Limits for lax morphisms

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Abstract

We investigate limits in the 2-category of strict algebras and lax morphisms for a 2-monad. This includes both the 2-category of monoidal categories and monoidal functors as well as the 2-category of monoidal categories and opomonoidal functors, among many other examples.

1 Introduction

Ordinary algebra is based on equations, such as the associative law $x(yz) = (xy)z$ for groups or rings, or the equation $f(xy) = f(x)f(y)$ which asserts that f is a homomorphism of groups. When the elements of a group or ring are replaced by the objects of a category, such equations become rare; more often one has an isomorphism, itself subject to certain equations. In a monoidal category, for example, one has an associativity isomorphism $X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$ subject to Mac Lane's pentagon condition [12].

In fact it was recognized very early on [4, 1] that sometimes there is not even an isomorphism, but only a (non-invertible) comparison map (subject of course to certain conditions). Thus one has the notion of *monoidal functor* between monoidal categories $\mathscr{V}$ and $\mathscr{W}$: this is a functor $F : \mathscr{V} \to \mathscr{W}$ equipped with a natural transformation $FX \otimes FY \to F(X \otimes Y)$ and a map $I_{\mathscr{W}} \to F(I_{\mathscr{V}})$, subject to certain conditions called *coherence conditions* — see [4] or [1]. (Some authors use the term monoidal functor only in the case where the comparison maps are invertible; we shall distinguish this case with the name *strong monoidal functor*.) A basic example of a monoidal functor is the forgetful functor $U : \mathbf{Ab} \to \mathbf{Set}$ from abelian groups to sets. If we regard $\mathbf{Ab}$ as monoidal via the tensor product of abelian groups, and $\mathbf{Set}$ as monoidal via the cartesian product, then $U(G \otimes H)$ is not equal or even isomorphic to $UG \times UH$, but there is a canonical map $UG \times UH \to U(G \otimes H)$ (the universal bilinear map) and this satisfies the conditions to make U into a monoidal functor. Further motivation of the definition of monoidal functor comes from the fact that if $\mathscr{V}$ is an arbitrary monoidal category and $\mathbf{1}$ is the terminal category with its unique monoidal structure, then a monoidal

functor from $\mathbf{1}$ to $\mathscr{V}$ is precisely a monoid in $\mathscr{V}$. (Strong monoidal functors from $\mathbf{1}$ to $\mathscr{V}$ are much less common and much less interesting.)

Following [9], we use the prefix "pseudo" to refer to notions involving (suitably coherent) isomorphisms rather than equalities, and "lax" for the case of (coherent, but) not necessarily invertible comparison maps. Thus a monoidal functor might also be called a lax morphism of monoidal categories. We use the prefix "strict" to specify that there are genuine equalities, rather than isomorphisms or arbitrary comparison maps. There is also a notion of "colax", once again involving not necessarily invertible comparisons, but in the opposite direction to the lax case. If one works with pseudo or lax morphisms rather than strict ones, it is no longer the case that one can form all limits of algebras: see [2].

The paper [2] contained a detailed study of those limits which do exist in the pseudo case; here we turn to the case of lax morphisms. As we shall see, in this case there are even fewer limits than in the pseudo case of [2], but there nonetheless are a few important limits which do exist. The limits we shall discuss are genuine 2-categorical limits, whose defining universal properties involve an isomorphism (not just an equivalence) of categories. If we speak of lax or pseudo limits, this refers not to a weakening of the isomorphism, but rather to a change of the categories which are required to be isomorphic. The precise definitions will be recalled as needed.

The algebraic structures we consider will be defined in terms of 2-monads, for which [2] is once again a suitable reference. The precise definitions are recalled in Section 2 below, in the meantime we give some basic examples.

Example 1.1 As observed above, if T is the 2-monad on **Cat** whose (strict!) algebras are the monoidal categories, a lax T-morphism is precisely a monoidal functor. A pseudo T-morphism, in which the comparison maps are invertible, is a strong monoidal functor, and a strict T-morphism is called a strict monoidal functor. The T-transformations are the monoidal natural transformations [4].

Example 1.2 In the previous example, the colax T-morphisms involve functors $F : \mathscr{V} \to \mathscr{W}$ with suitably coherent maps $F(X \otimes Y) \to FX \otimes FY$ and $FI_{\mathscr{V}} \to I_{\mathscr{W}}$. These are called *opmonoidal functors*, and have arisen, for example, in connection with "Hopf monads": see [15, 14]. In fact if we reverse the orientation of the 2-cells in **Cat**— the resulting 2-category is called $\mathbf{Cat}^{\mathrm{co}}$ — then opmonoidal functors themselves become an example of lax morphisms.

Example 1.3 For another key example, consider the 2-monad T on **Cat** whose algebras are the categories with chosen colimits of some particular type (for example finite coproducts) and whose strict morphisms are the functors which not only preserve these colimits in the usual sense, but actually preserve the chosen colimits. Then the pseudo morphisms are the functors preserving the colimits in the usual sense, while the lax morphisms are just arbitrary functors. The T-transformations are arbitrary natural transformations.

Example 1.4 If, dually, one considers chosen limits of some particular type, the pseudo morphisms once again are the functors preserving the limits in the usual sense, but now

every lax morphism is pseudo: see [7]. (This time arbitrary functors between such categories are recovered as the *colax* morphisms.)

See [2] and Remark 4.7 below for further examples.

The theorems we prove about limits for lax morphisms are more delicate to state than those of the pseudo case of [2] because many require restrictions that some morphisms in the diagram be strict. An alternative approach to these theorems involves the *horizontal double limits* of [5]. This would involve working in the double category with T-algebras as objects, strict morphisms as horizontal morphisms, and lax morphisms as vertical morphisms. This is approach is very good at isolating which diagrams have limits; it suffers, however, in that the actual universal property of the limits in the double category sense is too weak with respect to horizontal morphisms. This shortcoming was recognized in [5], where the notion of "functorial choice of limits" was introduced to try to overcome the problem.

The results of this paper were largely complete in 1997, and presented in [10], but the writing up was delayed for various reasons. The appearance of [14] and [15], and a general increase in interest in these matters were the recent spur to pubishing them at last.

