Dualizations and antipodes

Brian Day, Paddy McCrudden and Ross Street

January 2002

Abstract

Because an exact pairing between an object and its dual is extraordinarily natural in the object, ideas of the paper [St4] apply to yield a definition of dualization for a pseudomonoid in any autonomous monoidal bicategory as base; this is an improvement on [DS; Definition 11, page 114]. We analyse the dualization notion in depth. An example is the concept of autonomous (which, usually in the presence of a symmetry, also has been called "rigid" or "compact") monoidal category. The antipode of a quasi-Hopf algebra H in the sense of Drinfeld [Maj] is another example obtained using a different base monoidal bicategory. We define right autonomous monoidal functors and their higher-dimensional analogue. Our explanation of why the category Comod_f (H) of finite dimensional representations of H is autonomous is that the Comod_f operation is autonomous and so preserves dualization.

Mathematics Subject Classifications (2000): 18D05, 18D10, 18D20, 16W30.

Key words: monoidal bicategory, enriched category, bidual, antipode, quantum group, quasi-Hopf algebra, braided group, comodule.

Introduction

This paper defines and studies (left) autonomous pseudomonoids in any (right) autonomous monoidal bicategory. As examples we obtain autonomous monoidal $\mathcal{V}$-categories and quasi-Hopf algebras in the sense of Drinfel'd [Dd] (see [Maj; Section 2.4, page 62-63]); ordinary Hopf algebras are special cases of the latter. A more theoretical example, in one of a number of possible variants of slice monoidal bicategories, is what we call autonomous monoidal functors; these have escaped mention in the literature perhaps because strong monoidal functors automatically preserve duals. We use this to motivate our notion of autonomous monoidal lax functor which is of more interest to us here.

We provide a formal representation theory whereby objects of a monoidal bicategory are represented by certain morphisms from the tensor unit. This leads us, for example, to the higher-dimensional categorical ingredients making the monoidal category of (finite dimensional) representations of a quasi-Hopf algebra autonomous.

This paper can be considered as the next in the progression [St3], [JS3], [DS], [MC0–3]. However, we hope it can be read independently, apart from a reference to [St4] for our approach to monoidal bicategories. Because of a coherence theorem [GPS], we freely speak as if we were dealing with a monoidal bicategory $\mathcal{M}$ yet do our work in a Gray monoid. When $\mathcal{M}$ is right autonomous, we write A° for the right bidual of an object A; our notation for the unit and counit morphisms is $n : I \longrightarrow A^\circ \otimes A$ and $e : A \otimes A^\circ \longrightarrow I$. Also recall from [St4] that, for each good monoidal category $\mathcal{V}$, we have an autonomous monoidal bicategory $\mathcal{V}$–**Mod** whose objects are $\mathcal{V}$-categories and whose morphisms are two-sided modules.

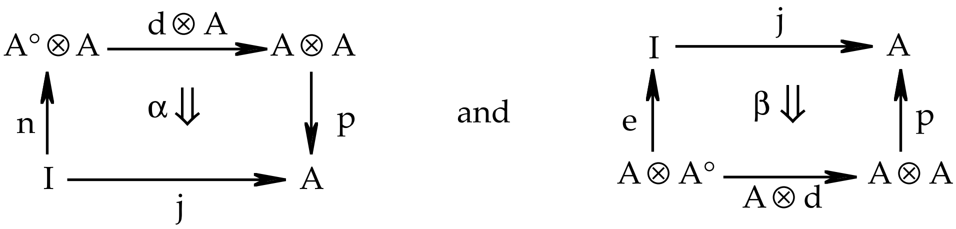
1. Dualization for pseudomonoids

Extraordinary 2-cells can be used to express rich structures in a right autonomous monoidal bicategory $\mathcal{M}$. In particular, we can express dualization. Suppose A is a pseudomonoid in $\mathcal{M}$ in the sense of [DS]; that is, we have a *multiplication* $p : A \otimes A \longrightarrow A$ and a *unit* $j : I \longrightarrow A$ which are associative and unital up to coherent isomorphisms

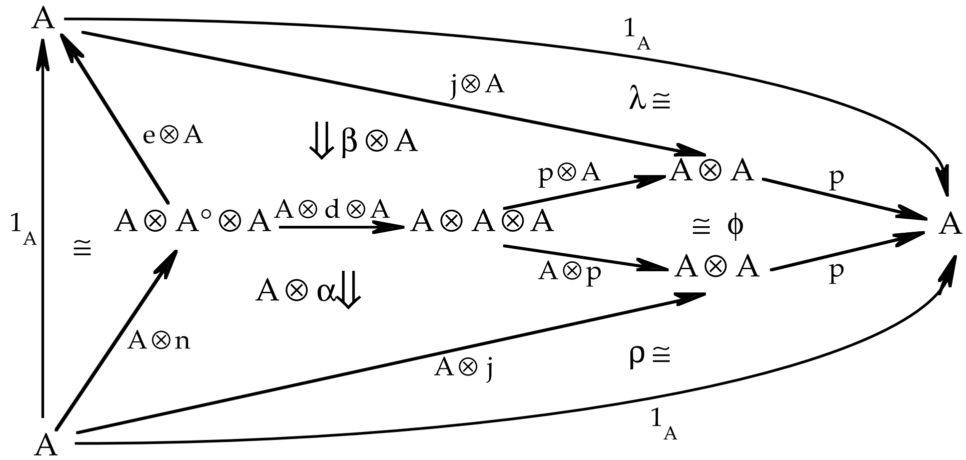
$$\phi : p \circ (p \otimes 1_A) \cong p \circ (1_A \otimes p), \quad \lambda : p \circ (j \otimes 1_A) \cong 1_A, \quad \text{and} \quad \rho : p \circ (1_A \otimes j) \cong 1_A .$$

By way of example, we remind the reader that pseudomonoids A in $\mathcal{V}$–**Mod** are promonoidal $\mathcal{V}$-categories[1] in the sense of [Dy1]. A promonoidal $\mathcal{V}$-category is a monoidal $\mathcal{V}$-category precisely when its multiplication and unit are representable by $\mathcal{V}$-functors.

A morphism $d : A^\circ \longrightarrow A$ is called *left dualization* for the pseudomonoid A in $\mathcal{M}$ when it is equipped with an extraordinary 2-cell α from $p \circ (d \otimes 1_A) : A^\circ \otimes A \longrightarrow A$ to $j : I \longrightarrow A$, and an extraordinary 2-cell β from $j : I \longrightarrow A$ to $p \circ (1_A \otimes d) : A \otimes A^\circ \longrightarrow A$ satisfying two conditions. Explicitly, we have 2-cells

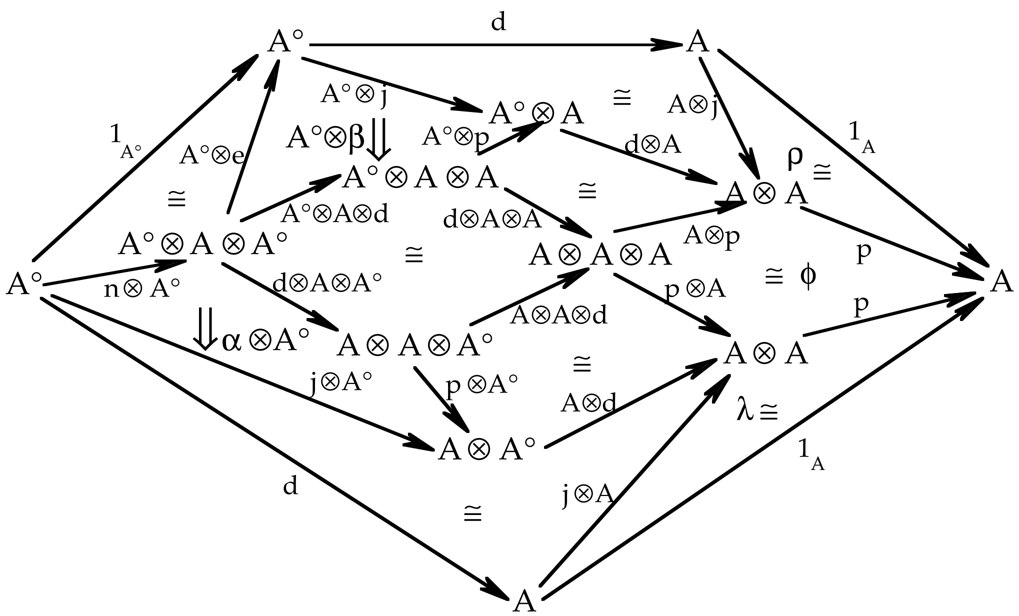


satisfying the condition that the pasted composite



is the identity 2-cell of 1_A, and the condition that the pasted composite

[1] Technically, it is a promonoidal structure on A^{op} in the sense of [Dy1].



is the identity 2-cell of d. It would be nice to see the surface diagrams for these conditions, but this would strain our technology. By mimicking the argument of Paré [ML; IV.1 Exercise 4, p.84], if we have d, α and β as above, but only satisfying the first condition, and if idempotents split in the category $\mathcal{M}(A^\circ, A)$, then the pasted composite of the second condition gives an idempotent on d whose splitting gives a left dualization.

Proposition 1.1 *Let* A *be a pseudomonoid in a right autonomous bicategory* $\mathcal{M}$. *There are bijections among the following structures on a morphism* $d : A^\circ \longrightarrow A$:

(a) *pairs* (α, β) *making* d *a left dualization for* A;

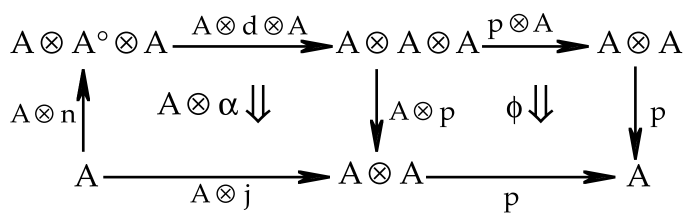
(b) *adjunctions* $p \dashv (p\otimes A)\circ(A\otimes d\otimes A)\circ(A\otimes n)$;

(c) *adjunctions* $p\circ(d\otimes A) \dashv (A^\circ\otimes p)\circ(n\otimes A)$.

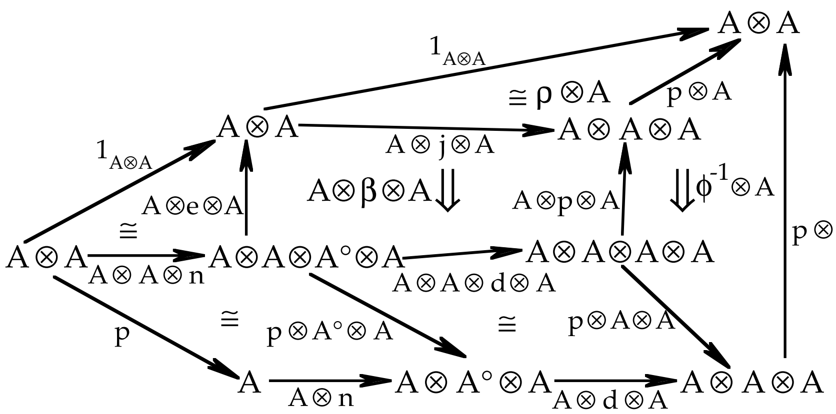
Proof (a) ⇒ (b) Write p^* for the composite

$$A \xrightarrow{A\otimes n} A\otimes A^\circ\otimes A \xrightarrow{A\otimes d\otimes A} A\otimes A\otimes A \xrightarrow{p\otimes A} A\otimes A.$$

The counit $p\circ p^* \Rightarrow 1_A$ is obtained by pasting ϕ and $A\otimes\alpha$ as follows.

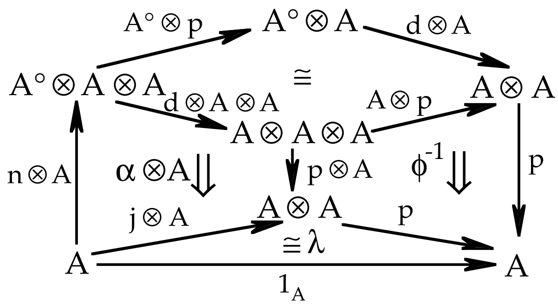


The unit $1_{A\otimes A} \Rightarrow p^*\circ p$ is the following pasting composite.

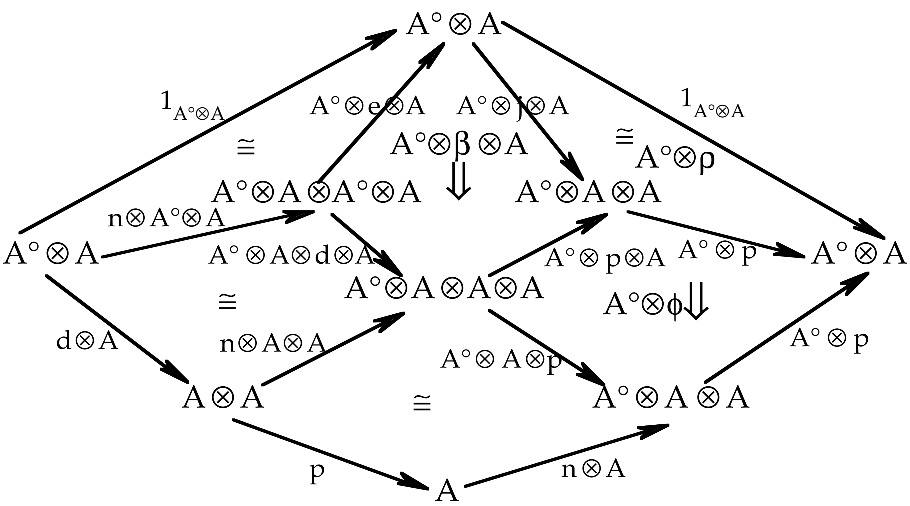


The two conditions on α and β give the adjunction conditions on this counit and unit.

(a) ⇒ (c) The counit for the adjunction in (c) is the following pasting composite.



The unit for the adjunction in (c) is the following pasting composite.



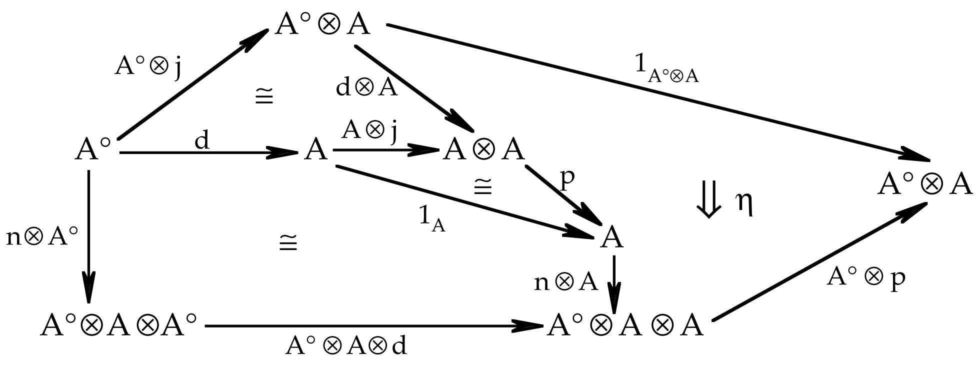
The two conditions on α and β give the adjunction conditions on this counit and unit.

