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Higher Categories, Strings, Cubes and Simplex Equations

ROSS STREET
Mathematics Department, Macquarie University, NSW 2109, Australia (E-mail: street@mpce.mq.edu.au)

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Abstract. This survey of categorical structures, occurring naturally in mathematics, physics and computer science, deals with monoidal categories; various structures in monoidal categories; free monoidal structures; Penrose string notation; 2-dimensional categorical structures; the simplex equations of field theory and statistical mechanics; higher-order categories and computads; the (v,d)-cube equations; the simplex equations as cubical cocycle equations; and, cubes, braids and higher braids.

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Key words: Monoidal category, tensor category, n-category, bicategory, string notation, d-simplex equation, braiding, tangles, cubes, cocycle, higher braids, pasting.

Introductory Remarks

Categories, functors and natural transformations are tools of many mathematicians, and they occur somewhere in most graduate programs, often with [32] as text. A typical example of a functor is

$$\pi_1 : \mathit{Top} \longrightarrow \mathcal{G}pd$$

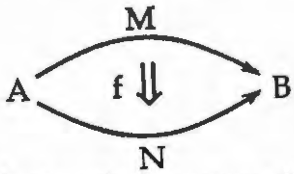
which assigns, to each topological space X, its fundamental groupoid π_1(X). The original purpose of categories was to provide a setting for discussing such constructions of one kind of mathematical structure out of another, and the relations between these constructions.

Categories are themselves algebraic structures which actually occur as mathematical objects, not only as organizational tools for large collections of structures. For example, a groupoid G is a special kind of category; this has the virtue that representations can be dealt with as functors

$$\mathrm{G} \longrightarrow \mathcal{V}ect$$

and intertwining operators as natural transformations between these. Also, posets are special categories, certain of which (locales, for instance) can sometimes serve as convenient spaces. While these observations are interesting for category theorists, other mathematicians have been able to ignore them and use the tools of their own areas. Perhaps the main point of these lectures is to present some structures, thrust upon us by mathematics, physics and computer science, which are more profitably viewed as categorical structures than more traditional "universal algebras". Consequently, these lectures will be about categorical structures, and not really about category theory. Yet important new examples will surely provide an opportunity to apply old theory and to suggest new directions.

The abstract notion of category allows the possibility that arrows are not necessarily functions; algebraic or geometric structures can occur as arrows, not only as objects. As an example, consider the category $\mathcal{R}el$ whose objects are sets and whose arrows R: X $\rightarrow$ Y are relations R $\subset$ X $\times$ Y. Indeed, the arrows can be equivalence classes of mathematical structures. For example, we have the category $\mathcal{M}od$ whose objects are rings and whose arrows [M]: A $\rightarrow$ B are isomorphism classes of left A-, right B-bimodules; composition here is given by tensoring the modules over the common ring. In order to take full account of the structure at hand and avoid taking the equivalence classes for arrows, one needs to introduce a structure, more general than a category, called a *bicategory* [3], in which there are arrows (called 2-*cells*) between the arrows (as shown below).



This kind of 2-*dimensional categorical structure* will be discussed. An example of a category in which geometric objects occur as arrows is the braid category [22] defined in Section 1.

The theme of this paper is the *free categorical structures are geometric*. The only structures intended are those with an established practical value. Of course, I cannot produce the appropriate geometry in *all* such cases, but this is likely to be lack of wit on my part. The connection with geometry is generally not an easy business.

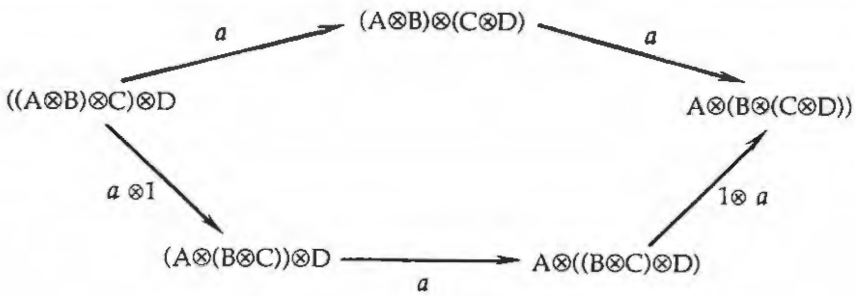
1. Monoidal Categories

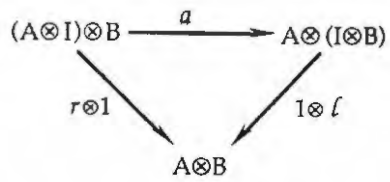
Large categories, such as the category $\mathcal{V}ect$ of (say, complex) vector spaces, often possess much more structure than a mere category. Some of this, such as direct sum, is determined by the category via a universal property or equations. Other structure is genuinely extra. Perhaps the most commonly used extra structure on a category is an abstract *tensor product*; we now make this precise while thinking of the case of $\mathcal{V} = \mathcal{V}ect$.

DEFINITION [31, 25]. A *monoidal* (or "tensor") *category* $\mathcal{V} = (\mathcal{V}, \otimes, \mathrm{I}, a, l, r)$ consists of a category $\mathcal{V}$, a functor $\otimes : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$ (called the *tensor product*), an object $I \in \mathcal{V}$ (called the *unit object*) and natural isomorphisms

$$a = a_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C),$$
$$l = l_A : I \otimes A \xrightarrow{\sim} A, \qquad r = r_A : A \otimes I \xrightarrow{\sim} A$$

(called the *associativity, left unit, right unit constraints*, respectively), such that, for all objects $A, B, C, D \in \mathcal{V}$, the following two diagrams (called the *associativity pentagon* and the *triangle for unit*) commute.





The monoidal category is called *strict* when all the constraints $a_{A,B,C}$, l_A, r_A are identity arrows.

There is a bonus: extra structures on categories motivated by large examples are also present on small geometric examples of importance in physics and computer science. While large examples are seldom strict, small ones often are. We provide some examples.

Category of matrices. Let $\mathcal{M}at$ be the category whose objects are natural numbers, whose arrows $a : n \to m$ are $m \times n$-matrices (with complex entries), and whose composition is matrix multiplication. The tensor product is multiplication on objects and Kronecker product on arrows. $\mathcal{M}at$ is a strict monoidal category equivalent to the full subcategory $\mathcal{V}ect_{fin}$ of $\mathcal{V}ect$ consisting of the finite dimensional vector spaces.

Simplicial category. Let Δ denote the category whose objects are finite linearly ordered sets

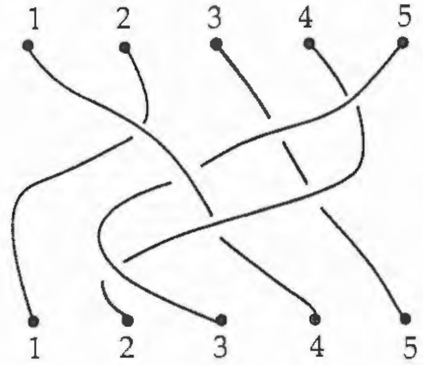
$$\underline{n} = \{0, 1, ..., n-1\}$$

and whose arrows are order-preserving functions. The tensor product is ordered sum:

$$\underline{m} + \underline{n} = \underline{m+n}.$$

Braid category. [22] Let P denote a Euclidean plane and let $C_n(P)$ be the space of subsets of P of cardiality n. *Artin's braid group* $\mathfrak{B}_n$ is the fundamental group of $C_n(P)$. Denoting some n distinct collinear points of P by 1, 2, ..., n, we can describe a loop $\omega : [0, 1] \to C_n(P)$ at the point $\{1, 2, ..., n\}$ of $C_n(P)$ by its graph in $[0, 1] \times P$; for example,

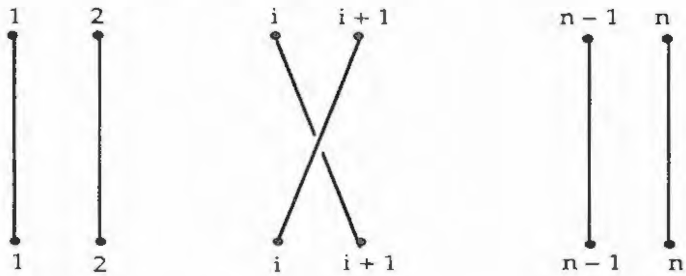
t = 1



t = 0

where a horizontal cross section by P at level $t \in [0, 1]$ intersects the curves (called the *strings*) in the subset $\omega(t)$ of P. Let s_i be the braid depicted by

s_i



A presentation for $\mathfrak{B}_n$ is given by the generators $s_1, ..., s_{n-1}$ and the relations

$$\begin{aligned} s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \quad \text{for } 1 \le i \le n-2, \\ s_i s_j &= s_j s_i \quad \text{for } 1 \le i < j-1 \le n-2. \end{aligned}$$

The first significant use of the braid groups in category theory was by John Gray [14]. The *braid category* $\mathfrak{B}$ is the disjoint union of the groups $\mathfrak{B}_n$ regarded as one-object categories. More explicitly, the objects of $\mathfrak{B}$ are the natural numbers $0, 1, 2, ...$, the homsets are given by

$$\mathfrak{B}(m, n) = \begin{cases} \mathfrak{B}_n & \text{when } m = n \\ \emptyset & \text{otherwise,} \end{cases}$$

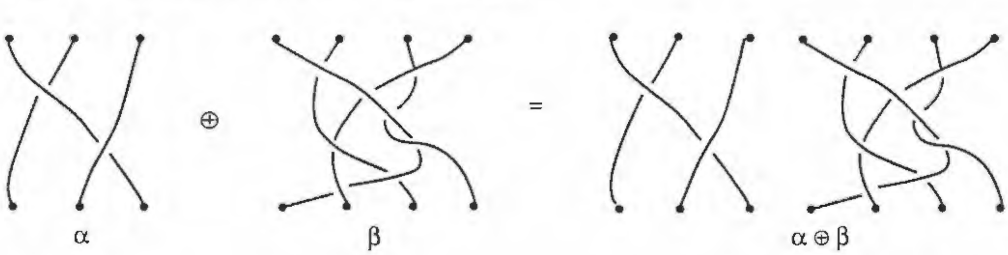
and composition is the multiplication of the braid groups. The category $\mathfrak{B}$ is equipped with a strict tensor structure defined by *addition of braids*

$$\oplus : \mathfrak{B}_m \times \mathfrak{B}_n \longrightarrow \mathfrak{B}_{m+n}$$

which is algebraically described by the equation

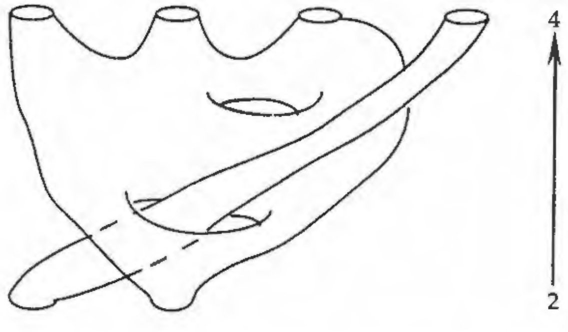
$$s_i \oplus s_j = s_i s_{m+j} (= s_{m+j} s_i),$$

and is pictured as in the following example.



We point out that, as a category, $\mathfrak{B}$ is just made up of non-interacting groups: it is the tensor product which allows interaction. Also, the arrows of $\mathfrak{B}$ are interesting equivalence classes of geometric structures.

Category of surfaces. There is a monoidal category *Surf* whose objects are natural numbers and whose arrows $m \rightarrow n$ are deformation classes of compact surfaces in $\mathbb{R}^3$ with boundary consisting of $m + n$ disjoint circles as below. The composition and tensor product are much as for braids. There are various variants of this example, and there is no need to be more precise for our purposes here.



Derivation category of a rewrite system. Consider the following notions from computational algebra [15]. A *rewrite system* $\mathcal{R}$ is a set Σ together with a directed graph whose vertices are words in the alphabet Σ. The edges $r : w \rightarrow w'$ of the graph are called the *rewrite rules*. An *application* of the rule $r : w \rightarrow w'$ is a formal expression

$$urv : uwv \longrightarrow uw'v$$

for any words u, v in the alphabet Σ: it is sometimes written as a deduction

$$\frac{\mathrm{uwv}}{\mathrm{uw'v}}\,\mathrm{r}$$

where the name of the rule being applied is recorded at the side. We have a new directed graph whose vertices are words in Σ and whose edges are applications of rules in $\mathcal{R}$. A directed path in this new graph is called a *derivation* in $\mathcal{R}$. A word in Σ is called *stable* when there is no rule which can be applied with that word as source. Interest often centres on the question of whether, for each word w, there exists a unique stable word v for which there is some derivation d: w $\rightsquigarrow$ v; this produces *normal forms* for the obvious equivalence relation on the words induced by the rewrite system. However, there are other questions of independent interest concerning derivations. For example, when should we regard two derivations as being equivalent? The most simple-minded notion of equivalence is the one generated by requiring the derivations

$$\begin{array}{c} \mathrm{us_1vs_2w} \\ \hline \mathrm{ut_1vs_2w} \\ \hline \mathrm{ut_1vt_2w} \end{array}\begin{array}{l} \\ \mathrm{r_1} \\ \\ \mathrm{r_2} \\ \\ \end{array} \quad \text{and} \quad \begin{array}{c} \mathrm{us_1vs_2w} \\ \hline \mathrm{us_1vt_2w} \\ \hline \mathrm{ut_1vt_2w} \end{array}\begin{array}{l} \\ \mathrm{r_2} \\ \\ \mathrm{r_1} \\ \\ \end{array}$$

to be equivalent where $r_1 : s_1 \rightsquigarrow t_1, r_2 : s_2 \rightsquigarrow t_2$ are any two rewrite rules.

The *derivation category der$\mathcal{R}$* on $\mathcal{R}$ has the words in Σ as objects and the equivalence classes of derivations as arrows. In fact, this is a strict monoidal category whose tensor product is given on objects by juxtaposition of words; the equivalence relation on derivations is precisely what is needed for this tensor product to be functorial.

Firing category of a Petri net. Write $\Pi^\dagger$ for the free commutative monoid on the set Π; the elements of $\Pi^\dagger$ are functions $u : \Pi \rightsquigarrow \mathbb{N}$ of finite support and the operation is pointwise addition. A *Petri net* $\mathcal{P}$ is a set Π together with a directed graph

$$s, t : R \longrightarrow \Pi^\dagger;$$

the elements of Π are called *places* and the elements of R are called *transitions*. An element of $\Pi^\dagger$ tells how many *tokens* should be in each place. There is a strict monoidal category *fir$\mathcal{P}$* of *firings of P*. Consider the directed graph

$$s', t' : \Pi^\dagger \times R \longrightarrow \Pi^\dagger$$

where $s'(u, r) = u + s(r)$ and $t'(u, r) = u + t(r)$. The two paths

$$u + s(r) + s(r') \overset{(u+s(r'),r)}{\longrightarrow} u + t(r) + s(r') \overset{(u+t(r),r')}{\longrightarrow} u + t(r) + t(r')$$
$$u + s(r) + s(r') \overset{(u+s(r),r')}{\longrightarrow} u + s(r) + t(r') \overset{(u+t(r'),r)}{\longrightarrow} u + t(r) + t(r')$$

in this graph are considered elementarily equivalent for all $u \in \Pi^\dagger$ and $r, r' \in R$. Any two paths in the graph are called *congruent* when they can be obtained from each other by a finite sequence of replacements of pairs of consecutive arrows using elementary equivalence. The category $fir\mathcal{P}$ has elements of $\Pi^\dagger$ as objects and congruence classes (denoted by square brackets) of paths in the directed graph $s', t' \colon \Pi^\dagger \times R \to \Pi^\dagger$ as arrows. A significant problem of Petri net theory is the determination of whether or not there is an arrow in $fir\mathcal{P}$ with assigned source and target. However, the point for us is that $fir\mathcal{P}$ is a strict monoidal category: the tensor product is given on objects by addition in $\Pi^\dagger$ and on arrows is determined by

$$[(u, r)] \otimes v = [(u + v, r)] = v \otimes [(u, r)].$$

(This is only a suggestion of the rich monoidal categories occurring in full *linear logic*; see [36] for more details and references.)

2. Structures in Monoidal Categories

Based on examples of large monoidal categories such as $\mathcal{V}ect$, it is natural to abstract extra structure which certain objects might possess. Again it turns out that the small geometric monoidal categories often possess such structured objects.

DEFINITION. A *monoid* in a monoidal category $\mathcal{V}$ is an object A equipped with arrows

$$\mu : A \otimes A \longrightarrow A, \quad \eta : I \longrightarrow A$$

satisfying the associativity condition

$$\mu \circ (\mu \otimes 1_A) = \mu \circ (1_A \otimes \mu)$$

and the unit conditions

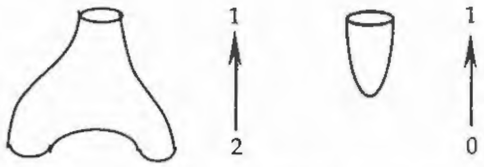
$$\mu \circ (\eta \otimes 1_A) = r_A, \quad \mu \circ (1_A \otimes \eta) = l_A.$$

A monoid in the opposite category $\mathcal{V}^{op}$ (keeping the same tensor product) is called a *comonoid* in $\mathcal{V}$.