Section 2 recalls the relevant 2-categories of algebras, Section 3 constructs the most important single limit, the oplax limit of an arrow, while Section 4 looks at other sorts of limits. Section 5 considers the special case of monoidal categories, while Section 6 contains some of the more technical proofs.

2 The 2-categories of interest

In this section we recall the basic definitions of 2-dimensional monad theory. We shall consider algebraic structures defined in terms of (strict) 2-monads. A 2-monad $T = (T, m, i)$ on a 2-category $\mathscr{K}$ is defined exactly as for ordinary monads, except that T is required to be functorial — and m and i to be natural — with respect to both 1-cells and 2-cells. In particular, such a 2-monad T induces an underlying ordinary monad T_0 on the underlying ordinary category $\mathscr{K}_0$ of $\mathscr{K}$ obtained by discarding the 2-cells. We work only with the *strict* algebras of T, which are nothing but the ordinary algebras of T_0, consisting of an object A of $\mathscr{K}$ equipped with a map $a : TA \to A$ of $\mathscr{K}$ satisfying the usual equations for an algebra. Notice that although it may seem to be a restriction to consider strict algebras rather than pseudo ones, this restriction is in fact illusory, since for reasonable 2-monads T, the pseudo T-algebras are precisely the strict T'-algebras for a different 2-monad T'. In fact the results of this paper remain true in the context of pseudo algebras, but the proofs become more complicated. See [2] for general information about 2-monads; see [8] for information about how to recognize when structure can be described in terms of 2-monads, and see [11] and the references therein for more about the construction T'.

It is when we come to the morphisms that we depart the strict world. If (A, a) and (B, b) are T-algebras (meaning, as we said, strict ones), a *lax T-morphism* from (A, a) to (B, b)

consists of a morphism $f : A \to B$ equipped with a 2-cell $\bar{f} : b.Tf \to fa$ as in

$$\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\bar{f}"] & TB \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

subject to the coherence conditions

$$\begin{tikzcd}
T^2A \arrow[rr, "T^2f"] \arrow[d, "mA"'] \arrow[dr, "Ta"] & & T^2B \arrow[dr, "Tb"] \arrow[d, phantom, "\Downarrow T\bar{f}"] & \\
TA \arrow[dr, "a"'] & TA \arrow[rr, "Tf"] \arrow[d, "a"] \arrow[drr, phantom, "\Downarrow\bar{f}"] & & TB \arrow[d, "b"] \\
& A \arrow[rr, "f"'] & & B
\end{tikzcd}
=
\begin{tikzcd}
T^2A \arrow[rr, "T^2f"] \arrow[d, "mA"'] & & T^2B \arrow[d, "mB"'] \arrow[dr, "Tb"] & \\
TA \arrow[rr, "Tf"] \arrow[dr, "a"'] & \arrow[d, phantom, "\Downarrow\bar{f}"] & TB \arrow[dr, "b"] & TB \arrow[d, "b"] \\
& A \arrow[rr, "f"'] & & B
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[rr, "1"] \arrow[dr, "iA"] \arrow[ddr, bend right, "1"'] & & B \arrow[dr, "iB"] & \\
& TA \arrow[rr, "Tf"] \arrow[d, "a"'] \arrow[drr, phantom, "\Downarrow\bar{f}"] & & TB \arrow[d, "b"] \\
& A \arrow[rr, "f"'] & & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[rr, "1"] \arrow[ddr, bend right, "1"'] & & B \arrow[dr, "iB"] \arrow[ddr, bend right, "1"'] & \\
& & & TB \arrow[d, "b"] \\
& A \arrow[rr, "f"'] & & B
\end{tikzcd}$$

wherein regions with no 2-cell are understood to commute and are deemed to contain the identity.

If the 2-cell $\bar{f}$ is invertible, then we speak instead of a *pseudo* morphism; if it goes in the opposite direction then we have a *colax* morphism; if it is an equality, then we have a *strict morphism*: of course this means that $b.Tf = fa$, so that f is a morphism of algebras in the usual sense. Finally, if $(f, \bar{f})$ and $(g, \bar{g})$ are two lax morphisms from (A, a) to (B, b), then a *T-transformation* from $(f, \bar{f})$ to $(g, \bar{g})$ is a 2-cell $\rho : f \to g$ in $\mathscr{K}$ satisfying the single condition

$$\begin{tikzcd}
TA \arrow[r, bend left, "Tf"] \arrow[r, bend right, "Tg"'] \arrow[r, phantom, "\Downarrow T\rho"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\bar{g}"] & TB \arrow[d, "b"] \\
A \arrow[r, bend right, "g"'] & B
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, bend left, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\bar{f}"] & TB \arrow[d, "b"] \\
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\rho"] & B.
\end{tikzcd}$$

For a 2-monad T on $\mathscr{K}$, the (strict) T-algebras, lax T-morphisms, and T-transformations form themselves into a 2-category $T\text{-}\mathrm{Alg}_\mathrm{l}$ with a forgetful 2-functor $U_l : T\text{-}\mathrm{Alg}_\mathrm{l} \to \mathscr{K}$,

with sub-2-categories T-Alg and T-$\mathrm{Alg_s}$ containing only the pseudo, respectively the strict, morphisms; in each case they contain all the objects, and all the 2-cells between the relevant morphisms. We write $J_l : T\text{-}\mathrm{Alg_s} \to T\text{-}\mathrm{Alg_l}$ for the inclusion, and write $U : T\text{-}\mathrm{Alg} \to \mathscr{K}$ and $U_s : T\text{-}\mathrm{Alg_s} \to \mathscr{K}$ for the forgetful 2-functors. Similarly we have T-$\mathrm{Alg_c}$ and $U_c : T\text{-}\mathrm{Alg_c} \to \mathscr{K}$, involving colax rather than lax morphisms. For examples of such 2-categories, see the introduction.

