Kan extensions and cartesian monoidal categories

Ross Street*

Mathematics Department, Macquarie University, NSW 2109 Australia

<ross.street@mq.edu.au>

18 September 2012

2010 Mathematics Subject Classification: 18D10; 18D20; 18A40; 18C10
Key words and phrases: Lawvere theory; monoidal category; enriched category; pointwise Kan extension.

Abstract

The existence of adjoints to algebraic functors between categories of models of Lawvere theories follows from finite-product-preservingness surviving left Kan extension. A result along these lines was proved in Appendix 2 of Brian Day's PhD thesis [1]. His context was categories enriched in a cartesian closed base. A generalization is described here with essentially the same proof. We introduce the notion of cartesian monoidal category in the enriched context. With an advanced viewpoint, we give a result about left extension along a promonoidal module and further related results.

Contents

*The author gratefully acknowledges the support of Australian Research Council Discovery Grant DP1094883.

1 Introduction

The pointwise left Kan extension, along any functor between categories with finite products, of a finite-product-preserving functor into a cartesian closed category is finite-product-preserving. This kind of result goes back at least to Bill Lawvere's thesis [8] and some 1966 ETH notes of Fritz Ulmer. Eduardo Dubuc and the author independently provided Saunders Mac Lane with a proof along the lines of the present note at Bowdoin College in the Northern Hemisphere Summer of 1969. Brian Day's thesis [1] gave a generalization to categories enriched in a cartesian closed base. Also see Kelly-Lack [7] and Day-Street [3]. Our purpose here is to remove the restriction on the base and, to some extent, the finite products.

2 Weighted colimits

We work with a monoidal category $\mathscr{V}$ as used in Max Kelly's book [9] as a base for enriched category theory.

Recall that the *colimit* of a $\mathscr{V}$-functor $F : \mathscr{A} \longrightarrow \mathscr{X}$ *weighted by* a $\mathscr{V}$-functor $W : \mathscr{A}^{\mathrm{op}} \longrightarrow \mathscr{V}$ is an object

$$\mathrm{colim}(W, F) = \mathrm{colim}_A(WA, FA)$$

of $\mathscr{X}$ equipped with an isomorphism

$$\mathscr{X}(\mathrm{colim}(W, F), X) \cong [\mathscr{A}^{\mathrm{op}}, \mathscr{V}](W, \mathscr{X}(F, X))$$

$\mathscr{V}$-natural in X.

Independence of naturality in the two variables of two variable naturality, or Fubini's theorem [9], has the following expression in terms of weighted colimits.

Nugget 1. *For $\mathscr{V}$-functors*

$$W_1 : \mathscr{A}_1^{\mathrm{op}} \longrightarrow \mathscr{V},\ W_2 : \mathscr{A}_2^{\mathrm{op}} \longrightarrow \mathscr{V},\ F : \mathscr{A}_1 \otimes \mathscr{A}_2 \longrightarrow \mathscr{X},$$

if $\mathrm{colim}(W_2, F(A, -))$ *exists for each* $A \in \mathscr{A}$ *then*

$$\mathrm{colim}(W_1, \mathrm{colim}(W_2, F)) \cong \mathrm{colim}(W_1 \otimes W_2, F)\ .$$

Here the isomorphism is intended to include the fact that one side exists if and only if the other does. Also $(W_1 \otimes W_2)(A, B) = W_1 A \otimes W_2 B$.

Proof. Here is the calculation:

$$\begin{aligned}
\mathscr{X}(\mathrm{colim}(W_1 \otimes W_2, F), X) &\cong [(\mathscr{A}_1 \otimes \mathscr{A}_2)^{\mathrm{op}}, \mathscr{V}](W_1 \otimes W_2, \mathscr{X}(F, X)) \\
&\cong [\mathscr{A}_1^{\mathrm{op}}, \mathscr{V}](W_1, [\mathscr{A}_2^{\mathrm{op}}, \mathscr{V}](W_2, \mathscr{X}(F, X))) \\
&\cong [\mathscr{A}_1^{\mathrm{op}}, \mathscr{V}](W_1, \mathscr{X}(\mathrm{colim}(W_2, F), X)) \\
&\cong \mathscr{X}(\mathrm{colim}(W_1, \mathrm{colim}(W_2, F)), X)\ .
\end{aligned}$$

□

Here is an aspect of the calculus of mates expressed in terms of weighted colimits. Note that $S \dashv T : \mathscr{A} \longrightarrow \mathscr{C}$ means $T^{\mathrm{op}} \dashv S^{\mathrm{op}} : \mathscr{A}^{\mathrm{op}} \longrightarrow \mathscr{C}^{\mathrm{op}}$.