A monoid in $\mathcal{V}ect$ is precisely an *algebra* where $\mu(x \otimes y) = xy$ and $\eta(1) = 1$. A comonoid in $\mathcal{V}ect$ is a *coalgebra*. Examples of algebras and coalgebras abound in mathematics.

The category *Cat* of categories is monoidal with cartesian product as tensor product. A monoid in *Cat* is a strict monoidal category; an example is the derivation category $der\mathcal{R}$ of a rewrite system $\mathcal{R}$.

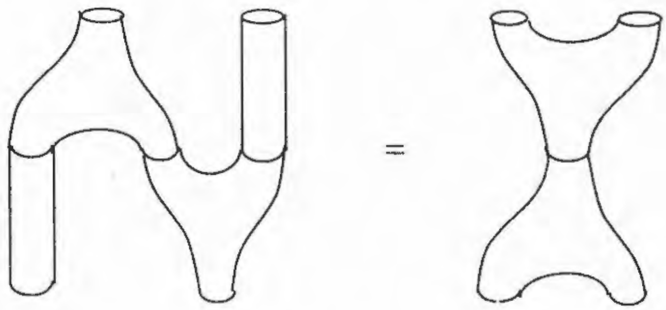
In the simplicial category Δ, the object $\underline{1} = \{0\}$ has a monoid structure where $\mu : \underline{2} \to \underline{1}$, $\eta : \underline{0} \to \underline{1}$ are the unique such arrows. In the category *Surf* of surfaces, 1 has a monoid structure with $\mu : 2 \to 1$ and $\eta : 0 \to 1$ illustrated below (the pair of pants and the bowl).



By standing these diagrams on their heads, 1 also becomes a comonoid in $\mathcal{S}urf$. The monoid multiplication $\mu : 2 \to 1$ and comonoid comultiplication $\delta : 1 \to 2$ are related by the equations

$$(\mu \otimes 1) \circ (1 \otimes \delta) = \delta \circ \mu = (1 \otimes \mu) \circ (\delta \otimes 1)$$

of Carboni–Walters [5]; as pointed out by Joyal, the geometric interpretation of (the first of) these equations is as illustrated below.



DEFINITION. An arrow $\varepsilon : \mathrm{A} \otimes \mathrm{B} \to \mathrm{I}$ in a monoidal category $\mathcal{V}$ is called an *exact pairing* when, for all objects X, Y, the assignment

$$\mathrm{f} \mapsto \mathrm{f}^{\#} = l_{\mathrm{Y}} \circ (\varepsilon \otimes \mathrm{I}_{\mathrm{Y}}) \circ (1_{\mathrm{A}} \otimes \mathrm{f})$$

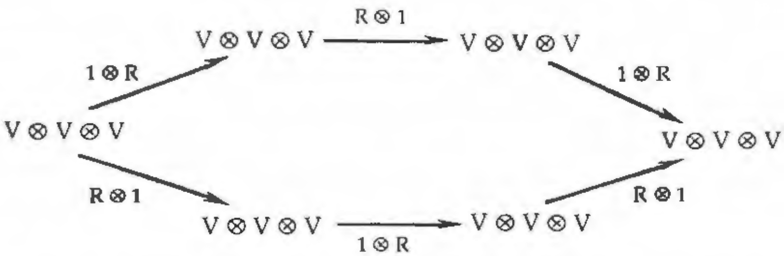
gives a bijection between arrows $\mathrm{f} : \mathrm{X} \longrightarrow \mathrm{B} \otimes \mathrm{Y}$ and arrows $\mathrm{g} : \mathrm{A} \otimes \mathrm{X} \longrightarrow \mathrm{Y}$ (so that we have an adjunction $\mathrm{A} \otimes - \dashv - \otimes \mathrm{B}$). When such an exact pairing exists, we call B a *(right) dual* for A. Note that we obtain $\eta : \mathrm{I} \longrightarrow \mathrm{B} \otimes \mathrm{A}$ with $\eta^{\#} = r_{\mathrm{A}}$.

In $\mathcal{V}ect$, the evaluation map Hom $(\mathrm{V}, \mathbb{C}) \otimes \mathrm{V} \to \mathbb{C}$ is an exact pairing iff V is finite dimensional.

DEFINITION. A *Yang–Baxter operator* [51, 19] on an object V of a monoidal category $\mathcal{V}$ is an invertible arrow

$$\mathrm{R} : \mathrm{V} \otimes \mathrm{V} \xrightarrow{\sim} \mathrm{V} \otimes \mathrm{V}$$

such that the following hexagon commutes.



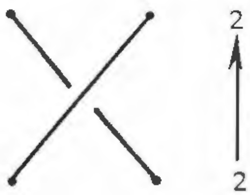
The equation $(R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R)$ is one form of the *Yang–Baxter equation* [16].

We can provide examples of Yang–Baxter operators on any finite dimensional object V of $\mathcal{V}ect$ as follows. Let $e_1, \ldots, e_n$ be a basis for V, and let q be a non-zero scalar, and define the linear isomorphism $R = R_q : V \otimes V \xrightarrow{\sim} V \otimes V$ by:

$$R(e_i \otimes e_j) = \begin{cases} e_j \otimes e_i & \text{for } i > j \\ e_j \otimes e_i + (q - q^{-1}) e_i \otimes e_j & \text{for } i < j \\ q e_i \otimes e_i & \text{for } i = j. \end{cases}$$

The operator R satisfies the equation $(R - q)(R + q^{-1}) = 0$. If $q = 1$ then R is the usual switch map $R(x \otimes y) = y \otimes x$. We can also regard R_q as a Yang–Baxter operator on n in $\mathcal{M}at$.

In $\mathfrak{B}$, there is a Yang–Baxter operator $s_1 : 1 \oplus 1 \longrightarrow 1 \oplus 1$ on the object 1 illustrated by the following braid.



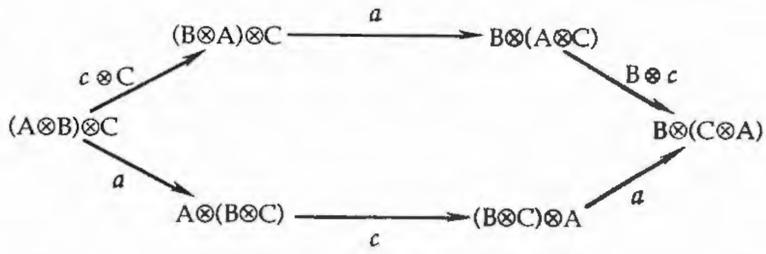
It is also possible to contemplate further structure on the monoidal category itself, and then more complicated structures on the objects therein.

DEFINITION. A *braiding* [20, 22] for a monoidal category $\mathcal{V}$ consists of a natural family of isomorphisms

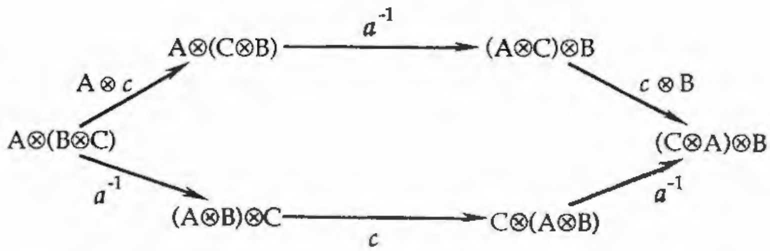
$$c = c_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$$

in $\mathcal{V}$ such that the two diagrams (B1) and (B2) commute.

(B 1)



(B 2)



If c is a braiding then so too is c' given by $c'_{A,B} = (c_{B,A})^{-1}$ since (B2) is just obtained from (B1) by replacing c with c'. A *symmetry* [31] is a braiding for which $c = c'$. A *braided [symmetric] monoidal category* is a monoidal category $\mathcal{V}$ with a chosen braiding [symmetry] c.

Carefully notice the difference between a braiding and a Yang–Baxter operator. A braiding is a piece of extra structure on the whole monoidal category $\mathcal{V}$ whereas a YB-operator is a piece of extra structure on a single object V in $\mathcal{V}$. A braiding gives a YB-operator on *every* object of $\mathcal{V}$ and more besides. This distinction will be important when we look at universal properties in the next section.

The monoidal category $\mathcal{V}ect$ has an obvious symmetry given by the switch map. There are interesting large examples of braided monoidal categories which are not symmetric; we provide two examples.

Super vector spaces. [21, 29] Consider the category $\mathbf{Z}_2\mathcal{V}ect_{\mathbf{C}}$ of $\mathbf{Z}_2$-graded complex vector spaces. The objects are pairs (A_0, A_1) of vector spaces and the arrows are pairs (f_0, f_1) of linear maps. There is a familiar tensor product on this category given by

$$(A_0, A_1) \otimes (B_0, B_1) = (A_0 \otimes B_0 \oplus A_1 \otimes B_1, A_0 \otimes B_1 \oplus A_1 \otimes B_0).$$

However, apart from the familiar associativity constraint and symmetry, we also have the associativity constraint and braiding which, for homogeneous x, y, z, are given by

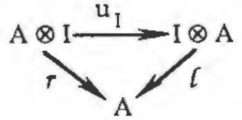
$$a((x \otimes y) \otimes z) = \begin{cases} (-1) x \otimes (y \otimes z) & \text{for x, y, z all odd,} \\ x \otimes (y \otimes z) & \text{otherwise,} \end{cases}$$

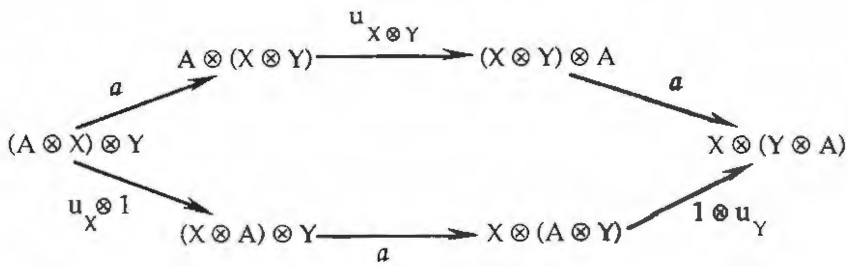
$$c(x \otimes y) = \begin{cases} \sqrt{-1}\, y \otimes x & \text{for x, y both odd,} \\ y \otimes x & \text{otherwise.} \end{cases}$$

The centre of a tensor category. [19, 28] The *centre* $\mathcal{Z}_\mathcal{V}$ of $\mathcal{V}$ is the category whose objects are pairs (A, u) where $A \in \mathcal{V}$ and

$$u : A \otimes - \xrightarrow{\sim} - \otimes A$$

is a natural isomorphism such that the following two conditions hold:





An arrow $f : (A, a) \longrightarrow (B, b)$ in $\mathcal{Z}_\mathcal{V}$ is an arrow $f : A \longrightarrow B$ such that, for all $X \in \mathcal{V}$,

$$b_X \circ (f \otimes 1) = (1 \otimes f) \circ a_X.$$

Moreover, $\mathcal{Z}_\mathcal{V}$ becomes a braided tensor category with tensor product given by

$$(A, a) \otimes (B, b) = (A \otimes B, (a \otimes 1) \circ (1 \otimes b)),$$

and braiding given by

$$c_{(A,a),(B,b)} = a_B : (A, a) \otimes (B, b) \longrightarrow (B, b) \otimes (A, a).$$

As an example, if H is a finite dimensional Hopf algebra with invertible antipode, there is an equivalence of monoidal categories

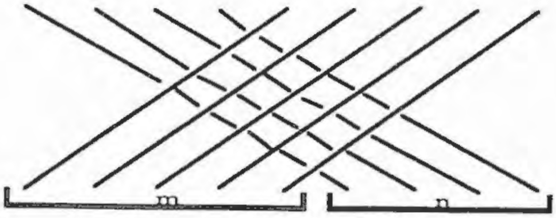
$$\mathcal{Z}_{H-\mathrm{Mod}} \xrightarrow{\sim} D(H) - \mathrm{Mod}$$

where D(H) is the Drinfeld double of H.

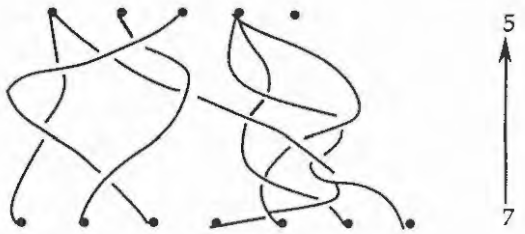
Some of the monoidal categories already defined above have natural braidings. A braiding for $\mathfrak{B}$ is given by the elements

$$c = c_{m,n} : m + n \longrightarrow n + m$$

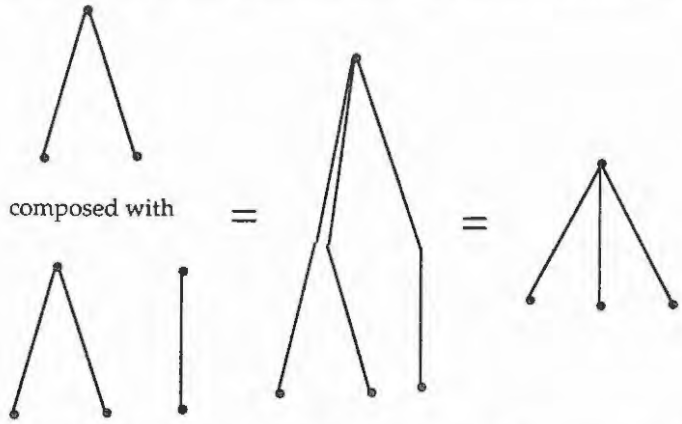
illustrated by the following figure.



There is a variant $\mathfrak{B}^{\#}$ of the braid category $\mathfrak{B}$; the objects are still the natural numbers, but there are arrows $m \longrightarrow n$ even when $m \neq n$, since different strings can end *flatly* at the same point as illustrated below.

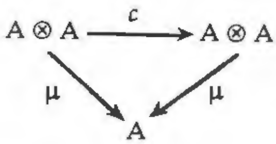


Composition is achieved by stacking the generalised braids vertically as before; but this time, if there is a string in the top "braid" beginning at a point of intersection of r strings from the bottom "braid", that top string must be replaced by r parallel strings which are spliced to the bottom r strings in order. For example,



Then $\mathfrak{B}^{\#}$ becomes braided monoidal as for $\mathfrak{B}$. Furthermore, the object $1 \in \mathfrak{B}^{\#}$ becomes a monoid by using the multiplication which looks like an upper-case lambda; the last diagram is half the proof of associativity.

DEFINITION. A monoid (A, μ, η) in a braided monoidal category $\mathcal{V}$ is called *commutative* when the following triangle commutes.



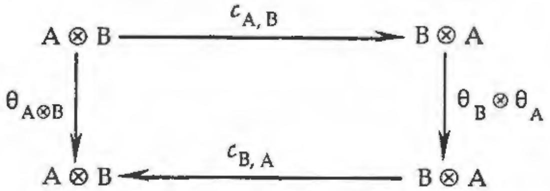
The firing category *firP* of a Petri net $\mathcal{P}$ is a commutative monoid in the monoidal category *Cat* of categories where the tensor is cartesian product. From another point of view, *firP* is a symmetric strict-monoidal category whose symmetry is an identity; such *strictly symmetric* monoidal categories are rather rare.

DEFINITION. [20] Suppose $\mathcal{V}$ is a braided monoidal category. A *(full) twist* for $\mathcal{V}$ is a natural family of isomorphisms

$$\theta = \theta_A : A \xrightarrow{\sim} A$$

such that $\theta_I = 1_I$ and the following diagram (T) commutes.

(T)

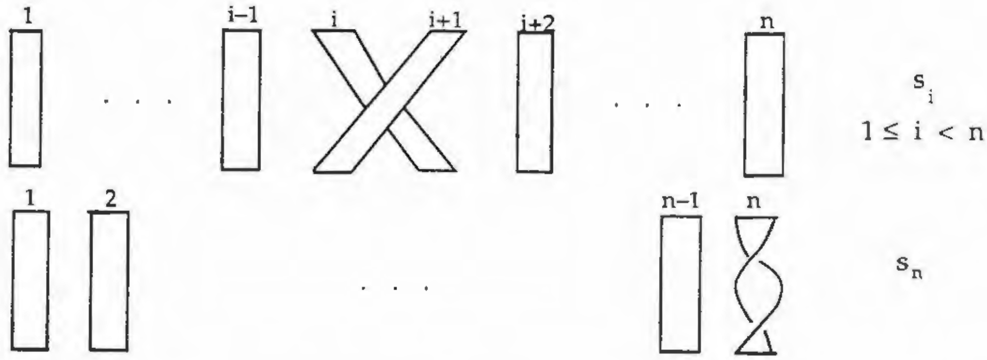


A monoidal category equipped with a braiding and a twist is called *balanced.* Note that the identity arrows $1_A : A \longrightarrow A$ form a twist iff the braiding is a symmetry.

