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Fusion Operators and Cocycloids in Monoidal Categories

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Abstract. The Yang–Baxter equation has been studied extensively in the context of monoidal categories. The fusion equation, which appears to be the Yang–Baxter equation with a term missing, has been studied mainly in the context of Hilbert spaces. This paper endeavours to place the fusion equation in an appropriate categorical setting. Tricocycloids are defined; they are new mathematical structures closely related to Hopf algebras.

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Key words: bialgebra, Hopf algebra, fusion equation, 3-cocycle, monoidal category, tensor category, braiding, string diagram, Tannaka duality.

Introduction

This note was inspired by reading [2] and [14]. I am very grateful to Dr Huu Hung Bui for showing me those papers which already hint at connections between the fusion equation and monoidal categories. The fusion equation as written there certainly has five terms and so suggests a geometric interpretation as a pentagon as for the axiom on the associativity constraint of a monoidal category. It is also pointed out that the fusion equation is the Yang–Baxter equation with the middle term missing on one side and that the fusion equation is somehow the more basic. My intention here is to clarify these relationships.

It should be recalled that the equation satisfied by a Yang–Baxter operator can be written in a form which makes sense in any monoidal category [7, 9]. However, the corresponding version of the fusion equation has an inextricable term involving the switch map and, instead of five terms, there are six. In this form, the fusion equation can be expressed in any braided monoidal category. I contend that this is the appropriate level of generality; indeed, the equation is then none other than the 3-cocycle condition. I would say that the fusion equation is more basic in the sense that it is an expression of an associativity constraint rather than a commutativity constraint. Yet it is more sophisticated because it requires the context of a commutativity constraint for its expression.

We begin by establishing the bijection between fusion operators and 3-cocycles, and show how these arise from bialgebras. Some constructions with 3-cocycles are described showing, in particular, the relation to the Stasheff–Mac Lane pentagon. An object equipped with an invertible 3-cocycle is called a *tricocycloid*; it should be regarded as a generalised Hopf algebra, and hence, as a generalised (quantum) group. By considering representations and using techniques of Tannaka duality, Hopf algebras are associated with tricocycloids. We make some speculations on higher cocycloids. Finally, in Section 6, we construct the generic tricocycloid.

1. Definition and Examples of Fusion Operators and 3-Cocycles

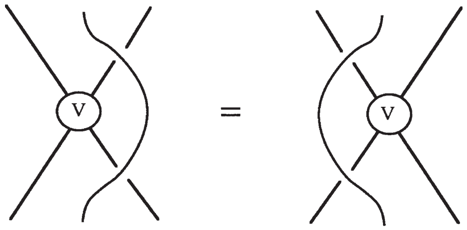
We work in a fixed braided monoidal category $\mathcal{V}$ in the sense of Joyal–Street, and we freely appeal to the coherence theorems [9]; in particular, we write as if $\mathcal{V}$ were strictly associative. We also provide proofs using string diagrams which was shown to be rigorous in [8].

For any arrow $V: A \otimes A \to A \otimes A$ in $\mathcal{V}$, we put

$$V_{12} = V \otimes 1_A, \qquad V_{23} = 1_A \otimes V, \quad \text{and}$$

$$V_{13} = (1_A \otimes c_{A,A})^{-1}(V \otimes 1_A)(1_A \otimes c_{A,A})\colon\ A \otimes A \otimes A \to A \otimes A \otimes A.$$

From the equality



we see that we also have the formula

$$V_{13} = (c_{A,A} \otimes 1_A)(1_A \otimes V)(c_{A,A} \otimes 1_A)^{-1}\colon\ A \otimes A \otimes A \to A \otimes A \otimes A.$$

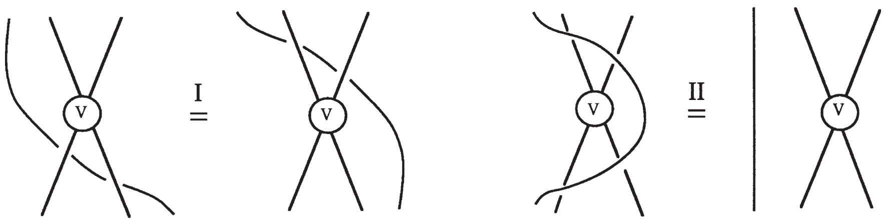
PROPOSITION 1.1. *An arrow* $V\colon\ A \otimes A \to A \otimes A$ *in* $\mathcal{V}$ *satisfies the 'fusion equation'*

$$V_{23}V_{12} = V_{12}V_{13}V_{23}$$

if and only if the arrow $v = c_{A,A}V$ *satisfies the* '3-*cocycle condition'*

$$(v \otimes 1_A)(1_A \otimes c_{A,A})(v \otimes 1_A) = (1_A \otimes v)(v \otimes 1_A)(1_A \otimes v).$$

Proof. We shall use the following two identities I and II.