Nugget 2. *For $\mathscr{V}$-functors $W : \mathscr{A}^{\mathrm{op}} \longrightarrow \mathscr{V}$, $G : \mathscr{C} \longrightarrow \mathscr{X}$, and a $\mathscr{V}$-adjunction*

$$S \dashv T : \mathscr{A} \longrightarrow \mathscr{C},$$

there is an isomorphism

$$\mathrm{colim}(WS^{\mathrm{op}}, G) \cong \mathrm{colim}(W, GT)\ .$$

Proof. Here is the calculation:

$$\begin{aligned}
\mathscr{X}(\mathrm{colim}(W, GT), X) &\cong [\mathscr{A}^{\mathrm{op}}, \mathscr{V}](W, \mathscr{X}(GT, X)) \\
&\cong [\mathscr{A}^{\mathrm{op}}, \mathscr{V}](W, \mathscr{X}(G, X)T^{\mathrm{op}}) \\
&\cong [\mathscr{C}^{\mathrm{op}}, \mathscr{V}](WS^{\mathrm{op}}, \mathscr{X}(G, X)) \\
&\cong \mathscr{X}(\mathrm{colim}(WS^{\mathrm{op}}, G), X)\ .
\end{aligned}$$

□

Recall that a *pointwise left Kan extension* of a $\mathscr{V}$-functor $F : \mathscr{A} \longrightarrow \mathscr{X}$ along a $\mathscr{V}$-functor $J : \mathscr{A} \longrightarrow \mathscr{B}$ is a $\mathscr{V}$-functor $K = \mathrm{Lan}_J(F) : \mathscr{B} \longrightarrow \mathscr{X}$ such that there is a $\mathscr{V}$-natural isomorphism

$$KB \cong \mathrm{colim}_A(\mathscr{B}(JA, B), FA)\ .$$

3 Cartesian monoidal enriched categories

A monoidal $\mathscr{V}$-category $\mathscr{A}$ will be called *cartesian* when the tensor product and unit object have left adjoints. That is, $\mathscr{A}$ is a map pseudomonoid in the monoidal 2-category $\mathscr{V}$-$\mathrm{Cat}^{\mathrm{co}}$ in the sense of [5].

Let us denote the tensor product of $\mathscr{A}$ by $-\star- : \mathscr{A} \otimes \mathscr{A} \longrightarrow \mathscr{A}$ with left adjoint $\Delta : \mathscr{A} \longrightarrow \mathscr{A} \otimes \mathscr{A}$ and the unit by $N : \mathscr{I} \longrightarrow \mathscr{A}$ with left adjoint $E : \mathscr{A} \longrightarrow \mathscr{I}$. (Here $\mathscr{I}$ is the unit $\mathscr{V}$-category: it has one object 0 and $\mathscr{I}(0,0) = I$.) It is clear that these right adjoints make $\mathscr{A}$ a comonoidal $\mathscr{V}$-category; that is, a pseudomonoid in $\mathscr{V}$-Cat$^{\text{op}}$. Since ob : $\mathscr{V}$-Cat $\longrightarrow$ Set is monoidal, we see that $\Delta : \mathscr{A} \longrightarrow \mathscr{A} \otimes \mathscr{A}$ is given by the diagonal on objects. We have

$$\mathscr{A}(A, A_1 \star A_2) \cong \mathscr{A}(A, A_1) \otimes \mathscr{A}(A, A_2) \ ,$$

where $\mathscr{V}$-functoriality in A on the right-hand side uses Δ.

If $\mathscr{A}$ is cartesian, the $\mathscr{V}$-functor category $[\mathscr{A}, \mathscr{V}]$ becomes monoidal under convolution using the comonoidal structure on $\mathscr{A}$. This is a pointwise tensor product in the sense that, on objects, it is defined by:

$$(M * N)A = MA \otimes NA \ .$$

On morphisms it requires the use of Δ. Indeed, the Yoneda embedding

$$\mathrm{Y} : \mathscr{A}^{\text{op}} \longrightarrow [\mathscr{A}, \mathscr{V}]$$

is strong monoidal.