There is a monoidal category $\tilde{\mathfrak{B}}$ defined similarly to $\mathfrak{B}$ except that the arrows are braids on ribbons (instead of on strings) and it is permissible to twist the ribbons through full turns. The objects of $\tilde{\mathfrak{B}}$ are the natural numbers, while the only non-empty homsets are the endomorphism monoids. The monoid $\tilde{\mathfrak{B}}(n, n)$ is the group generated by $s_1, ..., s_{n-1}, s_n$ subject to the usual braid relations for $s_1, ..., s_{n-1}$ and the further relation

$$s_{n-1} s_n s_{n-1} s_n = s_n s_{n-1} s_n s_{n-1}.$$

The pictorial representation of the generators is:



The braiding and twist are determined by $c_{1,1} = s_1 \in \tilde{\mathfrak{B}}(2,2)$ and $\theta_{1,1} = s_1 \in \tilde{\mathfrak{B}}(1,1)$. In this way, $\tilde{\mathfrak{B}}$ becomes a balanced monoidal category.

DEFINITION. [4] Suppose $\mathcal{V}$ is a balanced monoidal category. A *separable algebra* in $\mathcal{V}$ is a monoid A, $\mu : A \otimes A \longrightarrow A$, $\eta : I \longrightarrow A$ together with an arrow $\tau : A \longrightarrow I$ (called *trace*) such that

$$\varepsilon = \tau \circ \mu : A \otimes A \longrightarrow I$$

is a "symmetric bilinear form", by which we mean ε is an exact pairing and

$$\varepsilon \circ (\theta_A \otimes 1_A) = \varepsilon \circ c_{A,A}.$$

In any balanced monoidal category, if $\tau : X \otimes Y \longrightarrow I$ is an exact pairing then $A = X \otimes Y$ becomes a separable algebra with trace τ. Finite dimensional central separable algebras over a field are separable algebras in the category of vector spaces over that field [7; pp. 40, 49].

3. Free Monoidal Structures

Considerable mathematics is involved in passing from the geometric description of braids to Emil Artin's presentation of the braid groups. The result is a theorem which can be used to find representations of the geometry in algebra. In the last decade, many other geometric situations have been found to be describable algebraically, not by confining ourselves to groups, but in terms of other, naturally occurring, categorical structures. Category theorists had been looking at presentations of categorical structures in earlier decades under the general title of *coherence theorems*; but that work had a decidedly combinatorial flavour.

In order to discuss free structures and presentations, we need to look at the appropriate "homomorphisms" of monoidal structures. For expository purposes here, we shall make do with the strictest such notion of homomorphism. While there certainly are non-strict large monoidal categories, this approach may seem more reasonable if we point out that one coherence theorem [31] implies that each monoidal category is *equivalent* to a strict monoidal category.

Suppose $\mathcal{V}$, $\mathcal{W}$ are strict monoidal categories. A functor $F : \mathcal{V} \longrightarrow \mathcal{W}$ is called *strict monoidal* when $F(A \otimes B) = F(A) \otimes F(B)$ and $F(I) = I$. Note that strict monoidal functors take monoids to monoids and Yang–Baxter operators to Yang–Baxter operators. If $\mathcal{V}$, $\mathcal{W}$ are braided [symmetric], we call F *braided [symmetric]* strict monoidal when, furthermore, $F(c_A) = c_{F(A)}$. If $\mathcal{V}$, $\mathcal{W}$ are balanced, we call F *balanced* strict monoidal when, furthermore, $F(\theta_A) = \theta_{F(A)}$.

There is no mystery about the free strict monoidal category on one generating object. It is the discrete category N whose objects are the natural numbers and whose tensor product is addition. Explictly, this means that, for all strict monoidal categories $\mathcal{V}$ and all objects $X \in \mathcal{V}$, there exists a unique strict monoidal functor $F : N \longrightarrow \mathcal{V}$ with $F(1) = X$.

The useful general kind of free monoidal categories are those of the form *der*$\mathcal{R}$ for a rewrite system $\mathcal{R}$. There is a natural definition of morphism of rewrite system $P : \mathcal{R} \longrightarrow \mathcal{R}'$; namely, a morphism of directed graphs which preserves word length on vertices. Each monoidal category $\mathcal{V}$ determines a rewrite system *re*$\mathcal{V}$ whose alphabet is the set of objects of $\mathcal{V}$, and whose rewrite rules $A_1A_2...A_m \longrightarrow B_1B_2...B_n$ are arrows $A_1 \otimes A_2 \otimes ... \otimes A_m \longrightarrow B_1 \otimes B_2 \otimes ... \otimes B_n$ in $\mathcal{V}$. For each rewrite system $\mathcal{R}$, there is an inclusion morphism $J : \mathcal{R} \longrightarrow$ *reder*$\mathcal{R}$ such that, for all strict monoidal categories $\mathcal{V}$ and morphism $P : \mathcal{R} \longrightarrow$ *re*$\mathcal{V}$, there exists a unique strict monoidal functor $F :$ *der*$\mathcal{R} \longrightarrow \mathcal{V}$ such that *re*$F \circ J = P$.

The free strict monoidal category containing a monoid is the monoidal category Δ together with the monoid $\underline{1}$. This means that, given any monoid A in a strict monoidal category $\mathcal{V}$, there exists a unique strict monoidal functor $F : \Delta \longrightarrow \mathcal{V}$ with $F(\underline{1}) = A$ (as monoids). While this result is not as hard to prove as Artin's result on the braid groups, some works does need to be done, and the result has applications to cohomology and homotopy theory.

The free strict monoidal category containing an object equipped with a Yang–Baxter operator is the braid category $\mathfrak{B}$ containing 1 equipped with s_1. This is really a restatement of Artin's result. In particular, it means that there is a unique strict monoidal functor $F : \mathfrak{B} \longrightarrow \mathcal{M}at$ given by $F(1) = n$ and $F(s_1) = R_q$. This is the first step in the construction of the new knot polynomials from a Yang–Baxter operator [8, 50, 21].

In fact, there are *two* freeness properties for $\mathfrak{B}$. It is also the free braided strict monoidal category on one generating object [22]. This means that, for all braided strict monoidal categories $\mathcal{V}$ and all objects $A \in \mathcal{V}$, there exists a unique braided strict monoidal functor $F : \mathfrak{B} \longrightarrow \mathcal{V}$ with $F(1) = A$.

Similiarly [22], $\tilde{\mathfrak{B}}$ is the free balanced strict monoidal category on one generating object. It is also possible to define a *twist* on a Yang–Baxter operator, and another freeness property of $\tilde{\mathfrak{B}}$ is that it contains the "generic" object with such an operator.

Also, $\mathfrak{B}^{\#}$, and the monoid 1 therein, provide the free braided strict monoidal category containing a monoid.

There are many more results along these lines, perhaps the most important of these involving the category $\mathcal{T}ang$ of *tangles on ribbons* [8, 19, 42]. The endomorphisms of the unit object in this monoidal category are *isotopy classes of framed oriented links.* There are two freeness characterizations of $\mathcal{T}ang$ in terms of extra natural structure on the monoidal category itself, and in terms of extra structure on a Yang–Baxter operator. These results can be viewed as a sculpturing of the *Reidemeister moves,* which were devised specifically for knot theory, into a form immediately representable in algebraic systems. The extra structure needed on the balanced monoidal category is *duality* which is familiar for vector spaces; when appropriate axioms are satisfied we have a *tortile* (or *ribbon*) monoidal category. Turaev [51] has used tortile monoidal categories (and the *modular monoidal categories* of Lyubashenko–Turaev [30] which are additive tortile monoidal categories with extra axioms) to provide a precise mathematical foundation for the new invariants of 3-manifolds constructed by Witten using the physical ideas of 3D topological quantum field theorem.

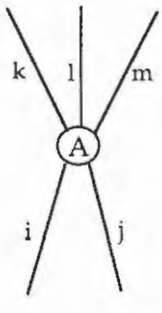
There is also a tangled version $\mathcal{T}ang\mathcal{S}urf$ on the monoidal category $\mathcal{S}urf$; the objects can be regarded as isotopy classes of Seifert surfaces [4]. Boyer–Joyal have shown that $\mathcal{T}ang\mathcal{S}urf$ is the free balanced monoidal category equipped with a separable algebra.

4. Penrose String Notation

Given that geometric figures arise in free monoidal structures, it is to be expected that these figures should appear in calculations within such structures. Perhaps the figures can aid the intuition more than algebraic equations. Penrose [38] introduced a variety of figures to help in the manipulation of tensors with many subscripts and superscripts in Einstein's general theory of relativity. The starting point is the representation of a tensor such as

$$A_{ij}^{klm}$$

by a string diagram

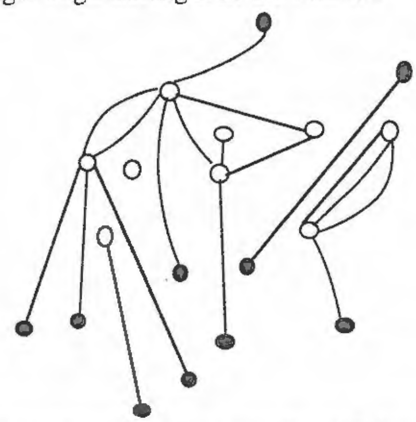


with one node labelled by A, strings below labelled by the subscripts, and strings above labelled by the superscripts. Such a tensor represents a linear map f :

$U \otimes V \longrightarrow W \otimes X \otimes Y$ with respect to chosen bases in the vector spaces U, V, W, X, Y. So this suggests the use of such string diagrams in arbitrary monoidal categories. This is indeed possible, as we now explain; for more details, see [20].

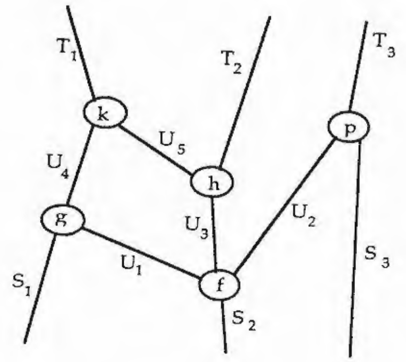
A *plane graph* Γ is a compact topological subspace of $\mathbb{R}^2$ with a distinguished set Γ_0 of points whose complement $\Gamma - \Gamma_0$ in Γ is homeomorphic to a finite union of disjoint open intervals. The elements of Γ_0 are called *vertices* and the connected components of $\Gamma - \Gamma_0$ are called *edges*. We say that (x, y) is *above* (x′, y′) in $\mathbb{R}^2$ when $y' \leq y$; *below* means the reverse. The plane graph Γ is called *progressive* when aboveness is a total (linear) order on each edge. Progressive plane graphs are directed graphs: the source and target of an edge are the vertices in the closure of the edge; the source is below the target.

A *progressive plane graph with boundary* consists of a progressive plane graph Γ with a distinguished set $i\Gamma$ of vertices such that each vertex in $\partial\Gamma = \Gamma_0 - i\Gamma$ is in the closure of precisely one edge, and $i\Gamma$ is an interval in the aboveness order on Γ_0 (that is, if p, q, r are vertices with p above q and q above r, then $p, r \in i\Gamma$ implies $q \in i\Gamma$). Notice that $\partial\Gamma$ is the disjoint union of the subset $s\Gamma$ of those vertices which are sources and the subset $t\Gamma$ of those vertices which are targets. For example, in the progressive plane graph depicted below, the white nodes provide an acceptable set $i\Gamma$; so the black nodes constitute $\partial\Gamma$, the cardinality of $s\Gamma$ is eight, and the cardinality of $t\Gamma$ is two. Of course, the size of the nodes is exaggerated for visibility. It is customary to omit the boundary (black) nodes from the picture, leaving loose the single edge having it in the closure.



Suppose Γ, Γ' are progressive plane graphs with boundary. We say that Γ is a *deformation* of Γ' when there exists a homeomorphism $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $h(\Gamma) = \Gamma'$, $h(\partial\Gamma) = \partial\Gamma'$, and h preserves the aboveness order on edges.

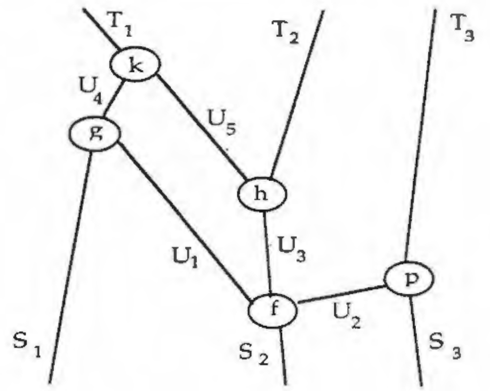
Suppose $\mathcal{V}$ is a monoidal category which we shall suppose is strict to simplify the discussion. A *string diagram* in $\mathcal{V}$ is a progressive plane graph with boundary labelled in $\mathcal{V}$ as below.



Each node of Γ is labelled by an arrow f, g, h, k, p and each string by an object S_i, U_j, T_r in $\mathcal{V}$ subject to the local compatibility condition that the source [respectively, target] of the arrow labelling a given node is the tensor product from left to right of the objects labelling the strings attached to and below [respectively, above] the node. In particular, $f : S_2 \longrightarrow U_1 \otimes U_3 \otimes U_2$ and $g : S_1 \otimes U_1 \longrightarrow U_4$. Each such string diagram can be assigned a *value* $\Gamma(v)$ which is an arrow in $\mathcal{V}$ with source equal to the tensor product of the bottom strings and target equal to the tensor product of the top strings. This is done by breaking the diagram up into layers containing only nodes which are on the same level, tensoring the arrows from left to right in each level, and then composing the results vertically. The value of the above diagram is therefore the composite:

$$\begin{aligned} &S_1 \otimes S_2 \otimes S_3 \xrightarrow{1\otimes f\otimes 1} S_1 \otimes U_1 \otimes U_3 \otimes U_2 \otimes S_3 \xrightarrow{g\otimes 1\otimes 1\otimes 1} \\ &\quad U_4 \otimes U_3 \otimes U_2 \otimes S_3 \xrightarrow{1\otimes h\otimes 1\otimes 1} U_4 \otimes U_5 \otimes T_2 \otimes U_2 \otimes S_3 \\ &\quad \xrightarrow{k\otimes 1\otimes p} T_1 \otimes T_2 \otimes T_3. \end{aligned}$$

The main result is that *the value of a diagram is invariant under deformation.* For example, the following diagram is a deformation of the diagram discussed above.



The value of this diagram is the composite

$$\begin{aligned} &S_1 \otimes S_2 \otimes S_3 \xrightarrow{1\otimes f\otimes 1} S_1 \otimes U_1 \otimes U_3 \otimes U_2 \otimes S_3 \xrightarrow{1\otimes 1\otimes 1\otimes p} \\ &\quad S_1 \otimes U_1 \otimes U_3 \otimes T_3 \xrightarrow{1\otimes 1\otimes h\otimes 1} S_1 \otimes U_1 \otimes U_5 \otimes T_2 \otimes T_3 \xrightarrow{g\otimes 1\otimes 1\otimes 1} \\ &\quad U_4 \otimes U_5 \otimes T_2 \otimes T_3 \xrightarrow{k\otimes 1\otimes 1} T_1 \otimes T_2 \otimes T_3. \end{aligned}$$

We leave it as an exercise to show that this is the same as the value of the earlier diagram.

There is an alternative description of the derivation category *der*$\mathcal{R}$ on a rewrite system $\mathcal{R}$ in terms of string diagrams. It is possible to define a *labelling* $v : \Gamma \longrightarrow \mathcal{R}$ of a progressive plane graph Γ with boundary in $\mathcal{R}$. The nodes of Γ are assigned edges of $\mathcal{R}$ and the edges of Γ are assigned vertices subject to the local compatibility condition explained above where now we use juxtaposition of words instead of tensor. Call the pair (Γ, v) a *(planar progressive) string diagram in* $\mathcal{R}$. The category *der*$\mathcal{R}$ has the words in elements of Σ as objects, has deformation classes of string diagrams in $\mathcal{R}$, has composition given by appropriate vertical stacking, and has tensor product given by horizontal placement. This description is isomorphic to the previous one in terms of derivations. Hence these string diagrams are correct for computation in a monoidal category.

There are also string diagrams perfectly adapted to braided monoidal categories. These are 3-dimensional instead of plane. For balanced monoidal categories there are 3-dimensional ribbon diagrams. For symmetry, combinatorial string diagrams are used. To account for duality in a monoidal category, progressiveness must be abandoned. The details of all this would take us too long to explain here (see [20] and its yet unpublished sequel); but the above cases should suffice to make the point.