The 3-cocycle condition for $v = c_{A,A}V$ is satisfied if and only if

$$\begin{aligned}&(c \otimes 1)(V \otimes 1)(1 \otimes c)(c \otimes 1)(V \otimes 1)\\ &\quad = (1 \otimes c)(1 \otimes V)(c \otimes 1)(V \otimes 1)(1 \otimes c)(1 \otimes V)\\ &\quad = (1 \otimes c)(1 \otimes V)(c \otimes 1)(1 \otimes c)V_{13}V_{23},\end{aligned}$$

which, using equation I, is equivalent to

$$\begin{aligned}&(c \otimes 1)(V \otimes 1)(1 \otimes c)(c \otimes 1)(V \otimes 1)\\ &\quad = (1 \otimes c)(c \otimes 1)(1 \otimes c)V_{12}V_{13}V_{23}\\ &\quad = (c \otimes 1)(1 \otimes c)(c \otimes 1)V_{12}V_{13}V_{23}.\end{aligned}$$

Multiplying both sides by the inverse of $(c \otimes 1)(1 \otimes c)(c \otimes 1)$, we see that this is equivalent to

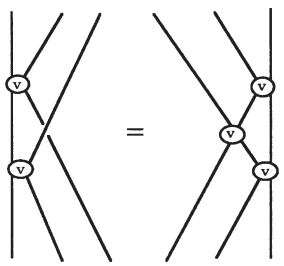
$$V_{12}V_{13}V_{23} = (c \otimes 1)^{-1}(1 \otimes c)^{-1}(V \otimes 1)(1 \otimes c)(c \otimes 1)(V \otimes 1).$$

Using equation II, we see that this is equivalent to $V_{12}V_{13}V_{23} = V_{23}V_{12}$, as required. □

REMARKS. (a) If $v: A \otimes A \to A \otimes A$ satisfies the 3-cocycle condition and is invertible then v^{-1} satisfies the 3-cocycle condition for the *inverse braiding*.

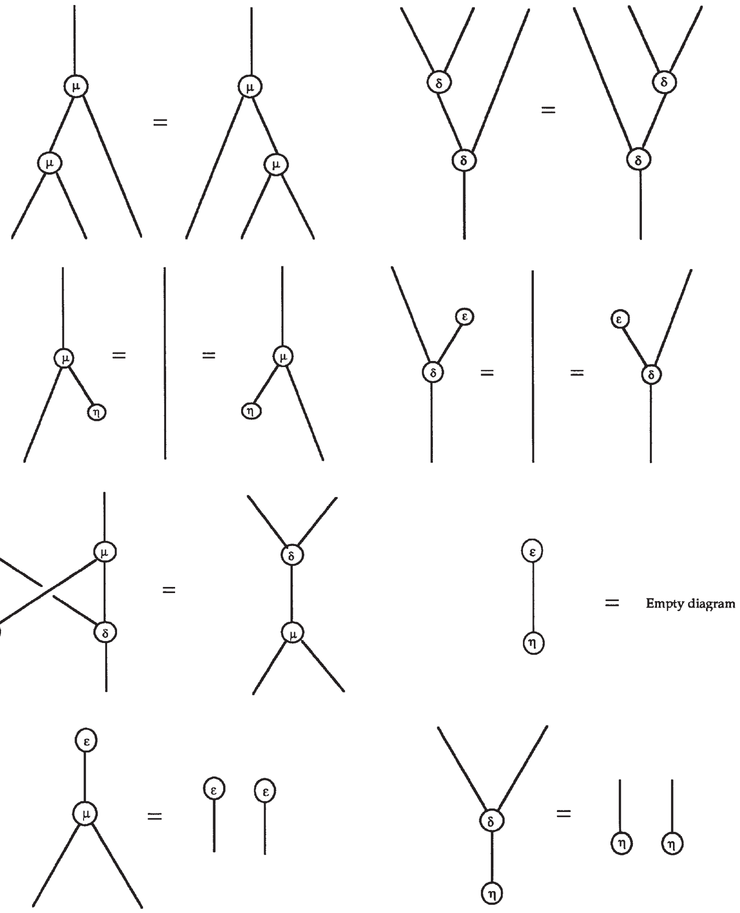
(b) It is possible to define the *nerve* of a tricategory [6]. A braided monoidal category $\mathcal{V}$ (or rather, its 'double suspension') is a tricategory $\Sigma^2\mathcal{V}$ with only one object and only one arrow. A 3-cocycle v yields a commutative 4-simplex in $\Sigma^2\mathcal{V}$; that is, an element of dimension 4 of the nerve of $\Sigma^2\mathcal{V}$. This is a reason why braided monoidal categories are an appropriate setting for 3-cocycles.

(c) The string diagram for the 3-cocycle condition is below.

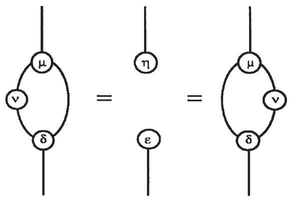


(d) Every object A is equipped with the canonical 3-cocycle $c_{A,A}: A \otimes A \to A \otimes A$.