Remark 2.1 Note that there is no need to develop separately a theory of limits for colax morphisms. For if $\mathscr{K}$ is a 2-category, then there is a 2-category $\mathscr{K}^{\mathrm{co}}$ obtained by formally reversing the direction of the 2-cells (but not the 1-cells), and a 2-monad T on $\mathscr{K}$ induces 2-monad T^{co} on $\mathscr{K}^{\mathrm{co}}$ with $T\text{-}\mathrm{Alg_c}^{\mathrm{co}} = T^{\mathrm{co}}\text{-}\mathrm{Alg_l}$.

3 The oplax limit of an arrow

In this section we recall the definition of a fundamental 2-categorical limit called the *oplax limit of an arrow.* There are corresponding notions called the pseudo limit and the lax limit of an arrow. A key result of [2] is that pseudo morphisms have pseudo limits: in other words that the 2-category T-Alg admits the pseudo limit of any arrow (supposing that the base 2-category $\mathscr{K}$ does so). This is false in the lax case: T-$\mathrm{Alg_l}$ does not in general admit lax limits of arrows, even in the case where $\mathscr{K}$ is **Cat**: see Example 3.3. In this section we shall see that T-$\mathrm{Alg_l}$ does admit the *oplax* limit of an arrow (supposing once again that the base 2-category does so). Thus if one tries to "lift" the result of [2] from the pseudo world to the lax world, there is a twist: if the pseudo morphisms are to be replaced by lax morphisms, then the pseudo limits should be replaced by oplax limits.

Recall that for a functor $f : A \to B$ the comma category B/f is the category whose objects consist of triples (b, φ, a) where a is an object of A and $\varphi : b \to fa$ a morphism in B; and whose morphisms $(b, \varphi, a) \to (b', \varphi', a')$ consist of a morphism $\alpha : a \to a'$ in A and a morphism $\beta : b \to b'$ in B such that $\varphi'\beta = f\alpha.\varphi$. There are evident projections $u : B/f \to A$ and $v : B/f \to B$ sending (b, φ, a), respectively to a and to b, and there is an evident natural transformation $\lambda : v \to fu$ whose component at (b, φ, a) is

$$v(b, \varphi, a) = b \xrightarrow{\varphi} fa = fu(b, \varphi, a).$$

Furthermore, B/f is universal, in the sense that given functors $r : C \to A$ and $s : C \to B$ with a natural transformation $\rho : s \to fr$, there is a unique induced functor $x : C \to B/f$ satisfying $ux = r$, $vx = s$, and $\lambda x = \rho$. Explicitly, x is given on objects by $xc = (sc, \rho c, rc)$. There is also a 2-dimensional aspect to the universality, involving functors $x, x' : C \to B/f$ and a natural transformation $x \to x'$. It states that for every pair of natural transformations

$(\alpha : ux \to ux', \beta : vx \to vx')$ for which the square

$$\begin{tikzcd} vx \arrow[r, "\beta"] \arrow[d, "\lambda x"'] & vx' \arrow[d, "\lambda x'"] \\ fux \arrow[r, "f\alpha"'] & fux' \end{tikzcd}$$

commutes, there is a unique natural transformation $\xi : x \to x'$ for which $u\xi = \alpha$ and $v\xi = \beta$.

This construction is the basis for a fundamental 2-categorical limit, called the oplax limit of an arrow. Given an arrow $f : A \to B$ in a 2-category $\mathscr{K}$, its oplax limit is defined to be a universal diagram consisting of an object B/f, morphisms $u : B/f \to A$ and $v : B/f \to B$, and a 2-cell $\lambda : v \to fu$. The universal property can be made more precise by saying that composition with u, v, and λ induces an isomorphism of categories

$$\mathscr{K}(C, B/f) \cong \mathscr{K}(C, B)/\mathscr{K}(C, f)$$

for any object C of $\mathscr{K}$. (Here $\mathscr{K}(C, B)$ is the hom-category, and $\mathscr{K}(C, f) : \mathscr{K}(C, A) \to \mathscr{K}(C, B)$ the functor given by composition with f.)

We shall use the following:

Lemma 3.1 *Let $u : B/f \to A$, $v : B/f \to B$ and $\lambda : v \to fu$ exhibit B/f as the oplax limit of the arrow $f : A \to B$. Suppose that arrows $x, x' : C \to B/f$ and a 2-cell $\xi : x \to x'$ are given. If $u\xi$ and $v\xi$ are both identities, then so is ξ.*

PROOF: By naturality of λ, the square

$$\begin{tikzcd} vx \arrow[r, "v\xi"] \arrow[d, "\lambda x"'] & vx' \arrow[d, "\lambda x'"] \\ fux \arrow[r, "fu\xi"'] & fux' \end{tikzcd}$$

commutes. Since $u\xi$ and $v\xi$ are identities, it follows that $ux = ux'$, $vx = vx'$, and $\lambda x = \lambda x'$. The fact that $x = x'$ now follows from the 1-dimensional aspect of the universal property, while the fact that ξ is an identity follows from the 2-dimensional aspect. □

Suppose now that $T = (T, m, i)$ is a 2-monad on a 2-category $\mathscr{K}$. We say that strict morphisms $u : L \to A$ and $v : L \to B$ *jointly detect strictness*, if for any (lax) morphism $x : C \to L$, we have x strict if both ux and vx are so.

Theorem 3.2 *The forgetful 2-functor $U_l : T\text{-Alg}_l \to \mathscr{K}$ creates oplax limits of arrows. In particular, if $\mathscr{K}$ has such limits then so does $T\text{-Alg}_l$, and U_l preserves them. The projections of an oplax limit are strict, and jointly detect strictness.*

PROOF: Let $(f, \bar{f}) : (A, a) \to (B, b)$ be a morphism in $T\text{-Alg}_{\mathrm{l}}$, and let u, v, and $\lambda : v \to fu$ exhibit L as the oplax limit of the arrow f in $\mathscr{K}$. Consider the composite 2-cell

$$\begin{tikzcd}
& TL \arrow[dl, "Tu"'] \arrow[dr, "Tv"] \arrow[d, phantom, "\Downarrow T\lambda"] & \\
TA \arrow[rr, "Tf"] \arrow[d, "a"'] & & TB \arrow[d, "b"] \\
A \arrow[rr, "f"'] & \Downarrow \bar{f} & B.
\end{tikzcd}$$

By the universal property of L, there is a unique induced map $l : TL \to L$ satisfying $ul = a.Tu$ and $vl = b.Tv$ with λl equal to the pasting composite displayed above. We shall show that: (a) l makes L into a T-algebra, (b) u and v become strict T-morphisms, (c) λ becomes a T-transformation, (d) u and v jointly detect strictness, and (e) these data satisfy the universal property of the oplax limit of the arrow $(f, \bar{f})$.