5. 2-Dimensional Categorical Structures

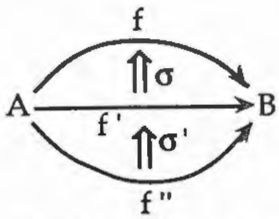
Bicategories [3] are to monoidal categories what categories are to monoids. Each monoid M (in the category of sets) can be regarded as a category ΣM with one object whose endoarrows are the elements of M; composition is the monoid multiplication. Similarly, each monoidal category $\mathcal{V}$ gives a bicategory $\Sigma\mathcal{V}$. However, just as we considered *strict* monoidal categories for simplicity, we shall consider only special bicategories called "2-categories" (see [26] for an introduction); again there is a coherence theorem which shows this loses little generality.

DEFINITION (Charles Ehresmann). A *2-category* $\mathcal{K}$ consists of a set of objects; a category $\mathcal{K}(A, B)$ (in which the composition is called *vertical* and denoted by $\bullet$) for each pair of objects A, B; functor

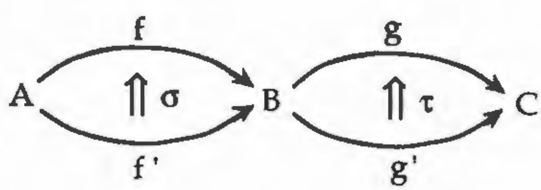
$$\circ : \mathcal{K}(B, C) \times \mathcal{K}(A, B) \longrightarrow \mathcal{K}(A, C)$$

(called *horizontal composition*); and objects $1_A \in \mathcal{K}(A, A)$ (called *horizontal identities*); subject to the conditions that horizontal composition is associative and the horizontal identities are two-sided identities for horizontal composition. The objects of $\mathcal{K}$ are sometimes called *0-cells*. The objects of categories $\mathcal{K}(A, B)$ are called *arrows* or *1-cells* of $\mathcal{K}$. The arrows of categories $\mathcal{K}(A, B)$ are called *2-cells* of $\mathcal{K}$.

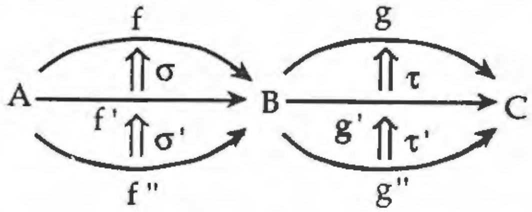
Vertical composition assigne a 2-cell $\sigma \bullet \sigma' : f'' \Rightarrow f : A \longrightarrow B$ to the following diagram,



while horizontal composition assigns a 2-cell $\tau \circ \sigma : g' \circ f' \Rightarrow g \circ f : A \longrightarrow C$ to the following diagram



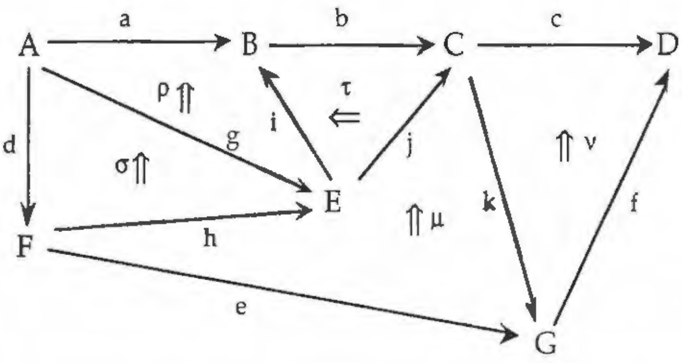
Functoriality of horizontal composition implies that the diagram



determines a unique 2-cell $g'' \circ f'' \Rightarrow g \circ f$ given by either side of the equation

$$(\tau \circ \sigma) \bullet (\tau' \circ \sigma') = (\tau \bullet \tau') \circ (\sigma \bullet \sigma');$$

this equation is called *the middle-four-interchange law.* It is possible to assign composites to far more complicated 2-dimensional cell complexes, called *pasting diagrams.* An example is below.



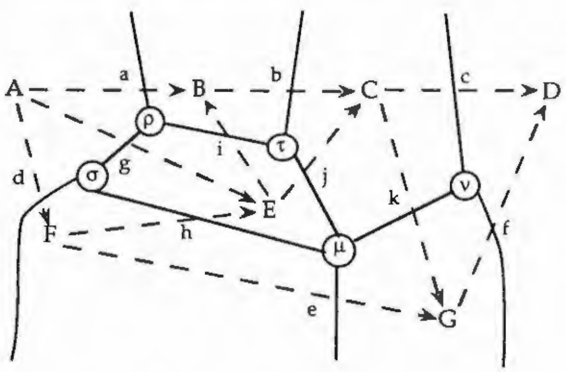
A 2-cell in a region means that it has source the (horizontal) composite of the arrows in the indicated source path, and target the composite of the arrows in the indicated target path: we could not have the 2-cell ρ as a horizontal double arrow since this would not properly indicate a source or target path. Also, the diagram needs to be "well formed": for example, if ρ were reversed, we would no longer have a pasting diagram. It is possible, using iterated horizontal and vertical pasting, to assign to each pasting diagram a uniquely determined 2-cell called its *pasting composite*. In our example, each 2-cell is "whiskered" by arrows on either side so that it is between arrows from A to B; this must be done in such a way that the resultant 2-cells are vertically composable; the pasting composite is that vertical composite. One way of obtaining the pasting composite of our diagram is as the composite

$$c \circ b \circ a \xleftarrow{b \circ b \circ \rho} c \circ b \circ i \circ g \xleftarrow{c \circ b \circ i \circ \sigma} c \circ b \circ i \circ h \circ d \xleftarrow{c \circ \tau \circ h \circ d}$$
$$c \circ j \circ h \circ d \xleftarrow{\nu \circ j \circ h \circ d} f \circ k \circ j \circ h \circ d \xleftarrow{f \circ \mu \circ d} f \circ e \circ d$$

in the category $\mathcal{K}(A, D)$. There are other ways, and they all lead to the same 2-cell from $f \circ e \circ d$ to $c \circ b \circ a$.

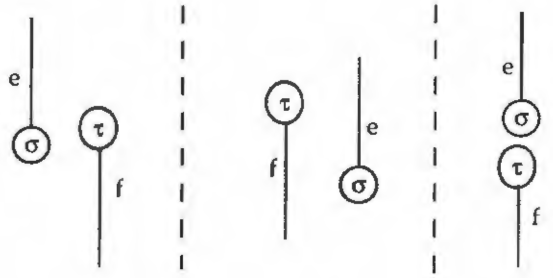
If $\mathcal{K}$ and $\mathcal{L}$ are 2-categories, a *2-functor* $F : \mathcal{K} \longrightarrow \mathcal{L}$ consists of three functions, between the sets of objects, arrows and 2-cells, respectively, such that sources, targets, horizontal identities and composition, and vertical identities and composition are all preserved.

We have pointed out that strict monoidal categories can be regarded as one-object 2-categories. In fact, the plane string diagrams used for monoidal categories can be used in the general setting of 2-categories. The translation from pasting diagrams to string diagrams is achieved by planar Poincaré duality. For example, in the above pasting diagram, each 2-cell ρ, σ, τ, ν, μ becomes a node labelled by the same symbol; each arrow a, b, c, ... becomes a string. A string is attached to a node when the original arrow formed part of the boundary of the region containing the 2-cell. Moreover, we require that the strings progress up the page from a node that had the arrow replaced by the string in its target towards a node that had the arrow replaced by the string in its source. The resultant graph, embedded in the plane and labelled in the 2-category, is a string diagram. The following diagram illustrates this process: the dotted arrows are the remnant of the pasting diagram while the solid parts make up the string diagram.



The technique for finding the value of a string diagram in a 2-category is precisely as in a monoidal category except that we now have 2-cells, arrows, horizontal composition, vertical composition (respectively) from the 2-category in place of arrows, objects, tensor product, composition from the monoidal category. The objects of the 2-category which appear in the pasting diagram correspond to regions bounded by the strings in the string diagram, but we usually do not mention them.

It turns out that the string diagrams have advantages over the pasting diagrams, especially in dealing with identity arrows which can occur in the diagrams. For example, the following three string diagrams (divided by the dotted partition lines) are deformations of each other for 2-cells $\sigma : 1_A \Rightarrow e : A \longrightarrow A$ and $\tau : f \Rightarrow 1_A : A \longrightarrow A$.



The ambiguity for pasting diagrams arises from the equations

$$\sigma \bullet \tau = \sigma \circ \tau = \tau \circ \sigma$$

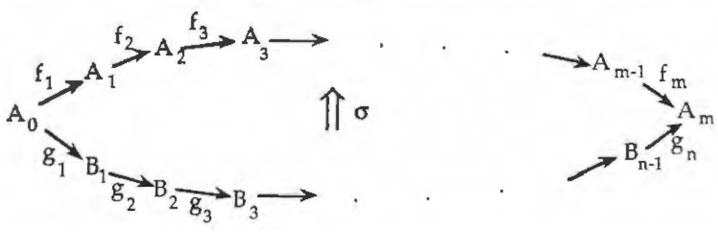
which follow from the middle-four-interchange law when σ, τ involve an identity in the indicated manner.

We now come to consider free 2-categories generated by combinatorial structures called "computads" (which are to 2-categories what rewrite systems are to monoidal categories). Recall that we write the edges of a directed graph G as arrows $f : A \longrightarrow B$ where A, B are the source, target vertices of f. A *(directed) path* from A_0 to A_n of length $n \geq 0$ in G is a diagram

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \ldots \xrightarrow{f_n} A_n.$$

Paths can be composed by concatenation, yielding the *category* $\mathcal{F}G$ *of paths* in the graph; the objects are the vertices of G and the arrows are the paths. The notation comes from the fact that $\mathcal{F}G$ is the *free category* on the graph in the obvious sense.

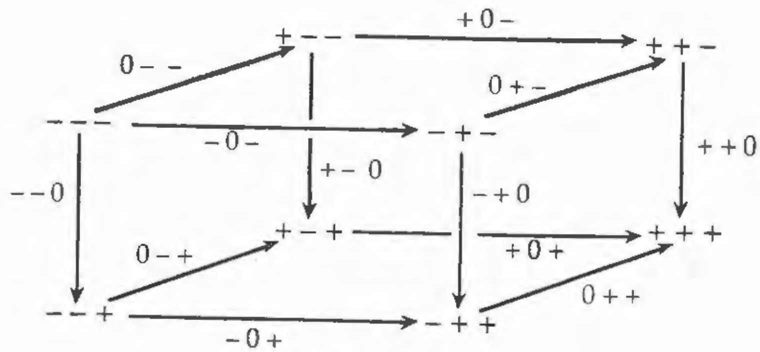
DEFINITION. [46] A *computad* C consists of a directed graph $C^{(1)}$ together with a set C_2, whose elements are called *2-cells*, and with an assignment to each 2-cell σ a source and target path in $C^{(1)}$; the source and target paths of each σ must have the same source vertex and the same target vertex as illustrated below.



Each rewrite system can be regarded as a computad with one object. The *free 2-category* $\mathcal{F}C$ on a computad C can be constructed in much the same way as the derivation monoidal category on a rewrite system. The objects $\mathcal{F}C$ are the vertices of C. The arrows are the paths of edges in C. The 2-cells can be taken to be deformation classes of plane progressive string diagrams labelled in the computad C. Sometimes we regard the 2-cells of $\mathcal{F}C$ as pasting diagrams labelled in C via planar Poincaré duality; but care needs to be taken with this when identity arrows are involved.

A *presentation* of a 2-category $\mathcal{A}$ consists of a computad C, a set R of pairs of 2-cells of $\mathcal{F}C$, and a 2-functor $\mathcal{F}C \longrightarrow \mathcal{A}$ which is universal amongst those which equate the 2-cells in each pair in R.

Commutative n-cube 2-category. [48] There is a 2-category $\mathit{Cub}[n,2]$ of "commutative" n-cubes defined as follows. The objects are words α in the symbols $-$, $+$ of length n (which we think of as vertices of an n-cube).



The 3-cube

For words α, β of length n in the symbols $-$, $+$, write $\alpha \leq \beta$ when α has the symbol $-$ in every position that β does, and let $\alpha\backslash\beta$ denote the set of positions where α has $-$ and β has $+$. There are no arrows in $\mathit{Cub}[n,2]$ from α to β unless $\alpha \leq \beta$ in which case an arrow $\alpha \longrightarrow \beta$ is a listing $u = u_1 u_2 \ldots u_k$ of the elements of $\alpha\backslash\beta$. With this notation, the source and target of $u : \alpha \longrightarrow \beta$ must be specified in order to fully determine the derivation. Put

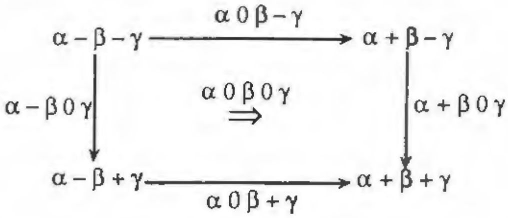
$$\sqrt{(u)} = \{(u_i, u_j) : i < j \text{ and } u_i < u_j\}.$$

Notice that, for arrows $u : \alpha \longrightarrow \beta$, $v : \beta \longrightarrow \gamma$, there is a partition of $\sqrt{(uv)}$ as

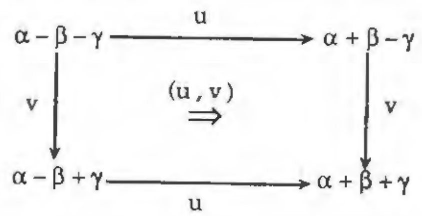
$$\sqrt{(uv)} = \sqrt{(u)} + \{(u_i, v_j) : u_i < v_j\} + \sqrt{(v)}.$$

For $u, u' : \alpha \longrightarrow \beta$, there is one, and only one, 2-cell $u \Rightarrow u'$ when $\sqrt{u'} \subseteq \sqrt{u}$, and none otherwise. Horizontal composition $\mathit{Cub}[n,2](\alpha,\beta) \times \mathit{Cub}[n,2](\beta,\gamma) \longrightarrow \mathit{Cub}[n,2](\alpha,\gamma)$ is concatenation of listings which is functorial (by the formula for $\sqrt{(uv)}$).

We shall provide a presentation of $\mathcal{C}ub[n, 2]$ which justifies the name and relates it to the commutative cube 2-category of Gray [14]. First, we describe a computad $\mathbb{I}[n, 2]$. The vertices are the objects of $\mathcal{C}ub[n, 2]$; that is, words α in the symbols $-$, $+$. The edges $e : \alpha \longrightarrow \beta$ are words e in the symbols $-$, $+$, 0 with exactly one occurrence of the symbol 0, where α is obtained from e by replacing the 0 by $-$ and β is obtained from e by replacing the $-$ by $+$. The 2-cells are oriented 2-faces of the n-cube which can be illustrated as below.



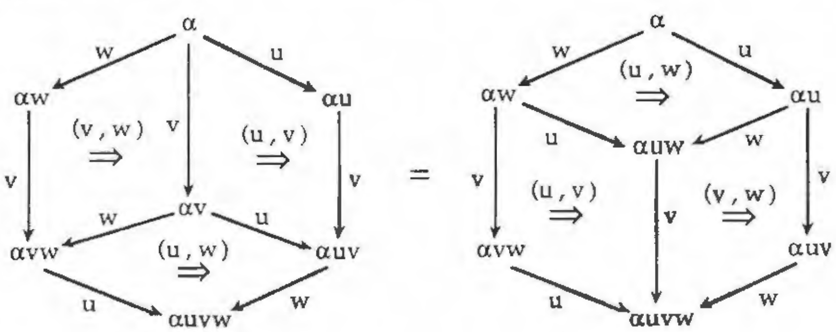
Each edge $e : \alpha \longrightarrow \beta$ of $\mathbb{I}[n, 2]$ can be regarded as an arrow $u : \alpha \longrightarrow \beta$ where u is the unique listing of the singleton set $\alpha \backslash \beta$ (u is the position where the 0 occurs in e). Then we can rewrite the above square as follows.



