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THE COMPREHENSIVE CONSTRUCTION OF FREE COLIMITS

Ross Street

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AND GLADLY LERNE

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(1) Let Γ denote a set of categories. A *Γ-colimit* in a category M is a colimit of a functor into M with domain in Γ. When all Γ-colimits in M exist then M is said to be *Γ-cocomplete*. A functor $f : M \longrightarrow N$ which preserves Γ-colimits is said to be *Γ-cocontinuous*. (See Mac Lane [4] for unexplained terminology.)

(2) This article asserts the existence of Γ-cocompletions and provides a construction:

<u>*Theorem.*</u> *Let Γ be any small set of small categories. For each small category X, there exist a small Γ-cocomplete category $\tilde{X}$ and a functor $n : X \longrightarrow \tilde{X}$ with the property that, for each Γ-cocomplete category M, composition with n yields an equivalence between the category of Γ-cocontinuous functors from $\tilde{X}$ to M and the category of all functors from X to M.*

(3) The problem of freely adjoining colimits has been investigated by Kock [3] and Wood [8] who, because of combinatorial difficulties created by the formation of the free categories on certain graphs, required conditions of *stability* on Γ. There is compelling *a priori* evidence that no conditions on Γ (apart from size) should be necessary. To wit, for category-valued 2-functors J,S with the same domain and such that S lands in the 2-category of Γ-cocomplete categories and Γ-cocontinuous functors, the category of pseudo-natural transformations (Kelly-Street [2]) from J to S is Γ-cocomplete; in other words, Γ-cocomplete categories are closed under "indexed bilimits" in the sense of Street [6].

(4) The case where Γ is the set of categories which have cardinality less than some regular cardinal γ has been dealt with by Gabriel-Ulmer [1]; regularity is itself a stability condition. (In this case we use the prefix "γ-" rather than "Γ-" in the above definitions.) They show that $\tilde{X}$ can be taken to be the skeleton of the full sub-category $K_\gamma(X)$ of $[X^{op}, \mathit{Set}]$ consisting of the γ-colimits of representable functors (= the γ-presentable objects). Clearly each object of $K_\gamma(X)$ can be obtained as a coequalizer of two arrows between γ-coproducts of representables in $[X^{op}, \mathit{Set}]$. If γ is small, so too then is $\tilde{X}$.

(5) Before proceeding with the general construction, we must recall some details from Street-Walters [7] and Street [5]. Each functor $w : C \longrightarrow X$ can be factored as a composite

$$C \xrightarrow{j_w} E(w) \xrightarrow{p_w} X$$

where j_w is a final functor and p_w is a discrete 1-fibration. If C, X are small, $E(w)$ is the category of elements of $\underset{c \in C}{\mathrm{col}}\, X(-, wc) : X^{op} \longrightarrow \mathit{Set}$ and so is also small. For each commutative square

$$\begin{array}{ccc} A & \xrightarrow{\ j\ } & C \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ E & \xrightarrow[\ p\ ]{} & B \end{array}$$

in which j is final and p is a discrete 1-fibration, there exists a unique functor $f : C \longrightarrow E$ such that $fj = u$ and $pf = v$. The pointwise left Kan extension k of a functor $h : C \longrightarrow M$ along a 0-fibration $q : C \longrightarrow A$ is given on objects by the formula

$$ka = \mathrm{col}\,(C_a \longrightarrow C \xrightarrow{h} M),$$

where C_a is the fibre of q over a.

(6) Suppose Γ is any set of small categories. For each ordinal θ, a set Γ_θ of small categories is recursively defined as follows:

— Γ_0 consists of the terminal categories (one of which is denoted by $\underset{\sim}{1}$);

— for each ordinal θ, $\Gamma_{\theta+1}$ consists of the small categories C for which there exists a 0-fibration $q : C \longrightarrow A$ such that A is in $\Gamma \cup \{\underset{\sim}{1}\}$ and each fibre C_a of q is the codomain of some final functor with domain in Γ_θ;

— for each limit ordinal θ, $\Gamma_\theta = \bigcup_{\phi<\theta} \Gamma_\phi$.

Observe that $\Gamma \subset \Gamma_1$ and $\Gamma_\phi \subset \Gamma_\phi$ for $\phi \leq \theta$.

