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THE COMPREHENSIVE CONSTRUCTION OF FREE COLIMITS

Ross Street
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(1) Let \( \Gamma \) denote a set of categories. A \( \Gamma \)-colimit in a category \( M \) is a colimit of a functor into \( M \) with domain in \( \Gamma \). When all \( \Gamma \)-colimits in \( M \) exist then \( M \) is said to be \( \Gamma \)-complete. A functor \( f : M \to N \) which preserves \( \Gamma \)-colimits is said to be \( \Gamma \)-cocontinuous. (See Mac Lane [4] for unexplained terminology.)

(2) This article asserts the existence of \( \Gamma \)-cocompletions and provides a construction:

**Theorem.** Let \( \Gamma \) be any small set of small categories. For each small category \( X \), there exist a small \( \Gamma \)-cocomplete category \( \bar{X} \) and a functor \( n : X \to \bar{X} \) with the property that, for each \( \Gamma \)-cocomplete category \( M \), composition with \( n \) yields an equivalence between the category of \( \Gamma \)-cocontinuous functors from \( \bar{X} \) to \( M \) and the category of all functors from \( X \) to \( M \).

(3) The problem of freely adjoining colimits has been investigated by Kock [3] and Wood [8] who, because of combinatorial difficulties created by the formation of the free categories on certain graphs, required conditions of stability on \( \Gamma \). There is compelling a priori evidence that no conditions on \( \Gamma \) (apart from size) should be necessary. To wit, for category-valued 2-functors \( J,S \) with the same domain and such that \( S \) lands in the 2-category of \( \Gamma \)-cocomplete categories and \( \Gamma \)-cocontinuous functors, the category of pseudo-natural transformations (Kelly-Street [2]) from \( J \) to \( S \) is \( \Gamma \)-cocomplete; in other words, \( \Gamma \)-cocomplete categories are closed under "indexed bilimits" in the sense of Street [6].
2.

(4) The case where \( \Gamma \) is the set of categories which have cardinality less than some regular cardinal \( \gamma \) has been dealt with by Gabriel-Ulmer [1]; regularity is itself a stability condition. (In this case we use the prefix "\( \gamma^- \)" rather than "\( \Gamma^- \)" in the above definitions.) They show that \( \mathfrak{K} \) can be taken to be the skeleton of the full subcategory \( K_\gamma(X) \) of \([X^{op}, Set]\) consisting of the \( \gamma \)-colimits of representable functors (= the \( \gamma \)-presentable objects). Clearly each object of \( K_\gamma(X) \) can be obtained as a coequalizer of two arrows between \( \gamma \)-coproducts of representables in \([X^{op}, Set]\). If \( \gamma \) is small, so too then is \( \mathfrak{K} \).

(5) Before proceeding with the general construction, we must recall some details from Street-Walters [7] and Street [5]. Each functor \( w: C \rightarrow X \) can be factored as a composite

\[
\begin{array}{ccc}
C & \xrightarrow{j_w} & E(w) & \xrightarrow{p_w} & X \\
\end{array}
\]

where \( j_w \) is a final functor and \( p_w \) is a discrete \( 1 \)-fibration. If \( C, X \) are small, \( E(w) \) is the category of elements of \( \col X(-, wc): X^{op} \rightarrow \text{Set} \) and so is also small. For each commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{j} & C \\
\downarrow{u} & & \downarrow{v} \\
E & \xrightarrow{p} & B \\
\end{array}
\]

in which \( j \) is final and \( p \) is a discrete \( 1 \)-fibration, there exists a unique functor \( f: C \rightarrow E \) such that \( fj = u \) and \( pf = v \). The pointwise left Kan extension \( k \) of a functor \( h: C \rightarrow M \) along a \( 0 \)-fibration \( q: C \rightarrow A \) is given on objects by the formula

\[
ka = \col(C_a \rightarrow C \xrightarrow{h} M),
\]

where \( C_a \) is the fibre of \( q \) over \( a \).
Suppose $\Gamma$ is any set of small categories. For each ordinal $\theta$, a set $\Gamma_\theta$ of small categories is recursively defined as follows:

- $\Gamma_0$ consists of the terminal categories (one of which is denoted by $1$);

- for each ordinal $\theta$, $\Gamma_{\theta+1}$ consists of the small categories $C$ for which there exists a $0$-fibration $q : C \rightarrow A$ such that $A$ is in $\Gamma_\theta \cup \{1\}$ and each fibre $C_a$ of $q$ is the codomain of some final functor with domain in $\Gamma_\theta$;

- for each limit ordinal $\theta$, $\Gamma_\theta = \bigcup_{\phi < \theta} \Gamma_\phi$.

Observe that $\Gamma = \Gamma_1$ and $\Gamma_\phi \subseteq \Gamma_\theta$ for $\phi \leq \theta$.

(7) Suppose $M$ is a $\Gamma$-cocomplete category. For all ordinals $\theta$, $M$ is $\Gamma_\theta$-cocomplete and any $\Gamma$-cocontinuous functor $f : M \rightarrow N$ is $\Gamma_\theta$-cocontinuous. For $\theta = 0$ this is trivial. Suppose $M$ is $\Gamma_\theta$-cocomplete and take a functor $h : C \rightarrow M$ with $C$ in $\Gamma_{\theta+1}$. There is a $0$-fibration $q : C \rightarrow A$ as in the definition of $\Gamma_{\theta+1}$ so that the left Kan extension $k$ of $h$ along $q$ can be calculated by the formula

$$ka = \text{colim}(B_a \longrightarrow C_a \longrightarrow C \xrightarrow{h} M)$$

where $B_a \longrightarrow C_a$ is final and $B_a$ is in $\Gamma_\theta$. Since $A$ is in $\Gamma_\theta \cup \{1\}$, the colimit of $k : A \rightarrow M$ exists. The left Kan extension along the composite $C \xrightarrow{q} A \rightarrow 1$ can be obtained by first left Kan extending along $q$ and then left Kan extending the result along $A \rightarrow 1$. So the colimit of $k$ is the colimit of $h$. So $M$ is $\Gamma_{\theta+1}$-cocomplete. If $\theta$ is a limit ordinal and $M$ is $\Gamma_\phi$-cocomplete for all $\phi < \theta$, clearly $M$ is $\Gamma_\theta$-cocomplete. So $M$ is $\Gamma_\phi$-cocomplete for all $\theta$ asserted. The statement about $f$ is now clear from the above construction of $f$. 


4.

$\Gamma_\theta$-colimits in $M$.

(8) For each small category $X$ and each ordinal $\theta$, let $X_\theta$ denote the category whose objects are functors $w : C \to X$ with $C$ in $\Gamma_\theta$ and whose arrows $f : w \to w'$ are commutative triangles:

```
      E(w')
     /    \\
    f     \\
   / \    \\
P_w  \downarrow   \downarrow P_{w'}
  \  /  \\
X  /  \\
```

For $\phi \leq \theta$, $X_\phi$ is a full subcategory of $X_\theta$. There is an equivalence of categories $r_\phi : X \to X_\phi$ which takes $x$ to $x : 1 \to X$ and takes $\xi : x \to x'$ to

$$E(x) = x \downarrow x \xrightarrow{\xi} x \downarrow x' = E(x').$$

The composite $X \xrightarrow{r_\phi} X_\phi \subseteq X_\theta$ is denoted by $r_\theta$.

(9) There is a fully faithful functor $t_\theta : X_\theta \to [X^{op}, Set]$ which is given on objects by:

$$t_\theta(w) = \text{colim}_{C} X(-, wc).$$

This is because $E(w)$ is just the category of elements of $t_\theta(w)$ and because taking categories of elements gives an equivalence between the category $[X^{op}, Set]$ and the category of discrete $1$-fibration over $X$ with small fibres.