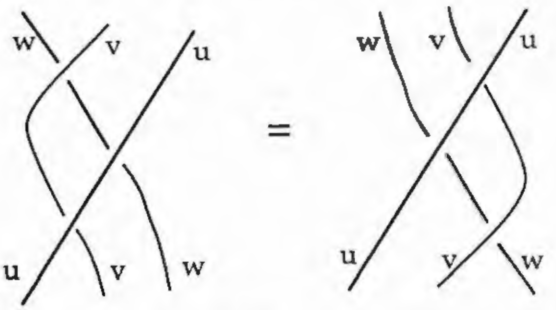
Notice in the last square that

$$\sqrt{(vu)} = \emptyset \subset \{(u, v)\} = \sqrt{(uv)},$$

so the 2-cells of the computad $\mathbb{I}[n, 2]$ give 2-cells in the 2-category $\mathcal{C}ub[n, 2]$. Thus we do have a 2-functor $\mathcal{F}\mathbb{I}[n, 2] \longrightarrow \mathcal{C}ub[n, 2]$. In $\mathcal{C}ub[n, 2]$, we have the following equality between pasted composites: for each object α with the symbol $-$ in positions $u < v < w$,

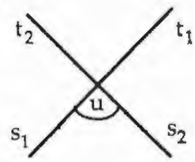


where αu denotes the result of changing $-$ to $+$ in position u of α. The corresponding diagrams drawn in $\mathcal{F}\mathbb{I}[n,2]$ (with $-$, $+$, 0 notation) are the relations R, called the *commuting 3-face relations*, for our presentation of the 2-category $\mathcal{C}ub[n,2]$. The proof of this is quite interesting, but will not be included here. The 2-category $\mathcal{C}ub[n,2]$ was given in terms of generators and relations in [14] who used the positive part of the braid groups to show its homcategories were ordered. To make a connection here with positive braids notice that the string diagrams for the commuting 3-face relations are as follows provided we depict the nodes as crossovers. (More will be said on this in Section 10.)



6. The Simplex Equations of Field Theory and Statistical Mechanics

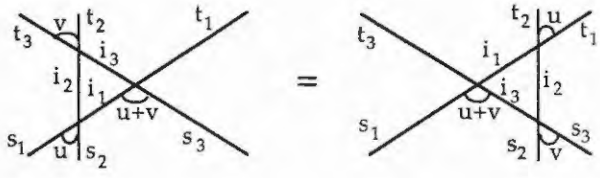
It is necessary to provide a little physical background from [16] before presenting the simplex equations. In field theory of one space and one time dimensions, two particles head towards a collision with inner states (such as charge or spin) s_1, s_2, have rapidity difference u at the point of collision, and exit with the inner states t_1, t_2. This is illustrated as follows.



The scattering amplitudes are recorded by a tensor

$$S^{t_2t_1}_{s_1s_2}(u)$$

called the *S-matrix*; also, u is called the *spectral parameter*. When there are more than two particles, there is a constraining condition for each set of three particles illustrated by

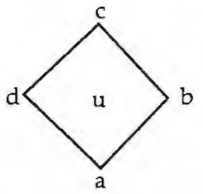


and expressed in terms of scattering amplitudes by the *factorization equation*:

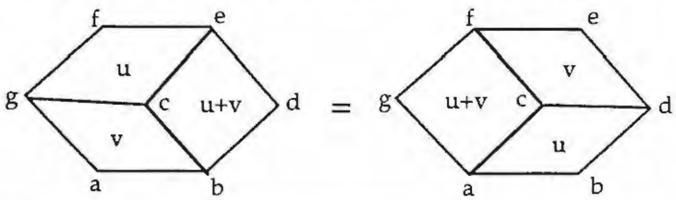
$$S^{i_2 i_1}_{s_1 s_2}(u) S^{i_3 t_1}_{i_1 s_3}(u+v) S^{t_3 t_2}_{i_2 i_3}(v) = S^{i_3 i_2}_{s_2 s_3}(v) S^{t_3 i_1}_{s_1 i_3}(u+v) S^{t_2 t_1}_{i_1 i_2}(u)$$

which is another form of the Yang–Baxter equation with a spectral parameter.

In statistical mechanics in two dimensions, spin variables are located on the sites of a square lattice. A Boltzmann weight w(a, b, c, d; u) is assigned to each spin configuration (a, b, c, d) round a unit square (the abbreviation IRF is used for "interaction round a face").



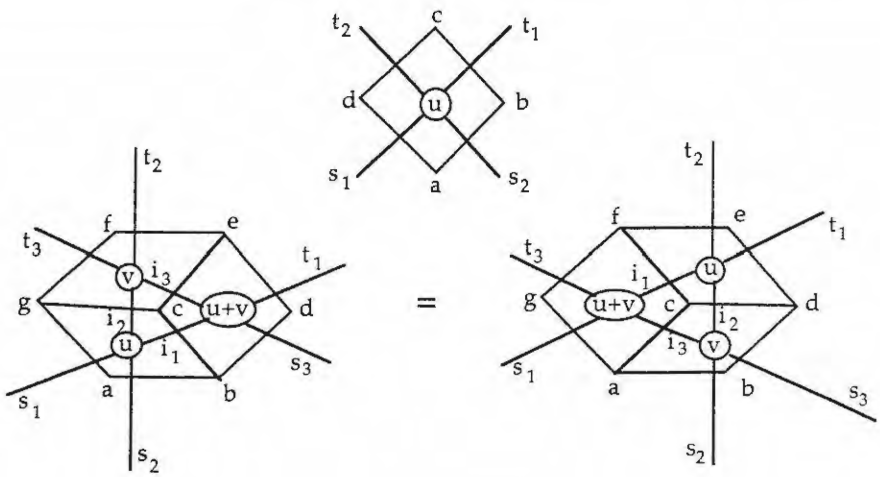
The compatibility condition is expressed diagrammatically by



resulting in the following equation, called the *star-triangle equation*:

$$\sum_c w(a,b,c,g;v)w(b,d,e,c;u+v)w(g,c,e,f;u)$$
$$= \sum_c w(a,b,d,c;u)w(a,c,f,g;u+v)w(c,d,e,f;v).$$

We notice here too that the relation between the factorization equation and the star-triangle equation comes via planar Poincaré duality.



The known solutions of the Yang–Baxter equation are in terms of elementary or, at worst, elliptic functions of the spectral parameter. A generalisation of the factorization equation has also been considered; this has the form

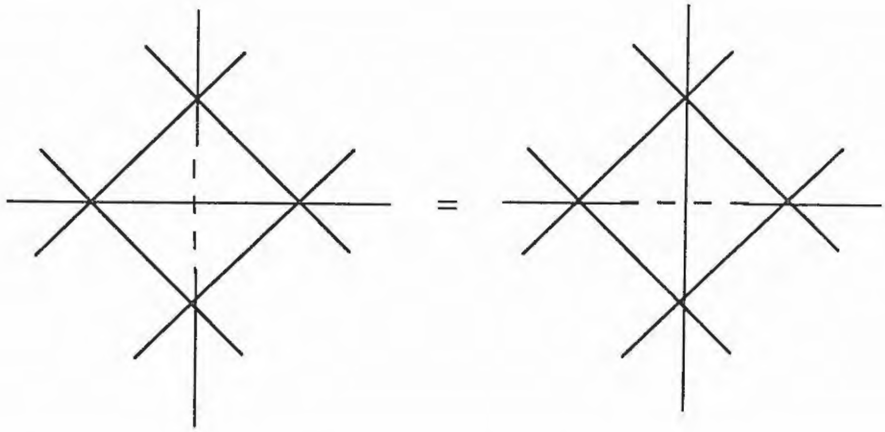
$$S^{i_2 i_1}_{s_1 s_2}(\theta_1, \theta_2) S^{i_3 t_1}_{i_1 s_3}(\theta_1, \theta_3) S^{t_3 t_2}_{i_2 i_3}(\theta_2, \theta_3) = S^{i_3 i_2}_{s_2 s_3}(\theta_2, \theta_3) S^{t_3 i_1}_{s_1 i_3}(\theta_1, \theta_3) S^{t_2 t_1}_{i_1 i_2}(\theta_1, \theta_2)$$

so that the earlier version is the special case where the S-matrix depends only on the difference of its two parameters. Another way of expressing this condition for the earlier version is

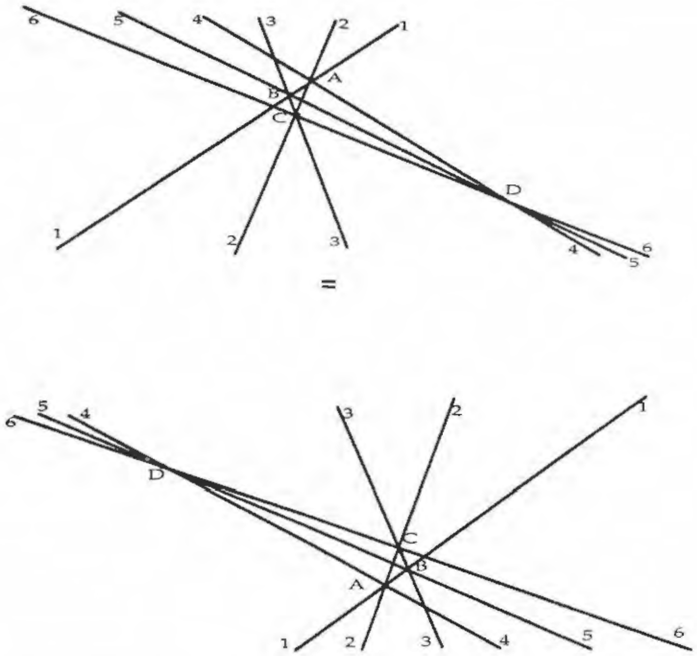
$$\det \begin{pmatrix} 1 & c_1 & c_2 \\ c_1 & 1 & c_3 \\ c_2 & c_3 & 1 \end{pmatrix} = 0,$$

where $c_i = \cos(\pi - \theta_i)$. New solutions of the generalised equation have been found in which the spectral parameters live on curves of genus > 1; however, these new solutions have been found to be related to new algebraic equations on "conventional" integrable models (the six-vertex model, in fact).

In field theory of two space and one time dimensions, instead of particles as points, we have particles represented by straight lines sweeping out planes ("world sheets"). The self-consistency of the factorization condition for the straight-string S-matrix requires the equality of the different formal expressions for the multiple string S-matrix in terms of the three-string amplitudes, corresponding to the different successions of three-string collisions. This is illustrated by Zamolodchikov as an equation between tetrahedra (thought of in three dimensional Euclidean space) as follows.



Each vertex of each tetrahedron represents the collision of three particles, and we use the letters A, B, C, D for the S-matrices of these respective collisions. The actual equation he derives for the condition that the S-matrix be factorizable can be obtained using a plane projection of the two tetrahedra



and then interpreting the result as Penrose diagrams for the four tensors. This gives the *tetrahedron (or 3-simplex) equation* of Zamolodchikov in the form:

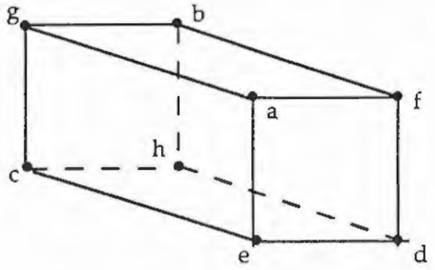
$$\begin{aligned} &A^{i_4 i_2 i_1}_{s_1 s_2 s_4}(\theta_1, \theta_2, \theta_4) B^{i_5 i_3 t_1}_{i_1 s_3 s_5}(\theta_1, \theta_3, \theta_5) C^{i_6 t_3 t_2}_{i_2 i_3 s_6}(\theta_2, \theta_3, \theta_6) D^{t_6 t_5 t_4}_{i_4 i_5 i_6}(\theta_4, \theta_5, \theta_6) \\ &= D^{i_6 i_5 i_4}_{s_4 s_5 s_6}(\theta_4, \theta_5, \theta_6) C^{t_6 i_3 i_2}_{s_2 s_3 i_6}(\theta_2, \theta_3, \theta_6) B^{t_5 t_3 i_1}_{s_1 i_3 i_5}(\theta_1, \theta_3, \theta_5) A^{t_4 t_2 t_1}_{i_1 i_2 i_4}(\theta_1, \theta_2, \theta_4) \end{aligned}$$

where the spectral parameters are related by the condition

$$\det \begin{pmatrix} 1 & c_1 & c_2 & c_4 \\ c_1 & 1 & c_3 & c_5 \\ c_2 & c_3 & 1 & c_6 \\ c_4 & c_5 & c_6 & 1 \end{pmatrix} = 0$$

with $c_i = \cos(\pi - \theta_i)$.

As with the Yang–Baxter (or *2-simplex*) equation, the tetrahedron equation has its translation into statistical mechanics. It becomes the condition that the transfer matrices of three-dimensional models commute. In the *interactions-round-a-cube* *model*, the following diagram gives the arrangements of spins at corners of a cube to which a Boltzmann weight w(a | efg | bcd | h) is assigned.

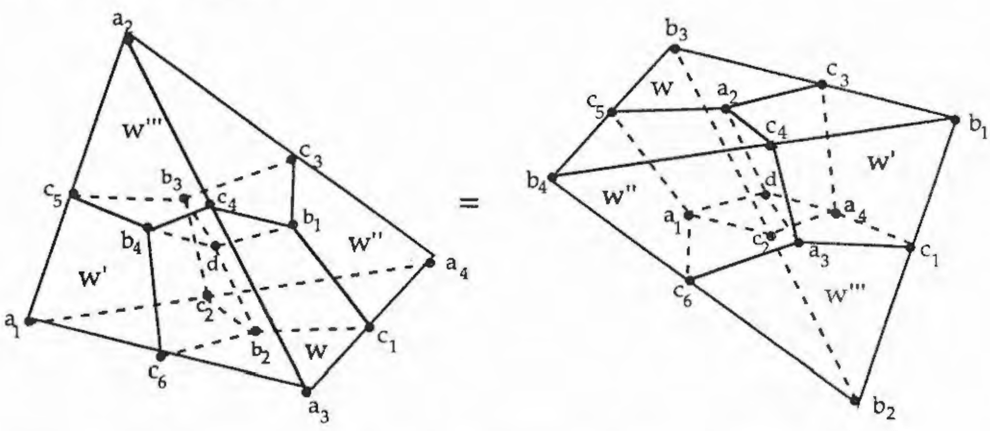


The layer-to-layer transfer matrix of a model having weight function w commutes with that of another model with weights w' if there exist two other weight functions w'', w''' such that the following *three-dimensional star-triangle equation* holds:

$$\sum_{d} w(a_4 \mid c_2\, c_1\, c_3 \mid b_1\, b_3\, b_2 \mid d)w'(c_1 \mid b_2\, a_3\, b_1 \mid c_4\, d_3\, c_6 \mid b_4)$$
$$w''(b_1 \mid dc_4\, c_3 \mid a_2\, b_3\, b_4 \mid c_5)w'''(d \mid b_2\, b_4\, b_3 \mid c_5\, c_2\, c_6 \mid a_1) =$$

$$\sum_{d} w'''(b_1 \mid c_1\, c_4\, c_3 \mid a_2\, a_4\, a_3 \mid d)w''(c_1 \mid b_2\, a_3\, a_4 \mid dc_2\, c_6 \mid a_1)$$
$$w'(a_4 \mid c_2\, dc_3 \mid a_2\, b_3\, a_1 \mid c_5)w(d \mid a_1\, a_3\, a_2 \mid c_4\, c_5\, c_6 \mid b_4)$$

for all values of the 14 "external" spins $a_1, \ldots, a_4, b_1, \ldots, b_4, c_1, \ldots, c_6$, and the summation is over the possible "internal" spins d. The pictorial representation of this is an equality between the following rhombic dodecahedra (or barycentrically subdivided tetrahedra).



There are 2^{14} equations here for the Ising-type model (as against the 2^6 in the 2-dimensional case). Remarkably, Zamolodchikov was able to conjecture a specific solution which Baxter later verified to be correct. It seems that this remains the only known solution although the techniques of Frenkel–Moore [2] may provide others.

Bazhanov–Stroganov proposed equations for general d-dimensions; these are the *d-simplex equations*. To discuss these, we need some abbreviations. Note that the information recorded in the Yang–Baxter and Zamolodchikov equations is completely contained in the subscripts of the tensors since the superscripts are obtained by reversal of order of subscripts on the superscripts and an obvious letter substitution; the spectral parameters are numbered in the same way as the subscripts. Moreover, the information in the subscripts is contained in their subscripts together with the information of where to change the letter from i to s on the left-hand side and from s to i on the right-hand side. Using this shorthand, the first four dimensions of the simplex equations can be written as

$$\begin{aligned} d &= 1: A_{*1}B_{1*} = B_{1*}A_{*1} \\ d &= 2: A_{*12}B_{1*3}C_{23*} = C_{23*}B_{1*3}A_{*12} \\ d &= 3: A_{*124}B_{1*35}C_{23*6}D_{456*} = D_{456*}C_{23*6}B_{1*35}A_{*124} \\ d &= 4: A_{*1247}B_{1*358}C_{23*69}D_{456*p} = E_{789p*}D_{456*p}C_{23*69}B_{1*358}A_{*1247} \end{aligned}$$

where we write $*$ to indicate where the letter change should occur, and we write p for the integer 10.

Despite the lack as yet of solutions for these equations in dimensions $d \geq 3$, it has been argued by Maillet–Nijhoff [33] that "... it is useful to consider the complete hierarchy of the d-simplex equations in order to get an insight into the algebraic structure of its individual members". I will show below that, in fact, these equations are part of a two-dimensional hierarchy depending not only on d but also v; the d-simplex equations correspond to the case where v = d+1. It is to be hoped that this will provide even more insight into the individual equations.