Recall that a *bialgebra* in $\mathcal{V}$ is an object A together with a multiplication $\mu\colon\ A \otimes A \to A$, a unit $\eta\colon\ I \to A$, a comultiplication $\delta\colon\ A \to A \otimes A$, and a counit $\varepsilon\colon\ A \to I$ which satisfy the following ten identities.



The bialgebra is called a *Hopf algebra* when there exists an arrow $\nu\colon\ A \to A$ satisfying the two identities below.



The arrow $\nu\colon\ A \to A$ is unique and is called the *antipode.*

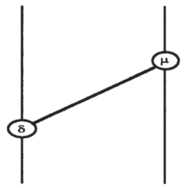
PROPOSITION 1.2. *If A is a bialgebra in $\mathcal{V}$ with the inverse braiding then the arrow*

$$V = (1_A \otimes \mu)(\delta \otimes 1_A)\colon\ A \otimes A \to A \otimes A$$

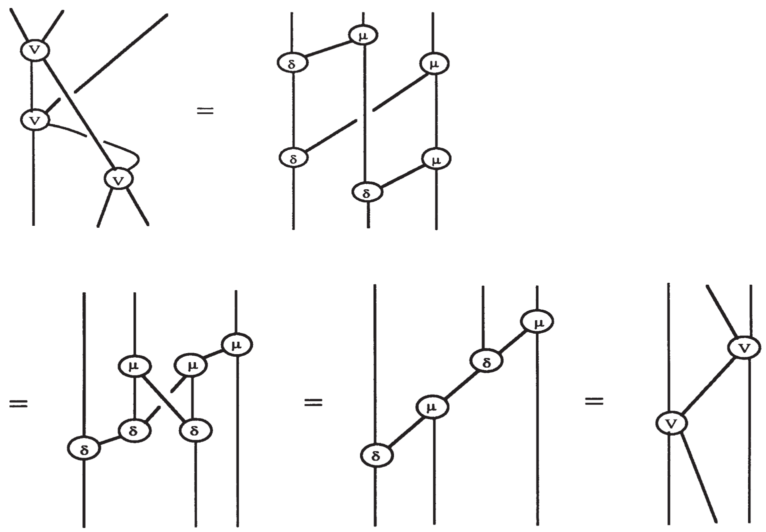
satisfies the fusion equation in $\mathcal{V}$ with the original braiding. Moreover, if A is a Hopf algebra then V is invertible with inverse

$$V^{-1} = (1_A \otimes \mu)(1_A \otimes \nu \otimes 1_A)(\delta \otimes 1_A)\colon\ A \otimes A \to A \otimes A.$$

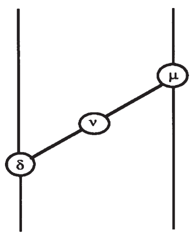
Proof. The string diagram for V is below.



So the following calculation provides the proof that V satisfies the fusion equation.

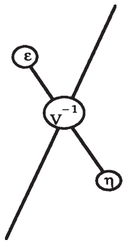


If A is a Hopf algebra, a short direct diagrammatic calculation shows that the following depicts an inverse for V.

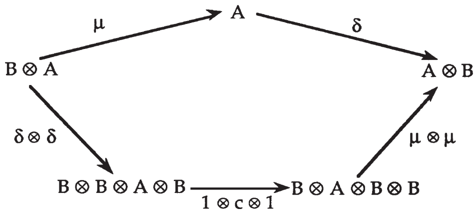


□

Notice that, for a Hopf algebra, the antipode can be recaptured from the inverse of the corresponding fusion operator as the value of the string diagram below.



Proposition 1.2 can be generalised somewhat. Suppose B is a bialgebra in $\mathcal{V}$. A *left B-module* is an object A together with an action $\mu\colon\ B \otimes A \to A$ which is compatible with the multiplication and unit of B in the usual way. A *right B-comodule* is an object A together with a coaction $\delta\colon\ A \to A \otimes B$ which is compatible with the comultiplication and counit of B in the usual way. A *B-mixed module* is an object A with a left B-module and right B-comodule structure related by the commutativity of the following diagram.