We defer the proof of (a) and (e) to Section 6. Then (b) amounts to the equations $ul = a.Tu$ and $vl = b.Tv$ forming part of the characterization of l, while the remaining part, asserting that λl is the displayed pasting composite, is precisely (c). If $(x, \bar{x}) : (C, c) \to (L, l)$ is a lax morphism whose composites with u and v are strict, then $\bar{x} : l.Tx \to xc$ is a 2-cell for which $u\bar{x}$ and $v\bar{x}$ are identities, and so $\bar{x}$ is an identity by Lemma 3.1, and this proves (d). $\square$

Example 3.3 Let T be the 2-monad on **Cat** whose algebras are the categories with finite coproducts, as in Example 1.3, and whose lax morphisms are arbitrary functors. Thus $T\text{-Alg}_{\mathrm{l}}$ is the *full* sub-2-category of **Cat** consisting of those categories which have finite coproducts. Since this full sub-2-category contains the terminal category **1**, which is dense in **Cat** (as a 2-category!), it follows that the inclusion must preserve and reflect any existing limits.

Let $\mathscr{B}$ be a category with finite coproducts, and let B be an object of $\mathscr{B}$. Then **1** and $\mathscr{B}$ are objects of $T\text{-Alg}_{\mathrm{l}}$, and B determines a (lax) morphism $B : \mathbf{1} \to \mathscr{B}$. By the previous paragraph, this will have a lax limit in $T\text{-Alg}_{\mathrm{l}}$ if and only if its lax limit in **Cat** actually lies in $T\text{-Alg}_{\mathrm{l}}$. The lax limit in **Cat** is of course $B/\mathscr{B}$, and this lies in $T\text{-Alg}_{\mathrm{l}}$ precisely when $\mathscr{B}$ has pushouts over B. Thus $T\text{-Alg}_{\mathrm{l}}$ will fail to have a lax limit of $B : \mathbf{1} \to \mathscr{B}$ whenever $\mathscr{B}$ fails to have such pushouts.

4 Other limits

The 2-category $T\text{-Alg}_{\mathrm{s}}$ of T-algebras and *strict* morphisms, behaves very much like the category of algebras for an ordinary monad. In particular, the forgetful 2-functor $U_s : T\text{-Alg}_{\mathrm{s}} \to \mathscr{K}$ creates limits, and so $T\text{-Alg}_{\mathrm{s}}$ has whatever limits $\mathscr{K}$ has. Although $T\text{-Alg}_{\mathrm{l}}$ has relatively few limits, we collect together here a list of certain types of limit which it does have. In particular, it does have limits of strict morphisms (once again, provided that $\mathscr{K}$ has the relevant limits); this is due to Blackwell, Kelly, and Power [2]:

Proposition 4.1 *The inclusion T-$\mathrm{Alg}_s \to T$-Alg_l preserves all existing limits. The projections of such limits in T-Alg_l are strict and they jointly detect strictness.*

PROOF: The first sentence can be proved by direct calculation. Alternatively, in good cases, the inclusion T-$\mathrm{Alg}_\mathrm{s} \to T$-$\mathrm{Alg}_\mathrm{l}$ has a left adjoint [2], and so certainly preserves limits. The fact that the projections are strict is part of the first statement; the fact that they jointly detect strictness is an easy consequence: see [2]. □

Remark 4.2 We will use the proposition to build up various other limits from the oplax limit of an arrow. Most of the proofs amount to well-known constructions of one type of limit from another, such as are given in [6]; the only thing preventing us from simply quoting such results is the need to check that certain morphisms constructed along the way are strict, to allow the use of the proposition. In each case a direct proof is also possible, if a little tedious. There is one advantage, however of the direct proof: it allows a slightly stronger result in some cases. Specifically, in Proposition 4.3, Proposition 4.4, and Proposition 4.6, the result remains true if g is required only to be a pseudo morphism, not a strict one. We sketch how this would go in Section 6, as Proposition 6.1.

Given 1-cells $f, g : A \to B$ and 2-cells $\alpha, \beta : g \to f$ in a 2-category, the *equifier* of α and β is the universal 1-cell $j : K \to A$ for which $\alpha j = \beta j$; see [6].

Proposition 4.3 *The forgetful 2-functor $U_l : T$-$\mathrm{Alg}_l \to \mathscr{K}$ creates equifiers of 2-cells $\alpha, \beta : (g, id) \to (f, \bar{f})$, where g is strict. In particular, if $\mathscr{K}$ has such equifiers then so does T-Alg_l, and U_l preserves them. The projection of the equifier is strict, and detects strictness.*

PROOF: This can be proved directly; instead, we suppose that $\mathscr{K}$ admits finite conical limits and oplax limits of arrows, and use Theorem 3.2 and Proposition 4.1. Form the oplax limit (L, l) of $(f, \bar{f})$, with projections u and v, and 2-cell λ. There is a unique strict morphism $m : (A, a) \to (L, l)$ satisfying $um = 1$, $vm = g$, and $\lambda m = \alpha$; similarly, a unique strict $n : (A, a) \to (L, l)$ induced by β. The desired equifier is the equalizer of the strict morphisms $m, n : (A, a) \to (L, l)$. □

Given 1-cells $g, f : A \to B$ in a 2-category, the *inserter* of the pair is the universal 1-cell $j : K \to A$ equipped with a 2-cell $\kappa : gj \to fj$; see [6].