Maillet–Nijhoff explain a technique they call *breaking* (which is familiar from cohomology theory) for creating a hierarchy of equations from the first. We illustrate this to derive the form of the Yang–Baxter equation from the matrix commutativity equation. Begin by "breaking" the precise matrix commutativity (or 1-simplex) equality

$$A_s^i B_i^t = B_s^i A_i^t$$

by introducing an obstruction tensor R which is to measure the failure of equality. That is, let

$$R_{\alpha\beta}^{\gamma\delta} A_s^{\alpha i} B_i^{\beta t} = B_s^{\gamma i} A_i^{\delta t}.$$

To discover what conditions must be satisfied by the tensors R one calculates the result of associativity:

$$R_{\alpha\beta}^{\varepsilon\delta} R_{\delta\gamma}^{\zeta\eta} R_{\varepsilon\zeta}^{\lambda\kappa} A_s^{\alpha i} B_i^{\beta j} C_j^{\gamma t} \; = \; R_{\delta\gamma}^{\zeta\eta} R_{\varepsilon\zeta}^{\lambda\kappa} B_s^{\varepsilon i} A_i^{\delta j} C_j^{\lambda t}$$

$$= R^{\lambda\kappa}_{\varepsilon\zeta} B^{\varepsilon i}_{s} C^{\zeta j}_{i} A^{\eta t}_{j}$$
$$= C^{\lambda i}_{s} B^{\kappa j}_{i} A^{\eta t}_{j}$$
$$= R^{\kappa\eta}_{\delta\varepsilon} C^{\lambda i}_{s} A^{\delta j}_{i} B^{\varepsilon t}_{j} = R^{\lambda\delta}_{\alpha\zeta} R^{\kappa\eta}_{\delta\varepsilon} A^{\alpha i}_{s} C^{\zeta j}_{i} B^{\varepsilon t}_{j}$$
$$= R^{\zeta\varepsilon}_{\beta\gamma} R^{\lambda\delta}_{\alpha\zeta} R^{\kappa\eta}_{\delta\varepsilon} A^{\alpha i}_{s} B^{\beta j}_{i} C^{\gamma t}_{j}.$$

Since this holds for all A, B, C, this leads us to the Yang–Baxter equation

$$R^{\varepsilon\delta}_{\alpha\beta} R^{\zeta\eta}_{\delta\gamma} R^{\lambda\kappa}_{\varepsilon\zeta} = R^{\zeta\varepsilon}_{\beta\gamma} R^{\lambda\delta}_{\alpha\zeta} R^{\kappa\eta}_{\delta\varepsilon}.$$

We shall show below that breaking corresponds to "laxification" in higher-order category theory; that is, the process of putting higher-order cells into diagrams which previously commuted and then finding the appropriate coherence conditions on these cells. Which reminds me of a small anecdote. In the early 1970's when I published my paper "Two constructions on lax functors" [45], Professor Fred Chong at Macquarie University asked me whether this had anything to do with the mathematician Peter D. Lax. I denied the connection. I was wrong! Maillet–Nijhoff also use the breaking technique on the 2-dimensional Lax linear system

$$\phi^{\ell}(m, n) = L_{\ell}(m, n)\phi(m, n)$$

to develop higher-order Lax equations. So, in fact, laxification applies to the Lax equation!

Before closing this section, it should be pointed out that the d-simplex equation is easily obtained for any d by using matrices. The idea should be clear from the following equality which represents the 4-simplex equation (the earlier equations are obtained from square top left-hand blocks of the left-hand matrix and square bottom left-hand blocks of the right-hand matrix).

$$\begin{bmatrix} * & 1 & 2 & 4 & 7 \\ 1 & * & 3 & 5 & 8 \\ 2 & 3 & * & 6 & 9 \\ 4 & 5 & 6 & * & 10 \\ 7 & 8 & 9 & 10 & * \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 & 10 & * \\ 4 & 5 & 6 & * & 10 \\ 2 & 3 & * & 6 & 9 \\ 1 & * & 3 & 5 & 8 \\ & 1 & 2 & 4 & 7 \end{bmatrix}$$

The 2-dimensional hierarchy involving these matrices is under examination in joint work of Iain Aitchison and the author. We conclude this section by remarking on the similarity between these formal matrices and the determinantal conditions, mentioned earlier, on the spectral parameters.

7. Higher-Order Categories and Computads

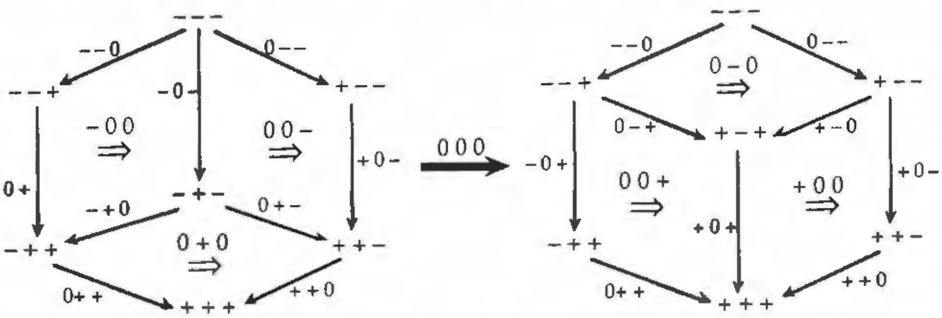
This section is based on papers [47, 1, 18, 2, 17, 39, 49, 40, 44, 12].

DEFINITION. A *3-category* $\mathcal{T}$ is defined exactly as a 2-category (see Section 5) except that each hom $\mathcal{T}(A, B)$ is a 2-category and the composition $\mathcal{T}(B, C) \times$

$\mathcal{T}(A,B) \longrightarrow \mathcal{T}(A,C)$ is a 2-functor. The 2-cells of the 2-category $\mathcal{T}(A,B)$ are called *3-cells* of $\mathcal{T}$; ignoring these 3-cells, what remains of $\mathcal{T}$ is a 2-category whose 0-cells, 1-cells, 2-cells are given the same name in $\mathcal{T}$. At this stage it is probably necessary to be systematic about the notation for the compositions: the vertical composition of $\mathcal{T}(A,B)$ is denoted by $\circ_2$, the horizontal composition of $\mathcal{T}(A,B)$ is denoted by $\circ_1$, and the composition $\mathcal{T}(B,C) \times \mathcal{T}(A,B) \longrightarrow \mathcal{T}(A,C)$ is denoted by $\circ_0$. A *3-functor* $F : \mathcal{T} \longrightarrow \mathcal{U}$ consists of four functions taking r-cells of $\mathcal{T}$ to r-cells of $\mathcal{U}$, r = 0, 1, 2, 3, in such a way as to preserve all sources and targets, and all compositions and their identities. The inductive definition of *n-category* and *n-functor*, for n = 0, 1, 2, 3, ..., should now be clear.

DEFINITION. A *3-computad* E consists of a (2-)computad $E^{(2)}$ together with a set E_3, whose elements are called *3-cells*, and with an assignment to each 3-cell x a source and target 2-cell in the free 2-category on $E^{(2)}$; the source and target paths of each 3-cell x must have the same source arrow and the same target arrow.

Here is an example of a 3-computad $\mathbb{I}[3]$ with one 3-cell called 0 0 0.

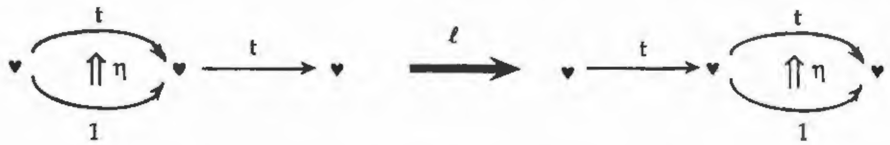


This 3-cell laxifies or "breaks" the equality between these two pasting composites in *Cub* [3, 2].

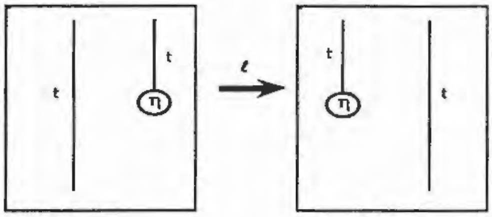
It is possible to construct the free 3-category $\mathcal{F}E$ on a 3-computad E. The 2-category obtained from $\mathcal{F}E$ by ignoring 3-cells is just the free 2-category $\mathcal{F}E^{(2)}$ on the (2-)computad $E^{(2)}$. The 3-cells are generated by the basic 3-cells of E under the three compositions, factored out by relations to force the various middle-four-interchange laws. (The idea that the 3-cells should be realized geometrically is even harder to make precise here than in the 2-category case.) Then it should be clear how to inductively define the concepts of *n-computad* and *free n-category on an n-computad.*

Simplicial 3-category. [27] The simplicial category Δ is in fact a 2-category: for order-preserving functions $f, g : \underline{m} \longrightarrow \underline{n}$, there is a 2-cell $f \Rightarrow g$ iff $f \leq g$ (in the pointwise order); and then there is only one such 2-cell. The tensor product of Δ is a 2-functor, and so Δ becomes a strict monoidal 3-category. Extending the ideas at the beginning of Section 5, we obtain a 3-category $\Sigma\Delta$ with a single object

♡; this is the *simplicial 3-category*. There is a nice presentation of this 3-category. Consider the 3-computad E with one vertex ♡, one edge $t : ♡ \longrightarrow ♡$, two 2-cells $\eta : 1 \Rightarrow T : ♡ \longrightarrow ♡$, $\mu : t \circ_0 t \Rightarrow t : ♡ \longrightarrow ♡$, and the following single 3-cell.



We can think of ℓ as a surgery rule which can be applied locally to string diagrams if we illustrate it as follows.



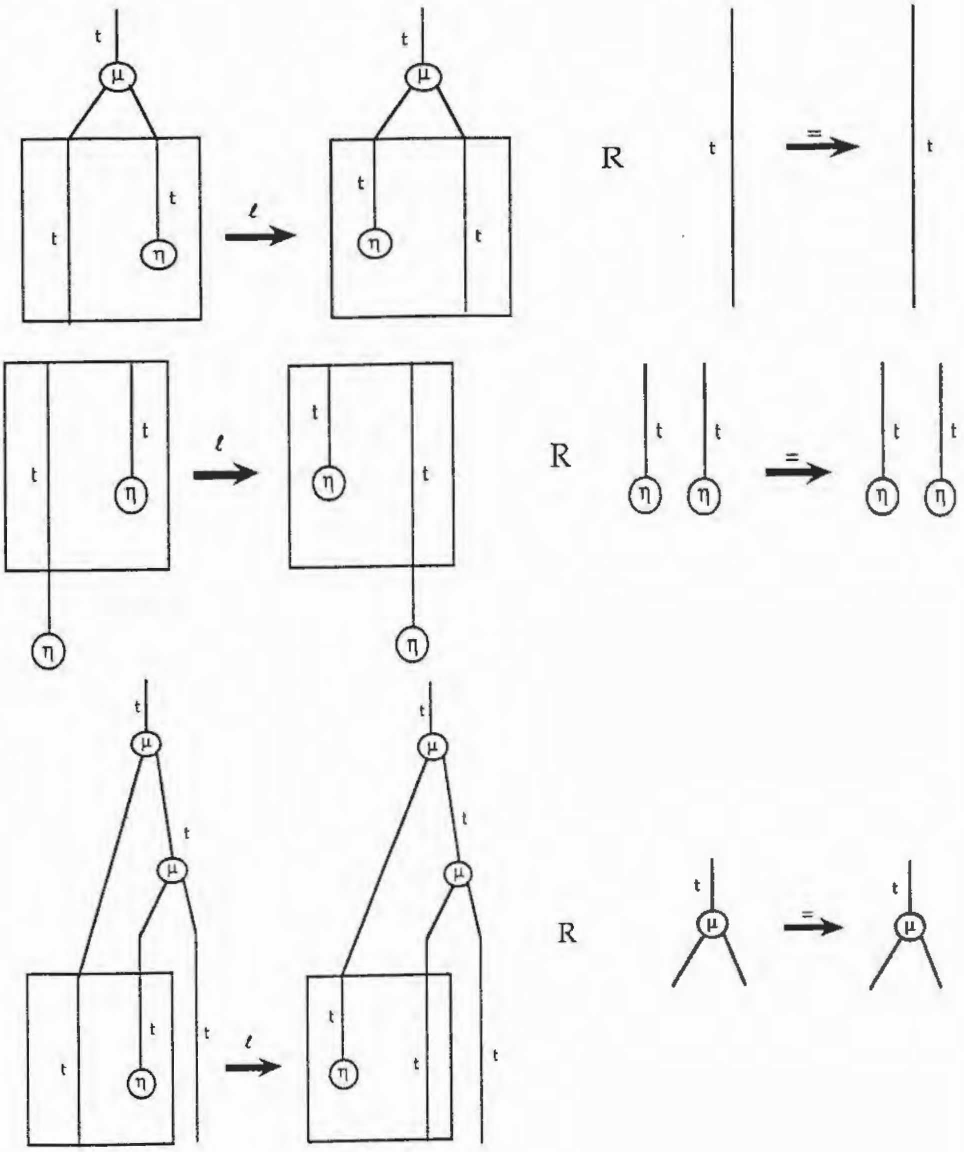
In the free 3-category $\mathcal{F}$E, we consider the following three relations R. (see next page)
The 3-computad E together with the relations R provide a presentation of the simplicial 3-category $\Sigma\Delta$.

The problem with the concept of n-computad for $n > 3$ is that the definition involves free r-categories for $r < n$ and these are generally difficult to describe. Fortunately, in good geometric examples coming from convex polytopes and the like, there is a great deal of redundancy in the data needed to fully specify the n-computad. We shall explain this in more detail.

Each 3-cell x of a 3-computad E determines a source 2-cell $s_2(x)$ and a target 2-cell $t_2(x)$ in the free 2-category $\mathcal{F}E^{(2)}$. These 2-cells can be represented by string diagrams in $E^{(2)}$. Write x^- for the *set* of 2-cells of $E^{(2)}$ which label the nodes of a string diagram for $s_2(x)$, and write x^+ for the *set* of 2-cells of $E^{(2)}$ which label the nodes of a string diagram for $t_2(x)$. (These two sets are independent of the choices of string diagrams in the deformation classes.) So we have two functions

$$(-)^-, (-)^+ : E_3 \longrightarrow \mathcal{P}(E_2)$$

where $\mathcal{P}(E_2)$ is the power set of the set of 2-cells of $E^{(2)}$. In considering only the labels on nodes of a string diagram, we are, in general, disregarding quite a lot of information about the string diagram. Hence, the following result may come as a surprise:



PROPOSITION. *The 3-computads* E *arising from many convex polytopes such as the cubes are uniquely determined by the 2-computad* $E^{(2)}$ *and the functions* $(-)^{-}$, $(-)^{+} : E_3 \longrightarrow \mathcal{P}(E_2)$.

At the lower dimension, the corresponding result is easily understood. For, suppose C is a (2-)computad. Then, for each 2-cell $\sigma \in C_2$, we have source, target paths $s_1(\sigma)$, $t_1(\sigma)$, and we can write σ^-, σ^+ for the *sets* of 1-cells in the directed graph $C^{(1)}$ which occur in the respective paths. Provided the graph $C^{(1)}$ has no circuits, the only information we need to reconstruct the paths from the set is the order. However, the order is forced by knowledge of the source, target functions $s_0, t_0 : C_1 \longrightarrow C_0$ of $C^{(1)}$. So the 2-dimensional version of the Proposition is true. To be consistent at even the lowest dimension, we can define $f^- = \{A\}$, $f^+ = \{B\}$ for each 1-cell $f : A \longrightarrow B$ of C.

In this way, each n-computad E leads to a graded set E_k, $0 \le k \le n$, whose elements are called *k-cells*, together with functions $(-)^-, (-)^+ : E_k \longrightarrow \mathcal{P}(E_{k-1})$, $0 < k \le n$. This is the basic structure involved in the higher-dimensional combinatorial notion of circuit-free graph which I have called *parity complex* and Michael Johnson has called *pasting scheme*. However, a parity complex (and likewise, a pasting scheme) is to satisfy some axioms which are not true of all such structures underlying n-computads. The axiom which somewhat reflects the source-target equations in a computad is, for all cells x of dimension ≤ 2, the equality of sets

$$x^{--} \cup x^{++} = x^{-+} \cup x^{+-},$$

where the unions are disjoint, and, for example, S^- is the union of the sets x^-, $x \in S$, for any $S \subset E_k$.