PROPOSITION 1.3. *If A is a B-mixed module in $\mathcal{V}$ with the inverse braiding then the arrow*

$$V = (1_A \otimes \mu)(\delta \otimes 1_A)\colon\ A \otimes A \to A \otimes A$$

satisfies the fusion equation in $\mathcal{V}$ with the original braiding. Moreover, if B is a Hopf algebra then V is invertible with inverse

$$V^{-1} = (1_A \otimes \mu)(1_A \otimes \nu \otimes 1_A)(\delta \otimes 1_A)\colon\ A \otimes A \to A \otimes A.$$

Proof. The proof uses the same string diagrams as Proposition 1.2 with strings labelled by A and B instead of A only. □

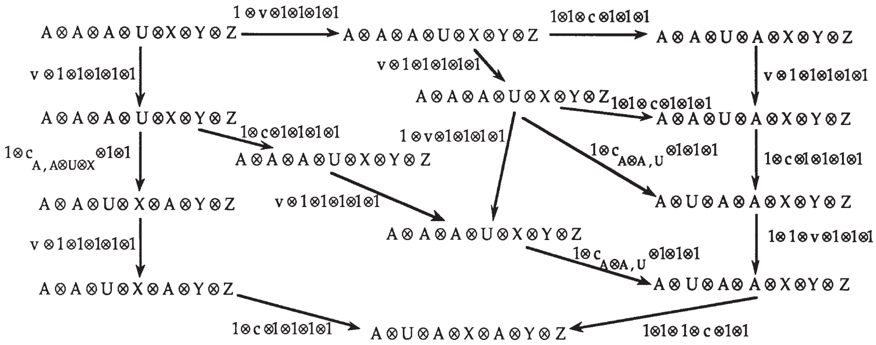
2. Constructions on 3-Cocycles

PROPOSITION 2.1. *If $v\colon\ A \otimes A \to A \otimes A$ is an invertible arrow satisfying the 3-cocycle condition then a monoidal structure without unit is defined on $\mathcal{V}$ as follows*:

the tensor product is given by $X * Y = A \otimes X \otimes Y$;
the associativity constraint $a\colon (X * Y) * Z \to X * (Y * Z)$ *is the composite*

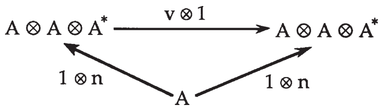
$$A \otimes A \otimes X \otimes Y \otimes Z \xrightarrow{v\otimes 1\otimes 1\otimes 1} A \otimes A \otimes X \otimes Y \otimes Z \xrightarrow{1\otimes c\otimes 1\otimes 1} A \otimes X \otimes A \otimes Y \otimes Z.$$

Proof. We must show that the proposed associativity constraint satisfies the usual pentagon condition. This amounts to commutativity of the outside of the diagram below. The top left region in the diagram commutes by the 3-cocycle condition for v; the other regions can be seen to commute by using string diagrams in the braided monoidal category.



□

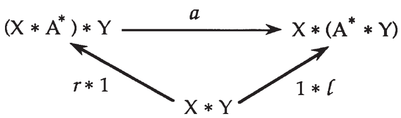
REMARKS. (a) In the setting of Proposition 2.1, suppose we have an object A^* and an arrow $n\colon I \to A \otimes A^*$ such that the following triangle commutes.



Then A^* acts as a 'lax' unit for the tensor product $*$ in the sense that we have natural transformations

$$l = n \otimes 1_X\colon X \to A^* * X, \qquad r = (1_A \otimes c)(n \otimes 1_X)\colon X \to X * A^*$$

such that the following triangle commutes.

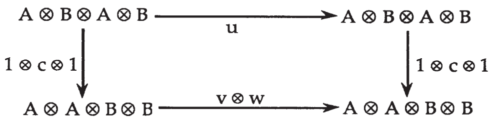


In particular, if n is invertible, this defines a monoidal structure on $\mathcal{V}$ with tensor product $*$.

(b) Let $\mathcal{I}$ denote the $\mathcal{V}$-category with one object 0 and $\mathcal{V}$-valued hom $\mathcal{I}(0, 0)$ equal to the unit I for the tensor product of $\mathcal{V}$. One interpretation of Proposition 2.1

is that an invertible 3-cocycle $v\colon A\otimes A \to A\otimes A$ provides a nonunital promonoidal structure on $\mathcal{I}$. Then the new nonunital tensor product of Proposition 2.1 is the convolution structure on $\mathcal{V} = \mathcal{V}^{\mathcal{I}}$ [3]. Moreover, the tensor product of promonoidal structures from [4, p. 313] suggests the following 'bicrossed product' (compare [14, Proposition 5.2]).

PROPOSITION 2.2. *Provided $\mathcal{V}$ is symmetric, if $v\colon A \otimes A \to A \otimes A$ and $w\colon B \otimes B \to B \otimes B$ satisfy the* 3*-cocycle condition then so does the arrow u: $A \otimes B \otimes A \otimes B \to A \otimes B \otimes A \otimes B$ defined by commutativity of the following square.*



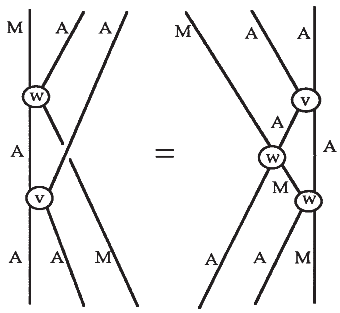
3. Tricocycloids and Their Modules

DEFINITION. A *tricocycloid* is an object A of $\mathcal{V}$ together with an invertible arrow $v\colon A\otimes A \to A\otimes A$ satisfying the 3-cocycle condition. We denote a tricocycloid by its underlying object A and use the same letter v for the 3-cocycle unless confusion seems possible.

