Limits of small functors *

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Abstract

For a small category $\mathscr{K}$ enriched over a suitable monoidal category $\mathscr{V}$, the free completion of $\mathscr{K}$ under colimits is the presheaf category $[\mathscr{K}^{\mathrm{op}}, \mathscr{V}]$. If $\mathscr{K}$ is large, its free completion under colimits is the $\mathscr{V}$-category $\mathscr{P}\mathscr{K}$ of small presheaves on $\mathscr{K}$, where a presheaf is small if it is a left Kan extension of some presheaf with small domain. We study the existence of limits and of monoidal closed structures on $\mathscr{P}\mathscr{K}$.

A fundamental construction in category theory is the category of presheaves $[\mathscr{K}^{\mathrm{op}}, \mathbf{Set}]$ on a small category $\mathscr{K}$. Among many other important properties, it is the free completion of $\mathscr{K}$ under colimits. If the category $\mathscr{K}$ is large, then the full presheaf category $[\mathscr{K}^{\mathrm{op}}, \mathbf{Set}]$ is not the free completion of $\mathscr{K}$ under colimits; indeed it is not even a legitimate category, insofar as its hom-sets are not in general small.

In some contexts it is more appropriate to consider not *all* the presheaves on $\mathscr{K}$, but only the *small* ones: a presheaf $F : \mathscr{K}^{\mathrm{op}} \to \mathbf{Set}$ is said to be small if it is the left Kan extension of some presheaf whose domain is small. This is equivalent to F being the left Kan extension of its restriction to some small full subcategory of its domain, or equally to its being a small colimit of representables. The natural transformations between two small presheaves on $\mathscr{K}$ do form a small set, and so the totality of small presheaves on $\mathscr{K}$ forms a genuine category $\mathscr{P}\mathscr{K}$ with small hom-sets. Furthermore, $\mathscr{P}\mathscr{K}$ is in fact the free completion of $\mathscr{K}$ under colimits. Of course if $\mathscr{K}$ is small, then every presheaf on $\mathscr{K}$ is small, and so $\mathscr{P}\mathscr{K}$ is just $[\mathscr{K}^{\mathrm{op}}, \mathbf{Set}]$, but in general this is not the case.

Although $\mathscr{P}\mathscr{K}$ is the free completion of $\mathscr{K}$ under colimits, it does not have all the good properties of $[\mathscr{K}^{\mathrm{op}}, \mathbf{Set}]$ for small $\mathscr{K}$. For example it is not necessarily complete or cartesian closed. In this paper we study, among other things, when $\mathscr{P}\mathscr{K}$ does have such good properties.

In fact we work not just with ordinary categories, but with categories enriched over a suitable monoidal category $\mathscr{V}$. Once again, if $\mathscr{K}$ is small then $[\mathscr{K}^{\mathrm{op}}, \mathscr{V}]$ is the free completion of $\mathscr{K}$ under colimits, but for large $\mathscr{K}$ this is no longer the case; the illegitimacy of $[\mathscr{K}^{\mathrm{op}}, \mathscr{V}]$ in that case is more drastic: it is not even a $\mathscr{V}$-category. The free completion of $\mathscr{K}$ under colimits is the $\mathscr{V}$-category $\mathscr{P}\mathscr{K}$ of small presheaves on $\mathscr{K}$, where once again a presheaf is small if it is the left Kan extension of some presheaf with small domain; and once again the two reformulations of this notion can be made.

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The case $\mathscr{V} = \mathbf{Set}$ is closely related to work by various authors. Freyd [7] introduced two smallness notions for presheaves on large categories. He called a functor $F : \mathscr{K}^{\mathrm{op}} \to \mathbf{Set}$ *petty* if there is a small family $(C_\lambda \in \mathscr{K})_{\lambda \in \Lambda}$ with an epimorphism

$$\sum_\lambda \mathscr{K}(-, C_\lambda) \to F;$$

and *lucid* if it is petty and for any representable $\mathscr{K}(-, A)$ and any pair of maps $u, v : \mathscr{K}(-, A) \to F$, their equalizer is petty. Freyd studied when the category of petty presheaves on $\mathscr{K}$ is complete, and when the category of lucid presheaves on $\mathscr{K}$ is complete, obtaining results similar to our Theorem 3.8 below. Rosický [15] showed that if $\mathscr{K}$ is complete, then a presheaf F on $\mathscr{K}$ is lucid if and only if it is small; one can then deduce our Corollary 3.9 from the results of Freyd. Rosický also characterized, in the case $\mathscr{V} = \mathbf{Set}$, when $\mathscr{P}\mathscr{K}$ is cartesian closed; see Example 7.4 below. In a slightly different direction, the existence of limits in free completions under some class of colimits was studied in [9].