Let H denote a parity complex of dimension n; thus H consists of a graded set $H = \sum_{0 \le k \le n} H_k$ and functions $(-)^-, (-)^+ : H_k \longrightarrow \mathcal{P}(H_{k-1})$ for $0 < k \le n$. The free n-category $\mathcal{O}$H on H will now be succinctly described; the purpose of providing the precise description here is to show that it is purely combinatorial. An n-cell of $\mathcal{O}$H is a pair (M, P) of non-empty finite subsets M, P of H such that the following conditions hold (where $\neg S$ means the complement of S in H):

(i) each of M and P contains at most one element of H_0 and, for all $x \ne y$ in H_K with $k > 0$, if both $x, y \in M$ or if both $x, y \in P$, then the set $(x^- \cap y^-) \cup (x^+ \cap y^+)$ is empty;

(ii) $P = (M \cup M^+) \cap \neg M^-$, $M = (P \cup M^-) \cap \neg M^+$, $P = (M \cup P^+ \cap \neg P^-$, $M = (P \cup P^-) \cap \neg P^+$.

The *k-source* and *k-target* of (M, P) are defined as follows (where $S_k = H_k \cap S$ and $S^{(k)} = \sum_{h \le k} S_h$ for any subset S of H);

$$s_k(M, P) = (M^{(k)}, M_k \cup P^{(k-1)}), \quad t_k(M, P) = (M^{(k-1)} \cup P_k, P^{(k)}).$$

An ordered pair of cells (M, P), (N, Q) is called *k-composable* when

$$t_k(M, P) = s_k(N, Q),$$

in which case their *k-composite* is defined by

$$(M, P) \circ_k (N, Q) = (M \cup (N \cap \neg N_k), (P \cap \neg P_k) \cup Q).$$

The *k-cells* of OH are the n-cells (M, P) with $s_k(M, P) = (M, P)$. The proof that OH is an n-category is non-trivial. There is a dimension preserving injective function

$$x \mapsto \langle x \rangle : H \longrightarrow OH$$

given inductively as follows: for $x \in H_k$, put $\langle x \rangle = (M, P)$ where

$$\begin{aligned} M_k &= P_k = \{x\}, \\ M_{r-1} &= (M_r)^- \cap \neg(M_r)^+, \quad P_{r-1} = (P_r)^- \cap \neg(P_r)^+ \quad \text{for } 0 < r \leq k. \end{aligned}$$

It is also non-trivial to prove that OH is the *free* n-category generated by the cells $\langle x \rangle$, $x \in H$.

Following Aitchison's ideas for cubes and simplexes, we note that it is possible to use string-like diagrams to keep track of facial relations in consecutive dimensions of parts of a parity complex. Specifically, suppose we have disjoint finite sets M, X and functions

$$(-)^-, (-)^+ : M \longrightarrow \mathcal{P}(X)$$

such that, for all $m \neq n$ in M,

$$(m^- \cap n^-) \cup (m^+ \cap n^+) = \emptyset.$$

Put

$$\partial M = \{(-, x) : x \in X, x \notin M^+\} \cup \{(+, x) : x \in X, x \notin M^-\}.$$

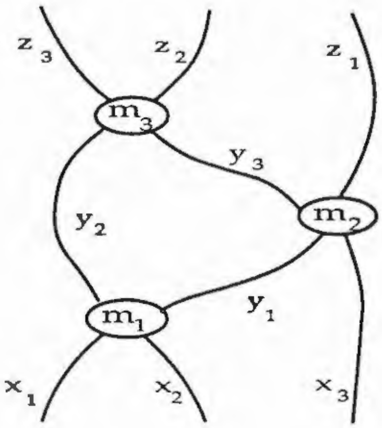
Then there is a graph $s, t : X \longrightarrow M \cup \partial M$ given by

$$\begin{aligned} &x \in s(x)^- \cap t(x)^+ \quad \text{for } x \in M^- \cap M^+, \\ &s(x) = (-, x) \quad \text{for } x \notin M^+, \quad \text{and} \\ &t(x) = (+, x) \quad \text{for } x \notin M^-. \end{aligned}$$

There is no reason why such a graph should be planar; however, we do draw it in the plane, with edges directed up, sometimes crossing at non-nodes, with each *inner node* $m \in M$ labelled by m, with each *outer node* in ∂M left undistinguished, and with each edge $x \in X$ labelled by x. For example, if we have

$$\begin{aligned} M &= \{m_1, m_2, m_3\}, \quad X = \{x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3\}, \\ m_1^- &= \{x_1, x_2\}, \quad m_2^- = \{y_1, x_3\}, \quad m_3^- = \{y_1, y_3\}, \\ m_1^+ &= \{y_1, y_2\}, \quad m_2^+ = \{y_3, z_1\}, \quad m_3^+ = \{z_2, z_3\}, \end{aligned}$$

then the string diagram is our old friend below.



It turns out that reasonable polytopes such as cubes and simplexes give rise to parity complexes: the axioms are satisfied. the previous Proposition generalises.

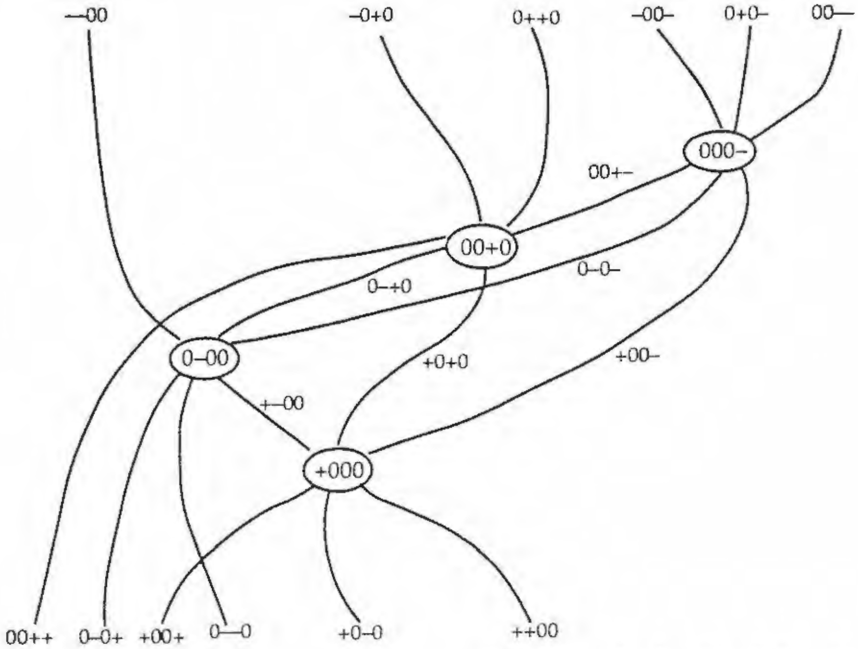
PROPOSITION. *The n-computads arising from many convex polytops (such as the cubes and simplexes) are uniquely determined by their parity complexes.*

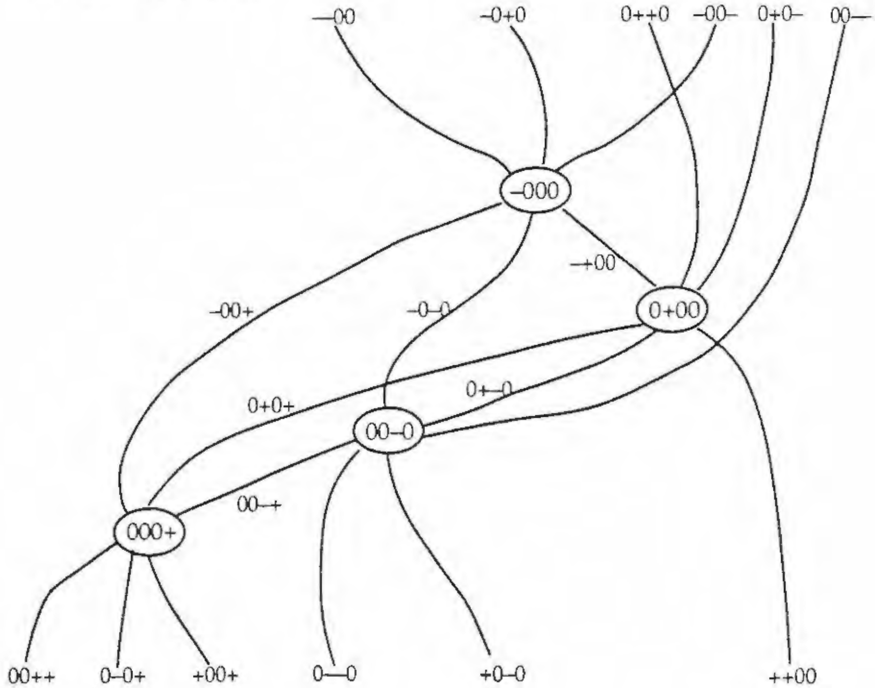
8. The (v, d)-Cube Equations

There is a v-computad $\mathbb{I}[v]$ determined by the v-cube with an appropriately oriented d-cell in each d-face. The underlying d-computad $\mathbb{I}[v]^{(d)}$ of $\mathbb{I}[v]$ is denoted by $\mathbb{I}[v,d]$. In particular, we look at the 3-computad $\mathbb{I}[4,3]$. The set $\mathbb{I}[4,3]_k$ of k-cells contains the words of length 4 in the symbols $-$, 0, $+$ where the symbol 0 occurs precisely k times. For example,

$$\mathbb{I}[4,3]_3 = \{-000, 0-00, 00-0, 000-, +000, 0+00, 00+0, 000+\}$$

and the parity complex structure is recorded by the string-like diagrams as shown below. The resemblance to the string diagrams for the 3-simplex equation should now come as no surprise since the Boltzmann weights viewpoint already led us to cubes. By the Proposition of last section, each of these string-like diagrams represents a 3-cell in the 3-category $\mathcal{F}\mathbb{I}[4,3]$. The *commuting 4-face relation* is the equality between these two 3-cells. The 3-computad $\mathbb{I}[4,3]$ together with the commuting 4-face relation provides a presentation of a 3-category $Cub[4,3]$.





From the last section, as with any parity complex, we have a combinatorial model $O\mathbb{I}[v, d]$ for the free d-category $\mathcal{F}\mathbb{I}[v, d]$ on the d-skeleton of the v-cube (where we are abusively using the notation $\mathbb{I}[v,d]$ for the d-computad and its parity complex of dimension d). There is also the *commutative v-cube d-category* $\mathit{Cub}[v,d]$ obtained from the d-category $\mathcal{F}\mathbb{I}[v,d]$ by imposing the *commuting (d+1)-face relations*. We do not have a nice model for $\mathit{Cub}[v,d]$ as for the case of $\mathit{Cub}[v,2]$ given in Section 5; however, to give a representation of a commutative v-cube in a d-category $\mathcal{X}$, we just take a (d+1)-functor $\mathcal{F}\mathbb{I}[v,d+1] \longrightarrow \mathcal{X}$; the non-identity (d+1)-cell of $\mathcal{F}\mathbb{I}[v,d+1]$ is necessarily taken to an identity (d+1)-cell in $\mathcal{X}$ (since the way we regard d-categories as (d+1)-categories is by giving them only identity (d+1)-cells).

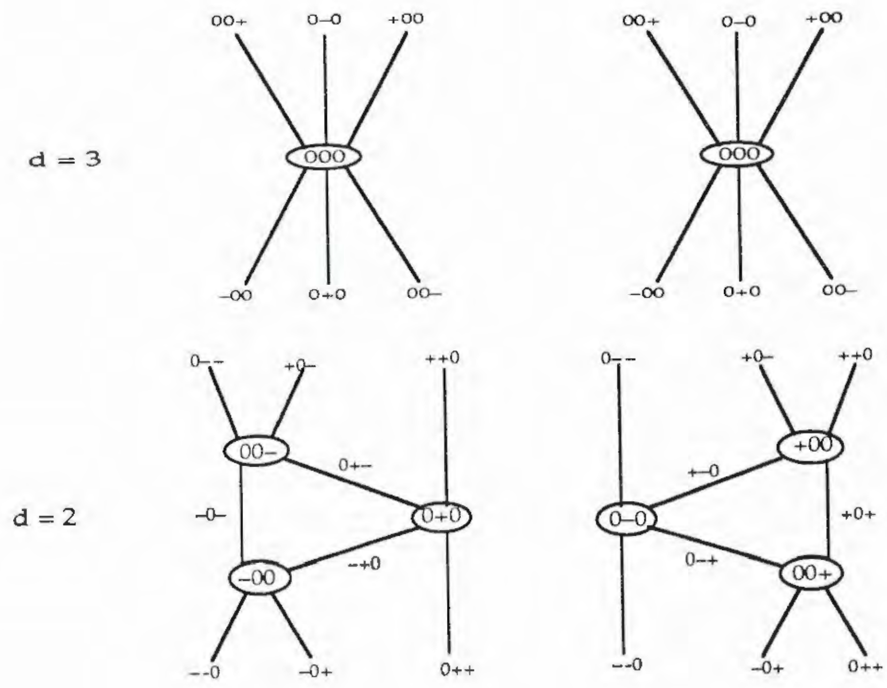
We now form a "Pascal Triangle" which has two entries in position (v, d); namely, the string diagrams for

$$(-)^-, (-)^+ : M_d \longrightarrow \mathcal{P}\mathbb{I}[v,d]_{d-1} \text{ and } (-)^-, (-)^+ : P_d \longrightarrow \mathcal{P}\mathbb{I}[v,d]_{d-1}$$

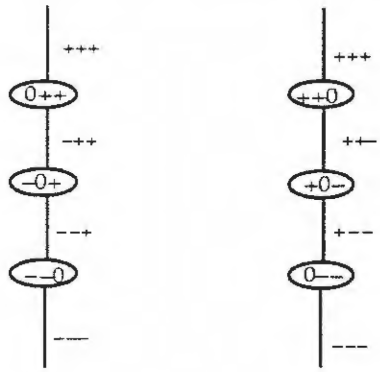
where $\langle x \rangle = (M, P)$ for $x = 00...0 \in \mathbb{I}[v, v]$. This triangle of string diagrams is due to Iain Aitchison, so we call it the *Aitchison–Pascal Triangle*. For example, take v = 3 so that x = 0 0 0, and

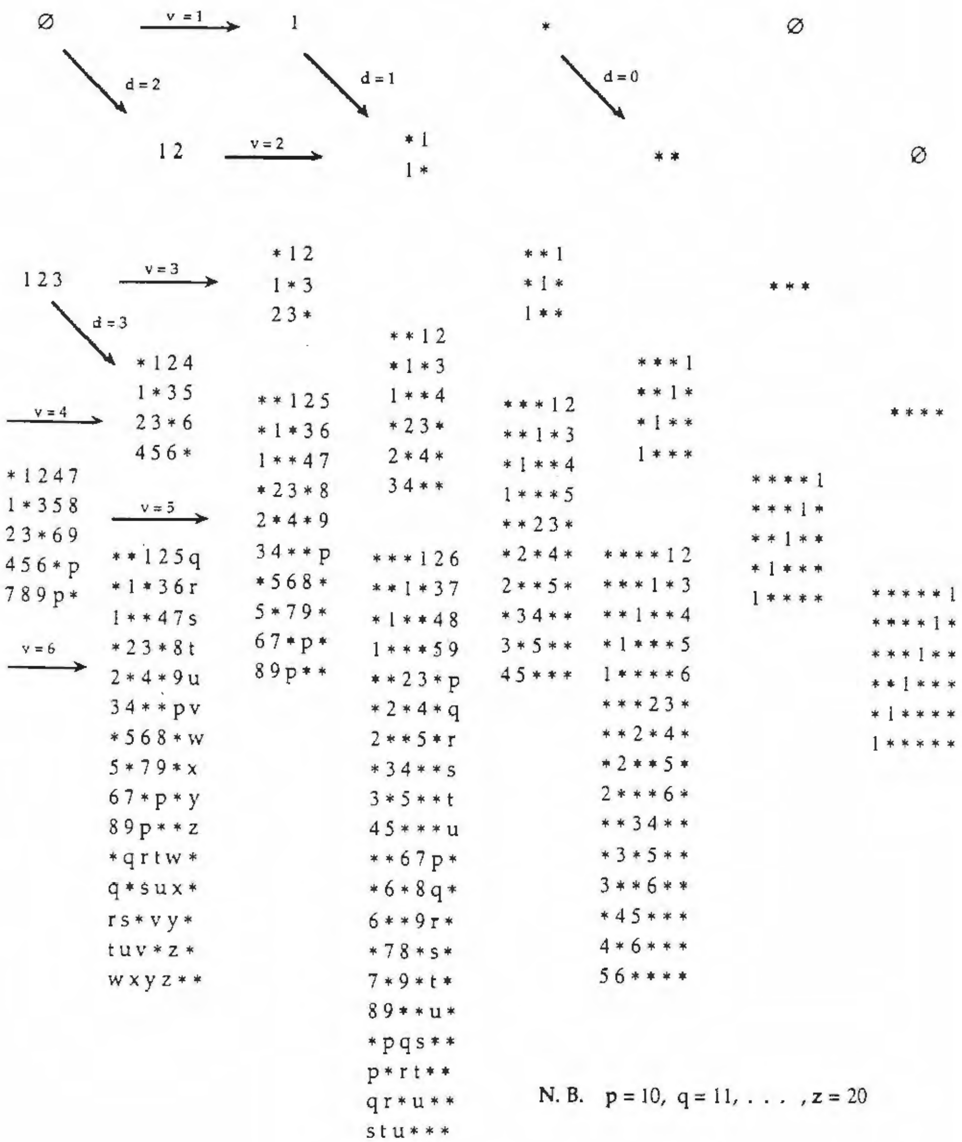
$$\begin{aligned}\langle x \rangle = (&\{000, -00, 0+0, 00-, --0, -0+, 0++, ---\},\\ &\{000, 00+, 0-0, +00, ++0, +0-, 0--, +++\});\end{aligned}$$

then we have the following "source-target" pairs of string diagrams



d = 1





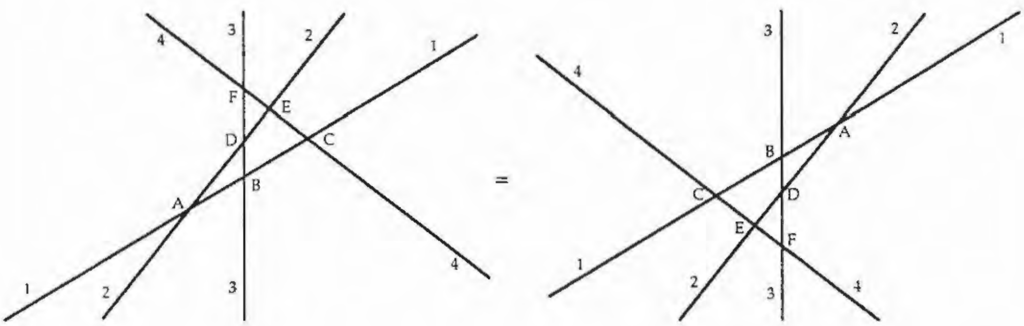
N. B. p = 10, q = 11, . . . , z = 20

Notice that the target diagrams are easily obtained from the source diagrams; so we centre attention only on the Aitchison–Pascal Triangle of source diagrams. Furthermore, this Triangle can be expressed (perhaps more compactly) in terms of matrices using the idea at the end of Section 6; the triangle begins as above (the rows of the triangle are enumerated by the variable v, while the left-to-right diagonals by d).