We can regard tricocycloids as generalised Hopf algebras (Proposition 1.2). It therefore makes sense to consider the possibility of modules (or representations) over tricocycloids.

DEFINITION. Suppose A is a tricocycloid. An *A-module* is an object M of $\mathcal{V}$ together with an invertible arrow $w\colon A \otimes M \to M \otimes A$ satisfying the condition

$$(w \otimes 1_A)(1_A \otimes c_{A,M})(v \otimes 1_M) = (1_M \otimes v)(w \otimes 1_A)(1_A \otimes w).$$



A *morphism* of A-modules is an arrow $f\colon M \to N$ such that $(f \otimes 1_A)w = w(1_A \otimes f)$. We denote the category of A-modules by $\mathrm{Mod}(A)$. Dually, an *A-comodule* is an object P of $\mathcal{V}$ together with an invertible arrow $w\colon P\otimes A \to A\otimes P$ satisfying the condition

$$(v \otimes 1_P)(1_A \otimes c_{P,A})(w \otimes 1_A) = (1_A \otimes w)(w \otimes 1_A)(1_P \otimes v).$$

There is a category Comod(A) of A-comodules.

PROPOSITION 3.1. *Suppose A is a Hopf algebra in $\mathcal{V}$ with the inverse braiding regarded as a tricocycloid via Propositions* 1.1 *and* 1.2*. Each left A-module $(M, \mu\colon A \otimes M \to M)$ becomes a module over the tricocycloid A when equipped with the arrow*

$$w = c_{A,M}(1_A \otimes \mu)(\delta \otimes 1_M)\colon\ A \otimes M \to M \otimes A.$$

Each right A-comodule $(P, \delta\colon P \to P \otimes A)$ becomes a comodule over the tricocycloid A when equipped with the arrow

$$w = c_{P,A}(1_P \otimes \mu)(\delta \otimes 1_A)\colon\ P \otimes A \to A \otimes P.$$

Proof. Put $W = (1_A \otimes \mu)(\delta \otimes 1_M)$ and proceed as in the proof of Proposition 1.2 to show that

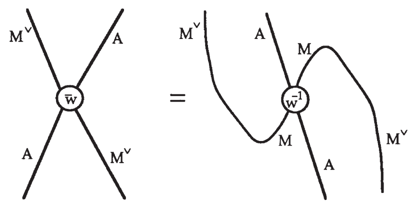
$$W_{23}V_{12} = V_{12}W_{13}W_{23}.$$

(Appropriate occurrences of A need to be replaced by M.) Then proceed as in the proof of Proposition 1.1 to show that $w = c_{A,M}W$ satisfies the module condition. The comodule case is similar. □

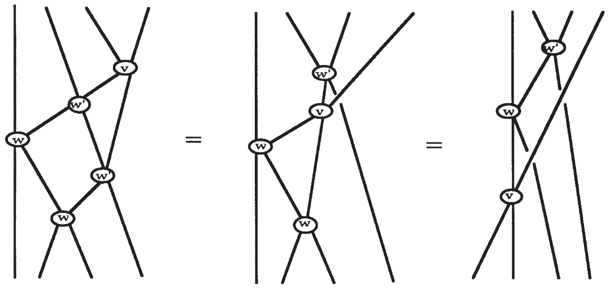
THEOREM 3.2. *For any tricocycloid A, there is a canonical monoidal structure on the category* Mod(A) *of A-modules such that the forgetful functor* Mod$(A) \to \mathcal{V}$ *preserves the tensor product. Explicitly, if M, M' are A-modules then $M \otimes M'$ is canonically an A-module when equipped with the composite arrow*

$$A \otimes M \otimes M' \xrightarrow{w \otimes 1} M \otimes A \otimes M' \xrightarrow{1 \otimes w'} M \otimes M' \otimes A.$$

If M is an A-module whose underlying object has a right dual $M^\vee$ in $\mathcal{V}$ then M has a right dual in Mod(A) *whose underlying object is $M^\vee$ and whose module structure is provided by the mate $\overline{w}\colon\ A \otimes M^\vee \to M^\vee \otimes A$ of $w^{-1}\colon\ M \otimes A \to A \otimes M$.*



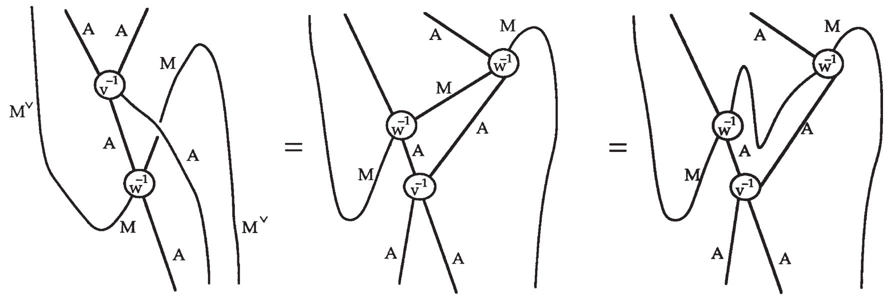
Proof. The following calculation shows that the displayed arrow satisfies the module condition.