(10) Notice that $t_\theta r_\theta$ is isomorphic to the Yoneda embedding $y_X : X \to [X^{op}, Set]$.

(11) For each ordinal $\theta$ and each functor $u : A \to X_\theta$ with $A$ in $\Gamma$, we shall now construct a colimit for the composite $A \xrightarrow{u} X_\theta \subseteq X_{\theta+1}$.

Write $E : X_\theta \to \text{Cat}$ for the functor which takes $w$ to its "comprehensive image" $E(w)$ and takes $f : w \to w'$ to $f : E(w) \to E(w')$. Let $L$ be the
category obtained from the composite \( A \xrightarrow{u} X_0 \xrightarrow{E} \text{Cat} \) via the Grothendieck construction; explicitly, an object of \( L \) is a pair \((a,e)\) where \( a,e \) are objects of \( A, E(ua) \), respectively, and an arrow \((\alpha,\eta) : (a,e) \rightarrow (a',e') \) in \( L \) consists of arrows \( \alpha : a \rightarrow a' \), \( \eta : (ua)e \rightarrow e' \) in \( A, E(ua') \), respectively. The first projection \( d : L \rightarrow A \) is a 0-fibration with \( E(ua) \) as its fibre over \( a \). Since \( ua \) is in \( X_0 \), there is a final functor \( j_{ua} \) into \( E(ua) \) with domain in \( \Gamma_0 \). It follows that \( L \) is in \( \Gamma_{0+1} \). This means that the functor \( s : L \rightarrow X \) given by \( s(a,e) = p^{uo} e \), \( s(\alpha,\eta) = p^{uo} \eta \) is an object of \( X_{0+1} \).

We shall show that \( s \) is a colimit for the composite \( A \xrightarrow{u} X_0 \xrightarrow{E} X_{0+1} \). Let \( \lambda_a : ua \rightarrow s \) in \( X_{0+1} \) be the inclusion \( i_a : E(ua) \rightarrow L \) composed with \( j_s : L \rightarrow E(s) \). The following composite is the identity natural transformation.

\[
\begin{array}{ccc}
E(ua) & \xrightarrow{j_a} & L \\
\downarrow{ua} & & \downarrow{j_s} \\
E(ua') & \xrightarrow{i_a'} & E(s) \\
\end{array}
\]

Since \( p_s \) is discrete it follows that \( j_s i_a = j_s i_a (ua) \) which means that the \( \lambda_a \) are the components of a cocone with vertex \( s \). To see that this cocone is universal, suppose \( w : C \rightarrow X \) is in \( X_{0+1} \) and \( \mu_a : ua \rightarrow w \) are the components of a cocone with vertex \( w \). This means we have commuting diagrams:

\[
\begin{array}{ccc}
E(ua) & \xrightarrow{j_a} & L \\
\downarrow{ua} & & \downarrow{j_w} \\
E(ua') & \xrightarrow{i_a'} & E(w) \\
\end{array}
\]
Let \( g : L \rightarrow E(w) \) be the functor given by \( g(a,e) = \mu_a e \), \( g(a,n) = \mu_a n \). Then \( P_w g = s = P_s j_s \), so there exists a unique functor \( f \) such that the following commutes.

\[
\begin{array}{ccc}
L & \xrightarrow{j_s} & E(s) \\
\downarrow{g} & & \downarrow{f} \\
E(w) & \xrightarrow{P_w} & X
\end{array}
\]

It is easily seen now that \( f : s \rightarrow w \) in \( X_{n+1} \) is unique with the property that \( \mu_a = f \lambda_a \) for all \( a \) of \( A \).