(c) ⇒ (a) Under an adjunction $p^\circ(d \otimes A) \dashv (A^\circ \otimes p)^\circ (n \otimes A)$, the isomorphism

$$n \;\cong\; (A^\circ \otimes p)^\circ (A^\circ \otimes j \otimes A)^\circ n \;\cong\; (A^\circ \otimes p)^\circ (n \otimes A)^\circ j$$

corresponds to a 2-cell $\alpha : p^\circ(d \otimes A)^\circ n \Rightarrow j$ as required in (a). Since A° is a right bidual for A, 2-cells $\beta : j^\circ e \Rightarrow p^\circ (A \otimes d) : A \otimes A^\circ \longrightarrow A$ are in bijection with 2-cells $\beta' : A^\circ \otimes j \Rightarrow (A^\circ \otimes p)^\circ (A^\circ \otimes A \otimes d)^\circ (n \otimes A^\circ)$, and for β' we take the following composite in which η

denotes the unit for the adjunction of (c).

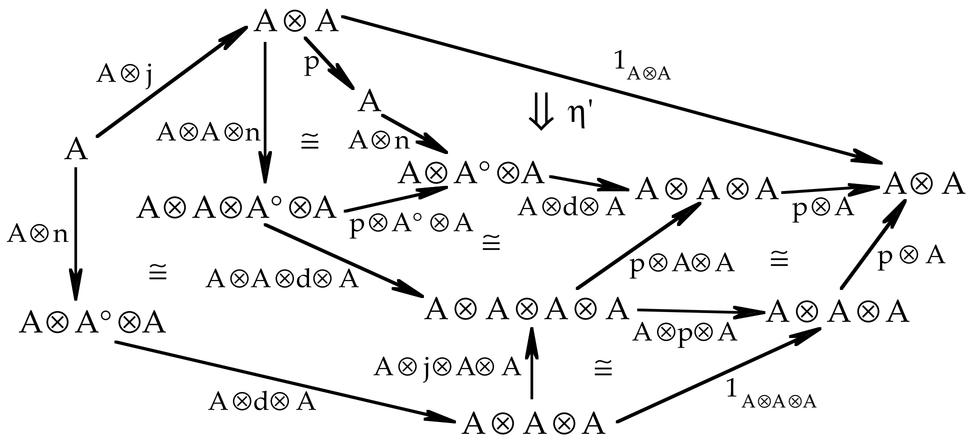


The conditions on α and β can be deduced from the adjunction conditions.

(b) ⇒ (a) Under an adjunction $p \dashv (p\otimes A)^\circ(A\otimes d\otimes A)^\circ(A\otimes n)$, the isomorphism

$$(d\otimes A)^\circ n \;\cong\; (p\otimes A)^\circ(j\otimes A\otimes A)^\circ(d\otimes A)^\circ n \;\cong\; (p\otimes A)^\circ(A\otimes d\otimes A)^\circ(A\otimes n)^\circ j$$

corresponds to a 2-cell $\alpha : p^\circ(d\otimes A)^\circ n \Rightarrow j$ as required in (a). Since A° is a right bidual for A, 2-cells $\beta : j^\circ e \Rightarrow p^\circ(A\otimes d) : A\otimes A^\circ \longrightarrow A$ are in bijection with 2-cells $\beta'' : A\otimes j \Rightarrow (p\otimes A)^\circ(A\otimes d\otimes A)^\circ(A\otimes n) : A \longrightarrow A\otimes A$, and for β'' we take the following composite in which η' denotes the unit for the adjunction of (b).



The conditions on α and β can be deduced from the adjunction conditions. **Q.E.D.**

Proposition 1.1 (c) is precisely the condition defining "right antipode" in [DS; Definition 11, page 114]. In [DS] we presumed the multiplication p, the unit j, and the dualization d had right adjoints; we shall now see that p necessarily has a right adjoint, while d has a right adjoint if j does.

Proposition 1.2 *Let A be a pseudomonoid with left dualization* $d : A^\circ \longrightarrow A$. *Then* $p : A\otimes A \longrightarrow A$ *has a right adjoint* p^* *and there is an isomorphism*

$$d \;\cong\; (A\otimes e)^\circ(p^*\otimes A^\circ)^\circ(j\otimes A^\circ).$$

Furthermore, if j *has a right adjoint* j^* *then*

(i) d *has a right adjoint* $d^* = (A^\circ \otimes j^*) \circ (A^\circ \otimes p) \circ (n \otimes A)$,

(ii) $(j^* \otimes A) \circ (p \otimes A) \circ (A \otimes p^*) \circ (A \otimes j) \cong 1_A$,

(iii) *the unit of the adjunction* $d \dashv d^*$ *is invertible (so that* d *is fully faithful), and*

(iv) *there is an isomorphism* $j^* \cong e \circ (A \otimes d^*) \circ (A \otimes j)$.

Proof From Proposition 1.1 (b), we have $p^* = (p \otimes A) \circ (A \otimes d \otimes A) \circ (A \otimes n)$. So

$$
\begin{aligned}
(A \otimes e) \circ (p^* \otimes A^\circ) \circ (j \otimes A^\circ) &\cong \\
&\cong (A \otimes e) \circ (p \otimes A \otimes A^\circ) \circ (A \otimes d \otimes A \otimes A^\circ) \circ (A \otimes n \otimes A^\circ) \circ (j \otimes A^\circ) \\
&\cong p \circ (A \otimes d) \circ (A \otimes A^\circ \otimes e) \circ (A \otimes n \otimes A^\circ) \circ (j \otimes A^\circ) \\
&\cong p \circ (A \otimes d) \circ (j \otimes A^\circ) \\
&\cong p \circ (j \otimes A) \circ d \\
&\cong d .
\end{aligned}
$$

If $j \dashv j^*$ then we can compose the adjunction of Proposition 1.1 (c) with $A^\circ \otimes j \dashv A^\circ \otimes j^*$ to obtain $p \circ (d \otimes A) \circ (A^\circ \otimes j) \dashv (A^\circ \otimes j^*) \circ (A^\circ \otimes p) \circ (n \otimes A)$; however, $p \circ (d \otimes A) \circ (A^\circ \otimes j) \cong p \circ (A \otimes j) \circ d \cong d$, which proves (i). Then

$$
\begin{aligned}
d^* \circ j &= (A^\circ \otimes j^*) \circ (A^\circ \otimes p) \circ (n \otimes A) \circ j \\
&\cong (A^\circ \otimes j^*) \circ (A^\circ \otimes p) \circ (A^\circ \otimes A \otimes j) \circ n \\
&\cong (A^\circ \otimes j^*) \circ n \\
&\cong j^{*\circ},
\end{aligned}
$$

from which (iv) immediately follows. To prove (ii) we use the formula for p^*:

$$
\begin{aligned}
(j^* \otimes A) \circ (p \otimes A) \circ (A \otimes p^*) \circ (A \otimes j) & \\
&\cong (j^* \otimes A) \circ (p \otimes A) \circ (A \otimes p \otimes A) \circ (A \otimes A \otimes d \otimes A) \circ (A \otimes A \otimes n) \circ (A \otimes j) \\
&\cong (j^* \otimes A) \circ (p \otimes A) \circ (p \otimes A \otimes A) \circ (A \otimes A \otimes d \otimes A) \circ (A \otimes A \otimes n) \circ (A \otimes j) \\
&\cong (j^* \otimes A) \circ (p \otimes A) \circ (A \otimes d \otimes A) \circ (p \otimes A^\circ \otimes A) \circ (A \otimes A \otimes n) \circ (A \otimes j) \\
&\cong (j^* \otimes A) \circ (p \otimes A) \circ (A \otimes d \otimes A) \circ (A \otimes n) \circ p \circ (A \otimes j) \\
&\cong (j^* \otimes A) \circ (p \otimes A) \circ (A \otimes d \otimes A) \circ (A \otimes n) \\
&\cong (j^* \otimes A) \circ p^* \\
&\cong (p \circ (j \otimes A))^* \\
&\cong 1_A .
\end{aligned}
$$

Finally, to verify (iii) we note that an inverse of the unit for $d \dashv d^*$ is provided by the composite isomorphism:

$$
\begin{aligned}
d^* \circ d &\cong (A^\circ \otimes j^*) \circ (A^\circ \otimes p) \circ (n \otimes A) \circ (A \otimes e) \circ (p^* \otimes A^\circ) \circ (j \otimes A^\circ) \\
&\cong (A^\circ \otimes j^*) \circ (A^\circ \otimes p) \circ (A^\circ \otimes A \otimes A \otimes e) \circ (n \otimes A \otimes A \otimes A^\circ) \circ (p^* \otimes A^\circ) \circ (j \otimes A^\circ) \\
&\cong (A^\circ \otimes j^*) \circ (A^\circ \otimes p) \circ (A^\circ \otimes A \otimes A \otimes e) \circ (A^\circ \otimes A \otimes p^* \otimes A^\circ) \circ (A^\circ \otimes A \otimes j \otimes A^\circ) \circ (n \otimes A^\circ) \\
&\cong (A^\circ \otimes e) \circ (A^\circ \otimes j^* \otimes A \otimes A^\circ) \circ (A^\circ \otimes p \otimes A \otimes A^\circ) \circ (A^\circ \otimes A \otimes p^* \otimes A^\circ) \circ (A^\circ \otimes A \otimes j \otimes A^\circ) \circ (n \otimes A^\circ) \\
&\cong (A^\circ \otimes e) \circ (A^\circ \otimes 1_A \otimes A^\circ) \circ (n \otimes A^\circ) \qquad \text{using (ii)} \\
&\cong (A^\circ \otimes e) \circ (n \otimes A^\circ)
\end{aligned}
$$

$\cong 1_{A^\circ}$. **Q.E.D.**

Corollary 1.3 *Left dualizations are unique up to isomorphism in* $\mathcal{M}(A^\circ, A)$ *compatible with* α *and* β.

Proof This follows from the formula for d in Proposition 1.2 and the fact that p^* is uniquely determined up to isomorphism by p. **Q.E.D.**

A pseudomonoid is called *left autonomous* when it admits a left dualization. When $\mathcal{M}$ is left autonomous, we define *right dualization* to be dualization in $\mathcal{M}^{\mathrm{rev}}$. When $\mathcal{M}$ is autonomous, we call a pseudomonoid *autonomous* when it admits both a left and a right dualization.

We write $A^\vee$ for the left bidual of A.

Proposition 1.4 *If* A *is an autonomous pseudomonoid in an autonomous monoidal bicategory* $\mathcal{M}$ *then its right dualization* $d' : A^\vee \longrightarrow A$ *is the mate* $d^\vee : A^\vee \longrightarrow A$ *of its left dualization* $d : A^\circ \longrightarrow A$. *If further* $j : I \longrightarrow A$ *has a right adjoint then* $d : A^\circ \longrightarrow A$ *is an equivalence.*

Proof By the dual of the formula for d in Proposition 1.2,

$$d' \cong (n^\vee \otimes A) \circ (A^\vee \otimes p^*) \circ (A^\vee \otimes j).$$

So

$$\begin{aligned} d'^\circ &\cong (j^\circ \otimes A) \circ (p^{*\circ} \otimes A) \circ (A^\circ \otimes n) \\ &\cong (e \otimes A) \circ (A \otimes e \otimes A^\circ \otimes A) \circ (p^* \otimes A^\circ \otimes A^\circ \otimes A) \circ (j \otimes A^\circ \otimes A^\circ \otimes A) \circ (A^\circ \otimes n) \\ &\cong (e \otimes A) \circ (A \otimes n) \circ (A \otimes e) \circ (p^* \otimes A^\circ) \circ (j \otimes A^\circ) \\ &\cong (A \otimes e) \circ (p^* \otimes A^\circ) \circ (j \otimes A^\circ) \\ &\cong d\,. \end{aligned}$$

This proves the first sentence of the Proposition.

If $j \dashv j^*$ then by Proposition 1.2 (iii), the unit of $d \dashv d^*$ is invertible; so the counit of $d^{*\vee} \dashv d^\vee$ is invertible. By the dual of Proposition 1.2 (iii), the unit of $d^\vee \dashv d^{\vee *}$ is invertible. So $d^\vee$ has both a left and right quasi-inverse and hence is an equivalence. So d is an equivalence. **Q.E.D.**

Proposition 1.5 *In an autonomous monoidal bicategory* $\mathcal{M}$, *a left autonomous pseudomonoid is autonomous if its left dualization* $d : A^\circ \longrightarrow A$ *is an equivalence.*

Proof We begin by proving the existence of a canonical isomorphism

$$\begin{array}{ccc} I & \xrightarrow{\;e^{\vee}\;} & A\otimes A^{\vee} \\ {\scriptstyle n}\downarrow & \cong & \downarrow{\scriptstyle A\otimes d^{\vee}} \\ A^{\circ}\otimes A & \xrightarrow[\;d\otimes A\;]{} & A\otimes A \end{array}$$

by showing that there is an isomorphism after applying the biequivalence $(-)^{\circ}$; we have:

$$\begin{aligned} e\circ(d\otimes A^{\circ}) &\cong e\circ(d\otimes A^{\circ})\circ(A\otimes e\otimes A^{\circ})\circ(n\otimes A^{\circ}\otimes A^{\circ}) \\ &\cong e\circ(A\otimes e\otimes A^{\circ})\circ(d\otimes A\otimes A^{\circ}\otimes A^{\circ})\circ(n\otimes A^{\circ}\otimes A^{\circ}) \\ &\cong ((d\otimes A)\circ n)^{\circ} \cong n^{\circ}\circ(A^{\circ}\otimes d^{\circ}). \end{aligned}$$

From the above square and Proposition 1.1 (b), we have

$$\begin{aligned} p^{*} &\cong (p\otimes A)\circ(A\otimes d\otimes A)\circ(A\otimes n) \\ &\cong (p\otimes A)\circ(A\otimes A\otimes d^{\vee})\circ(A\otimes e^{\vee}) \\ &\cong (A\otimes d^{\vee})\circ(p\otimes A^{\vee})\circ(A\otimes e^{\vee}). \end{aligned}$$

However, if d is an equivalence, so is $A\otimes d^{\vee}$ and it follows from the last isomorphism that $p\circ(A\otimes d^{\vee})$ is left adjoint to $(p\otimes A^{\vee})\circ(A\otimes e^{\vee})$. By the dual of Proposition 1.4 (c), we have shown that $d^{\vee} : A^{\vee} \longrightarrow A$ is a right dualization for A. **Q.E.D.**

