

Talk at ANZMC \subset ICIAM

Ross Street
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UTS Rm UB112

Categorical and Combinatorial Aspects of Descent Theory

Descent Theory \doteq Theory of Stacks

An *n-stack* is a weak morphism from a weak n -category to the weak n -category of weak $(n-1)$ -categories, satisfying a gluing condition. (The French word for "stack" is "champ".)

Category theory and combinatorics are involved in the correct definitions of both

- weak n -categories, and
- the gluing condition.

See the article, based on the remarkable work of Michael Batanin (our third speaker), of the same title as this lecture:

Categorical and combinatorial aspects of descent theory,
Applied Categorical Structures (to appear);
<<http://www.maths.mq.edu.au/~street/DescFlds.pdf>>

Here I shall concentrate on $n = 0, 1, 2$; already a challenge for a half-hour talk.

weak 0-category = set
weak 1-category = category
weak 2-category = bicategory

Bicategories will be the subject of Steve Lack (our fourth speaker).

They have objects A, B, \dots , morphisms $f: A \longrightarrow B$, and 2-cells $A \begin{array}{c} \xrightarrow{f} \\ \Downarrow \theta \\ \xrightarrow{g} \end{array} B$,

with compositions as you might expect, however, composition of morphisms is only up to coherent natural isomorphism. (A 2-category has strict associativity.)

Cat is the category of categories and functors, made a 2-category using natural transformations as the 2-cells.

A *groupoid* is a category in which all morphisms are invertible. Every groupoid is equivalent to a disjoint union of groups by identifying isomorphic objects: however, *this does not kill off all isomorphisms* as the automorphisms remain.

Gpd is the 2-category of groupoids. All the 2-cells are invertible.

weak 0-morphism = function
 weak 1-morphism = functor
 weak 2-morphism = pseudofunctor

(the "pseudo" means composition of morphisms is preserved only up to coherent natural isomorphisms)

1-stack = sheaf
 2-stack = stack in the "classical" sense

A *presheaf* is a functor $P : C^{op} \rightarrow Set$ where C is a category. To express the gluing condition that makes a presheaf a sheaf we need the specification J of a set of covering families

$$\mathcal{U} = \left(U_i \longrightarrow U \right)_{i \in I} \quad (\text{each family is "a cover of } U \text{"})$$

of morphisms in C for all objects U . Technically, we need that the singleton family consisting of the identity morphism of each object U should be a cover of U , and each pullback of a cover of U along a morphism $V \rightarrow U$ is a cover of V . The pair (C, J) is called a *site*. A *sheaf on the site* is a presheaf $F : C^{op} \rightarrow Set$ such that, for all covers \mathcal{U} , the diagram

$$FU \xrightarrow{w} \prod_i FU_i \begin{array}{c} \xrightarrow{u} \\ \xrightarrow{v} \end{array} \prod_{i,j} F\left(U_i \times_U U_j \right)$$

exhibits FU as the equalizer of u and v , where w , u , and v are induced by the morphisms $U_i \rightarrow U$, the first projections $U_i \times_U U_j \rightarrow U_i$, and the second projections $U_i \times_U U_j \rightarrow U_j$, respectively. We obtain a category $Sh(C, J)$ of sheaves; morphisms are natural transformations. Such categories are called *toposes*.

Example 1. *Sheaves on a topological space X*

$C = \text{Opn}(X)$ is the category whose objects are the open subsets U of X and whose morphisms are the inclusions. A "cover" $\mathcal{U} = \left(U_i \longrightarrow U \right)_{i \in I}$ is a cover: $\bigcup_i U_i = U$. Each continuous function

$h : T \rightarrow X$ determines a sheaf $F_h : \text{Opn}(X)^{op} \rightarrow Set$ defined by

$$F_h U = \text{Cts}_X(U \rightarrow X, T \rightarrow X).$$

There is a local homeomorphism $p : E_h \rightarrow X$ and a commutative diagram

$$\begin{array}{ccc} E_h & \xrightarrow{\quad} & T \\ p \searrow & & \swarrow h \\ & X & \end{array}$$

(in the category Top of topological spaces) inducing an isomorphism $F_p \cong F_h$. We write $Sh(X)$

for the topos $\text{Sh}(C, J)$ in this case. There is an equivalence of categories

$$\text{Sh}(X) \cong (\text{Local homeom. over } X).$$

A sober space X can be recaptured from $\text{Sh}(X)$. ///

Example 2. Giraud's Theorem

Suppose C itself is a topos and suppose the covers $\mathcal{U} = \left(U_i \longrightarrow U \right)_{i \in I}$ are the jointly epimorphic families; call this J_{can} . There is an equivalence of categories

$$\text{Sh}(C, J_{\text{can}}) \cong C. ///$$

Example 3. Top as a site

A cover of a space U is taken to be the family of inclusions $U_i \longrightarrow U$ of an open cover of U ; call this J_{cov} . For each space X , there is induced a notion of cover on Top / X . There is a close relationship between $\text{Sh}(X)$ and $\text{Sh}(\text{Top} / X, J_{\text{cov}})$.

This kind of thing is important in algebraic geometry where Top is replaced by the category of schemes. ///

A *prestack* is a pseudofunctor $P : C^{\text{op}} \longrightarrow \text{Cat}$ where, in general, C is a bicategory. Most attention has been paid to the case where C is a category (only identity 2-cells). A *stack on the site* (C, J) is a prestack $F : C^{\text{op}} \longrightarrow \text{Cat}$ such that the diagram

$$\begin{array}{ccccc} & & \longrightarrow & & \longrightarrow \\ \text{FU} & \xrightarrow{w} & \prod_i \text{FU}_i & \longleftarrow & \prod_{i,j} F\left(U_i \times_U U_j \right) & \longrightarrow & \prod_{i,j,k} F\left(U_i \times_U U_j \times_U U_k \right) \\ & & \longrightarrow & & \longrightarrow & & \longrightarrow \end{array}$$

exhibits FU as the "descent category" for the remainder of the diagram.