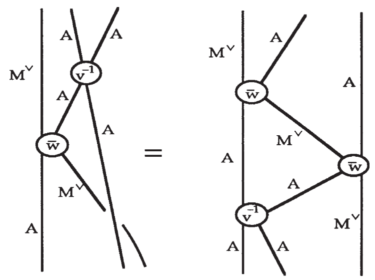
To see that $\overline{w}$ is an A-module structure, take the inverse of the module condition for w; that is,

$$(v^{-1} \otimes 1)(1 \otimes c^{-1})(w^{-1} \otimes 1) = (1 \otimes w^{-1})(w^{-1} \otimes 1)(1 \otimes v^{-1}).$$

This, together with the diagram calculus for duals, gives the following equalities.



Substituting in the diagram for $\overline{w}$, we obtain the equality:



which gives the equation

$$(1 \otimes v^{-1})(\overline{w} \otimes 1)(1 \otimes c) = (\overline{w} \otimes 1)(1 \otimes \overline{w})(v^{-1} \otimes 1),$$

and hence the desired equation

$$(\overline{w} \otimes 1)(1 \otimes c)(v \otimes 1) = (1 \otimes v)(\overline{w} \otimes 1)(1 \otimes \overline{w}).$$

The remaining details are even more straightforward. □

Let $\mathrm{Mod}(A)_r$ denote the full subcategory of $\mathrm{Mod}(A)$ consisting of those A-modules whose underlying object has a right dual in $\mathcal{V}$. Let $\mathrm{Comod}(A)_l$ denote the full subcategory of those A-comodules whose underlying object P has a left dual P^* in $\mathcal{V}$.

COROLLARY 3.3. *For any tricocycloid A, the category* $\mathrm{Mod}(A)_r$ (*respectively,* $\mathrm{Comod}(A)_l$) *is right-* (*respectively, left-*) *autonomous monoidal and the underlying functor* $U_A\colon \mathrm{Mod}(A)_r \to \mathcal{V}$ (*respectively,* $U_A^\wedge\colon \mathrm{Comod}(A)_l \to \mathcal{V}$) *preserves tensor product.*

This puts us in a position to apply Tannaka duality [10]. With appropriate assumptions on $\mathcal{V}$, we obtain Hopf algebras

$$\mathcal{S}(A) = \mathrm{End}^\vee(U_A), \qquad \mathcal{S}^\wedge(A) = \mathrm{End}^\vee(U_A^\wedge)$$

which should be compared with the Hopf algebras $S,\ S^\wedge$ of [14, Theorem 3.3].

COROLLARY 3.4. *For any tricocycloid A in the category $\mathcal{V} =$* Vect, *the monoidal category* $\mathrm{Mod}(A)_r$ (*respectively,* $\mathrm{Comod}(A)_l$) *is equivalent to the monoidal category* $\mathrm{Comod}(\mathcal{S}(A))_f$ (*respectively,* $\mathrm{Comod}(\mathcal{S}^\wedge(A))_f$) *of finite dimensional comodules over the Hopf algebra* $\mathcal{S}(A)$ (*respectively,* $\mathcal{S}^\wedge(A)$).

Proof. Since $A\otimes$- preserves colimits and is left exact, it is easy to see that $\mathrm{Mod}(A)_r$ is Abelian and U_A is exact. Clearly U_A is faithful. So the representation theorem [10, Section 7, Theorem 3] applies. □

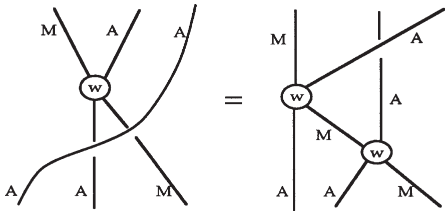
There is an interesting variant of Theorem 3.2. As mentioned in the remark of Section 1, each object A of the braided monoidal category $\mathcal{V}$ becomes a tricocycloid by equipping it with the braiding isomorphism $c_{A,A}$. We might consider objects of $\mathcal{V}$ which are modules for all these tricocycloids simultaneously in a natural way. Such objects can be thought of as modules for the bialgebra (used in [11, 12])

$$F = \int^A A \otimes A^\vee$$

even when the coend and the duals do not exist in $\mathcal{V}$. More precisely, we define a monoidal category $\mathcal{F}_\mathcal{V}$ whose objects are pairs (M, w) where M is an object of $\mathcal{V}$ and w is a family of invertible arrows

$$w_A\colon\ A \otimes M \xrightarrow{\sim} M \otimes A$$

such that the following equation holds.