Proposition 1.6 *A (left) autonomous monoidal $\mathcal{V}$-category is precisely a (left) autonomous pseudomonoid of $\mathcal{V}$–**Mod** such that the multiplication, unit and dualization modules are representable by $\mathcal{V}$-functors.*

Proof We take it as known that a monoidal $\mathcal{V}$-category A is precisely a pseudomonoid of $\mathcal{V}$–**Mod** such that the multiplication and unit modules are representable by $\mathcal{V}$-functors; we have $p(a,b,c) \cong A(c, a\otimes b)$ and $j(a) \cong A(a, j)$. Suppose we have a $\mathcal{V}$-functor $(-)^{*} : A^{op} \longrightarrow A$ and we let d be the corresponding represented $\mathcal{V}$-module given by $d(a,b) = A(a, b^{*})$. We need to see what it means for d to satisfy the equivalent conditions of Proposition 1.1; let us take condition (b) to be specific. The composite $(p\otimes A)\circ(A\otimes d\otimes A)\circ(A\otimes n)$ has value at $(a,b,c) \in A^{op}\otimes A^{op}\otimes A$ given by the coend

$$\int^{x,y,z,u,v,w} A(x,c)\otimes A(z,y)\otimes A(u,x)\otimes A(v,y^{*})\otimes A(w,z)\otimes A(a,u\otimes v)\otimes A(b,w)$$

which, by repeated application of the coend form of the Yoneda Lemma, reduces to

$$A(a, c\otimes b^{*}).$$

The condition that this should be $\mathcal{V}$-naturally isomorphic to $p^{*}(a,b,c) = A(a\otimes b, c)$ is precisely the condition that each b^{*} should be a left dual for b. **Q.E.D.**

Recall from [DS; Definition 21, page 142] that a Hopf $\mathcal{V}$-algebroid is a $\mathcal{V}$-category $\mathcal{H}$

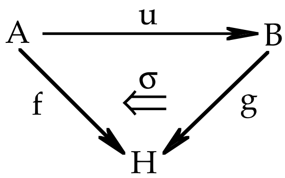
together with $\mathcal{V}$-functors $D : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$, $E : \mathcal{H} \longrightarrow I$ and $S : \mathcal{H} \longrightarrow \mathcal{H}^{op}$ whose right adjoints D^*, E^* and S^* in $\mathcal{V}$–**Mod** are equipped with the structure of a right autonomous pseudomonoid in $(\mathcal{V}\text{–}\mathbf{Mod})^{co}$. So S^* is a dualization for the pseudomonoid $\mathcal{H}$, D^*, E^* in $(\mathcal{V}\text{–}\mathbf{Mod})^{co}$. (Note that adjoints are reversed in $(\mathcal{V}\text{–}\mathbf{Mod})^{co}$, so that S is the right adjoint of S^* therein).

2. An autonomous construction and autonomous morphisms

Let $\mathcal{M}$ be a right autonomous monoidal bicategory.

A morphism in $\mathcal{M}$ with a right adjoint will be called a *map* [St1], [St2]; we write $h^* : B \longrightarrow A$ for the right adjoint of a map $h : A \longrightarrow B$. A *map* pseudomonoid A is one for which both $p : A \otimes A \longrightarrow A$ and $j : I \longrightarrow A$ are maps. Recall (Proposition 1.2) that it is a consequence that any left dualization for A is a map; also, a left autonomous pseudomonad A is a map pseudomonoid if and only if $j : I \longrightarrow A$ is a map.

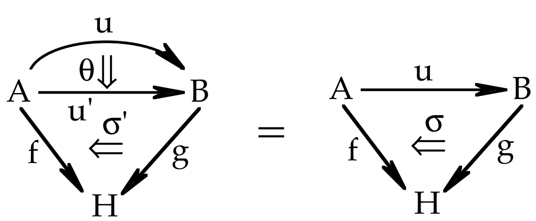
Suppose now that H is a left autonomous map pseudomonoid in $\mathcal{M}$. We explicitly describe an autonomous monoidal bicategory $\mathrm{Map}(\mathcal{M};H)$. The objects (A, f) are maps $f : A \longrightarrow H$ in $\mathcal{M}$. The morphisms $(u, \sigma) : (A, f) \longrightarrow (B, g)$ are diagrams



in $\mathcal{M}$. The 2-cells $\theta : (u, \sigma) \Rightarrow (u', \sigma') : (A, f) \longrightarrow (B, g)$ are 2-cells $\theta : u \Rightarrow u' : A \longrightarrow B$ (with no conditions relating them to σ and σ'). The compositions making this a bicategory are the obvious ones. The tensor product is given by

$$(A, f) \otimes (B, g) = (A \otimes B, p \circ (f \otimes g)), \quad (u, \sigma) \otimes (v, \tau) = (u \otimes v, p \circ (\sigma \otimes \tau)),$$

with unit object (I, j); so the first projection $\mathrm{Map}(\mathcal{M};H) \longrightarrow \mathcal{M}$ is strict monoidal. There is a monoidal subbicategory $\mathrm{Map}'(\mathcal{M};H)$ of $\mathrm{Map}(\mathcal{M};H)$ with the same objects and morphisms, however the 2-cells $\theta : (u, \sigma) \Rightarrow (u', \sigma') : (A, f) \longrightarrow (B, g)$ are those satisfying the following equation.



It is easy to see that $(u, \sigma) : (A, f) \longrightarrow (B, g)$ is a map in $\mathrm{Map}(\mathcal{M};H)$ if and only if $u : A \longrightarrow B$ is a map and there exists some 2-cell $f \Rightarrow g \circ u$ in $\mathcal{M}$; whereas, (u, σ) is a map in $\mathrm{Map}'(\mathcal{M};H)$ if and only if u is a map and $\sigma : g \circ u \Rightarrow f$ is invertible.

Proposition 2.1 *The monoidal bicategories* $\mathrm{Map}(\mathcal{M};H)$ *and* $\mathrm{Map}'(\mathcal{M};H)$ *are both right* *autonomous with the same right bidual* $(A, f)^{\circ} = (A^{\circ}, d \circ f^{*\circ})$ *for the object* (A, f).

Proof We shall check that a right bidual $(A, f)^{\circ}$ for (A, f) is $(A^{\circ}, d \circ f^{*\circ})$. Take a morphism $(v, \tau) : (B, g) \longrightarrow (C, h) \otimes (A, f)$. So we have $v : B \longrightarrow C \otimes A$ and $\tau : p \circ (H \otimes f) \circ (h \otimes A) \circ v \Rightarrow g$. Up to isomorphism there exists a unique $u : B \otimes A^{\circ} \longrightarrow C$ such that $v \cong (u \otimes A) \circ (B \otimes n)$. Moreover, since H has left dualization, Proposition 1.1 (b) gives the right adjoint of $p \circ (H \otimes f)$ as

$$(H \otimes f^{*}) \circ (p \otimes H) \circ (H \otimes d \otimes H) \circ (H \otimes n).$$

It follows that we have a natural bijection between 2-cells τ and 2-cells

$$\upsilon : (h \otimes A) \circ (u \otimes A) \circ (B \otimes n) \Rightarrow (H \otimes f^{*}) \circ (p \otimes H) \circ (H \otimes d \otimes H) \circ (H \otimes n) \circ g.$$

However, we have isomorphisms

$$\begin{aligned}
&(H \otimes f^{*}) \circ (p \otimes H) \circ (H \otimes d \otimes H) \circ (H \otimes n) \circ g \\
&\qquad \cong (p \otimes A) \circ (H \otimes H \otimes f^{*}) \circ (H \otimes d \otimes H) \circ (g \otimes H^{\circ} \otimes H) \circ (B \otimes n) \\
&\qquad \cong ((p \circ (H \otimes d) \circ (g \otimes H^{\circ})) \otimes A) \circ (B \otimes H^{\circ} \otimes f^{*}) \circ (B \otimes n) \\
&\qquad \cong ((p \circ (H \otimes d) \circ (g \otimes H^{\circ})) \otimes A) \circ (B \otimes f^{*\circ} \otimes A) \circ (B \otimes n) \\
&\qquad \cong (p \circ (g \otimes (d \circ f^{*\circ}))) \otimes A) \circ (B \otimes n);
\end{aligned}$$

so that 2-cells υ are in natural bijection with 2-cells

$$((h \circ u) \otimes A) \circ (B \otimes n) \Rightarrow (p \circ (g \otimes (d \circ f^{*\circ}))) \otimes A) \circ (B \otimes n).$$

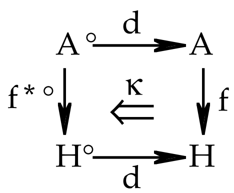
From the universal property of $n : I \longrightarrow A^{\circ} \otimes A$, these 2-cells are in turn in natural bijection with 2-cells $\sigma : h \circ u \Rightarrow p \circ (g \otimes (d \circ f^{*\circ}))$. The assignment $(v, \tau) \longmapsto (u, \sigma)$ clearly extends to equivalences of categories

$$\mathrm{Map}(\mathcal{M};H)((B, g), (C, h) \otimes (A, f)) \simeq \mathrm{Map}(\mathcal{M};H)((B, g) \otimes (A^{\circ}, d \circ f^{*\circ}), (C, h)) \text{ and}$$

$$\mathrm{Map}'(\mathcal{M};H)((B, g), (C, h) \otimes (A, f)) \simeq \mathrm{Map}'(\mathcal{M};H)((B, g) \otimes (A^{\circ}, d \circ f^{*\circ}), (C, h)).$$ **Q.E.D.**

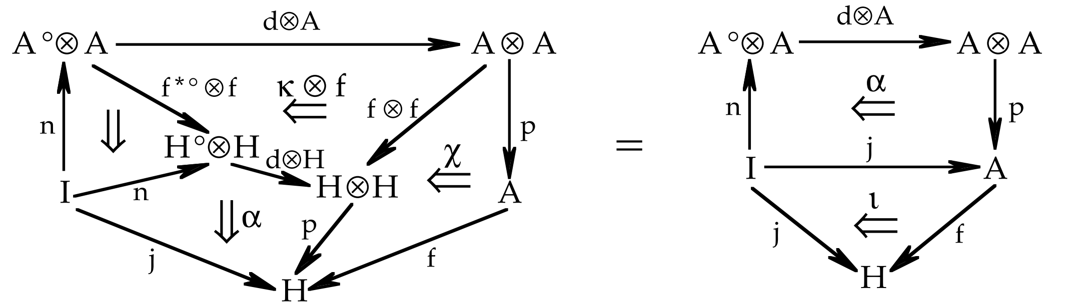
A pseudomonoid (A, f) in $\mathrm{Map}(\mathcal{M};H)$ consists of, in $\mathcal{M}$, a pseudomonoid A together with a map $f : A \longrightarrow H$ and 2-cells $\chi : f \circ p \longrightarrow p \circ (f \otimes f)$ and $\iota : f \circ j \longrightarrow j$ subject to no conditions. Yet, a pseudomonoid (A, f) in $\mathrm{Map}'(\mathcal{M};H)$ consists of a pseudomonoid A in $\mathcal{M}$ together with a map $f : A \longrightarrow H$ and 2-cells $\chi : f \circ p \longrightarrow p \circ (f \otimes f)$ and $\iota : f \circ j \longrightarrow j$ subject to the usual conditions making $f : A \longrightarrow H$ a colax morphism of pseudomonoids.

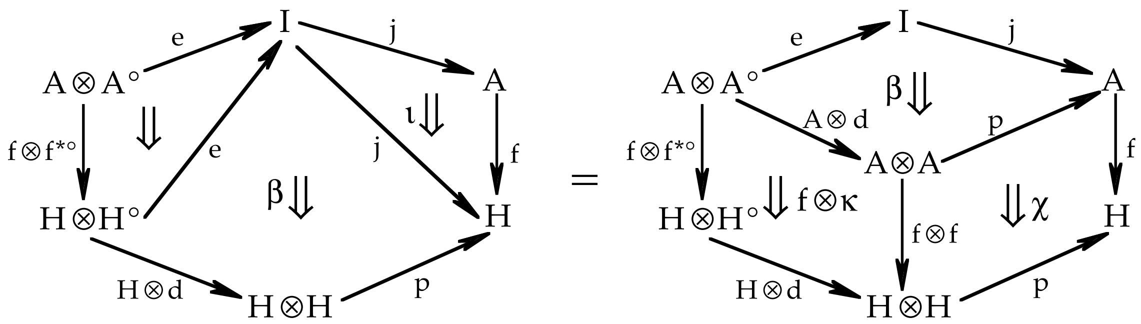
A left dualization for a pseudomonoid (A, f) in $\mathrm{Map}(\mathcal{M};H)$ is a left dualization d for A together with a 2-cell



satisfying no conditions. When this dualization is moreover a left dualization in $\mathrm{Map}'(\mathcal{M};H)$, we say $f : A \longrightarrow H$ is a *left autonomous* colax morphism of left autonomous pseudomonoids. Let us now be more explicit about this definition.

Suppose A and H are left autonomous pseudomonoids in $\mathcal{M}$. Suppose $f : A \longrightarrow H$ is a colax morphism of pseudomonoids such that f is a map as a morphism in $\mathcal{M}$. We say that f is *left autonomous* when there exists a 2-cell $\kappa : f \circ d \Rightarrow d \circ f^{*\circ}$ which is compatible with α and β in the sense that the following two equations hold. In these diagrams, the unlabelled 2-cells $(f^{*\circ} \otimes f) \circ n \Rightarrow n$ and $e \Rightarrow e \circ (f \otimes f^{*\circ})$ are the mates of the canonical isomorphisms $(A^\circ \otimes f) \circ n \cong n \circ (f^\circ \otimes H)$ and $e \circ (A \otimes f^\circ) \cong e \circ (f \otimes H^\circ)$ under the adjunction $f^{*\circ} \dashv f^\circ$.

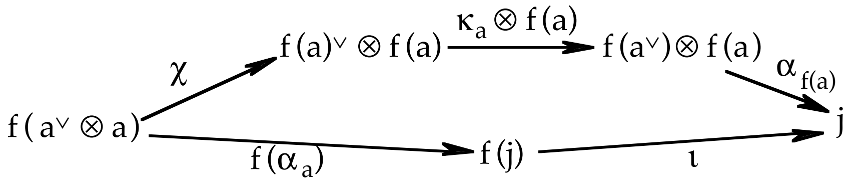


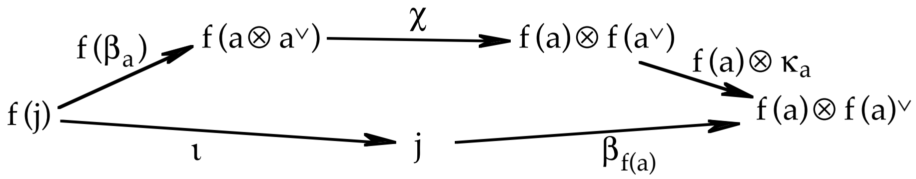


If $f : A \longrightarrow H$ is a pseudomorphism (that is, a colax morphism for which both χ and ι are invertible) then there is a unique 2-cell $\kappa : f \circ d \Rightarrow d \circ f^{*\circ}$ satisfying the first of these two conditions; moreover, this κ is invertible and also satisfies the second condition. So pseudomorphisms automatically have a unique left autonomous structure and preserve left dualization.