The tensor equation corresponding to the (v, d)-position of the Aitchison–Pascal triangle will be called the *(v,d)-cube equation.* For v = d+1, this is precisely the d-simplex equation (see Section 6). An example of one of the new equations is the (4,2)-cube equation):

$$A^{i_2 i_1}_{s_1 s_2} B^{i_3 j_1}_{i_1 s_3} C^{i_4 t_1}_{j_1 s_4} D^{j_3 j_2}_{i_2 i_3} E^{j_4 t_2}_{j_2 i_4} F^{t_4 t_3}_{j_3 j_4} = F^{i_4 i_3}_{s_3 s_4} E^{j_4 i_2}_{s_2 i_4} D^{j_3 j_2}_{i_2 i_3} C^{t_4 i_1}_{s_1 j_4} B^{t_3 j_1}_{i_1 j_3} A^{t_2 t_1}_{j_1 j_2}$$

whose string diagram is illustrated below.



We now make precise the Pascal construction of the above triangle of formal matrices. Let ${}^{v}C_{d}$ denote the binomial coefficient "v choose d" which is in the v d-position of the classic Pascal Triangle; by convention, ${}^{v}C_{-1} = 0$. The formal matrix S(v, d) in the v d-position of our triangle has ${}^{v}C_{d}$ rows and v columns with each entry either a strictly positive integer or a symbol *. The matrix S(v+1, d) is obtained from the two matrices S(v, d-1), S(v, d) in the row above it as the block matrix

$$S(v+1,d) = \begin{bmatrix} S(v,d-1) & X \\ S'(v,d) & \underline{*} \end{bmatrix}$$

where $\underline{*}$ denotes a column vector with all entries *, X is a column vector of length ${}^{v}C_{d-1}$ whose entries are increasing consecutive integers beginning with ${}^{v}C_{d-1}+1$, and $S'(v, d)$ is the matrix obtained from S(v, d) by adding ${}^{v}C_{d-2}$ to all the integer entries.

In fact, it is possible to obtain the entries of the matrix S(v, d) directly, without going through the Pascal algorithm. First we obtain the positions of the *'s as follows. Write down all words of length v in the symbols 0, 1 which have precisely d 0's. Assemble these words in a column so that, read from right to left and from bottom to top, they are in increasing order as binary numbers. (Check this from the triangle of matrices given above!) The 1's give the positions of *'s and the 0's give the positions of integers. It remains to obtain which integer goes in each integer position. We use the fact that we now know the position of the *'s in both S(v, d) and S(v, d-1). The list of integers in the j-th column of S(v, d) is equal to the list of row numbers of those rows in S(v, d-1) whose entry in the j-th column is *. For example, in the case of S(4,3) and S(4,2), we have

$$\begin{array}{cccc} & 1 & 2 & 4 \\ 1 & * & 3 & 5 \\ 2 & 3 & * & 6 \\ 4 & 5 & 6 & * \end{array} \qquad \begin{array}{cccc} * & * & 1 & 2 \\ * & 1 & * & 3 \\ 1 & * & * & 4 \\ * & 2 & 3 & * \\ 2 & * & 4 & * \\ 3 & 4 & * & * \end{array}$$

so it is indeed true that the second matrix has *'s in rows 1,3,5 of column 2, and that 1,3,5 are the integers (in order) in column 2 of the first matrix.

9. The Simplex Equations as Cubical Cocycle Equations

It can be argued that n-categories are appropriate algebraic coefficient structures for (non-abelian) cohomology. The idea is that a simplicial 1-cocycle condition is an equation of the form

$$a - b + c = 0$$

in some coefficient abelian group A. Rewriting this as

$$a + c = b,$$

we can regard the equation as a 2-functor

$$F : O\Delta[2] \longrightarrow \Sigma A$$

where ΣA is regarded as a 2-category all of whose 2-cells are identities.

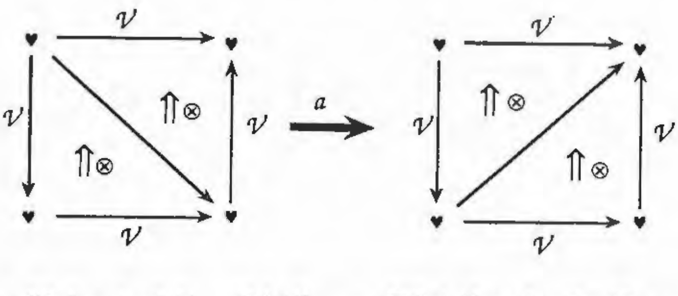
In general, it is necessary to take less strict structures than n-categories (in the way that bicategory is less strict than 2-category; see [11, 24] for the 3-dimensional case) as coefficient objects. However, it turns out that the very axioms needed for the less strict structures are recognizable as cocycle conditions obtained using the strict structures. (One of the motivations for obtaining the precise form of the general cocycle conditions in [47] was to prepare the way for the axioms for weak n-categories. The geometric form of these cocycle conditions is represented by the spaces called Stasheff associahedra [43].)

In more detail, consider the free n-category $\mathcal{O}\Delta[n+1]$ on the (n+1)-simplex $\Delta[n+1]$. This (n+1)-category has precisely one non-identity (n+1)-cell. Each n-category $\mathcal{K}$ can be regarded as an (n+1)-category with only identity (n+1)-cells. An instance of a *(non-abelian simplicial) n-cocycle equation in* $\mathcal{K}$ is an (n+1)-functor

$$F : \mathcal{O}\Delta[n+1] \longrightarrow \mathcal{K}.$$

The "equation" comes from the fact that F necessarily turns the non-identity (n+1)-cell of $\mathcal{O}\Delta[n+1]$ into an identity, so F provides an equality between two n-cells in $\mathcal{K}$.

As we mentioned above, strict n-categories $\mathcal{K}$ are not general enough to include all the examples. As a concrete example, we will now show how the associativity pentagon, in the definition of monoidal category (see Section 1), is an example of a 3-cocycle condition in an appropriately weak 3-category ΣCat. We have already pointed out that Cat is a 2-category: the objects are (small) categories, the arrows are functors, and the 2-cells are natural transformations. In fact, Cat is a monoidal 2-category if we take the tensor product to be cartesian product of categories. At the beginning of Section 5, we discussed how to bump monoids M up to single-object categories ΣM, and monoidal categories $\mathcal{V}$ up to single-object bicategories $\Sigma\mathcal{V}$. We now bump the monoidal 2-category Cat up a dimension to obtain a weak 3-category ΣCat with a single object $\heartsuit$, with the hom-2-category $\Sigma Cat(\heartsuit, \heartsuit) = Cat$, and with the composition $\circ_0$ taken to be cartesian product of categories. Given a category $\mathcal{V}$, a functor $\otimes : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$, and an associativity constraint $a : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$, we attempt to obtain a 3-functor $F : \mathcal{O}\Delta[4] \longrightarrow \Sigma Cat$ whose value on each 3-face of $\Delta[4]$ is the following oriented tetrahedron.



This yields a "3-functor" $F : \mathcal{O}\Delta[4] \longrightarrow \Sigma Cat$ if and only if a satisfies the associativity pentagon.

DEFINITION. A *cubical n-cocyle equation in an n-category* $\mathcal{K}$ is an (n+1)-functor

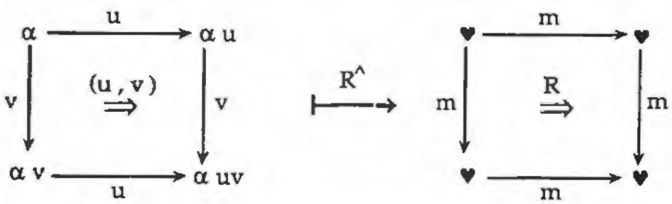
$$F : \mathcal{O}\mathbb{I}[n+1] \longrightarrow \mathcal{K}.$$

We shall show how the d-simplex equation is a cubical d-cocycle equation in an appropriately weak d-category $\Sigma^{d-1}\mathcal{M}at$.

In Section 1, we described the strict monoidal category $\mathcal{M}at$ of matrices (the matrices are the arrows!); the tensor product is Kronecker product of matrices. So there is no problem forming the 2-category $\Sigma\mathcal{M}at$ with a single object $\heartsuit$. Now suppose $R : mm \Rightarrow mm$ is an invertible $mm \times mm$-matrix. We wish to define a 3-functor

$$R^\wedge : O\mathbb{I}[3] \longrightarrow \Sigma\mathcal{M}at$$

determined by the following assignment on the 2-faces of the 3-cube $\mathbb{I}[3]$ (see Section 5 for notation).



The matrix R is a solution of the Yang–Baxter equation if and only if $R^\wedge : O\mathbb{I}[3] \longrightarrow \Sigma\mathcal{M}at$ is a 3-functor. So the Yang–Baxter equation is a cubical 2-cocycle equation. We further remark, in this case, that this induces a 2-functor $R^\wedge : \mathcal{C}ub[n, 2] \longrightarrow \Sigma\mathcal{M}at$ for all n.

Now consider applying the same ideas to the Zamolodchikov equation. On the geometric side there is no problem since we have the free 4-category $O\mathbb{I}[4]$. A small difficulty arises on the algebraic side when we try to bump the category of matrices up another dimension. This time we would like to consider a 3-category $\Sigma^2\mathcal{M}at$ whose only object is $\heartsuit$, whose only arrow is the identity of $\heartsuit$, whose 2-cells m are the objects of $\mathcal{M}at$, and whose 3-cells are matrices. This time two of the compositions are to be Kronecker product with the third taken to be multiplication of matrices, as before. The problem of non-strictness of associativity of tensor product of vector spaces has been avoided as before by the use of matrices instead of linear functions, however, now we also require the middle-four-interchange law:

$$(U \otimes V) \otimes (W \otimes X) = (U \otimes W) \otimes (V \otimes X)$$

which of course does not strictly hold; there is only a canonical isomorphism in place of the equality. This problem cannot be avoided. In fact, $\Sigma^2\mathcal{V}ect$ and $\Sigma^2\mathcal{M}at$ are examples of *tricategories* [11]. Using matrices, we obtain what is called a *Gray-category* $\Sigma^2\mathcal{M}at$. (It has been shown [11] that, more generally, every tricategory is "triequivalent" to a Gray-category.) Every 3-category is a Gray-category. It is therefore meaningful to consider Gray-functors from a 3-category to a Gray-category. In particular, each invertible $m^3 \times m^3$-matrix R induces such a Gray-functor

$$R^\wedge : O\mathbb{I}[n, 3] \longrightarrow \Sigma^2\mathcal{M}at;$$

the matrix R provides a solution to the Zamolodchikov equation when it identifies the commuting 4-face relations for some (and hence all) $n \geq 4$. So we have the appropriate kind of 4-functor $R^\wedge : \mathcal{O}\mathbb{I}[4] \longrightarrow \Sigma^2 \mathcal{M}at$. This is the sense in which the Zamolodchikov equation is a cubical 3-cocycle equation.

Higher dimensions offer no new problems. For the d-simplex equation, there is an appropriate structure $\Sigma^{d-1}\mathcal{M}at$ with precisely one i-cell for each $i \leq d-2$, whose (d-1)-cells are natural numbers, whose d-cells are matrices, whose first d-1 compositions are Kronecker product (among which the middle-four-interchange law holds only up to a coherent invertible d-cell), and whose remaining composition is usual matrix product (which strictly satisfies the middle-four-interchange law with each earlier composition). An invertible $m^d \times m^d$-matrix R which is a solution to the d-simplex equation induces a structure-preserving morphism

$$R^\wedge : \mathcal{C}ub[d+1, d] \longrightarrow \Sigma^{d-1}\mathcal{M}at,$$

and hence a "(d+1)-functor" $R^\wedge : \mathcal{O}\mathbb{I}[d+1] \longrightarrow \Sigma^{d+1}\mathcal{M}at$.

10. Cubes, Braids and Higher Braids

At the end of Section 5 we mentioned a relationship between cubes and braids which will now be made more precise. We shall explain how the braid category is universally obtained from the 2-category of cubes by identifying all the objects and inverting all the 2-cells.

Let $\mathcal{C}ub[\infty, 2]$ denote the union of all the 2-categories $\mathcal{C}ub[n, 2]$; so that $\mathcal{C}ub[\infty, 2]$ is the 2-category of cubes of all dimensions with commuting 3-faces: the objects are words α in $-$ and $+$, there are arrows only between words of the same length, and the 2-cells between these are as in the appropriate $\mathcal{C}ub[n, 2]$.

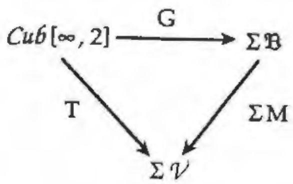
Because the braid category $\mathfrak{B}$ is strict monoidal, we can regard it as a 1-object 2-category $\Sigma\mathfrak{B}$. There is a 2-functor

$$G : \mathcal{C}ub[\infty, 2] \longrightarrow \Sigma\mathfrak{B}$$

$$\begin{array}{ccc} \alpha & \xrightarrow{\ u\ } & \alpha u \\ {\scriptstyle v}\downarrow & \Rightarrow & \downarrow{\scriptstyle v} \\ \alpha v & \xrightarrow[\ u\ ]{} & \alpha uv \end{array} \quad \longmapsto \quad \begin{array}{ccc} \heartsuit & \xrightarrow{\ 1\ } & \heartsuit \\ {\scriptstyle 1}\downarrow & \overset{c_{1,1}}{\Rightarrow} & \downarrow{\scriptstyle 1} \\ \heartsuit & \xrightarrow[\ 1\ ]{} & \heartsuit \end{array}$$

(where 1 here means the natural number 1 as an object of $\mathfrak{B}$). This 2-functor has a universal property which shows how *cubes determine the braid groups*.

PROPOSITION. *For all strict monoidal categories* $\mathcal{V}$ *and all 2-functors* T : $\mathcal{C}ub[\infty, 2] \longrightarrow \Sigma\mathcal{V}$ *which invert 2-cells, there exists a unique strict monoidal functor* M : $\mathfrak{B} \longrightarrow \mathcal{V}$ *such that the following triangle commutes.*



This can also be expressed in terms of Yang–Baxter operators: the Yang–Baxter operators in a strict monoidal category $\mathcal{V}$ are in natural bijection with 2-functors $\mathcal{C}ub[\infty, 2] \longrightarrow \Sigma\mathcal{V}$ which invert all 2-cells. This shows that the 2-category $\mathcal{C}ub[\infty, 2]$ can be used as a replacement for the braid groups in many situations.

This suggests that higher braid structures [34] can similarly be studied using the d-categories $\mathcal{C}ub[\infty, \mathrm{d}]$, as we hope to explain in a future paper.

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