faithful, and so get a coherence result for M, including and extending that of [19].

4.7 Coherence for monoidal functors between closed categories (Lewis)

The thesis being written up is an expansion and simplification of Lewis [26]. That dealt with the club K for two symmetric monoidal categories V_1 and V_2 and a monoidal functor $\Phi: V_1 \to V_2$; and the mixed-variance club L arising from this when V_1 and V_2 are not only monoidal but closed. Here it is no longer true even for K that "all diagrams commute"; rather we augment $\Gamma\colon K \to \underline{\underline{P}}$ by an extra functor $\Delta\colon K \to \underline{\underline{S}}$ and show that Γ and Δ are jointly faithful, thus giving a complete description of K. We extend Γ and Δ suitably to L, and prove the same result for those $f,g\colon P \to Q$ where P and Q are both Γ- proper and Δ- proper.

The expansion of [26] lies in our now considering, besides V_1, V_2, and Φ, a V_1-natural transformation between two V_1-functors between two V_1-categories, as considered in Kelly-Mac Lane [19].

There are various simplifications of [26], of which the most significant is the simplified proof that K can be described in terms of the simpler H, in which the monoidal categories V_1 and V_2 are strict. This is now derived from the following expected, but surprisingly untrivial, result:

Theorem (Kelly-Lewis) Let $\phi\colon K \to L$ be a map of covariant clubs, which as a map of categories is an equivalence. Let K' be the club for two K- categories together with a K-functor (in the sense of §2.1 above). Let L' be given similarly and let $\phi'\colon K' \to L'$ be the obvious induced map of clubs. Then ϕ' is an equivalence of categories.

4.8 Coherence for cartesian closed categories (Blackwell)

Kelly in [16] conjectured that the coherence result of Szabo ([32],[33]) might reduce to the assertion that $f = g$ whenever $\Gamma f = \Gamma g$, for a suitable definition of the graph Γf; and we set ourselves to look at this conjecture.

It turns out to be harder than anticipated to define Γf. Kelly had observed that the graphs in $\underline{\underline{P}}'$, as defined in §4.4 above, were bijections between the +'s and the -'s in $-P\ +Q$, where $f\colon P \to Q$ and ΓP = a string of +'s and -'s. He conjectured that in the analoguous $\underline{\underline{S}}^{op}$, the graphs were functions from the +'s to the -'s. It turns out to be harder than this: those functions, for cartesian closed categories, have to be replaced by relations; this involves at least all the difficulties mentioned in §4.2 and §4.3 above in extending the "club" idea to a very general G. Heaven knows at the moment what becomes of the club concept in this generality; but it certainly continues to make sense to talk of graphs. However we must certainly allow variables in g to be set equal before it can be compared with f, as in

$$f\colon [A,B] \times A \times C \xrightarrow[e\times 1]{} B \times C \xrightarrow[pr_2]{} C$$

$$g\colon [A,B] \times D \times C \xrightarrow[pr_3]{} C$$

$$g'\colon [A,B] \times A \times C \xrightarrow[pr_3]{} C;$$

but if we go too far in this, and set $A = B$ in $1: A\times B \to A\times B$ and $c: A\times B \to B\times A$, we get two maps $A\times A \to A\times A$ that are certainly different. We are still investigating to see whether anything can be salvaged from Kelly's conjecture, and whether Szabo's result admits any description of this kind.

We are also looking at Laplaza's result in [21] on coherence for distributive laws; here everything is covariant, the graphs are in $\underline{\underline{S}}^{op}$, and there is no problem in describing things in terms of clubs; we hope to describe his results in terms of the graph-functor Γ.

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