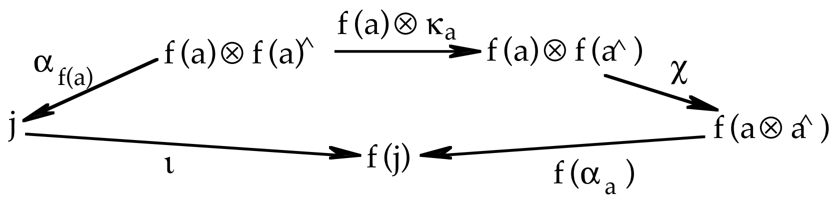
As examples in $\mathcal{V}$–**Mod**, we have the concepts of "left autonomous comonoidal $\mathcal{V}$-functor" and "right autonomous monoidal functor". We shall explain these explicitly because the concept somehow seems to have escaped consideration in the literature on monoidal categories; also we will need a higher-dimensional version described in the next section of this paper.

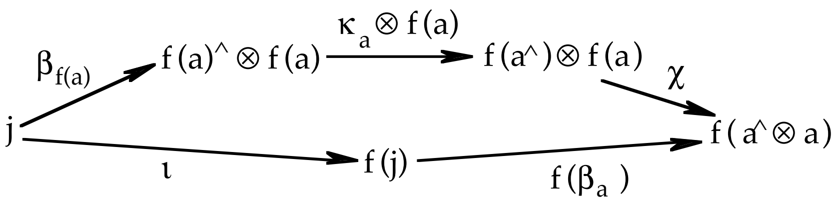
Let A and B denote left autonomous monoidal $\mathcal{V}$-categories. We write $(-)^\vee : A^{op} \longrightarrow A$ for left dualization, with counit $\alpha_a : a^\vee \otimes a \longrightarrow j$ and unit $\beta_a : j \longrightarrow a \otimes a^\vee$, and we use the same notation in B. A comonoidal $\mathcal{V}$-functor $f : A \longrightarrow B$ is said to be *left autonomous* when it is a left autonomous colax morphism in $\mathcal{V}$–**Mod**. Explicitly, it is equipped with a $\mathcal{V}$-natural family $\kappa_a : f(a^\vee) \longrightarrow f(a)^\vee$, $a \in A$, of morphisms in B such that the following two diagrams commute.





Now suppose A and B are right autonomous monoidal $\mathcal{V}$-categories and suppose $f : A \longrightarrow B$ is a monoidal functor (not necessarily strong). Define $f : A \longrightarrow B$ to be *right autonomous* when the comonoidal functor $f : A^\circ \longrightarrow B^\circ$ is left autonomous. (Note that we do not write $f^\circ : A^\circ \longrightarrow B^\circ$ since this would be confusing; indeed $f : A^\circ \longrightarrow B^\circ$ induces the module $f^{*\circ} : A^\circ \longrightarrow B^\circ$.) Explicitly, writing $a^\wedge$ for the right dual of $a \in A$ and using otherwise obvious notation, a right autonomous monoidal functor $f : A \longrightarrow B$ is equipped with a $\mathcal{V}$-natural family $\kappa_a : f(a)^\wedge \longrightarrow f(a^\wedge)$, $a \in A$, of morphisms in B such that the following two diagrams commute.





We call f *strong right autonomous* when κ is invertible. If $f : A \longrightarrow B$ is *strong* monoidal (so that the χ_i are invertible), there is a unique κ_a rendering the second of these diagrams commutative; moreover, it is invertible, $\mathcal{V}$-natural in a, and also renders the first diagram commutative. That is, strong monoidal functors between right autonomous monoidal categories are automatically strong right autonomous. However, a monoidal functor can be strong right autonomous without being strong monoidal.

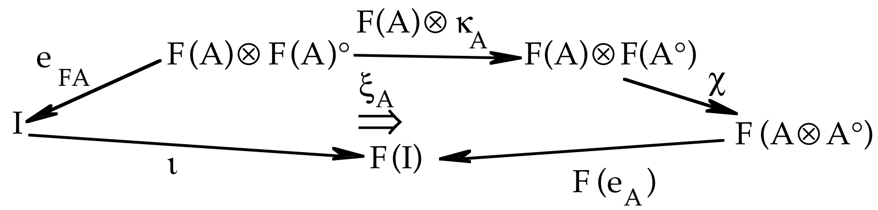
3. Autonomous monoidal lax functors

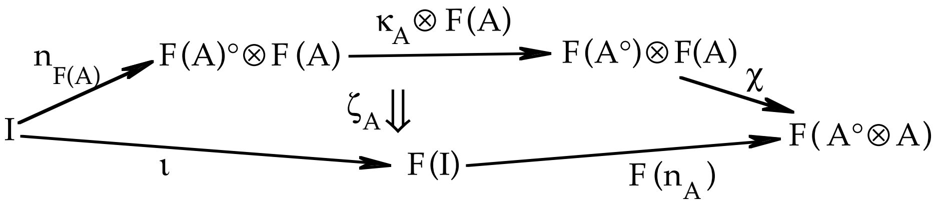
The purpose of this section is to "categorify" the notion of autonomous monoidal functor and to examine when such things preserve dualization.

Let $\mathcal{M}$ and $\mathcal{N}$ be right autonomous monoidal bicategories. A monoidal lax functor $F : \mathcal{M} \longrightarrow \mathcal{N}$ is called *right autonomous* when it is equipped with a pseudonatural family of morphisms

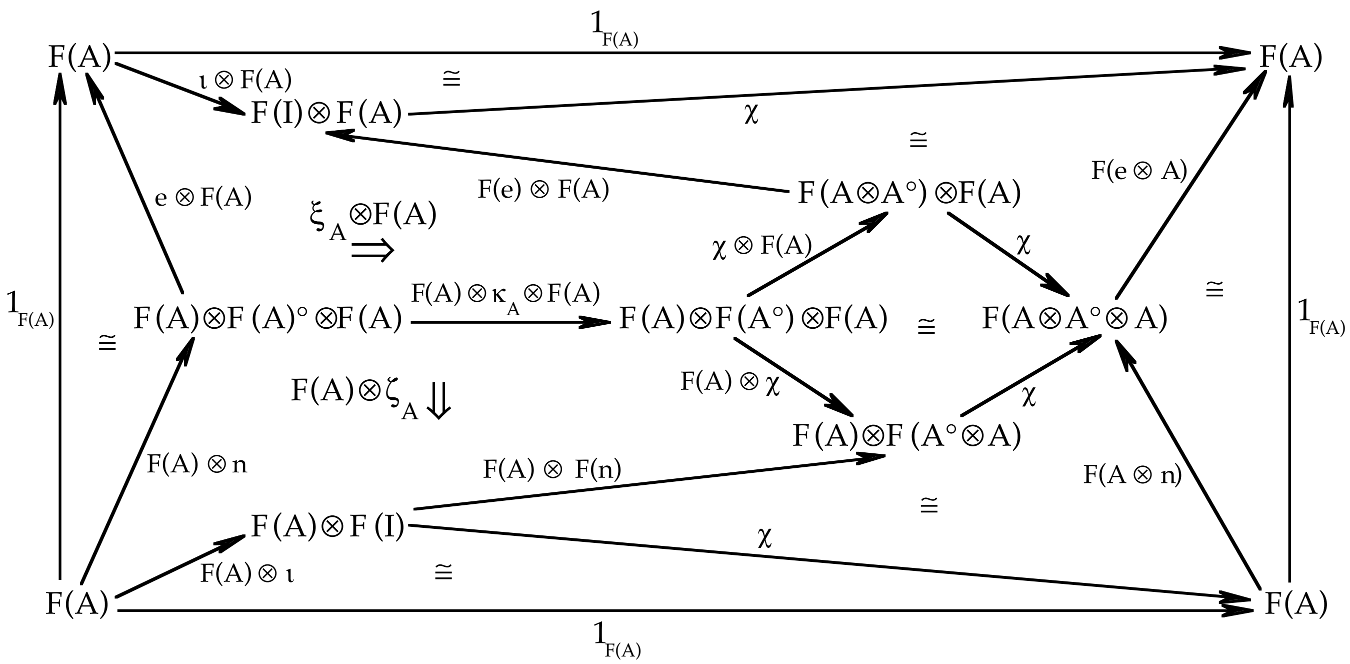
$$\kappa_A : (FA)^\circ \longrightarrow F(A^\circ)$$

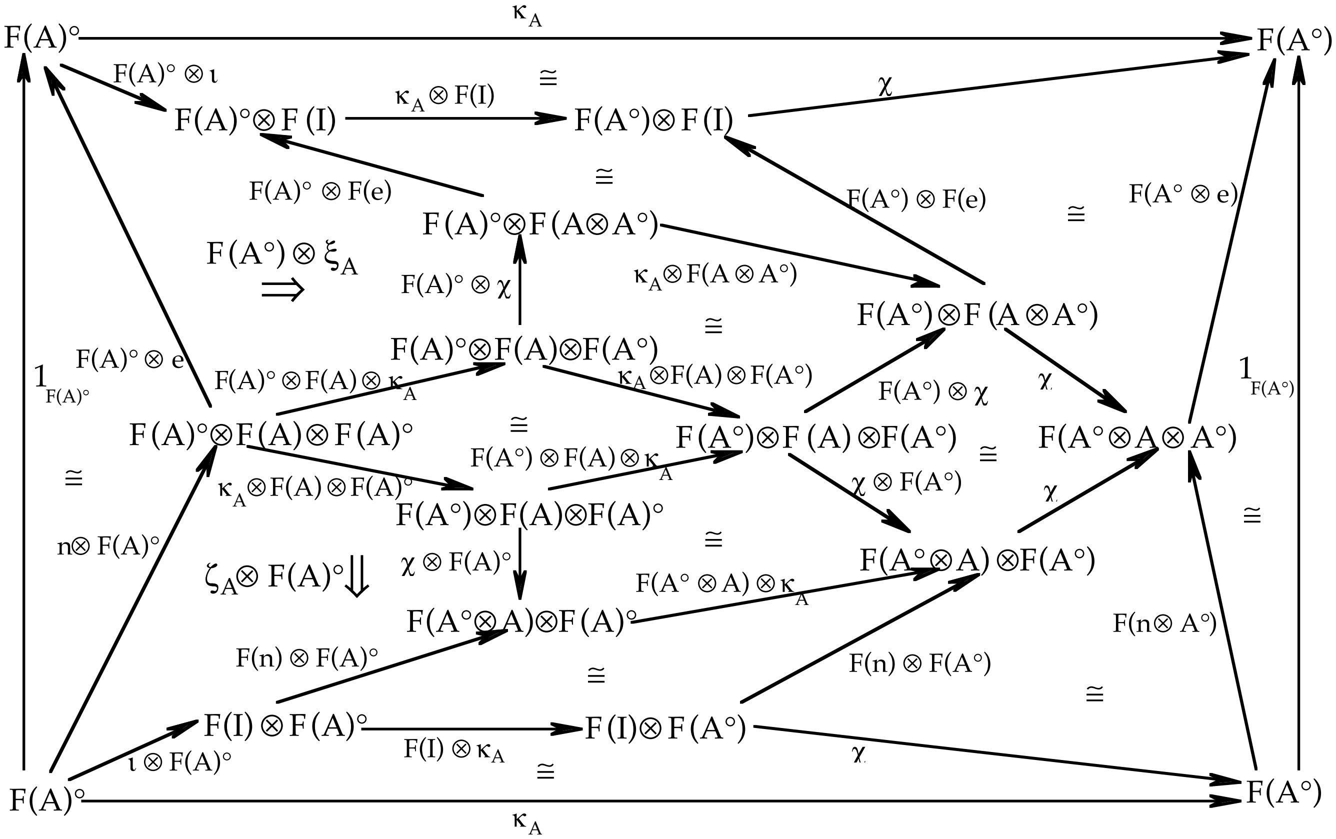
and modifications





such that the following two pasting composites are equal to the identities of $1_{F(A)}$ and κ_A, respectively.





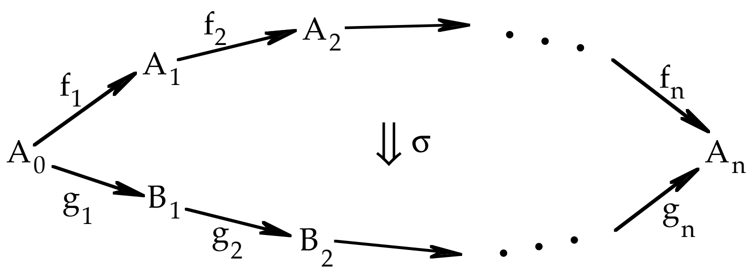
A lax functor $F : \mathcal{M} \longrightarrow \mathcal{N}$ will be said to be *special* when

the identity constraint $1_{FA} \longrightarrow F(1_A)$ is invertible for all objects A of $\mathcal{M}$; and

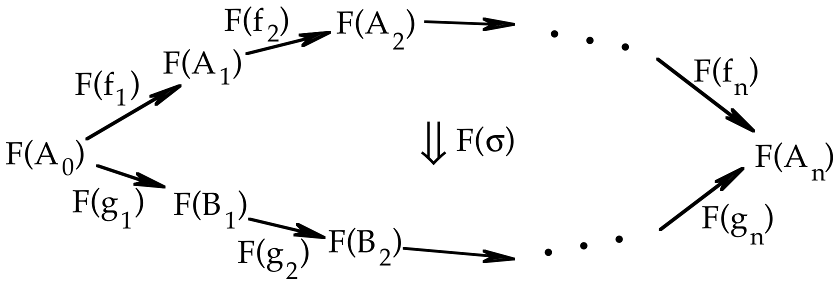
if $f : A \longrightarrow B$ is a map then the composition constraint $F(g) \circ F(f) \longrightarrow F(g \circ f)$ is invertible.

The first of these says that F is *normal* while the second implies that F is a pseudofunctor on the subbicategory of $\mathcal{M}$ consisting of the maps.