(7) *Suppose* M *is a* Γ*-cocomplete category. For all ordinals* θ, M *is* Γ_θ*-cocomplete and any* Γ*-cocontinuous functor* $f : M \longrightarrow N$ *is* Γ_θ*-cocontinuous.* For $\theta = 0$ this is trivial. Suppose M is Γ_θ-cocomplete and take a functor $h : C \longrightarrow M$ with C in $\Gamma_{\theta+1}$. There is a 0-fibration $q : C \longrightarrow A$ as in the definition of $\Gamma_{\theta+1}$ so that the left Kan extension k of h along q can be calculated by the formula

$$ka = \operatorname{col}(B_a \longrightarrow C_a \longrightarrow C \xrightarrow{h} M)$$

where $B_a \longrightarrow C_a$ is final and B_a is in Γ_θ. Since A is in $\Gamma \cup \{\underset{\sim}{1}\}$, the colimit of $k : A \longrightarrow M$ exists. The left Kan extension along the composite $C \xrightarrow{q} A \longrightarrow \underset{\sim}{1}$ can be obtained by first left Kan extending along q and then left Kan extending the result along $A \longrightarrow \underset{\sim}{1}$. So the colimit of k *is* the colimit of h. So M is $\Gamma_{\theta+1}$-cocomplete. If θ is a limit ordinal and M is Γ_ϕ-cocomplete for all $\phi < \theta$, clearly M is Γ_θ-cocomplete. So M is Γ_ϕ-cocomplete for all θ asserted. The statement about f is now clear from the above construction of

Γ_θ-colimits in M.

(8) For each small category X and each ordinal θ, let X_θ denote the category whose objects are functors $w : C \longrightarrow X$ with C in Γ_θ and whose arrows $f : w \longrightarrow w'$ are commutative triangles:

$$\begin{array}{ccc} E(w) & \xrightarrow{\ f\ } & E(w') \\ & {\scriptstyle p_w}\searrow \quad \swarrow{\scriptstyle p_{w'}} & \\ & X & \end{array}$$

For $\phi \le \theta$, X_ϕ is a full subcategory of X_θ. There is an equivalence of categories $r_0 : X \longrightarrow X_0$ which takes x to $x : \underset{\sim}{1} \longrightarrow X$ and takes $\xi : x \longrightarrow x'$ to

$$E(x) = X \downarrow x \xrightarrow{X \downarrow \xi} X \downarrow x' = E(x') .$$

The composite $X \xrightarrow{r_0} X_0 \subset X_\theta$ is denoted by r_θ.

(9) *There is a fully faithful functor* $t_\theta : X_\theta \longrightarrow [X^{op}, Set]$ *which is given on objects by:*

$$t_\theta(w) = \underset{c \in C}{\mathrm{col}}\, X(-,wc) .$$

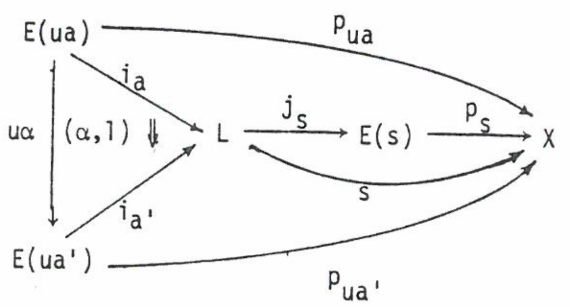
This is because $E(w)$ is just the category of elements of $t_\theta(w)$ and because taking categories of elements gives an equivalence between the category $[X^{op}, Set]$ and the category of discrete 1-fibration over X with small fibres.

(10) Notice that $t_\theta r_\theta$ is isomorphic to the Yoneda embedding $y_X : X \longrightarrow [X^{op}, Set]$.

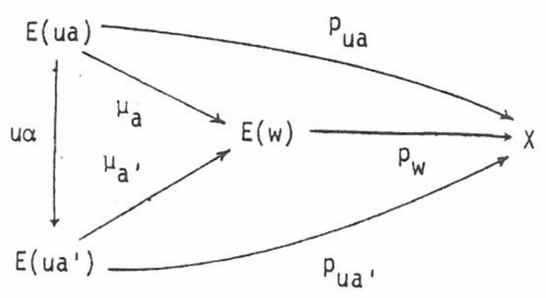
(11) *For each ordinal* θ *and each functor* $u : A \longrightarrow X_\theta$ *with* A *in* Γ, *we shall now construct a colimit for the composite* $A \xrightarrow{u} X_\theta \subset X_{\theta+1}$. Write $E : X_\theta \longrightarrow Cat$ for the functor which takes w to its "comprehensive image" $E(w)$ and takes $f : w \longrightarrow w'$ to $f : E(w) \longrightarrow E(w')$. Let L be the

category obtained from the composite $A \xrightarrow{u} X_\theta \xrightarrow{E} \mathcal{Cat}$ via the Grothendieck construction; explicitly, an object of L is a pair (a,e) where a,e are objects of A, $E(ua)$, respectively, and an arrow $(\alpha,\eta) : (a,e) \to (a',e')$ in L consists of arrows $\alpha : a \to a'$, $\eta : (u\alpha)e \to e'$ in A, $E(ua')$, respectively. The first projection $d : L \to A$ is a 0-fibration with $E(ua)$ as its fibre over a. Since ua is in X_θ, there is a final functor j_{ua} into $E(ua)$ with domain in Γ_θ. It follows that L is in $\Gamma_{\theta+1}$. This means that the functor $s : L \to X$ given by $s(a,e) = p_{ua}e$, $s(\alpha,\eta) = p_{ua'}\eta$ is an object of $X_{\theta+1}$.