An arrow $f\colon (M, w) \to (M', w')$ in $\mathcal{F}_\mathcal{V}$ is an arrow $f\colon M \to M'$ in $\mathcal{V}$ such that

$$w'_A(1_A \otimes f) = (f \otimes 1_A)w_A$$

for all $A \in \mathcal{V}$. The tensor product is given by

$$(M, w) \otimes (M', w') = (M \otimes M', (1_M \otimes w'_A)(w_A \otimes 1_{M'})).$$

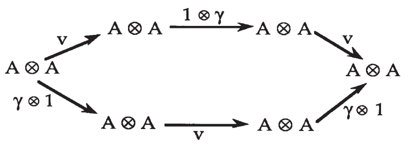
PROPOSITION 3.5. *For each braided monoidal category $\mathcal{V}$, there is a monoidal category $\mathcal{F}_\mathcal{V}$ as defined above. The forgetful functor $\mathcal{F}_\mathcal{V} \to \mathcal{V}$ is monoidal and has a monoidal section taking $X \in \mathcal{V}$ to $(X, c_{-,X}) \in \mathcal{F}_\mathcal{V}$. An object $(M, w) \in \mathcal{F}_\mathcal{V}$ has a right dual if and only if $M \in \mathcal{V}$ has a right dual.*

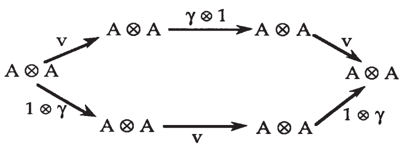
Proof. The same string diagrams as in the proof of Theorem 3.2 can be used. □

4. Abelian Tricocycloids

This short section is motivated by the cohomology of Abelian groups [5, 9, Section 3] and by the notion of symmetry [3, p. 23] (or more precisely, braiding [9, Section 5]) for a promonoidal category.

DEFINITION. An *Abelian tricocycloid* is a triplet (A, v, γ) where (A, v) is a tricocycloid and $\gamma\colon A \to A$ is an invertible arrow such that the following two diagrams commute.





PROPOSITION 4.1. *If (A, v, γ) is an Abelian tricocycloid then a braiding*

$$k_{X,Y}\colon X * Y \to Y * X$$

for the unitless monoidal structure of Proposition 2.1 *is defined by the composite*

$$A \otimes X \otimes Y \xrightarrow{\gamma \otimes 1_X \otimes 1_Y} A \otimes X \otimes Y \xrightarrow{1_A \otimes c_{X,Y}} A \otimes Y \otimes X.$$

5. Speculations on Higher Cocycloids

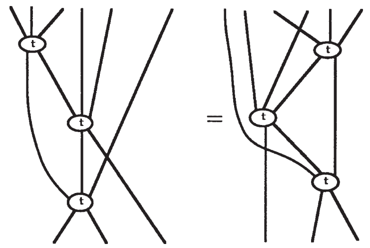
A nonunital coalgebra in a monoidal category $\mathcal{V}$ is precisely an object A together with an arrow $\delta\colon A \to A \otimes A$ which is coassociative. By referring to the non-Abelian cocycle conditions as expressed in [15], we see that coassociativity is precisely the 2-*cocycle condition.*

Recall that for a tricocycloid we asked that the 3-cocycle should be invertible. Invertibility can be regarded as amounting to an arrow in the opposite direction also satisfying a dual 3-cocycle condition together with a compatibility condition with the original 3-cocycle. I propose that the correct notion of *bicocycloid* is precisely a nonunital-or-counital bialgebra such that the arrow V of Proposition 1.2 is invertible. That is, rather than ask for the unlikely requirement that $\delta\colon A \to A \otimes A$ should be invertible, we ask for an arrow $\mu\colon A \otimes A \to A$ in the opposite direction which satisfies the dual 2-cocycle condition (associativity) and some compatibility requirements with δ.

Now suppose that $\mathcal{V}$ is a symmetric monoidal category (so that we can suspend $\mathcal{V}$ as many times as we wish). By referring to the diagram for a 5-simplex [15], we see that the 4-*cocycle condition* on an arrow $t\colon A \otimes A \to A \otimes A \otimes A$ is the equation

$$\begin{aligned}&(t \otimes 1_A \otimes 1_A \otimes 1_A)(1_A \otimes t \otimes 1_A)(1_A \otimes 1_A \otimes c_{A,A})(t \otimes 1_A)\\ &\quad = (1_A \otimes 1_A \otimes c_{A,A} \otimes 1_A \otimes 1_A)(1_A \otimes 1_A \otimes 1_A \otimes t)(1_A \otimes t \otimes 1_A)\\ &\qquad (c_{A,A} \otimes 1_A \otimes 1_A)(1_A \otimes t).\end{aligned}$$

The string diagram for this equation is as follows: it occurred in the simplicial case of the 'Pascal triangle' of [1].