If one is careful, it is possible to use pasting diagrams involving a special lax functor $F : \mathcal{M} \longrightarrow \mathcal{N}$. Given a 2-cell



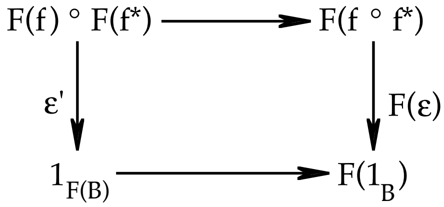
for which $g_1, g_2 \circ g_1, \ldots, g_{n-1} \circ g_{n-2} \circ \ldots \circ g_1$ are all maps, by abuse of language, we write

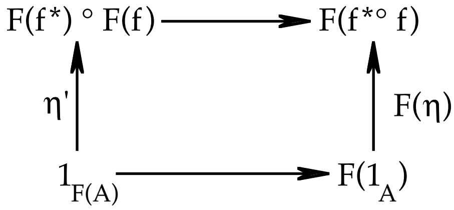


for the 2-cell $F(f_n)\circ \ldots \circ F(f_2)\circ F(f_1) \Rightarrow F(g_n)\circ \ldots \circ F(g_2)\circ F(g_1)$ obtained by conjugating the actual $F(\sigma) : F(f_n\circ \ldots \circ f_2 \circ f_1) \Rightarrow F(g_n\circ \ldots \circ g_2 \circ g_1)$ by the composition constraints of F.

Proposition 3.1 *Special lax functors take maps to maps. More precisely, if* $F : \mathcal{M} \longrightarrow \mathcal{N}$ *is a special lax functor and* $f : A \longrightarrow B$ *is a map in* $\mathcal{M}$ *then* $F(f) \dashv F(f^*)$ *in* $\mathcal{N}$.

Proof Let $\varepsilon : f \circ f^* \Rightarrow 1_B$ and $\eta : 1_A \Rightarrow f^* \circ f$ be the counit and unit for f. Define ε' and η' such that the following diagrams commute.



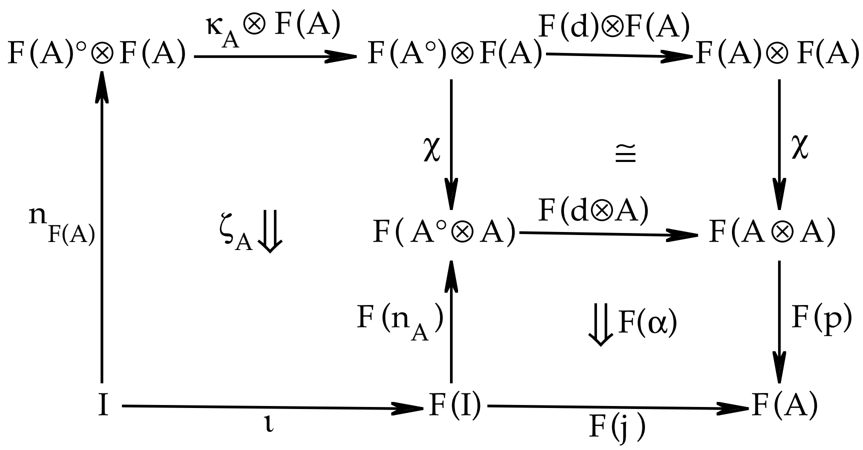


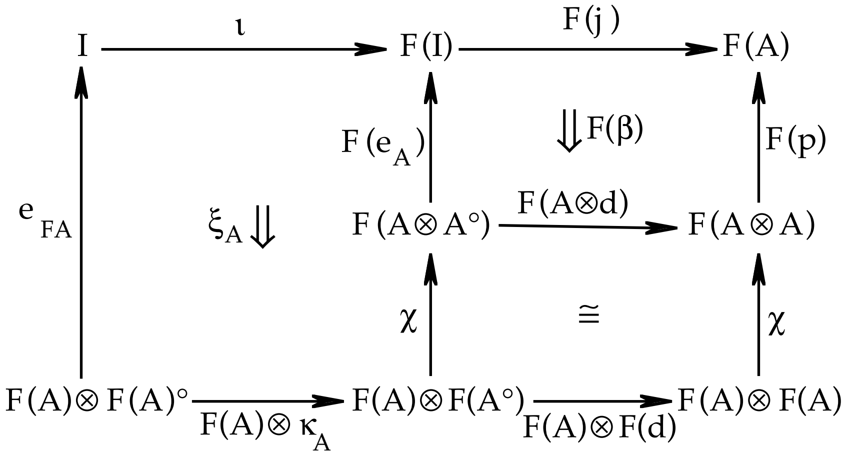
(The bottom and top-right arrows are invertible.) It is easily checked that ε' and η' are a counit and unit for $F(f) \dashv F(f^*)$. **Q.E.D.**

Theorem 3.2 *Suppose* $F : \mathcal{M} \longrightarrow \mathcal{N}$ *is a right autonomous monoidal special lax functor. If* A *is a left autonomous pseudomonoid in* $\mathcal{M}$ *then* $F(A)$ *is a left autonomous pseudomonoid in* $\mathcal{N}$ *with left dualization given by the composite*

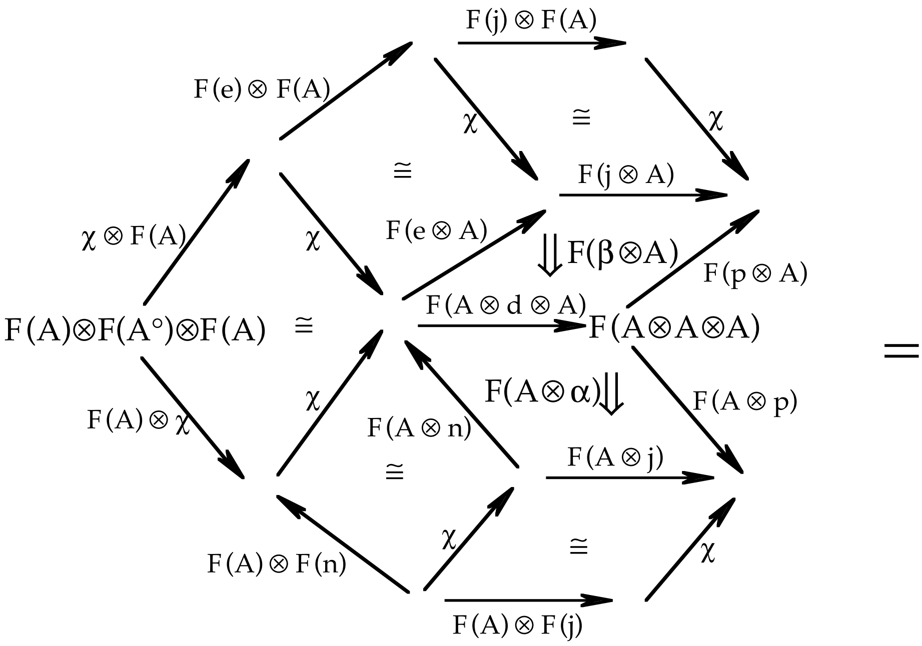
$$F(A)^\circ \xrightarrow{\kappa_A} F(A^\circ) \xrightarrow{F(d)} F(A)$$

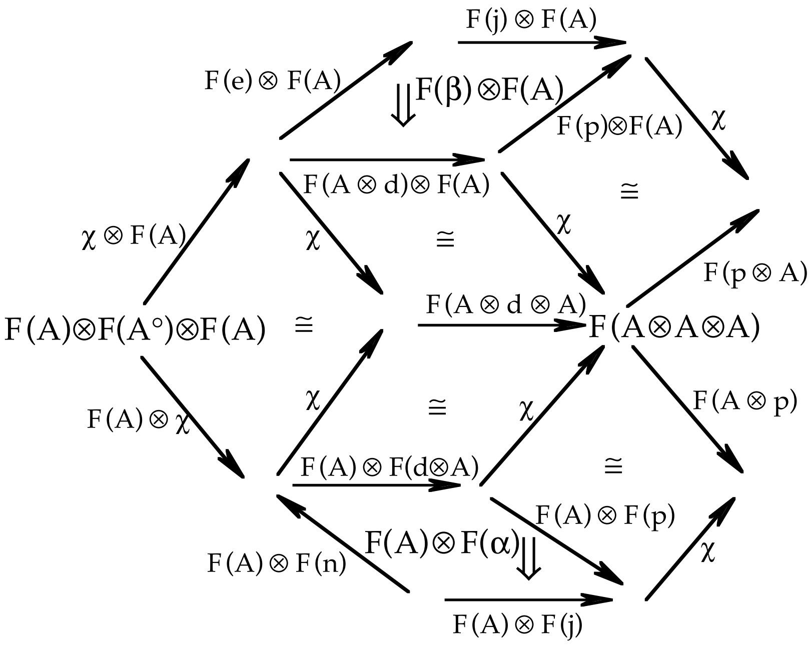
equipped with the following two 2-cells.



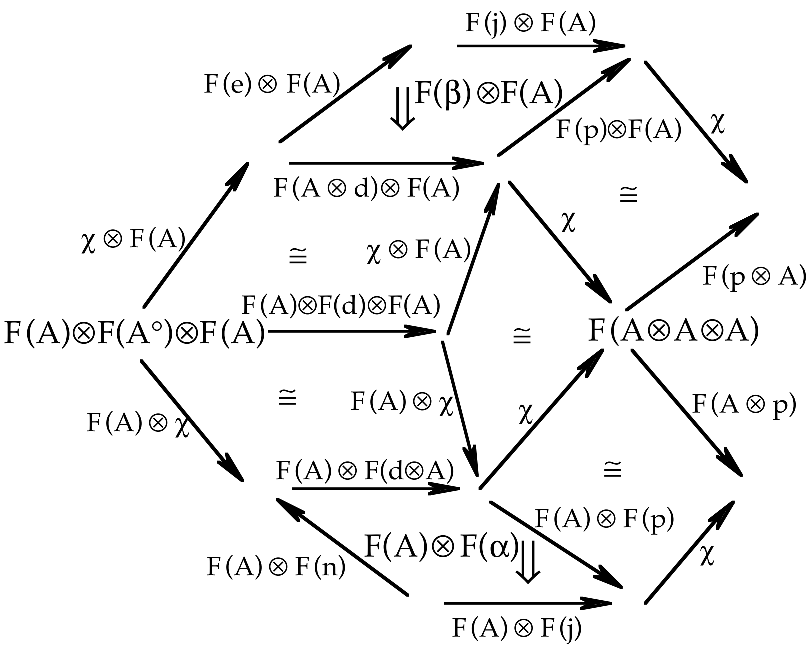


Proof The proof requires some rather large diagrams which make use of the two conditions on α and β and the two conditions on ξ_A and ζ_A. The crux of the argument for the proof of the first condition for a left dualization structure is supplied by the following equations in which most of the objects have been omitted to save space.





$$=$$



Q.E.D.

4. Formal representation theory

This section provides a conceptual setting for examining categories of representations of objects of $\mathcal{M}$. We are interested in "finite" representations.

Since the monoidal category $\mathcal{V} = \mathcal{M}(\mathrm{I}, \mathrm{I})$ is braided, if $u : \mathrm{I} \longrightarrow \mathrm{I}$ is a map then u^* is a map (in fact, $u^{**} \cong u$). We need to assume that, for all objects A of $\mathcal{M}$,

$$\textit{if } r \textit{ and } s : \mathrm{I} \longrightarrow \mathrm{A} \textit{ are maps then so is } r^* \circ s.$$

Write $\mathcal{V}_{\mathrm{fin}}$ for the monoidal full subcategory of $\mathcal{V} = \mathcal{M}(\mathrm{I}, \mathrm{I})$ consisting of the maps $u : \mathrm{I} \longrightarrow \mathrm{I}$; it is autonomous and braided. We shall suppose further that $\mathcal{V}$ is cocomplete as a monoidal category so we do indeed have the monoidal bicategory $\mathcal{V}$–**Mod**.

We shall describe three monoidal special lax functors

$$\mathrm{Rep} \ : \ \mathcal{M} \longrightarrow \mathcal{V}\text{–}\mathbf{Mod} \,,$$

$$\mathrm{Rep}_\omega : \mathrm{Map}(\mathcal{M};\mathrm{I}) \longrightarrow \mathrm{Map}(\mathcal{V}\text{–}\mathbf{Mod}\,;\mathcal{V}_{\mathrm{fin}}) \quad \text{and}$$
$$\mathrm{Rep}' : \mathrm{Map}'(\mathcal{M};\mathrm{I}) \longrightarrow \mathrm{Map}'(\mathcal{V}\text{–}\mathbf{Mod}\,;\mathcal{V}_{\mathrm{fin}}).$$

For an object A of $\mathcal{M}$, we have a $\mathcal{V}_{\mathrm{fin}}$-category Rep(A); the objects are the maps $r : I \longrightarrow A$, while, for such objects r and r', we put $\mathrm{Rep}(A)(r, r') = r^* \circ r' \in \mathcal{V}_{\mathrm{fin}}$. The unit of the adjunction $r \dashv r^*$ provides the identity $1_I \longrightarrow \mathrm{Rep}(A)(r, r)$ of r while the counit of $r' \dashv r'^*$ induces composition

$$\mathrm{Rep}(A)(r', r'') \circ \mathrm{Rep}(A)(r, r') \longrightarrow \mathrm{Rep}(A)(r, r'').$$

In particular, $\mathrm{Rep}(I) = \mathcal{V}_{\mathrm{fin}}$.

Objects $\varepsilon_A : A \longrightarrow I$ of $\mathrm{Map}(\mathcal{M};I)$ will be denoted by A, suppressing the augmentation ε_A. The value of Rep_ω at such an A is denoted by

$$\omega_A : \mathrm{Rep}(A) \longrightarrow \mathcal{V}_{\mathrm{fin}}\,.$$

Now ω_A is in fact a $\mathcal{V}$-functor, not merely a left adjoint $\mathcal{V}$-module; it is defined on objects by composition with ε_A, and its effect

$$\mathrm{Rep}(A)(r, r') \longrightarrow \mathcal{V}_{\mathrm{fin}}(\varepsilon_A \circ r, \varepsilon_A \circ r')$$

on homs is induced by the unit of $\varepsilon_A \dashv \varepsilon_A{}^*$. Of course, Rep' agrees with Rep_ω on objects.

Next we define Rep on a morphism $u : A \longrightarrow B$ of $\mathcal{M}$ to be the $\mathcal{V}$-module $\mathrm{Rep}(u) : \mathrm{Rep}(A) \longrightarrow \mathrm{Rep}(B)$ defined by

$$\mathrm{Rep}(u)(s, r) = s^* \circ u \circ r\,.$$

For a morphism $(u, \sigma) : A \longrightarrow B$ of $\mathrm{Map}(\mathcal{M};I)$, we define $\mathrm{Rep}_\omega(u, \sigma)$ to be the $\mathcal{V}$-module Rep(u) together with the $\mathcal{V}$-module morphism $\omega_B \circ \mathrm{Rep}(u, \sigma) \Rightarrow \omega_A$ whose component at (x, r) is the composite

$$\int^{s \in \mathrm{Rep}(B)} x^* \circ \varepsilon_B \circ s \circ s^* \circ u \circ r \xrightarrow{\int^s x^* \circ \varepsilon_B \circ \text{counit} \circ u \circ r} x^* \circ \varepsilon_B \circ u \circ r \xrightarrow{x^* \circ \sigma \circ r} x^* \circ \varepsilon_A \circ r\,.$$

Of course, Rep' agrees with Rep_ω on morphisms.