4 Main result

Theorem 3. *Suppose $J : \mathscr{A} \longrightarrow \mathscr{B}$ is a $\mathscr{V}$-functor between cartesian monoidal $\mathscr{V}$-categories. Assume also that J is strong comonoidal. Suppose $\mathscr{X}$ is a monoidal $\mathscr{V}$-category such that each of the $\mathscr{V}$-functors $- \otimes X$ and $X \otimes -$ preserves colimits. Assume the $\mathscr{V}$-functor $F : \mathscr{A} \longrightarrow \mathscr{X}$ is strong monoidal. If the pointwise left Kan extension $K : \mathscr{B} \longrightarrow \mathscr{X}$ of F along J exists then K too is strong monoidal.*

Proof. Using that tensor in $\mathscr{X}$ preserves colimits in each variable, the Fubini Theorem 1, that F is strong monoidal, Theorem 2 with the cartesian property of $\mathscr{A}$, and the cartesian property of $\mathscr{B}$, we have the calculation:

$$\begin{aligned}
KB_1 \otimes KB_2 &\cong \operatorname{colim}_{A_1}(\mathscr{B}(JA_1, B_1), FA_1) \otimes \operatorname{colim}_{A_2}(\mathscr{B}(JA_2, B_2), FA_2) \\
&\cong \operatorname{colim}_{A_1,A_2}(\mathscr{B}(JA_1, B_1) \otimes \mathscr{B}(JA_2, B_2), FA_1 \otimes FA_2) \\
&\cong \operatorname{colim}_{A_1,A_2}(\mathscr{B}(JA_1, B_1) \otimes \mathscr{B}(JA_2, B_2), F(A_1 \star A_2)) \\
&\cong \operatorname{colim}_{A}(\mathscr{B}(JA, B_1) \otimes \mathscr{B}(JA, B_2), FA) \\
&\cong \operatorname{colim}_{A}(\mathscr{B}(JA, B_1 \star B_2), FA) \\
&\cong K(B_1 \star B_2) \ .
\end{aligned}$$

For the unit part, for similar reasons, we have:

$$\begin{aligned}
N &\cong FN0 \\
&\cong \operatorname{colim}_{0}(\mathscr{I}(0, 0), FN0) \\
&\cong \operatorname{colim}_{A}(\mathscr{I}(EA, 0), FA) \\
&\cong \operatorname{colim}_{A}(\mathscr{I}(EJA, 0), FA) \\
&\cong KN \ .
\end{aligned}$$

□

5 An advanced viewpoint

In terminology of [4], suppose $H : \mathscr{M} \longrightarrow \mathscr{N}$ is a monoidal pseudofunctor between monoidal bicategories. The main point to stress here is that the constraints

$$\Phi_{A,B} : HA \otimes HB \longrightarrow H(A \otimes B)$$

are pseudonatural in A and B. Then we see that H takes pseudomonoids (= monoidales) to pseudomonoids, lax morphisms of pseudomonoids to lax morphisms, oplax morphisms of pseudomonoids to oplax morphisms, and strong morphisms of pseudomonoids to strong morphisms.

In particular, this applies to the monoidal pseudofunctor

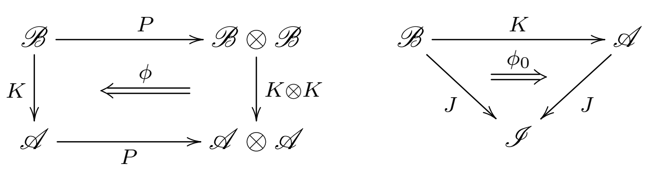
$$\mathscr{V}\text{-Mod}(-, \mathscr{I}) : \mathscr{V}\text{-Mod}^{\text{op}} \longrightarrow \mathscr{V}\text{-CAT}$$

which takes the $\mathscr{V}$-category $\mathscr{A}$ to the $\mathscr{V}$-functor $\mathscr{V}$-category $[\mathscr{A}, \mathscr{V}]$. Now pseudomonoids in $\mathscr{V}\text{-Mod}^{\text{op}}$ are precisely promonoidal (= premonoidal) $\mathscr{V}$-categories in the sense of Day [1, 2]. Therefore, for each promonoidal $\mathscr{V}$-category $\mathscr{A}$, we obtain a monoidal $\mathscr{V}$-category

$$\mathscr{V}\text{-Mod}(\mathscr{A}, \mathscr{I}) = [\mathscr{A}, \mathscr{V}]$$

which is none other than what is now called Day convolution since it is defined and analysed in [1, 2].