PROPOSITION 5.1. *Suppose $\mathcal{V}$ is symmetric monoidal. For any bialgebra A in $\mathcal{V}$, the arrow*

$$t = c_{321}(1_A \otimes \mu \otimes 1_A)(\delta \otimes \delta)\colon A \otimes A \to A \otimes A \otimes A,$$

where $c_{321} = (c_{A,A} \otimes 1_A)(1_A \otimes c_{A,A})(c_{A,A} \otimes 1_A)\colon A \otimes A \otimes A \to A \otimes A \otimes A$, *satisfies the* 4-*cocycle condition.*

A *tetracocycloid* should consist of an object A, an arrow $t\colon A \otimes A \to A \otimes A \otimes A$ satisfying the 4-cocycle condition, and an arrow $\bar{t}\colon A \otimes A \otimes A \to A \otimes A$ satisfying the dual 4-cocycle condition, subject to some compatibility condition which we shall not explore at this time. We would define an A-comodule to consist of an object M equipped with an arrow $t\colon M \otimes A \to A \otimes M \otimes M$ satisfying the string version of the 4-cocycle condition with strings appropriately labelled with A and M.

PROPOSITION 5.2. *If (M, t) and (N, t') are A-comodules for a tetracocycloid A then so is $M \otimes N$ equipped with the following composite arrow.*

$$M \otimes N \otimes A \xrightarrow{1\otimes t'} M \otimes A \otimes N \otimes N \xrightarrow{t\otimes 1} A \otimes M \otimes M \otimes N \otimes N$$
$$\xrightarrow{1\otimes 1\otimes c\otimes 1} A \otimes M \otimes N \otimes M \otimes N$$

We expect to be able to characterize those comodules with duals for the tensor product of comodules provided by Proposition 5.2 and then to apply Tannaka duality to obtain a Hopf algebra from the tetracocycloid A.

6. Fusion Groups

Recall [9] that the braid category **B** is the free monoidal category containing an object bearing a Yang–Baxter operator. The braid category is the disjoint union of the braid groups $\mathbf{B}_n$, $n \geqslant 0$. In this section we shall describe the free braided monoidal category **Fus** containing a tricocycloid.

Let $\mathbf{Fus}_n$ denote the group generated by symbols $y_1, y_2, \ldots, y_{n-1}$ and $v_1, v_2, \ldots, v_{n-1}$ subject to the relations

$$y_i y_{i+1} y_i = y_{i+1} y_i y_{i+1}, \qquad v_i v_{i+1} v_i = v_{i+1} y_i v_{i+1},$$
$$v_i y_{i+1} y_i = y_{i+1} y_i v_{i+1}, \qquad y_i y_{i+1} v_i = v_{i+1} y_i y_{i+1},$$
$$y_i y_j = y_j y_i, \quad v_i v_j = v_j v_i, \quad y_i v_j = v_j y_i,$$

for $j - i > 1$. There is a canonical homomorphism $\phi_n\colon \mathbf{B}_n \to \mathbf{Fus}_n$ taking the braid which passes the ith string over the $(i+1)$th string to the generator y_i. Let **Fus** be the groupoid obtained as the disjoint union of the groups $\mathbf{Fus}_n$, $n \geqslant 0$: the objects are the natural numbers and $\mathbf{Fus}(m, n)$ is empty unless $m = n$ in which case it is the group $\mathbf{Fus}_n$. There is a canonical strict monoidal structure $\oplus\colon \mathbf{Fus} \times \mathbf{Fus} \to \mathbf{Fus}$ on the category **Fus** given on objects by addition $m \oplus n = m + n$ and determined on arrows by $y_i \oplus 1_n = y_i$, $v_i \oplus 1_n = v_i$, $1_m \oplus y_i = y_{m+i}$, $1_m \oplus v_i = v_{m+i}$. There is a strict monoidal functor $\phi\colon \mathbf{B} \to \mathbf{Fus}$ which is the identity on objects and is given by the homomorphisms ϕ_n on arrows. Indeed, **Fus** admits a unique braiding such that $\phi\colon \mathbf{B} \to \mathbf{Fus}$ is braided. Furthermore, v_1 is a 3-cocycle on the object 1 of **Fus**.

PROPOSITION 6.1. *Suppose $y, v\colon A \otimes A \to A \otimes A$ are invertible arrows in a strict monoidal category $\mathcal{V}$ such that y is a Yang–Baxter-operator and v is a 3-cocycle (that is, the equations*

$$(1 \otimes y)(y \otimes 1)(1 \otimes y) = (y \otimes 1)(1 \otimes y)(y \otimes 1),$$

$$(1 \otimes v)(y \otimes 1)(1 \otimes v) = (v \otimes 1)(1 \otimes v)(v \otimes 1)$$

hold). *Then there exists a unique strict monoidal functor* $F\colon \mathbf{Fus} \to \mathcal{V}$ *such that*

$$F1 = A, \qquad Fy_1 = y \quad \textit{and} \quad Fv_1 = v.$$

It would be interesting to know whether there is a satisfactory geometric model of **Fus**.

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