The formula $s^* \circ u \circ r$ is appropriately functorial in the variable u so that Rep is easily defined on 2-cells. This lifts in the obvious way to 2-cells in both $\mathrm{Map}(\mathcal{M};I)$ and $\mathrm{Map}'(\mathcal{M};I)$, which, in the last case, sees the image 2-cell landing in $\mathrm{Map}'(\mathcal{V}\text{–}\mathbf{Mod}\,;\mathcal{V}_{\mathrm{fin}})$. So we have defined all three lax functors on objects, morphisms and 2-cells.

Now we need to see how Rep relates to horizontal composition. Take $u : A \longrightarrow B$ and $v : B \longrightarrow C$ in $\mathcal{M}$. We define the composition constraint

$$\mathrm{Rep}(v) \circ \mathrm{Rep}(u) \longrightarrow \mathrm{Rep}(v \circ u)$$

to have component at (t, r) given by the 2-cell

$$\int^{s \in \mathrm{Rep}(B)} t^* \circ v \circ s \circ s^* \circ u \circ r \xrightarrow{\int^s t^* \circ v \circ \text{counit} \circ u \circ r} t^* \circ v \circ u \circ r.$$

Clearly Rep is normal since $\mathrm{Rep}(1_A, 1_{\varepsilon_A})(s, r) = s^* \circ r = \mathrm{Rep}(A)(s, r)$ which is the component of the identity $\mathcal{V}$-module of Rep(A) at (s, r). The coherence conditions for a lax functor are easily checked. It is important to notice that the $\mathcal{V}$-module $\mathrm{Rep}(u) : \mathrm{Rep}(A) \longrightarrow \mathrm{Rep}(B)$ is actually representable by a $\mathcal{V}$-functor when u is a map. Similarly, the $\mathcal{V}$-module $\mathrm{Rep}(u)$ is the right adjoint of a $\mathcal{V}$-module represented by a $\mathcal{V}$-functor when u is the right adjoint w^* of a map $w : B \longrightarrow A$. It follows that the composition constraint

$$\mathrm{Rep}(v) \circ \mathrm{Rep}(u) \longrightarrow \mathrm{Rep}(v \circ u)$$

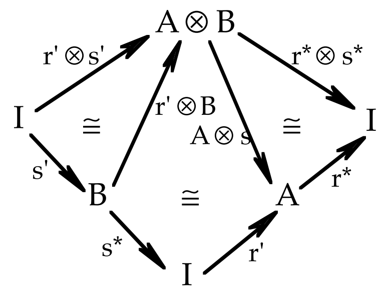
is invertible if either u is a map or v is a right adjoint. In particular, it follows that Rep is indeed a special lax functor.

It is easy to see that, if $(u, \sigma) : A \longrightarrow B$ and $(v, \tau) : B \longrightarrow C$ are morphisms of $\mathrm{Map}(\mathcal{M};I)$, then the composition constraint $\mathrm{Rep}(v) \circ \mathrm{Rep}(u) \longrightarrow \mathrm{Rep}(v \circ u)$ is in fact a 2-cell

$$\mathrm{Rep}_\omega(v, \tau) \circ \mathrm{Rep}_\omega(u, \sigma) \longrightarrow \mathrm{Rep}_\omega(v \circ u, (\sigma \circ u)\tau)$$

in $\mathrm{Map}'(\mathcal{V}\text{–}\mathbf{Mod}; \mathcal{V}_{\mathrm{fin}})$. It therefore follows that both Rep_ω and Rep' are special lax functors.

Next we describe the monoidal structure on Rep. In fact, there is a fully faithful $\mathcal{V}$-functor $\chi_{A,B} : \mathrm{Rep}(A) \otimes \mathrm{Rep}(B) \longrightarrow \mathrm{Rep}(A \otimes B)$. On objects we define $\chi_{A,B}(r, s) = r \otimes s : I \longrightarrow A \otimes B$. To define $\chi_{A,B}$ on homs, we make use of the canonical isomorphisms



to obtain the required invertible morphism

$$(\mathrm{Rep}(A) \otimes \mathrm{Rep}(B))((r, s), (r', s')) \xrightarrow{\ \cong\ } \mathrm{Rep}(A \otimes B)(r \otimes s, r' \otimes s').$$

There is also the fully faithful $\mathcal{V}$-functor $\omega_I : I \longrightarrow \mathcal{V}_{\mathrm{fin}}$ which picks out the unit object 1_I of $\mathcal{V}_{\mathrm{fin}}$; this gives a $\mathcal{V}$-functor $\iota : I \longrightarrow \mathrm{Rep}(I)$. The coherence conditions can be verified showing Rep to be monoidal. Furthermore, when A and B are in $\mathrm{Map}(\mathcal{M};I)$, we see that the $\mathcal{V}$-functor $\chi_{A,B}$ commutes up to isomorphism with the augmentations into $\mathcal{V}_{\mathrm{fin}}$. Certainly $\iota : I \longrightarrow \mathrm{Rep}(I)$ commutes with the augmentations. So Rep_ω and Rep' become monoidal.

We shall now see that our three monoidal lax functors are right autonomous. There is a canonical equivalence of $\mathcal{V}$-categories

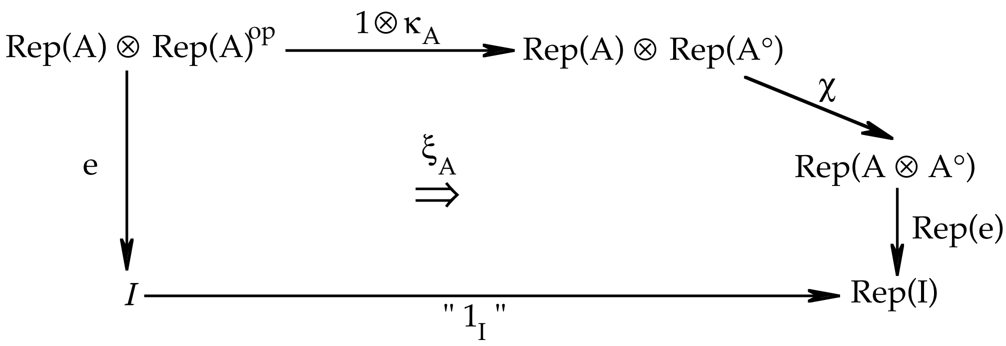
$$\kappa_A : \mathrm{Rep}(A)^{\mathrm{op}} \longrightarrow \mathrm{Rep}(A^\circ).$$

On objects it takes $r : I \longrightarrow A$ to $r^{*\circ} : I \longrightarrow A^\circ$ and on homs we have

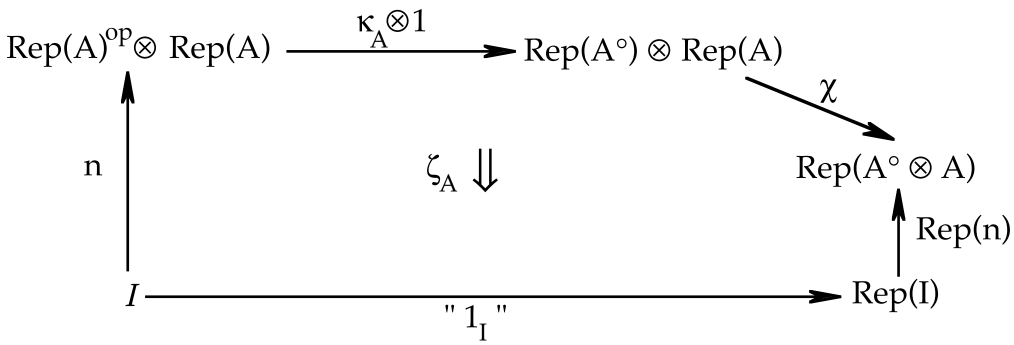
$$\mathrm{Rep}(A)^{\mathrm{op}}(r, s) = \mathrm{Rep}(A)(s, r) = s^{*\circ} r \cong (s^{*\circ} r)^\circ \cong r^{\circ\circ} s^{*\circ} \cong \mathrm{Rep}(A^\circ)(r^{*\circ}, s^{*\circ}),$$

using $x^\circ \cong x$ for $x : I \longrightarrow I$ and using $r^{*\circ} \dashv r^\circ$.

There is a canonical invertible $\mathcal{V}$-module morphism ξ_A :



since the components of both legs of the diagram at (x, r, s), for objects x of $\mathrm{Rep}(I) = \mathcal{V}_{\mathrm{fin}}$ and (r, s) of $\mathrm{Rep}(A) \otimes \mathrm{Rep}(A)^{\mathrm{op}}$, are isomorphic to $x^* \circ s^* \circ r$ (using the fact that $e \circ (A \otimes s^{*\circ}) \cong s^*$). There is a canonical $\mathcal{V}$-module morphism ζ_A :



;

the component of the domain leg of the diagram at an object t of $\mathrm{Rep}(A^\circ \otimes A)$ is the coend over all objects (r, s) of $\mathrm{Rep}(A)^{\mathrm{op}} \otimes \mathrm{Rep}(A)$ of the expression $t^* \circ (r^{*\circ} \otimes s) \circ s^* \circ r$, while the component of the codomain leg at t is $t^* \circ n$; so $\zeta_A(t)$ is induced by the composite with t^* of the mate $(r^{*\circ} \otimes s) \circ s^* \circ r \Rightarrow n$ of the isomorphism $s^* \circ r \cong (r^\circ \otimes s^*) \circ n$ under the adjunction $r^{*\circ} \otimes s \dashv r^\circ \otimes s^*$. As one might expect from the canonical nature of ξ and ζ, it can be seen that the two conditions required to make Rep right autonomous do indeed hold. Furthermore, if A is in $\mathrm{Map}(\mathcal{M}; I)$ then κ_A commutes with the augmentations, while ξ_A and ζ_A are 2-cells in $\mathrm{Map}'(\mathcal{V}\text{–}\mathbf{Mod}; \mathcal{V}_{\mathrm{fin}})$. So Rep_ω and Rep' become right autonomous.

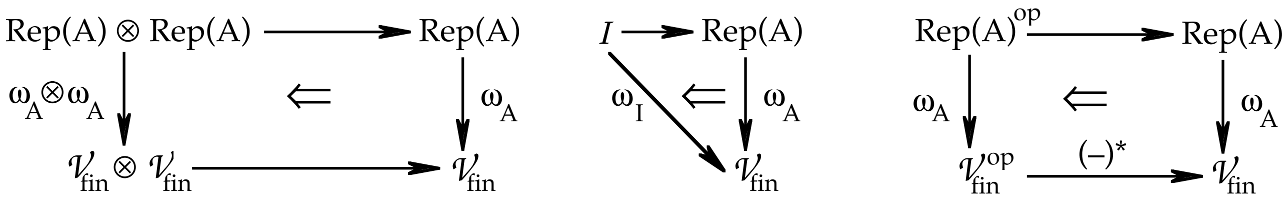
As corollaries of Theorem 3.2 we obtain a variety of results, some of which we list in:

Theorem 4.1 (i) *If* A *is a left autonomous pseudomonoid in* $\mathcal{M}$ *then* Rep(A) *is a left autonomous pseudomonoid in* $\mathcal{V}$–**Mod**.

(ii) *If* A *is a left autonomous map pseudomonoid in* $\mathcal{M}$ *then* Rep(A) *is a left autonomous monoidal* $\mathcal{V}$*-category.*

(iii) *If* $\varepsilon_A : A \longrightarrow I$ *is a left autonomous pseudomonoid in* $\mathrm{Map}(\mathcal{M}; I)$ *then the* $\mathcal{V}$*-functor* $\omega_A : \mathrm{Rep}(A) \longrightarrow \mathcal{V}_{\mathrm{fin}}$ *is a left autonomous pseudomonoid in* $\mathrm{Map}(\mathcal{V}\text{–}\mathbf{Mod}; \mathcal{V}_{\mathrm{fin}})$.

(iv) *If in* (iii) *one of the structure 2-cells* $\chi : \varepsilon_A \circ p \Rightarrow \varepsilon_A \otimes \varepsilon_A$, $\iota : \varepsilon_A \circ j \Rightarrow 1_I$ *or* $\kappa : \varepsilon_A \circ d \Rightarrow (\varepsilon_A)^{*\circ}$ *is invertible then the corresponding* $\mathcal{V}$*-module morphism*



is also invertible.

(v) *If* A *is as in* (ii) *and* $\varepsilon_A : A \longrightarrow I$ *is a map which is a pseudomorphism of pseudomonoids then the* $\mathcal{V}$*-functor* $\omega_A : \mathrm{Rep}(A) \longrightarrow \mathcal{V}_{fin}$ *is strong monoidal.*

5. Comonoids and pro-Hopf algebras

Let $\mathcal{V}$ be a braided monoidal category. The purpose of this section is to describe and analyse a monoidal bicategory **Comod**$(\mathcal{V})$ to be taken as $\mathcal{M}$ in Section 4 so that the formal representation theory applies. We identify the left autonomous pseudomonoids in **Comod**$(\mathcal{V})$; they generalize both the "braided groups" of Majid and the quasi-Hopf algebras of Drinfeld (see [Maj]).

In order to construct the right autonomous monoidal bicategory **Comod**$(\mathcal{V})$ we shall assume the condition:

each of the functors $X \otimes - : \mathcal{V} \longrightarrow \mathcal{V}$ *preserves equalizers.*

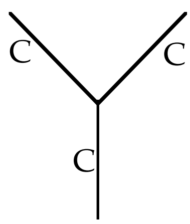
In order for **Comod**$(\mathcal{V})$ to satisfy the condition we needed on $\mathcal{M}$ in Section 5, we assume that $\mathcal{V}$ satisfies the condition:

every regular subobject of an object with a right dual has a right dual.

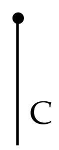
(A regular subobject is one that occurs as an equalizer.) Of course, because of the braiding, right duals in $\mathcal{V}$ are automatically left duals.

The quick description of **Comod**$(\mathcal{V})$ is that it is the monoidal full sub-bicategory of $(\mathcal{V}^{op}$–**Mod**$)^{op}$ consisting of the one-object $\mathcal{V}^{op}$-categories. There is the technical problem (which does not arise in the one-object case) that defining the horizontal composition in $(\mathcal{V}^{op}$–**Mod**$)^{op}$ requires stronger conditions of completeness on $\mathcal{V}$ than we have; and, in any case, to make calculations we will need to make the definition more explicit.