Proposition 4.4 *The forgetful 2-functor $U_l : T$-$\mathrm{Alg}_l \to \mathscr{K}$ creates inserters of pairs g,$(f, \bar{f}) : (A, a) \to (B, b)$, where $g = (g, id)$ is strict. In particular, if $\mathscr{K}$ has such inserters then so does T-Alg_l, and U_l preserves them. The projection of the inserter is strict, and detects strictness.*

PROOF: Once again we could prove this directly but instead use Theorem 3.2 and Proposition 4.1. We now build the desired inserter out of the oplax limit of $(f, \bar{f})$ and various limits of strict morphisms, much as in the prevous proposition. □

A *comonad* in a 2-category consists of a 1-cell $w : A \to A$ equipped with 2-cells $\varepsilon : w \to 1$ and $\delta : w \to w^2$ satisfying the usual comonad laws [16]. A co-Eilenberg-Moore object for the comonad is the universal map $u : B \to A$ equipped with a 2-cell $\gamma : u \to wu$ satisfying the usual coalgebra laws $w\gamma.\gamma = \delta u.\gamma$ and $\varepsilon u.\gamma = 1$. (In the 2-category **Cat**, this is just the usual category of coalgebras for the comonad.) See [16].

Proposition 4.5 *The forgetful 2-functor $U_l : T\text{-Alg}_l \to \mathscr{K}$ creates co-Eilenberg-Moore objects of comonads. In particular, if $\mathscr{K}$ has such objects then so does $T\text{-Alg}_l$, and U_l preserves them. The projection is strict and detects strictness.*

Proof: Once again we could prove this directly but instead use Theorem 3.2 and Proposition 4.1. We now build the co-Eilenberg-Moore object by first forming the inserter $(j, \kappa : j \to wj)$, then the equifier l of $\varepsilon j.\kappa : j \to j$ and the identity on j, and finally the equifier m of $w\kappa l.\kappa l$ and $\delta jl.\kappa l$. The desired Eilenberg-Moore object is given by $u = klm$ and $\gamma = \kappa lm$. □

Given 1-cells $f : A \to C$ and $g : B \to C$ in a 2-category, the *comma object* g/f is the universal diagram consisting of $u : D \to A$, $v : D \to B$, and $\varphi : gv \to fu$; see [6]. Observe that if g is in fact the identity then this is just the oplax limit of f, so that Theorem 3.2 is a special case of:

Proposition 4.6 *The forgetful 2-functor $U_l : T\text{-Alg}_l \to \mathscr{K}$ creates comma objects $g/(f, \bar{f})$ where $g = (g, id)$ is strict. In particular, if $\mathscr{K}$ has such comma objects then so does $T\text{-Alg}_l$, and U_l preserves them. The projections of the comma object are strict, and detect strictness.*

Proof: As usual, we omit the direct proof, and use Theorem 3.2 and Proposition 4.1. First form the oplax limit of $(f, \bar{f})$, then pull back the projection $v : (L, l) \to (C, c)$ along $g : (B, b) \to (C, c)$. □

Let $\mathscr{C}$ be a 2-category. Write $\text{Oplax}(\mathscr{C}, \mathbf{Cat})$ for the 2-category whose objects are the 2-functors from $\mathscr{C}$ to **Cat**, whose morphisms are the oplax natural transformations, and whose 2-cells are the modifications. (An oplax natural transformation between 2-functors $J, K : \mathscr{C} \to \mathbf{Cat}$ involves a functor $\varphi C : JC \to KC$ for every object C of $\mathscr{C}$, and a natural transformation $\varphi f : Kf.\varphi C \to \varphi D.Jf$ for every morphism $f : C \to D$ in $\mathscr{C}$, subject to various conditions [9].)

Remark 4.7 If $\mathscr{C}_0$ is the set of objects of $\mathscr{C}$, there is a 2-monad T on $\mathbf{Cat}^{\mathscr{C}_0}$ whose algebras are 2-functors from $\mathscr{C}$ to **Cat** and whose colax morphisms are oplax natural transformations; see [2, Section 6.6]. The 2-category $\text{Oplax}(\mathscr{C}, \mathbf{Cat})$ is what we have been calling $T\text{-Alg}_c$. Similarly $T\text{-Alg}_l$ is the 2-category $\text{Lax}(\mathscr{C}, \mathbf{Cat})$ of 2-functors, lax natural transformations, and modifications.

Given 2-functors $J : \mathscr{C} \to \mathbf{Cat}$ and $S : \mathscr{C} \to \mathscr{L}$, the oplax limit $\{J, S\}_{\text{oplax}}$ of S weighted by J is an object of $\mathscr{L}$ equipped with isomorphisms of categories

$$\mathscr{L}(X, \{J, S\}_{\text{oplax}}) \cong \text{Oplax}(\mathscr{C}, \mathbf{Cat})(J, \mathscr{L}(X, S))$$

natural in X; see [6]. As usual, one can take $X = \{J, S\}_{\text{oplax}}$, then corresponding to the identity on the left hand side is the unit $\eta : J \to \mathscr{L}(\{J, S\}_{\text{oplax}}, S)$, which will be an oplax natural transformation, called the unit. If $J : \mathscr{C} \to \mathbf{Cat}$ is the 2-functor constant at the terminal category $\mathbf{1}$, then we speak of the oplax limit of S, and denote it $\operatorname{oplaxlim} S$.

If $\mathscr{C}$ is the arrow-category $\mathbb{2}$, seen as a 2-category with no non-identity 2-cells, then to give S is to give an arrow $f : A \to B$ in $\mathscr{L}$. In this case $\operatorname{oplaxlim} S$ is just the oplax limit of the arrow f. For another example, let $\mathbf{\Delta}$ be the category of finite non-empty ordinals (the "augmented" or "algebraic" simplicial category). This is strict monoidal via ordinal sum, and so determines a 1-object 2-category $\mathbf{Mnd}$. A 2-functor from $\mathbf{Mnd}$ to $\mathscr{L}$ is precisely a monad in $\mathscr{L}$, while a 2-functor $S : \mathbf{Mnd}^{\text{co}} \to \mathscr{L}$ is precisely a comonad in $\mathscr{L}$, and in this latter case $\operatorname{oplaxlim} S$ is the co-Eilenberg-Moore object of S. Thus the following generalizes both Theorem 3.2 and Proposition 4.5:

Theorem 4.8 *The forgetful 2-functor $U_l : T\text{-}\mathrm{Alg}_l \to \mathscr{K}$ creates oplax limits. In particular, if $\mathscr{K}$ has such limits then so does $T\text{-}\mathrm{Alg}_l$, and U_l preserves them. The projections of an oplax limit are strict, and jointly detect strictness.*

Proof: Let $S : \mathscr{C} \to T\text{-}\mathrm{Alg}_{\mathrm{l}}$ and $J : \mathscr{C} \to \mathbf{Cat}$ be given. Let $L = \{J, U_l S\}_{\text{oplax}}$ be the oplax limit in $\mathscr{K}$, with unit $\eta : J \to \mathscr{K}(L, U_l S)$. Consider now the composite

$$J \xrightarrow{\eta} \mathscr{K}(L, U_l S) \xrightarrow{T_{L,U_l S}} \mathscr{K}(TL, TU_l S) \xrightarrow{\mathscr{K}(TL, uS)} \mathscr{K}(TL, U_l S)$$

where $T_{L,U_l S}$ is the action of the 2-functor T and is (strictly) 2-natural, while $u : TU_l \to U_l$ is the oplax natural transformation, whose components are the actions of T on the T-algebras in the image of S. By the universal property of the oplax limit L, this composite oplax natural transformation induces a map $l : TL \to L$. One now verifies, much as in the proof of Theorem 3.2 that (a) l makes L into a T-algebra, that (b,c) $\eta : J \to \mathscr{K}(L, U_l S)$ factorizes as $\zeta : J \to T\text{-}\mathrm{Alg}_{\mathrm{s}}((L, l), S)$ followed by $U_s : T\text{-}\mathrm{Alg}_{\mathrm{s}}((L, l), S) \to \mathscr{K}(L, U_s S)$, that (d) if $(x, \bar{x}) : (C, c) \to (L, l)$ is a lax morphism for which the composite

$$J \xrightarrow{\zeta} T\text{-}\mathrm{Alg}_{\mathrm{s}}((L, l), S) \xrightarrow{J_l} T\text{-}\mathrm{Alg}_{\mathrm{l}}((L, l), S) \xrightarrow{T\text{-}\mathrm{Alg}_{\mathrm{l}}((x, \bar{x}), S)} T\text{-}\mathrm{Alg}_{\mathrm{l}}((C, c), S)$$

factorizes through $T\text{-}\mathrm{Alg}_{\mathrm{s}}((C, c), S)$, then $(x, \bar{x})$ is strict, and finally that (e) the composite of $\zeta : J \to T\text{-}\mathrm{Alg}_{\mathrm{s}}((L, l), S)$ followed by the inclusion $T\text{-}\mathrm{Alg}_{\mathrm{s}}((L, l), S) \to T\text{-}\mathrm{Alg}_{\mathrm{l}}((L, l), S)$ exhibits (L, l) as the oplax limit $\{J, S\}_{\text{oplax}}$. $\Box$

If the 2-category $\mathscr{C}$ is discrete (meaning that it has no non-identity 1-cells or 2-cells) then $\mathrm{Oplax}(\mathscr{C}, \mathbf{Cat})$ is the same as $[\mathscr{C}, \mathbf{Cat}]$, the 2-category of (strict) 2-functors, (strict) 2-natural transformations, and modifications. Thus for such $\mathscr{C}$, oplax limits are just ordinary weighted limits. In particular products and cotensor products are oplax limits, and we recover the result of [2] that $T\text{-}\mathrm{Alg}_{\mathrm{l}}$ admits products and cotensor products when $\mathscr{K}$ does so:

Corollary 4.9 *The forgetful 2-functor $U_l : T\text{-}\mathrm{Alg}_l \to \mathscr{K}$ creates products and cotensor products. In particular, if $\mathscr{K}$ has such limits then so does $T\text{-}\mathrm{Alg}_l$, and U_l preserves them. The projections of a product or cotensor product are strict, and jointly detect strictness.*

For the record, we also state Theorem 4.8 in its dual form, involving lax limits. (Recall from Remark 2.1 that for a 2-category $\mathscr{K}$ there is a 2-category $\mathscr{K}^{\mathrm{co}}$ in which the 2-cells are reversed, and a 2-monad T on $\mathscr{K}$ induces a 2-monad T^{co} on $\mathscr{K}^{\mathrm{co}}$ with $T\text{-}\mathrm{Alg_c}^{\mathrm{co}} = T^{\mathrm{co}}\text{-}\mathrm{Alg_l}$.)

Theorem 4.10 *The forgetful 2-functor $U_c : T\text{-}\mathrm{Alg}_c \to \mathscr{K}$ creates lax limits. In particular, if $\mathscr{K}$ has such limits then so does $T\text{-}\mathrm{Alg}_c$, and U_c preserves them. The projections of lax limit are strict, and jointly detect strictness.*

We also give the dual form of Proposition 4.5; the dualization of the remaining results of the last two sections is left to the reader.

Proposition 4.11 *The forgetful 2-functor $U_c : T\text{-}\mathrm{Alg}_c \to \mathscr{K}$ creates Eilenberg-Moore objects of monads. In particular, if $\mathscr{K}$ has such objects then so does $T\text{-}\mathrm{Alg}_c$, and U_c preserves them. The projection is strict and detects strictness.*

5 Applications to monoidal categories, and dualizations

This brief section analyzes Proposition 4.5 in terms of Examples 1.1 and 1.2 of the introduction. In Example 1.1, we saw that there is a 2-monad T on **Cat** for which $T\text{-}\mathrm{Alg_l}$ is the 2-category of monoidal categories, monoidal functors, and monoidal natural transformations. A consequence of Proposition 4.5 is that if (W, ε, δ) is a comonad on a monoidal category $\mathscr{V}$, with W a monoidal functor, and ε and δ monoidal natural transformations, then the Eilenberg-Moore category $\mathscr{V}^W$ (the category of W-coalgebras) has a canonical monoidal structure for which the forgetful functor $U : \mathscr{V}^W \to \mathscr{V}$ and the cofree functor $G : \mathscr{V} \to \mathscr{V}^W$ are both monoidal, and the unit and counit of the adjunction $U \dashv G$ are monoidal natural transformations.