In the enriched case, the fact, mentioned above, that $\mathscr{P}\mathscr{K}$ is the free completion of $\mathscr{K}$ under colimits, is due to Lindner [14]. The existence of limits or monoidal closed structures on $\mathscr{P}\mathscr{K}$ seems not to have been considered in the enriched setting.

Some of our results have been used in abstract homotopy theory; for example Corollary 3.9 was used in [5]. The idea is that one wants to have a complete and cocomplete category of diagrams of some particular type, where the indexing category is large. In this context one is particularly interested in the case $\mathscr{V} = \mathbf{SSet}$, the category of simplicial sets.

In Section 1 we review the required background from enriched category theory, and in Section 2 the notion of small functor. Then in Section 3 we prove the fundamental result that $\mathscr{P}\mathscr{K}$ is complete if and only if it has limits of representables; thus in particular $\mathscr{P}\mathscr{K}$ is complete if $\mathscr{K}$ is so. In Section 4 we refine the results of the previous section to deal not with arbitrary (small) limits, but with limits of some particular type, such as finite limits or finite products. In Section 5 we deduce from the earlier results various known results about the case $\mathscr{V}_0 = \mathbf{Set}$ of ordinary categories, before extending them to the case where $\mathscr{V}_0$ is a presheaf category. Section 6 concerns not the existence of limits in $\mathscr{P}\mathscr{K}$ but the preservation of limits by functors $\mathscr{P}F : \mathscr{P}\mathscr{K} \to \mathscr{P}\mathscr{L}$ given by left Kan extension along $F^{\mathrm{op}} : \mathscr{K}^{\mathrm{op}} \to \mathscr{L}^{\mathrm{op}}$. In Section 7 we study monoidal closed structures on $\mathscr{P}\mathscr{K}$ using the notion of promonoidal category. In Section 8 we consider limits of small functors with codomain a locally presentable category $\mathscr{M}$, generalizing the earlier case of $\mathscr{M} = \mathscr{V}$. Finally in Section 9 we briefly discuss Isbell conjugacy for large categories.

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1 Review of relevant enriched category theory

We shall work over a symmetric monoidal closed category $\mathscr{V}$. The tensor product is denoted $\otimes$, the unit object I, and the internal hom $[\ ,\ ]$. Where necessary the underlying ordinary category is denoted $\mathscr{V}_0$.

We suppose that this underlying ordinary category is locally presentable [8, 2]: thus for some regular cardinal α and some small category $\mathscr{C}$ with α-small limits $\mathscr{V}_0$ is equivalent to the category of α-continuous functors from $\mathscr{C}$ to **Set**. It follows that $\mathscr{V}_0$ is complete and cocomplete, and it turns out that $\mathscr{C}$ is equivalent to the opposite of the full subcategory $(\mathscr{V}_0)_\alpha$ of $\mathscr{V}_0$ consisting of the α-presentable objects: these are the $X \in \mathscr{V}_0$ for which $\mathscr{V}_0(X,-) : \mathscr{V}_0 \to \mathbf{Set}$ preserves α-filtered colimits. By [13], after possibly changing α, we may suppose that $(\mathscr{V}_0)_\alpha$ is closed in $\mathscr{V}_0$ under the monoidal structure, so that $\mathscr{V}$ is *locally α-presentable as a closed category*, in the sense of [10].

We shall work throughout the paper over such a locally presentable closed category. This includes many important examples, such as the categories **Set**, **Ab**, R-**Mod**, **Cat**, **Gpd**, and **SSet**, of sets, abelian groups, R-modules (over a commutative ring R), categories, groupoids, and simplicial sets, as well as the two-element lattice $\mathbb{2}$. All these examples are locally *finitely* presentable (that is, locally $\aleph_0$-presentable) but there are further examples which require a higher cardinal than $\aleph_0$: for example any Grothendieck topos, the category **Ban** of Banach spaces and linear contractions, Lawvere's category $[0,\infty]$ of extended non-negative real numbers, or the first-named author's $*$-autonomous category $[-\infty,\infty]$ of extended real numbers. All categorical notions are understood to be enriched over $\mathscr{V}$, even if this is not explicitly stated. (Thus category means $\mathscr{V}$-category, functor means $\mathscr{V}$-functor, and so on.) We fix a regular cardinal α_0 for which $\mathscr{V}_0$ is locally α_0-presentable and $(\mathscr{V}_0)_{\alpha_0}$ is closed under the monoidal structure. Henceforth "α is a regular cardinal" will mean "α is a regular cardinal and $\alpha \geq \alpha_0$".