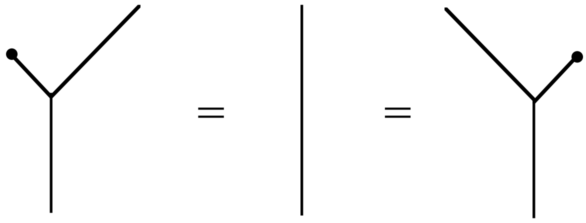
The objects of **Comod**$(\mathcal{V})$ are comonoids C in $\mathcal{V}$; we depict the comultiplication $\delta : C \longrightarrow C \otimes C$ as a string diagram of the form

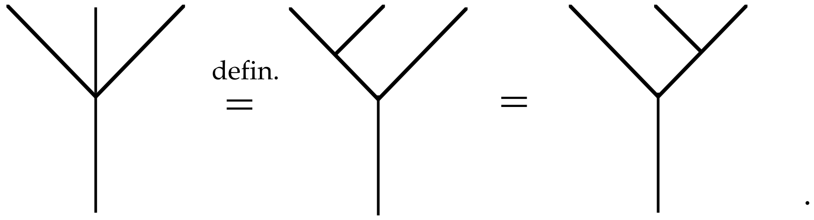


and the counit $\varepsilon : C \longrightarrow I$ as a string diagram

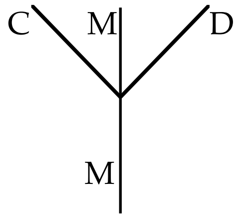


so that the comonoid axioms become the equations

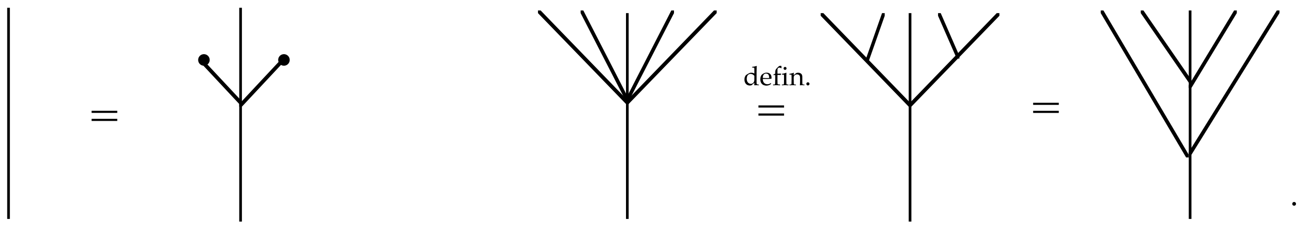




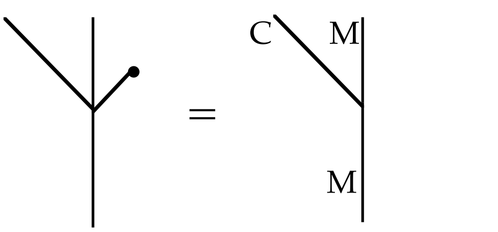
The hom-category **Comod**$(\mathcal{V})(C, D)$ is the category of Eilenberg-Moore coalgebras for the comonad $C \otimes - \otimes D$ on the category $\mathcal{V}$. This implies that the morphisms $M : C \longrightarrow\!\!\!\!\!\!\!\!+\;\;\; D$ in **Comod**$(\mathcal{V})$ are comodules from C to D; that is, left C–, right D-comodules. So M is an object of $\mathcal{V}$ together with a coaction $\delta : M \longrightarrow C \otimes M \otimes D$ depicted by



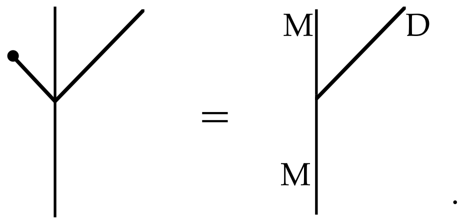
satisfying the equations



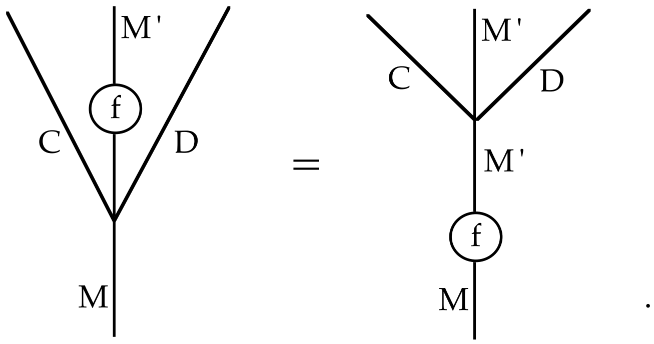
It is sometimes useful to deal with the left and right actions $\delta_{\ell} : M \longrightarrow C \otimes M$ and $\delta_r : M \longrightarrow M \otimes D$ which are depicted by



and



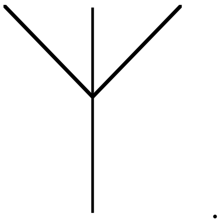
The 2-cells $f : M \Rightarrow M' : C \longrightarrow\!\!\!\!\!\!\!\!+\;\;\; D$ in **Comod**$(\mathcal{V})$ are morphisms $f : M \longrightarrow M'$ in $\mathcal{V}$ satisfying the equations



Composition of comodules $M : C \mathrel{-\!\!\!+\!\!\!\rightarrow} D$ and $N : D \mathrel{-\!\!\!+\!\!\!\rightarrow} E$ is given by the equalizer

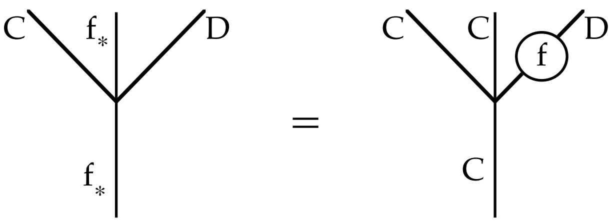
$$N \circ M \;=\; M \underset{D}{\otimes} N \longrightarrow M \otimes N \overset{\delta_r \otimes 1}{\underset{1 \otimes \delta_\ell}{\rightrightarrows}} M \otimes D \otimes N \,.$$

The identity comodule $1_C : C \mathrel{-\!\!\!+\!\!\!\rightarrow} C$ is C with coaction

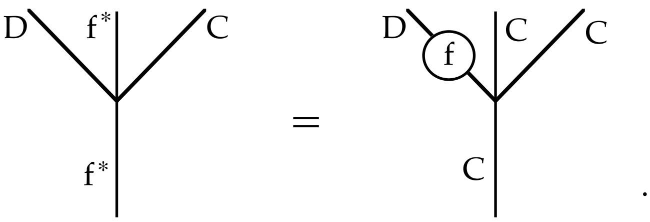


The remaining details describing $\mathbf{Comod}(\mathcal{V})$ as a bicategory should now be clear.

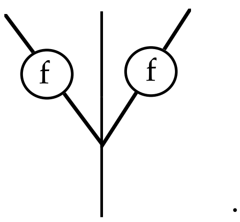
Each comonoid morphism $f : C \longrightarrow D$ determines a comodule $f_* : C \mathrel{-\!\!\!+\!\!\!\rightarrow} D$ defined to be C together with the coaction



and a comodule $f^* : D \mathrel{-\!\!\!+\!\!\!\rightarrow} C$ defined to be C together with the coaction



Notice that we have $\gamma_f : f_* \circ f^* \Rightarrow 1_D$ which is defined to be $f : C \longrightarrow D$ since $f_* \circ f^* = f^* \underset{C}{\otimes} f_* = C$ with coaction



Also, $f_* \underset{D}{\otimes} f^* = f^* \circ f_*$ is the equalizer

$$f_* \underset{D}{\otimes} f^* \longrightarrow C \otimes C \underset{(C\otimes f\otimes C)\circ(C\otimes\delta)}{\overset{(C\otimes f\otimes C)\circ(\delta\otimes C)}{\rightrightarrows}} C \otimes D \otimes C \quad ;$$

and, since

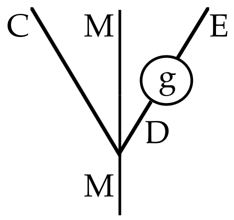
$$C \longrightarrow C \otimes C \underset{C\otimes\delta}{\overset{\delta\otimes C}{\rightrightarrows}} C \otimes C \otimes C$$

is an (absolute) equalizer, we have a unique morphism $C \longrightarrow f_* \underset{D}{\otimes} f^*$ commuting with the morphisms into $C \otimes C$; this gives us $\omega_f : 1_C \Rightarrow f^* \circ f_*$. Indeed, γ_f, ω_f are the counit and unit for an adjunction $f_* \dashv f^*$ in the bicategory **Comod**($\mathcal{V}$).

For any comodule $M : C \longrightarrow\!\!\!\!\!\!\!+\;\; D$ and any comonoid morphism $g : D \longrightarrow E$, we have

$$g_* \circ M = M \underset{D}{\otimes} g_* = M : C \longrightarrow\!\!\!\!\!\!\!+\;\; E$$

with coaction

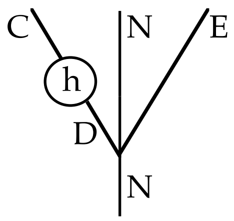


.

Similarly, for $h : D \longrightarrow C$ and $N : D \longrightarrow\!\!\!\!\!\!\!+\;\; E$, we have

$$N \circ h^* = h^* \underset{D}{\otimes} N = N : C \longrightarrow\!\!\!\!\!\!\!+\;\; E$$

with coaction



.

Proposition 5.1 *A comodule* $M : C \longrightarrow\!\!\!\!\!\!\!+\;\; D$ *has a right adjoint in* **Comod**($\mathcal{V}$) *if and only if its composite* $\varepsilon_* \circ M$ *with* $\varepsilon_* : D \longrightarrow\!\!\!\!\!\!\!+\;\; I$ *has a right adjoint. If* H *and* $K : I \longrightarrow\!\!\!\!\!\!\!+\;\; D$ *have right adjoints then so does* $H^* \circ K$.

Proof A comodule $M : C \longrightarrow\!\!\!\!\!\!\!+\;\; D$ has a right adjoint if and only if, for all comonoids E, the functor **Comod**($\mathcal{V}$)(E, M) : **Comod**($\mathcal{V}$)$(E, C) \longrightarrow$ **Comod**($\mathcal{V}$)(E, D) has a right adjoint. Consider the following commutative triangle of functors.

$$\begin{array}{ccc} \mathbf{Comod}(\mathcal{V})(E,C) & \xrightarrow{\mathbf{Comod}(\mathcal{V})(E,M)} & \mathbf{Comod}(\mathcal{V})(E,D) \\ & \searrow \qquad \swarrow {\scriptstyle \mathbf{Comod}(\mathcal{V})(E,\varepsilon_*)} & \\ & \mathbf{Comod}(\mathcal{V})(E,I) & \end{array}$$

The right side is comonadic via the comonad induced by tensoring with the comonoid D. So, by Dubuc's Adjoint Triangle Theorem [Dbc], the top side has a right adjoint if and only if the left side does.

The unit $1_D \longrightarrow \varepsilon^* \circ \varepsilon_*$ of the adjunction $\varepsilon_* \dashv \varepsilon^*$ is an equalizer so

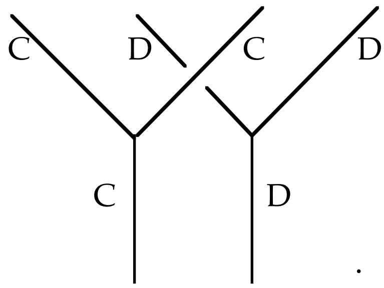
$$H^* \circ K \longrightarrow H^* \circ \varepsilon^* \circ \varepsilon_* \circ K \cong (\varepsilon_* \circ H)^* \circ (\varepsilon_* \circ K)$$

is a regular monomorphism. Now $\varepsilon_* \circ H$ and $\varepsilon_* \circ K$ have right adjoints (duals) since H, K and ε_* do. But, since $\varepsilon_* \circ H$ and $\varepsilon_* \circ K$ are in the braided monoidal $\mathcal{V}$, they also have left duals. So $(\varepsilon_* \circ H)^* \circ (\varepsilon_* \circ K)$ has a right dual. By our assumption on $\mathcal{V}$, it follows that $H^* \circ K$ has a right dual. **Q.E.D.**

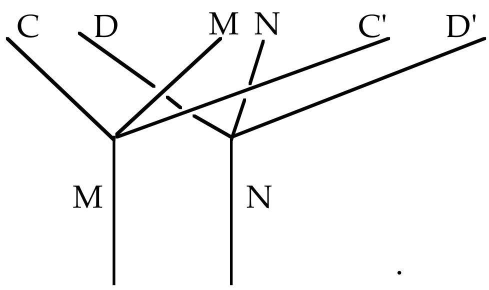
Suppose C, D are comonoids. Then $C \otimes D$ becomes a comonoid with coaction

$$C \otimes D \xrightarrow{\delta \otimes \delta} C \otimes C \otimes D \otimes D \xrightarrow{C \otimes c_{C,D} \otimes D} C \otimes D \otimes C \otimes D$$

where c is the braiding and, as justified by coherence theorems, we ignore associativity in $\mathcal{V}$; the string diagram for this coaction is

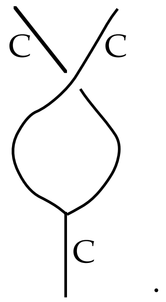


For comodules $M : C \longrightarrow\!\!\!\!\!\!\!+\;\; C'$ and $N : D \longrightarrow\!\!\!\!\!\!\!+\;\; D'$, we obtain a comodule $M \otimes N : C \otimes D \longrightarrow\!\!\!\!\!\!\!+\;\; C' \otimes D'$ where the coaction is given by

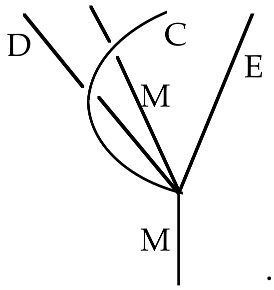


This extends to a pseudofunctor $\otimes : \mathbf{Comod}(\mathcal{V}) \times \mathbf{Comod}(\mathcal{V}) \longrightarrow \mathbf{Comod}(\mathcal{V})$. The remaining structure required to obtain $\mathbf{Comod}(\mathcal{V})$ as a monoidal bicategory should be obvious.

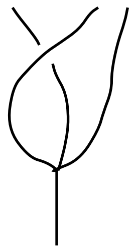
Write C° for C with the comultiplication



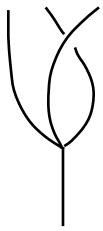
There is a pseudonatural equivalence between the category of comodules $M : C\otimes D \,-\!\!\!+\!\!\!\longrightarrow E$ and the category of comodules $\check{M} : D \,-\!\!\!+\!\!\!\longrightarrow C^{\circ}\otimes E$, where $M = \check{M}$ as objects, while the coaction of $\check{M}$ is given by



It follows that C° is a right bidual for C; the unit $n : I \,-\!\!\!+\!\!\!\longrightarrow C^{\circ}\otimes C$ is C with coaction



and the counit is $e : C\otimes C^{\circ} \,-\!\!\!+\!\!\!\longrightarrow I$ is C with coaction



This gives the structure of right autonomous monoidal bicategory to $\mathbf{Comod}(\mathcal{V})$.