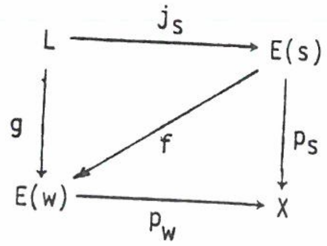
We shall show that s is a colimit for the composite $A \xrightarrow{u} X_\theta \subset X_{\theta+1}$. Let $\lambda_a : ua \to s$ in $X_{\theta+1}$ be the inclusion $i_a : E(ua) \to L$ composed with $j_s : L \to E(s)$. The following composite is the identity natural transformation.



Since p_s is discrete it follows that $j_s i_a = j_s i_{a'}(u\alpha)$ which means that the λ_a are the components of a cocone with vertex s. To see that this cocone is universal, suppose $w : C \to X$ is in $X_{\theta+1}$ and $\mu_a : ua \to w$ are the components of a cocone with vertex w. This means we have commuting diagrams:



Let $g : L \longrightarrow E(w)$ be the functor given by $g(a,e) = \mu_a e$, $g(\alpha,\eta) = \mu_{a'}\eta$. Then $p_w g = s = p_s j_s$, so there exists a unique functor f such that the following commutes.

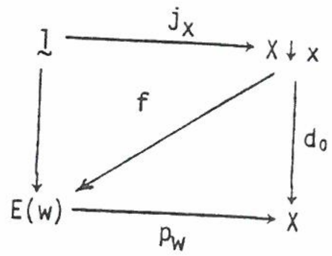


It is easily seen now that $f : s \longrightarrow w$ in $X_{\theta+1}$ is unique with the proper that $\mu_a = f\lambda_a$ for all a of A.

(12) *For all* Γ*-cocomplete categories* M, *each functor* $h : X \longrightarrow M$ *has a pointwise left Kan extension* k_θ *along* $r_\theta : X \longrightarrow X_\theta$ *whose value at an object* w *of* X_θ is given by:

$$k_\theta(w) = \mathrm{col}\,(C \xrightarrow{w} X \xrightarrow{h} M).$$

To see this notice that the colimit of hw does exist since C is in Γ_θ (7). Since j_w is final, the colimit is also the colimit of the composite $E(w) \xrightarrow{p_w} X \xrightarrow{h} M$. We shall show that $p_w : E(w) \longrightarrow X$ is isomorphic to $d_0 : r_\theta \downarrow w \longrightarrow X$ so that the above formula for $k(w)$ is isomorphic to the usual formula (see Mac Lane [4]) for the pointwise left Kan extension of h along r_θ. An object of $r_\theta \downarrow w$ is a pair (x,f) where x is an object of X and $f : r_0(x) \longrightarrow w$ is an arrow of X_θ. Since the top arrow of the square below is final and p_w is a discrete 1-fibration, to give such an object is precisely (see (5)) to give an object of $E(w)$.



The required isomorphism $r_\theta \downarrow w \cong E(w)$ is now clear.

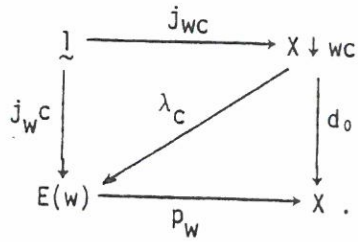
(13) *The left Kan extension* k_θ *of* (12) *has the following two properties:*

(i) $k_\theta r_\theta \cong h$;

(ii) k_θ *preserves the colimits of functors* $A \xrightarrow{u} X_\phi \subset X_\theta$ *with* A *in* Γ *and* $\phi < \theta$.

Since k_θ is pointwise and r_θ is fully faithful, property (i) follows (Mac Lane [4; Ch. X §3, Cor. 3, p. 235], Street [5; p. 129]). Since property (ii) is vacuous for $\theta = 0$ and since $X_\theta = \bigcup_{\phi<\theta} X_\phi$ for θ a limit ordinal, it suffices to show that, for all ordinals θ, $k_{\theta+1}$ preserves the colimit of each functor $A \xrightarrow{u} X_\theta \subset X_{\theta+1}$ with A in Γ. Such a colimit was constructed in (11) and denoted by s. What we must show is that $k_{\theta+1}(s)$ is canonically isomorphic to the colimit of the composite $A \xrightarrow{u} X_\theta \subset X_{\theta+1} \xrightarrow{k_{\theta+1}} M$; the latter composite is isomorphic to $k_\theta u$. Now $k_{\theta+1}(s) = \mathrm{col}\,(L \xrightarrow{s} X \xrightarrow{h} M)$ by (12), and this colimit can be calculated by first Kan extending along $d : L \longrightarrow A$ and then along $A \longrightarrow \underset{\sim}{1}$. Since d is a 0-fibration, the value at a of the left Kan extension of hs along d is $\mathrm{col}\,(E\,(ua) \xrightarrow{p_{ua}} X \xrightarrow{h} M) \cong \mathrm{col}\,(h.ua) =$ $= k_\theta(ua)$. So $k_\theta u$ is the left Kan extension of hs along d. So the colimit of hs is $\mathrm{col}\,(k_\theta u)$ as required.