(12) For all \( \Gamma \)-cocomplete categories \( M \), each functor \( h : X \rightarrow M \) has a pointwise left Kan extension \( k_\theta \) along \( r_\theta : X \rightarrow X_\theta \) whose value at an object \( w \) of \( X_\theta \) is given by:

\[
k_\theta(w) = \text{colim} \left( C \longrightarrow X \xrightarrow{h} M \right).
\]

To see this notice that the colimit of \( hw \) does exist since \( C \) is in \( \Gamma_\theta \) (7). Since \( j_w \) is final, the colimit is also the colimit of the composite \( E(w) \xrightarrow{P_w} X \xrightarrow{h} M \). We shall show that \( P_w : E(w) \rightarrow X \) is isomorphic to \( d_\theta : r_\theta \downarrow w \rightarrow X \) so that the above formula for \( k(w) \) is isomorphic to the usual formula (see Mac Lane [4]) for the pointwise left Kan extension of \( h \) along \( r_\theta \). An object of \( r_\theta \downarrow w \) is a pair \((x,f)\) where \( x \) is an object of \( X \) and \( f : r_\theta(x) \rightarrow w \) is an arrow of \( X_\theta \). Since the top arrow of the square below is final and \( P_w \) is a discrete 1-fibration, to give such an object is precisely (see (5)) to give an object of \( E(w) \).

\[
\begin{array}{ccc}
1 & \xrightarrow{j} & X \downarrow x \\
\downarrow{f} & & \downarrow{d_\theta} \\
E(w) & \xrightarrow{P_w} & X
\end{array}
\]
The required isomorphism \( r_\theta \cdot w \cong E(w) \) is now clear.

(13) The left Kan extension \( k_\theta \) of (12) has the following two properties:

(i) \( k_\theta \cdot r_\theta \cong h \);

(ii) \( k_\theta \) preserves the colimits of functors \( A \xrightarrow{u} X_\phi \cong X_\theta \) with \( A \) in \( \Gamma \) and \( \phi < \theta \).

Since \( k_\theta \) is pointwise and \( r_\theta \) is fully faithful, property (i) follows (Mac Lane [4; Ch. X §3, Cor. 3, p. 235], Street [5; p. 129]). Since property (ii) is vacuous for \( \theta = 0 \) and since \( X_\theta = \cup_{\phi < \theta} X_\phi \) for \( \theta \) a limit ordinal, it suffices to show that, for all ordinals \( \theta \), \( k_{\theta+1} \) preserves the colimit of each functor \( A \xrightarrow{u} X_\theta \cong X_{\theta+1} \) with \( A \) in \( \Gamma \).

Such a colimit was constructed in (11) and denoted by \( s \). What we must show is that \( k_{\theta+1}(s) \) is canonically isomorphic to the colimit of the composite \( A \xrightarrow{u} X_\theta \cong X_{\theta+1} \xrightarrow{k_{\theta+1}} M \); the latter composite is isomorphic to \( k_\theta u \). Now \( k_{\theta+1}(s) = \text{col}(L \xrightarrow{s} X \xrightarrow{h} M) \) by (12), and this colimit can be calculated by first Kan extending along \( d : L \rightarrow A \) and then along \( A \rightarrow X \). Since \( d \) is a 0-fibration, the value at \( a \) of the left Kan extension of \( hs \) along \( d \) is \( \text{col}(E(Ua) \xrightarrow{Pua} X \xrightarrow{h} M) \cong \text{col}(h(Ua)) = k_\theta(Ua) \). So \( k_\theta u \) is the left Kan extension of \( hs \) along \( d \). So the colimit of \( hs \) is \( \text{col}(k_\theta u) \) as required.

(14) Each object \( w : C \rightarrow X \) of \( X_\theta \) is the colimit of \( C \xrightarrow{r_\theta} X_\theta \). For each object \( c \) of \( C \), there is an arrow \( \lambda_c : r_\theta(wc) \rightarrow w \) in \( X_\theta \) uniquely determined by the commutativity of the diagram:
For each \( \zeta : c \rightarrow c' \) in \( C \), we have \( p_w^\lambda c : (X \downarrow wc) \downarrow wc = p_w^\lambda c j_{wc} \) (both sides equal \( wc : \top \rightarrow X \)). Since \( p_w \) is a discrete 1-fibration and \( j_{wc} \) is final, it follows that the arrows \( \lambda_c \) form a cocone over \( r_w^w \) with vertex \( w \). The universal property is easily checked.