A lax morphism of pseudomonoids in $\mathscr{V}\text{-Mod}^{\text{op}}$, as written in $\mathscr{V}$-Mod, is a module $K : \mathscr{B} \longrightarrow \mathscr{A}$ equipped with module morphisms



satisfying appropriate conditions. In other words, we have

$$\begin{aligned}\phi_{A_1,A_2,B} : \operatorname{colim}_{B_1,B_2}(K(A_1, B_1) \otimes K(A_2, B_2), P(B_1, B_2, B)) \\ \Longrightarrow \operatorname{colim}_A(K(A, B), P(A_1, A_2, A))\end{aligned}$$

and

$$\phi_{0B} : JB \Longrightarrow \operatorname{colim}_A(K(A, B), JA) \ .$$

We call such a K a *promonoidal module*. It is *strong* when ϕ and ϕ_0 are invertible.

We also have the $\mathscr{V}$-functor

$$\exists_K : [\mathscr{A}, \mathscr{X}] \longrightarrow [\mathscr{B}, \mathscr{X}]$$

defined by

$$(\exists_K)B = \operatorname{colim}_A(K(A, B), FA) \ .$$

By the general considerations on monoidal pseudofunctors, $\exists_K$ is a monoidal $\mathscr{V}$-functor when $\mathscr{X} = \mathscr{V}$. However, the same calculations needed to show this explicitly show that it works for any monoidal $\mathscr{V}$-category $\mathscr{X}$ for which each of the tensors $X \otimes -$ and $- \otimes X$ preserves colimits.

Theorem 4. *If $K : \mathscr{B} \longrightarrow \mathscr{A}$ is a promonoidal $\mathscr{V}$-module then $\exists_K : [\mathscr{A}, \mathscr{X}] \longrightarrow [\mathscr{B}, \mathscr{X}]$ is a monoidal $\mathscr{V}$-functor. If K is strong promonoidal then $\exists_K$ is strong monoidal.*

Proof. Although the result should be expected from our earlier remarks, here is a direct calculation.

$$\begin{aligned}
(\exists_K F_1 * \exists_K F_2)B \quad &\cong \quad \mathrm{colim}_{B_1,B_2}(P(B_1,B_2,B), (\exists_K F_1)B_1 \otimes (\exists_K F_2)B_2) \\
&\cong \quad \mathrm{colim}_{B_1,B_2}(P(B_1,B_2,B), \mathrm{colim}_{A_1}(K(A_1,B_1), F_1A_1) \otimes \\
&\qquad \mathrm{colim}_{A_2}(K(A_2,B_2), F_2A_2)) \\
&\cong \quad \mathrm{colim}_{B_1,B_2,A_1,A_2}(K(A_1,B_1) \otimes K(A_2,B_2) \otimes P(B_1,B_2,B), \\
&\qquad F_1A_1 \otimes F_2A_2) \\
&\Longrightarrow \quad \mathrm{colim}_{A,A_1,A_2}(K(A,B) \otimes P(A_1,A_2,A), F_1A_1 \otimes F_2A_2) \\
&\cong \quad \mathrm{colim}_A(K(A,B), \mathrm{colim}_{A_1,A_2}(P(A_1,A_2,A), F_1A_1 \otimes F_2A_2)) \\
&\cong \quad \mathrm{colim}_A(K(A,B), (F_1 * F_2)A)) \\
&\cong \quad \exists_K(F_1 * F_2)B \ .
\end{aligned}$$

The morphism on the fourth line of the calculation is induced by $\phi_{A_1,A_2,B}$ and so is invertible if K is strong promonoidal. We also have $\phi_{0B} : JB \Longrightarrow (\exists_K J)B$. □

For the corollaries now coming, assume as above that $\mathscr{X}$ is a monoidal $\mathscr{V}$-category such that $X \otimes -$ and $- \otimes X$ preserve existing colimits. Also $\mathscr{A}$ and $\mathscr{B}$ are monoidal $\mathscr{V}$-categories. The monoidal structure on $[\mathscr{A}^{\mathrm{op}}, \mathscr{X}]$ is convolution with respect to the promonoidal structure $\mathscr{A}(A, A_1 \star A_2)$ on $\mathscr{A}^{\mathrm{op}}$; similarly for $[\mathscr{B}^{\mathrm{op}}, \mathscr{X}]$.