Definition A *pro-Hopf comonoid* in the braided monoidal category $\mathcal{V}$ is a left autonomous pseudomonoid H in $\mathbf{Comod}(\mathcal{V})$. We denote the multiplication, unit and dualization comodules by $P : H\otimes H \,-\!\!\!+\!\!\!\longrightarrow H$, $J : I \,-\!\!\!+\!\!\!\longrightarrow H$ and $S : H^{\circ} \,-\!\!\!+\!\!\!\longrightarrow H$; indeed, we use the term *antipode* rather than "dualization" for S. A *quasi-Hopf comonoid in* $\mathcal{V}$ is a pro-Hopf comonoid for which the multiplication, unit and antipode comodules are of the form $P = p_*$, $J = j_*$ and $S = s_*$ for comonoid morphisms $p : H\otimes H \longrightarrow H$, $j : I \longrightarrow H$ and $s : H^{\circ} \longrightarrow H$, and the unit constraints λ and ρ are identities.

More explicitly, a pro-Hopf comonoid H is a comonoid together with a *multiplication comodule* $P : H \otimes H \longrightarrow\!\!\!\!\!\!\!+\;\; H$ and a *unit comodule* $J : I \longrightarrow\!\!\!\!\!\!\!+\;\; H$, which are associative and unital up to coherent isomorphisms ϕ, λ, ρ (as at the beginning of Section 1 or, more fully, in [DS; Section 3]), and an *antipode comodule* $S : H^\circ \longrightarrow\!\!\!\!\!\!\!+\;\; H$ such that the comodule

$$(H \otimes n) \underset{H\otimes H^\circ \otimes H}{\otimes} (H \otimes S \otimes H) \underset{H\otimes H\otimes H}{\otimes} (P \otimes H) \;:\; H \longrightarrow H \otimes H$$

is right adjoint to P. Alternatively to the last clause, we require comodule morphisms

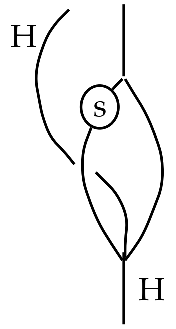
$$\alpha \;:\; n \underset{H^\circ\otimes H}{\otimes} (S \otimes H) \underset{H\otimes H}{\otimes} P \longrightarrow J \quad \text{and} \quad \beta \;:\; e \otimes J \longrightarrow (H \otimes S) \underset{H\otimes H}{\otimes} P$$

satisfying two conditions as given in Section 1.

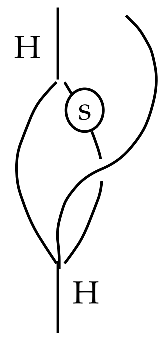
We can be even more explicit in the case of a quasi-Hopf comonoid H. We have comonoid morphisms $p : H \otimes H \longrightarrow H$, $j : I \longrightarrow H$ and $s : H^\circ \longrightarrow H$ together with comodule morphisms

$$\alpha \;:\; n \underset{H^\circ\otimes H}{\otimes} (p \circ (s \otimes 1_H))_* \longrightarrow j_* \quad \text{and} \quad \beta \;:\; (p \circ (1_H \otimes s))^* \underset{H\otimes H^\circ}{\otimes} e \longrightarrow j^*$$

satisfying two conditions. From our earlier observations about composing a comodule with an f_* or a g^*, we see that $n \underset{H^\circ\otimes H}{\otimes} (p \circ (s \otimes 1_H))_*$ is just $n = H : I \longrightarrow\!\!\!\!\!\!\!+\;\; H$ with coaction

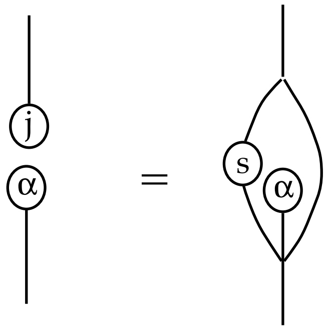


and $(p \circ (1_H \otimes s))^* \underset{H\otimes H^\circ}{\otimes} e$ is just $e = H : H \longrightarrow\!\!\!\!\!\!\!+\;\; I$ with coaction

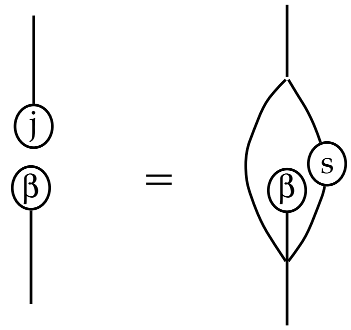


.

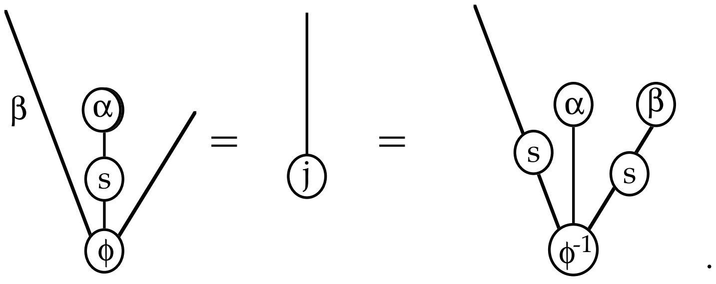
Therefore the condition that α and β are comodule morphisms is that they are morphisms α and $\beta : H \longrightarrow I$ in $\mathcal{V}$ satisfying



and



The other two conditions on α and β are:



It is now easy to see that a *quasi-Hopf algebra* is precisely a quasi-Hopf comonoid in $\mathcal{V} = \mathbf{Vect}^{\mathrm{op}}$, the dual of the category of vector spaces and linear functions. The reader can refer to [Maj; Section 2.4, page 62-63] for the definition of quasitriangular quasi-Hopf algebra; to obtain a definition of quasi-Hopf merely drop the "quasitriangular" element and the two axioms involving it; also he omits to explicitly say that the antipode should be an algebra antimorphism.

To digress a little, we should also point out that a quasi-Hopf algebra is a one-object example of a Hopf **Vect**-algebroid. Moreover, for any Hopf $\mathcal{V}$-algebroid $\mathcal{H}$, the convolution structure (see [DS; Proposition 19, page 143]) on the $\mathcal{V}$-functor $\mathcal{V}$-category $[\mathcal{H}, \mathcal{V}_{\mathrm{fin}}]$ is right autonomous: the tensor product is pointwise and dualization $M \longmapsto M^*$ is given by $M^*C = (MSC)^*$.

Return now to $\mathcal{M} = \mathbf{Comod}(\mathcal{V})$. We wish to apply the formal representation theory of Section 4 to this $\mathcal{M}$.

We need to identify the $\mathcal{V}_{\mathrm{fin}}$-category Rep(C) for any comonoid C in $\mathcal{V}$. The objects are right C-comodules M with right duals in $\mathcal{V}$ (Proposition 5.1). The $\mathcal{V}_{\mathrm{fin}}$-valued hom object $\mathrm{Rep(C)(M, N)} = M^* \circ N = N \otimes_C M^*$ is determined up to isomorphism as an equalizer

$$\mathrm{Rep(C)(M,N)} \longrightarrow \mathcal{V}(M,N) \underset{\mathcal{V}(\delta, N\otimes C)\circ(-\otimes C)}{\overset{\mathcal{V}(M,\delta)}{\rightrightarrows}} \mathcal{V}(M, N\otimes C),$$

where we write $\mathcal{V}(X, Y)$ for the internal hom in $\mathcal{V}$; the equalizer is induced by the isomorphisms of the form $\mathcal{V}(X, Y) \cong Y \otimes X^*$ when X has a right dual. So we can think of Rep(C)(M, N) as the "object of right-C-comodule homomorphisms from M to N". (When $\mathcal{V}$ is the category of complex vector spaces, Rep(C) is the complex-linear category denoted by

$\mathrm{Comod_f(C)}$ in [JS3].) Notice that each comonoid C can be canonically regarded as an object of $\mathrm{Map}(\mathcal{M};\mathrm{I})$ by augmenting C with the counit map $\varepsilon : C \longrightarrow I$.

This puts us in a position to apply Theorem 4.1. We highlight only a couple of cases which have occurred already in the literature. If H is a quasi-Hopf comonoid in $\mathcal{V}$ then $p : H \otimes H \longrightarrow H$, $j : I \longrightarrow H$ and $s : H^{\circ} \longrightarrow H$ are comonoid morphisms and so commute with the counit map; so we are in the position of Theorem 4.1 (iv). It follows that the forgetful (or "fibre") $\mathcal{V}$-functor $\omega_H : \mathrm{Rep}(H) \longrightarrow \mathcal{V}_{\mathrm{fin}}$ is "multiplicative" [Maj] or "quasi-strong-monoidal", meaning that it preserves the unit and tensor product up to $\mathcal{V}$-natural isomorphisms but these isomorphisms are not required to satisfy the coherence conditions for a monoidal $\mathcal{V}$-functor.

The other case is the more special case of a Hopf algebra H. Here the multiplication is associative so that $\phi : p \circ (p \otimes 1) \Rightarrow p \circ (1 \otimes p)$ is an identity. It follows that Theorem 4.1 (v) applies, implying that $\omega_H : \mathrm{Rep}(H) \longrightarrow \mathcal{V}_{\mathrm{fin}}$ is a strong monoidal $\mathcal{V}$-functor.

References

[B] J. Bénabou, Introduction to bicategories, *Lecture Notes in Math.* **47** (Springer, Berlin, 1967) 1- 77.

[BD] J. Baez and J. Dolan, From finite sets to Feynman diagrams, *Mathematics Unlimited - 2001 and Beyond,* eds. Bjorn Engquist and Wilfried Schmid (Springer-Verlag, Berlin, 2001) 29-50.

[Dd] V.G. Drinfel'd, Quasi-Hopf algebras (Russian), *Algebra i Analiz* **1** #6 (1989) 114-148; translation in *Leningrad Math. J.* **1** #6 (1990) 1419-1457.

[Dy1] B.J. Day, On closed categories of functors, *Lecture Notes in Math.* **137** (Springer, Berlin, 1970) 1- 38.

[Dy2] B.J. Day, Note on compact closed categories. *J. Austral. Math. Soc. Ser. A* **24** (1977) 309-311.

[DS] B.J. Day and R. Street, Monoidal bicategories and Hopf algebroids, *Advances in Math.* **129** (1997) 99-157.

[Dbc] E. Dubuc, Adjoint triangles, *Lecture Notes in Math.* **61** (Springer, Berlin, 1968) 69-91.

[EM1] S. Eilenberg and S. Mac Lane, Natural isomorphisms in group theory, *Proc. Nat. Acad. Sci. U.S.A.* **28** (1942) 537-543.

[EM2] S. Eilenberg and S. Mac Lane, General theory of natural equivalences. *Trans. Amer. Math. Soc.* **58** (1945) 231-294.

[EK1] S. Eilenberg and G.M. Kelly, A generalization of the functorial calculus, *J. Algebra* **3** (1966) 366-375.

[EK2] S. Eilenberg and G.M. Kelly, Closed categories, *Proceedings of the Conference on Categorical Algebra at La Jolla* (Springer, 1966) 421-562.

[GPS] R. Gordon, A.J. Power and R. Street, Coherence for tricategories, *Memoirs of the American Math. Society* **117** #558 (1995) (ISBN 0-8218-0344-1).

[G1] J.W. Gray, *Formal Category Theory: Adjointness for 2-Categories,* Lecture Notes in Math. **391** (Springer, Berlin, 1974).

[G2] J.W. Gray, *Coherence for the tensor product of 2-categories, and braid groups,* Algebra, Topology, and Category Theory (a collection of papers in honour of Samuel Eilenberg), (Academic Press, New York 1976) 63-76.

[JS1] A. Joyal and R. Street, Braided tensor categories, *Advances in Math.* **102** (1993) 20-78.

[JS2] A. Joyal and R. Street, The geometry of tensor calculus I, *Advances in Math.* **88** (1991) 55-112.

[JS3] A. Joyal and R. Street, An introduction to Tannaka duality and quantum groups; Part II of *Category Theory, Proceedings, Como 1990* (Editors A. Carboni, M.C. Pedicchio and G. Rosolini) Lecture Notes in Math. **1488** (Springer-Verlag Berlin, Heidelberg 1991) 411-492.

[K] G.M. Kelly, Tensor products in categories, *J. Algebra* **2** (1965) 15-37.

[KL] G.M. Kelly and M.L. Laplaza, Coherence for compact closed categories, *J. Pure Appl. Algebra* **19** (1980) 193-213.

[KS] G.M. Kelly and R. Street, Review of the elements of 2-categories, *Lecture Notes in Math.* **420** (Springer, Berlin, 1974) 75-103.

[ML] S. Mac Lane, *Categories for the Working Mathematician* , Graduate Texts in Math. **5** (Springer-Verlag, 1971).

[Maj] S. Majid, *Foundations of Quantum Group Theory* (Cambridge University Press, Cambridge, 1995. x+607 pp. ISBN: 0-521-46032-8).

[MC0] P. McCrudden, *Categories of Representations of Balanced Coalgebroids* (PhD Thesis, Macquarie University, 22 February 1999).

[MC1] P. McCrudden, Balanced coalgebroids, *Theory and Appl. of Categories* **7** (2000) 71-147.

[MC2] P. McCrudden, Categories of representations of coalgebroids, *Advances in Math.* **154** (2000) 299-332.

[MC3] P. McCrudden, Tannaka duality for Maschkean categories, Proceedings of the Category Theory Conference in Coimbra (July 1999), *J. Pure. Appl. Algebra* (to appear).

[MT] M. McIntyre and T. Trimble, The geometry of Gray monoids (preprint).

[St1] R. Street, Cauchy characterization of enriched categories, *Rendiconti del Seminario Matematico e Fisico di Milano* 51 (1981) 217-233.

[St2] R. Street, Enriched categories and cohomology, *Quaestiones Math.* 6 (1983) 265-283.

[St3] R. Street, *Quantum Groups : an entrée to modern algebra* (available at http://www-texdev.mpce.mq.edu.au/Quantum/Quantum.html)

[St4] R. Street, Functorial calculus in a monoidal bicategory, *Applied Categorical Structures* (to appear).

[SV] R. Street and D. Verity, Low–dimensional topology and higher-order categories, *Proceedings of CT95, Halifax, July 9-15, 1995* <http://www.maths.mq.edu.au/~street/LowDTop.pdf.

Centre of Australian Category Theory
Macquarie University
N. S. W. 2109
AUSTRALIA
email: street@math.mq.edu.au