(15) There is an ordinal \( \psi \) such that the inclusion \( X_\psi \subseteq X_{\psi + 1} \) is an equivalence of categories. If \( \Gamma \) is a small set of categories then the first such \( \psi \) is small and \( X_\psi \) has a small skeleton. Let \( \gamma \) be a regular cardinal which exceeds the cardinalities of all the categories in \( \Gamma \) and is small if \( \gamma \) is. The category \( K_\gamma(X) \) described in (4) is \( \Gamma \)-cocomplete. The Yoneda embedding \( y_X : X \rightarrow [X^{\text{op}}, \text{Set}] \) factors through \( K_\gamma(X) \) via a functor \( h : X \rightarrow K_\gamma(X) \) say. The left Kan extension \( k_\theta \) of \( h \) along \( r_\theta : X \rightarrow X_\theta \) exists and is given by \( k_\theta(w) = \text{colim}_{c \in C} X(-, wc) \) using (12). Since \( K_\gamma(X) \) is closed under \( \gamma \)-colimits in \( [X^{\text{op}}, \text{Set}] \), the composite \( X_\theta \rightarrow K_\gamma(X) \subseteq [X^{\text{op}}, \text{Set}] \) is just \( t_\theta \) as given in (9). Since \( t_\theta \) is fully faithful, so too is \( k_\theta \). Let \( \phi \) be the first ordinal of cardinality exceeding that of the skeleton of \( K_\gamma(X) \); by (4), \( \phi \) is small if \( \gamma \) is. There is no \( \phi \)-sequence of non-isomorphic objects in \( K_\gamma(X) \), so there exists \( \psi < \phi \) for which \( X_\psi \subseteq X_{\psi + 1} \) is an equivalence. We have shown that each \( X_\theta \) is equivalent to a full subcategory of \( K_\gamma(X) \) and so has a small skeleton when \( \gamma \) is small.

(16) For an ordinal \( \psi \) as in (15), \( X_\psi \) is \( \Gamma \)-cocomplete and, for each \( \Gamma \)-cocomplete category \( M \), composition with \( r_\psi : X \rightarrow X_\psi \) yields an equivalence between the category \( [X_\psi, M]_\Gamma \) of \( \Gamma \)-cocontinuous functors from \( X_\psi \) to \( M \) and the category of all functors from \( X \) to \( M \). Since \( X_\psi \subseteq X_{\psi + 1} \) is an equivalence, \( \Gamma \)-cocompleteness of \( X_\psi \) follows directly from (11). By (12), the functor \( [r_\psi, M] : [X_\psi, M] \rightarrow [X, M] \)
has a left adjoint which, by (13), is fully faithful and lands in 
$[X_\psi, M]_\Gamma$. It remains to prove that each $\Gamma$-cocontinuous functor 
k : $X_\psi \longrightarrow M$ is a left Kan extension of $kr_\psi$ along $r_\psi$. From (14) 
we have that each object $w$ of $X_\psi$ is the colimit of $r_\psi w$. Since $k$
is $\Gamma$-cocontinuous it is also $\Gamma_\psi$-cocontinuous by (7), so $k(w)$ is the 
colimit of $kr_\psi w$. By (12), $k$ is a left Kan extension of $kr_\psi$
along $r_\psi$.

(17) To complete the proof of the Theorem, let $\psi$ be as in the second 
sentence of (15), let $\bar{X}$ be the skeleton of $X_\psi$, and let $n : X \longrightarrow \bar{X}$
be induced by $r_\psi : X \longrightarrow X_\psi$.

(18) Notice that our construction gives the $\Gamma$-cocompletion of a small 
category $X$ even when $\Gamma$ is not small, however, the $\Gamma$-cocompletion 
in this case may not have a small skeleton.
REFERENCES