Corollary 5. *If $J : \mathscr{A} \longrightarrow \mathscr{B}$ is strong monoidal then so is*

$$\mathrm{Lan}_{J^{\mathrm{op}}} : [\mathscr{A}^{\mathrm{op}}, \mathscr{X}] \longrightarrow [\mathscr{B}^{\mathrm{op}}, \mathscr{X}] \ .$$

Proof. Apply Theorem 4 to the module $K : \mathscr{B}^{\mathrm{op}} \longrightarrow \mathscr{A}^{\mathrm{op}}$ defined by $K(A,B) = \mathscr{B}(B, JA)$. We see that K is strong promonoidal using Yoneda twice and strong monoidalness of J. □

Corollary 6. *If $W : \mathscr{A} \longrightarrow \mathscr{V}$ is strong monoidal then so is*

$$\mathrm{colim}(W, -) : [\mathscr{A}^{\mathrm{op}}, \mathscr{X}] \longrightarrow \mathscr{X} \ .$$

Proof. Take $\mathscr{B} = \mathscr{I}$ in Theorem 4. □

Corollary 7. *Suppose $\mathscr{A}$ is cartesian monoidal. If $F : \mathscr{A} \longrightarrow \mathscr{X}$ is strong monoidal then so is*

$$\mathrm{colim}(-, F) : [\mathscr{A}^{\mathrm{op}}, \mathscr{V}] \longrightarrow \mathscr{X} \ .$$

Proof. Here is the calculation for binary tensoring:

$$\begin{aligned}
\mathrm{colim}(W_1 \otimes W_2, F) &\cong \mathrm{colim}_A((W_1 \otimes W_2)\Delta A, FA) \\
&\cong \mathrm{colim}_{A_1,A_2}(W_1A_1 \otimes W_2A_2, F(A_1 \star A_2)) \\
&\cong \mathrm{colim}_{A_1,A_2}(W_1A_1 \otimes W_2A_2, FA_1 \otimes FA_2)) \\
&\cong \mathrm{colim}_{A_1}(W_1A_1, FA_1) \otimes \mathrm{colim}_{A_2}(W_2A_2, FA_2) \\
&\cong \mathrm{colim}(W_1, F) \otimes \mathrm{colim}(W_2, F) \ .
\end{aligned}$$

The unit preservation is easier. □

Corollary 8. *Suppose $\mathscr{A}$ and $\mathscr{B}$ are cartesian monoidal and $J : \mathscr{A} \longrightarrow \mathscr{B}$ is strong comonoidal. If $F : \mathscr{A} \longrightarrow \mathscr{X}$ is strong monoidal then so is*

$$\mathrm{Lan}_J F : \mathscr{B} \longrightarrow \mathscr{X} \ .$$

Proof. Notice that $\mathrm{Lan}_J F$ is the composite of $\mathscr{B}(J, 1) : \mathscr{B} \longrightarrow [\mathscr{A}^{\mathrm{op}}, \mathscr{V}]$ and $\mathrm{colim}(-, F) : [\mathscr{A}^{\mathrm{op}}, \mathscr{V}] \longrightarrow \mathscr{X}$. The first is strong monoidal by hypothesis on J. The second is strong monoidal by Corollary 7. □

References

[1] Brian J. Day, *Construction of Biclosed Categories* (PhD Thesis, UNSW, 1970) <http://www.math.mq.edu.au/~street/DayPhD.pdf>.

[2] Brian J. Day, *On closed categories of functors*, Lecture Notes in Mathematics **137** (Springer-Verlag, 1970) 1–38.

[3] Brian J. Day and Ross Street, *Kan extensions along promonoidal functors*, Theory and Applications of Categories **1(4)** (1995) 72–77.

[4] Brian J. Day and Ross Street, *Monoidal bicategories and Hopf algebroids*, Advances in Math. **129** (1997) 99–157.

[5] Brian J. Day, Paddy McCrudden and Ross Street, *Dualizations and antipodes*, Applied Categorical Structures **11** (2003) 229–260.

[6] Samuel Eilenberg and G. Max Kelly, *Closed categories*, Proceedings of the Conference on Categorical Algebra (La Jolla, 1965), (Springer-Verlag,1966) 421–562.

[7] G. Max Kelly and Stephen Lack, *Finite-product-preserving functors, Kan extensions, and strongly-finitary monads*, Applied Categorical Structures **1(1)** (1993) 84–94.

[8] F. William Lawvere, *Functorial Semantics of Algebraic Theories and Some Algebraic Problems in the context of Functorial Semantics of Algebraic Theories*, (Ph.D. thesis, Columbia University, 1963); Reports of the Midwest Category Seminar II (1968) 41–61; Reprints in Theory and Applications of Categories **5** (2004) 1–121.

[9] G. Max Kelly, *Basic concepts of enriched category theory*, London Mathematical Society Lecture Note Series **64** (Cambridge University Press, Cambridge, 1982).

[10] Saunders Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics **5** (Springer-Verlag, 1971).