Danny Stevenson (our second speaker) will be concerned with stacks that land in Gpd rather than Cat .

Example 4. Locally trivial structures

We illustrate this with the specific example of *vector bundles*. Consider the site $(\text{Top}, J_{\text{cov}})$ as in Example 3. Topological real vector spaces are an important structure much studied in mathematics and it is reasonable to consider the category of these. Deeper information is contained in the stack $\mathcal{V} : \text{Top}^{\text{op}} \longrightarrow \text{Cat}$ defined by

$$\mathcal{V}U = \left(\text{category of modules in Top} / U \text{ over the topol. ring } \mathbf{R} \times U \xrightarrow{\text{pr}_2} U \right).$$

For example, a general theorem about stacks yields, in this example, an equivalence of categories

$$\text{Cts functors } (\text{Nerve } \mathcal{U}, \text{Mat } \mathbf{R}) \cong (\mathbf{R} - \text{vector bundles over } U \text{ trivialized by } \mathcal{U}).$$

Here $\text{Nerve } \mathcal{U}$ and $\text{Mat } \mathbf{R}$ are topological categories (see below). By restricting to groupoids, this is the usual classification of vector bundles over U by Čech cocycles with coefficients in the general linear groupoid $\text{Gl}(\mathbf{R})$. ///

There is a 2-category Topos of toposes. The morphisms $T: \mathcal{E} \longrightarrow \mathcal{F}$ between toposes, called *geometric morphisms*, are functors with a left exact left adjoint. The 2-cells are natural transformations.

Example 5. Top in Topos

For sober spaces X and Y , the category $\text{Geom}(\text{Sh}(X), \text{Sh}(Y))$ is a groupoid which is equivalent to the discrete category of continuous functions from X to Y . ///

A topos \mathcal{E} is called an *étendue* when there is an epimorphism $C \longrightarrow 1$ and a topological space X such that $\mathcal{E} / C \cong \text{Sh}(X)$. This is saying that \mathcal{E} is "locally like sheaves on a space".

Topological groupoids: Moduli spaces and orbifolds force us to look beyond mere spaces when spaces of automorphisms need to be accommodated.

A groupoid G can be displayed as a diagram of sets and functions:

$$\begin{array}{ccccc} & \xrightarrow{\text{proj}_1} & & \xrightarrow{\text{source}} & \\ G_2 & \xrightarrow{\text{compos}} & G_1 & \xleftarrow{\text{inverse}} & G_1 & \xleftarrow{\text{ident}} & G_0 \\ & \xrightarrow{\text{proj}_2} & & \xrightarrow{\text{target}} & & & \end{array}$$

$$\begin{aligned} G_0 &= \text{set of objects,} & G_1 &= \text{set of morphisms,} \\ G_2 &= \text{set of composable pairs of morphisms.} \end{aligned}$$

The groupoid is *topological* when the diagram is equipped with the structure needed to put it in Top .

Example 6. Actions of a Lie group

A Lie group G acting on a space X yields a topological groupoid $G \backslash \backslash X$ whose objects are the points of X and whose morphisms are pairs $(g, x): x \longrightarrow y$ where $y = gx$:

$$\begin{array}{ccccc}
 & & \longrightarrow & & \\
 G \backslash \backslash X & : & G \times G \times X & \longrightarrow & G \times X & \xrightarrow{\text{pr}_1} & X \\
 & & \longrightarrow & & \xrightarrow{\text{action}} & &
 \end{array}$$

The passage from an orbifold to a groupoid is a little more involved. ///

A *sheaf* on a topological groupoid G is a topological groupoid E with a continuous functor $p: E \longrightarrow G$ such that $p_0: E_0 \longrightarrow G_0$ is a local homeomorphism and the following square is a pullback.

$$\begin{array}{ccc}
 E_1 & \xrightarrow{\text{source}} & E_0 \\
 p_1 \downarrow & & \downarrow p_0 \\
 G_1 & \xrightarrow{\text{source}} & G_0
 \end{array}$$

Write $\mathcal{B}G$ for the category of sheaves on G . It is very closely related to the category of sheaves on the classifying space BG of G .

Theorem (Grothendieck and Joyal-Tierney)

- $\mathcal{B}G$ is a topos.
- Essentially every topos is equivalent to a $\mathcal{B}G$. ///

If G comes from an orbifold then $\mathcal{B}G$ is an étendue.

A topological groupoid G is *étale* when the source and target functions $G_1 \longrightarrow G_0$ are both local homeomorphisms.

Theorem (Moerdijk-Pronk)

$$\text{ÉtaleGpd}[W^{-1}] \stackrel{\mathcal{B}}{\sim} \text{Étendue} \stackrel{\mathcal{S}}{\sim} \text{Stack}(\text{Top}, J_{\text{cov}})_{\text{gpd}} \quad ///$$

Here, for \mathcal{E} an étendue, the stack $\mathcal{S}(\mathcal{E}): \text{Top}^{\text{op}} \longrightarrow \text{Cat}$ is defined by

$$\mathcal{S}(\mathcal{E}) X = \text{Geom}(\text{Sh}(X), \mathcal{E}).$$