(14) *Each object* $w : C \longrightarrow X$ *of* X_θ *is the colimit of* $C \xrightarrow{w} X \xrightarrow{r_\theta} X_\theta$. For each object c of C, there is an arrow $\lambda_c : r_0(wc) \longrightarrow w$ in X_θ uniquely determined by the commutativity of the diagram:



For each $\zeta : c \longrightarrow c'$ in C, we have $p_w \lambda_{c'} (X \downarrow w\zeta) j_{wc} = p_w \lambda_c j_{wc}$ (both sides equal $wc : \underset{\sim}{1} \longrightarrow X$). Since p_w is a discrete 1-fibration and j_{wc} is final, it follows that the arrows λ_c form a cocone over $r_\theta w$ with vertex w. The universal property is easily checked.

(15) *There is an ordinal* ψ *such that the inclusion* $X_\psi \subset X_{\psi+1}$ *is an equivalence of categories. If* Γ *is a small set of categories then the first such* ψ *is small and* X_ψ *has a small skeleton.* Let γ be a regular cardinal which exceeds the cardinalities of all the categories in Γ and is small if γ is. The category $K_\gamma(X)$ described in (4) is Γ-cocomplete. The Yoneda embedding $y_X : X \longrightarrow [X^{op}, \mathcal{S}et]$ factors through $K_\gamma(X)$ via a functor $h : X \longrightarrow K_\gamma(X)$ say. The left Kan extension k_θ of h along $r_\theta : X \longrightarrow X_\theta$ exists and is given by $k_\theta(w) = \underset{c \in C}{\mathrm{col}}\, X(-, wc)$ using (12). Since $K_\gamma(X)$ is closed under γ-colimits in $[X^{op}, \mathcal{S}et]$, the composite $X_\theta \xrightarrow{k_\theta} K_\gamma(X) \subset [X^{op}, \mathcal{S}et]$ is just t_θ as given in (9). Since t_θ is fully faithful, so too is k_θ. Let ϕ be the first ordinal of cardinality exceeding that of the skeleton of $K_\gamma(X)$; by (4), ϕ is small if γ is. There is no ϕ-sequence of non-isomorphic objects in $K_\gamma(X)$, so there exists $\psi < \phi$ for which $X_\psi \subset X_{\psi+1}$ is an equivalence. We have shown that each X_θ is equivalent to a full subcategory of $K_\gamma(X)$ and so has a small skeleton when γ is small.

(16) *For an ordinal* ψ *as in (15),* X_ψ *is* Γ*-cocomplete and, for each* Γ*-cocomplete category* M*, composition with* $r_\psi : X \longrightarrow X_\psi$ *yields an equivalence between the category* $[X_\psi, M]_\Gamma$ *of* Γ*-cocontinuous functors from* X_ψ *to* M *and the category of all functors from* X *to* M*.* Since $X_\psi \subset X_{\psi+1}$ is an equivalence, Γ-cocompleteness of X_ψ follows directly from (11). By (12), the functor $[r_\psi, M] : [X_\psi, M] \longrightarrow [X, M]$

has a left adjoint which, by (13), is fully faithful and lands in $[X_\psi, M]_\Gamma$. It remains to prove that each Γ-cocontinuous functor $k : X_\psi \longrightarrow M$ is a left Kan extension of kr_ψ along r_ψ. From (14) we have that each object w of X_ψ is the colimit of $r_\psi w$. Since k is Γ-cocontinuous it is also Γ_ψ-cocontinuous by (7), so $k(w)$ is the colimit of $kr_\psi w$. By (12), k is a left Kan extension of kr_ψ along r_ψ.

(17) To complete the proof of the Theorem, let ψ be as in the second sentence of (15), let $\tilde{X}$ be the skeleton of X_ψ, and let $n : X \longrightarrow \tilde{X}$ be induced by $r_\psi : X \longrightarrow X_\psi$.

(18) Notice that our construction gives the Γ-cocompletion of a small category X even when Γ is not small, however, the Γ-cocompletion in this case may not have a small skeleton.

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