This is perhaps more familiar in the dual case of Example 1.2. The 2-category of monoidal categories, opmonoidal functors, and monoidal natural transformations is $T\text{-}\mathrm{Alg_c}$; Proposition 4.11 (the dual of Proposition 4.5) now gives the folklore result that monads in $T\text{-}\mathrm{Alg_c}$ have Eilenberg-Moore objects, formed as in **Cat**. A monad in $T\text{-}\mathrm{Alg_c}$ is called an *opmonoidal monad*; it consists of a monad on a monoidal category, where the endofunctor part is an opmonoidal functor, and the unit and multiplication are monoidal natural transformations. This folklore result formed the basis of [14], which in turn analyzed some of the formal aspects of [15]. The *original* (not dualized) Proposition 4.5 says that *monoidal comonads* have coEilenberg-Moore objects (in the 2-category of monoidal categories, monoidal functors, and monoidal natural transformations.)

Finally, we note that there is another dualization of the folklore result of this section, involving reversing not the 1-cells but the 2-cells. It also forms part of the folklore, but seems not to follow from the first. It states that the 2-category of monoidal categories, monoidal functors, and monoidal natural transformations has Kleisli objects. Kleisli objects [16] are the colimit-notion corresponding to the limit-notion of Eilenberg-Moore object: Kleisli objects in $\mathscr{L}$ are Eilenberg-Moore objects in $\mathscr{L}^{\mathrm{op}}$ (where $\mathscr{L}^{\mathrm{op}}$ is obtained from $\mathscr{L}$ by reversing the 1-cells but not the 2-cells).

6 Appendix

Here we complete the proof of Theorem 3.2, and sketch a proof of a strengthened form of Proposition 4.4.

Proof of Theorem 3.2

The remaining steps are (a) that $l : TL \to L$ makes L into a T-algebra, and (e) that (L, l), with the data described in the earlier part of the proof, satisfies the universal property of the oplax limit.

To check (a), we must show that l satisfies the two equations for a T-algebra: $l.iL = 1_L$ and $l.Tl = l.mL$. In each case these are equations involving maps with codomain L, so we may use the universal property of the oplax limit L in $\mathscr{K}$. Now $ul.iL = a.Tu.iL = a.iA.u = u = u.1_L$ and similarly $vl.iL = v.1_L$; while

$$
\begin{tikzcd}
& L \arrow[d, "iL"'] & \\
& TL \arrow[d, "l"'] & \\
& L \arrow[dl, "u"'] \arrow[dr, "v"] \arrow[d, phantom, "\Swarrow\lambda"] & \\
A \arrow[rr, "f"'] & {} & B
\end{tikzcd}
=
\begin{tikzcd}
& L \arrow[d, "iL"'] & \\
& TL \arrow[dl, "Tu"'] \arrow[dr, "Tv"] \arrow[d, phantom, "\Swarrow T\lambda"] & \\
TA \arrow[rr, "Tf"] \arrow[d, "a"'] & {} \arrow[d, phantom, "\Swarrow\bar{f}"] & TB \arrow[d, "b"] \\
A \arrow[rr, "f"'] & {} & B
\end{tikzcd}
=
\begin{tikzcd}
& L \arrow[dl, "u"'] \arrow[dr, "v"] \arrow[d, phantom, "\Swarrow\lambda"] & \\
A \arrow[rr, "f"'] \arrow[d, "iA"'] & {} & B \arrow[d, "iB"] \\
TA \arrow[rr, "Tf"] \arrow[d, "a"'] & {} \arrow[d, phantom, "\Swarrow\bar{f}"] & TB \arrow[d, "b"] \\
A \arrow[rr, "f"'] & {} & B
\end{tikzcd}
=
\begin{tikzcd}
& L \arrow[dl, "u"'] \arrow[dr, "v"] \arrow[d, phantom, "\Swarrow\lambda"] & \\
A \arrow[rr, "f"'] \arrow[dd, "1_A"'] & {} & B \arrow[dd, "1_B"] \\
& & \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}
$$

and so finally $\lambda.l.iL = \lambda.1_L$, thus $l.iL = 1_L$ by the universal property.

The associativity condition $l.Tl = l.mL$ is similar, but slightly harder. First we check the composites with u and v:

$$u.l.Tl = a.Tu.Tl = a.Ta.T^2u = a.mA.T^2u = a.Tu.mL = u.l.mL$$

and likewise $v.l.Tl = v.l.mL$. Finally

$$
\begin{tikzcd}
& T^2L \arrow[d, "Tl"'] & \\
& TL \arrow[d, "l"'] & \\
& L \arrow[dl, "u"'] \arrow[dr, "v"] \arrow[d, phantom, "\Swarrow\lambda"] & \\
A \arrow[rr, "f"'] & {} & B
\end{tikzcd}
=
\begin{tikzcd}
& T^2L \arrow[d, "Tl"'] & \\
& TL \arrow[dl, "Tu"'] \arrow[dr, "Tv"] \arrow[d, phantom, "\Swarrow T\lambda"] & \\
TA \arrow[rr, "Tf"] \arrow[d, "a"'] & {} \arrow[d, phantom, "\Swarrow\bar{f}"] & TB \arrow[d, "b"] \\
A \arrow[rr, "f"'] & {} & B
\end{tikzcd}
=
\begin{tikzcd}
& T^2L \arrow[dl, "T^2u"'] \arrow[dr, "T^2v"] \arrow[d, phantom, "\Swarrow T^2\lambda"] & \\
T^2A \arrow[rr, "T^2f"'] \arrow[d, "Ta"'] & {} \arrow[d, phantom, "\Swarrow T\bar{f}"] & T^2B \arrow[d, "Tb"] \\
TA \arrow[rr, "Tf"] \arrow[d, "a"'] & {} \arrow[d, phantom, "\Swarrow\bar{f}"] & TB \arrow[d, "b"] \\
A \arrow[rr, "f"'] & {} & B
\end{tikzcd}
$$

$$
= \begin{tikzcd}
 & T^2L \arrow[dl, "T^2u"'] \arrow[dr, "T^2v"] \arrow[d, phantom, "\Swarrow T^2\lambda"] & \\
T^2A \arrow[rr, "T^2f"] \arrow[d, "mA"'] & {} & T^2B \arrow[d, "mB"] \\
TA \arrow[rr, "Tf"] \arrow[d, "a"'] & \Swarrow \bar{f} & TB \arrow[d, "b"] \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}
=
\begin{tikzcd}
 & T^2L \arrow[d, "mL"] & \\
 & TL \arrow[dl, "Tu"'] \arrow[dr, "Tv"] \arrow[d, phantom, "\Swarrow T\lambda"] & \\
TA \arrow[rr, "Tf"] \arrow[d, "a"'] & \Swarrow \bar{f} & TB \arrow[d, "b"] \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}
=
\begin{tikzcd}
 & T^2L \arrow[d, "mL"] & \\
 & TL \arrow[d, "l"] & \\
 & TL \arrow[dl, "Tu"'] \arrow[dr, "Tv"] \arrow[d, phantom, "\Swarrow \lambda"] & \\
A \arrow[rr, "f"'] & {} & B
\end{tikzcd}
$$

and so $l.Tl = l.mL$. This completes the proof of (a), that (L, l) is a T-algebra.

As for (e), let $(r, \bar{r}) : (C, c) \to (A, a)$ and $(s, \bar{s}) : (C, c) \to (B, b)$ be lax morphisms, and $\rho : (s, \bar{s}) \to (f, \bar{f})(r, \bar{r})$ a T-transformation. By the universal property of the oplax limit L in $\mathscr{K}$, there is a unique $x : C \to L$ satisfying $ux = r$, $vx = s$, and $\lambda x = \rho$. The maps

$$ul.Tx = a.Tu.Tx = a.Tr \xrightarrow{\bar{r}} rc = uxc$$

$$vl.Tx = b.Tv.Tx = b.Ts \xrightarrow{\bar{s}} sc = vxc$$

will induce a unique $\bar{x} : l.Tx \to xc$ satisfying $u\bar{x} = \bar{r}$ and $v\bar{x} = \bar{s}$ provided that the square

$$
\begin{tikzcd}
vl.Tx \arrow[r, "\lambda l.Tx"] \arrow[d, "\bar{s}"'] & ful.Tx \arrow[d, "f\bar{r}"] \\
vxc \arrow[r, "\lambda xc"'] & fuxc
\end{tikzcd}
$$

commutes. Unravelling the definitions, this amounts to commutativity of

$$
\begin{tikzcd}
b.Ts \arrow[r, "b.T\rho"] \arrow[d, "\bar{s}"'] & b.Tf.Tr \arrow[r, "\bar{f}.Tr"] & fa.Tr \arrow[d, "f\bar{r}"] \\
sc \arrow[rr, "\rho c"'] & & frc
\end{tikzcd}
$$

which in turn amounts to the condition for ρ to be a T-transformation. Thus we have the required $\bar{x} : l.Tx \to xc$. To verify the one-dimensional aspect of the universal property, it remains to check that $\bar{x}$ satisfies the conditions to make x into a lax morphism from (C, c) to (L, l); but these follow easily from the fact that $u\bar{x}$ and $v\bar{x}$ make ux and vx into lax morphisms from (C, c) to (A, a) and from (C, c) to (B, b).

As for the two-dimensional aspect, suppose further that we have $(r', \bar{r'})$, $(s', \bar{s'})$, and $\rho' : s' \to fr'$ inducing $(x', \bar{x'}) : (C, c) \to (L, l)$, and T-transformations $\alpha : (r, \bar{r}) \to (r', \bar{r'})$ and

$\beta : (s, \bar{s}) \to (s', \bar{s'})$ making

$$\begin{tikzcd}
fr \arrow[r, "\rho"] \arrow[d, "f\alpha"'] & s \arrow[d, "\beta"] \\
fr' \arrow[r, "\rho'"'] & s'
\end{tikzcd}$$

commute. Then by the universal property of the oplax limit L in $\mathscr{K}$, there is a unique $\gamma : x \to x'$ satisfying $u\gamma = \alpha$ and $v\gamma = \beta$. It remains to check that γ is a T-transformation; but this follows easily from the fact that $u\gamma$ and $v\gamma$ are so. $\square$

Strengthening of Proposition 4.4

We now turn to the promised strengthening of Proposition 4.4. We state this as:

Proposition 6.1 *The forgetful 2-functor $U_l : T\text{-Alg}_l \to \mathscr{K}$ creates inserters of pairs $(g, \bar{g})$, $(f, \bar{f}) : (A, a) \to (B, b)$, where $\bar{g}$ is invertible, so that $(g, \bar{g})$ is pseudo. In particular, if $\mathscr{K}$ has such inserters then so does $T\text{-Alg}_l$, and U_l preserves them. The projection of the inserter is strict, and detects strictness.*

Proof: We only sketch the proof; the remaining details are similar to the proof of Theorem 3.2. Let $\kappa : gj \to fj$ exhibit $j : K \to A$ as the inserter in $\mathscr{K}$. We seek a T-algebra structure $k : TK \to K$ on K making (K, k) into the desired inserter. The composite 2-cell

$$\begin{tikzcd}
 & TA \arrow[rr, "a"] \arrow[dr, "Tg"] & & A \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow \bar{g}^{-1}"] & \\
TK \arrow[ur, "Tj"] \arrow[dr, "tj"'] \arrow[rr, phantom, "\Downarrow T\kappa"] & & TB \arrow[rr, "b"] & {} & B \\
 & TA \arrow[rr, "a"'] \arrow[ur, "Tf"] & & A \arrow[ur, "f"'] \arrow[u, phantom, "\Downarrow \bar{f}"] &
\end{tikzcd}$$

induces a unique arrow $k : TK \to K$ with $jk = a.Tj$ and κk equal to the displayed composite. One now verifies, much as in the proof of Theorem 3.2, that: (a) k makes K into a T-algebra, (b) j becomes a strict morphism, (c) κ becomes a T-transformation, (d) j detects strictness, and (e) j and κ exhibit (K, k) as the inserter. $\square$

The corresponding strengthenings of Propositions 4.3 and 4.6 are left to the reader.

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