For such a $\mathscr{V}$, it was shown in [10] that there is a good notion of locally α-presentable $\mathscr{V}$-category, for any regular cardinal $\alpha \geq \alpha_0$. A locally α-presentable $\mathscr{V}$-category $\mathscr{K}$ is complete and cocomplete, and is equivalent to the $\mathscr{V}$-category of α-continuous $\mathscr{V}$-functors from $\mathscr{C}$ to $\mathscr{V}$ for some small $\mathscr{V}$-category $\mathscr{C}$ with α-small limits. This $\mathscr{C}$ can be identified with the opposite of the category of α-presentable objects in $\mathscr{K}$.

A *weight* is a presheaf $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{V}$, usually, although not always with small domain. The *colimit* of a functor $S : \mathscr{C} \to \mathscr{K}$ is denoted by $F * S$, while the *limit* of a functor $S : \mathscr{C}^{\mathrm{op}} \to \mathscr{K}$ is denoted by $\{F, S\}$. When $\mathscr{C}^{\mathrm{op}}$ is the unit $\mathscr{V}$-category $\mathscr{I}$, we may identify F with an object of $\mathscr{V}$ and S with an object of $\mathscr{C}$; we sometimes write $F \cdot S$ for $F * S$ and call it a tensor, and we sometimes write $F \pitchfork S$ for $\{F, S\}$ and call it a cotensor.

2 Small functors

A functor $F : \mathscr{K} \to \mathscr{V}$ is said to be *small* if it is the left Kan extension of its restriction to some small full subcategory of $\mathscr{K}$. This will clearly be the case if F is a small colimit of representables, for then we may take as the subcategory precisely those objects corresponding to the representables in the colimit. On the other hand, if $F : \mathscr{K} \to \mathscr{V}$ is the left Kan extension of FJ along the inclusion $J : \mathscr{C} \to \mathscr{K}$ of some small full subcategory, then $F = (FJ) * \mathscr{K}(J, 1)$, and so F is a small colimit of representables. Thus the small functors are precisely the small colimits of representables.

Of course if $\mathscr{K}$ is itself small, then every functor from $\mathscr{K}$ to $\mathscr{V}$ is small. If on the other hand $\mathscr{K}$ is locally presentable, then a functor $F : \mathscr{K} \to \mathscr{V}$ is small if and only if it is accessible: that is, if and only if it preserves α-filtered colimits for some regular cardinal α. For if F is accessible, then we may choose α so that $\mathscr{K}$ is locally α-presentable and F preserves α-filtered colimits; then F is the left Kan extension of its restriction to the full subcategory of $\mathscr{K}$ consisting of the α-presentable objects. Conversely, if F is the left Kan extension of its restriction to a small full subcategory $\mathscr{C}$ of $\mathscr{K}$, then we may choose a regular cardinal α in such a way that $\mathscr{K}$ is locally α-presentable and every object in $\mathscr{C}$ is α-presentable in $\mathscr{K}$, and then F preserves α-filtered colimits.

Remark 2.1 There is a corresponding result for the case where $\mathscr{K}$ is accessible, but we have not taken the trouble to formulate it here, since as usual there is a greater sensitivity to the choice of regular cardinal in the accessible case than in the locally presentable one.

The totality of small functors from $\mathscr{K}^{\mathrm{op}}$ to $\mathscr{V}$ forms a $\mathscr{V}$-category $\mathscr{P}\mathscr{K}$ which is cocomplete and is in fact the free cocompletion of $\mathscr{K}$ via the Yoneda embedding $Y : \mathscr{K} \to \mathscr{P}\mathscr{K}$. In the case where $\mathscr{K}$ is small, $\mathscr{P}\mathscr{K}$ is simply the presheaf category $[\mathscr{K}^{\mathrm{op}}, \mathscr{V}]$, but in general not every presheaf is small.

Example 2.2 Let $\mathscr{V}$ be **Set**, and let $\mathscr{K}$ be any large set X, seen as a discrete category. Then a presheaf on $\mathscr{K}$ can be seen as an X-indexed set $A \to X$, and it is small if and only if A is so.

The construction $\mathscr{P}\mathscr{K}$ is pseudofunctorial in $\mathscr{K}$, and forms part of a pseudomonad $\mathscr{P}$ on $\mathscr{V}$-**Cat**. We shall also consider free completions under certain types of colimit. Let Φ be a class of weights with small domain. For a $\mathscr{V}$-category $\mathscr{K}$ write $\Phi(\mathscr{K})$ for the closure of $\mathscr{K}$ in $\mathscr{P}\mathscr{K}$ under Φ-colimits. The Yoneda embedding $Y : \mathscr{K} \to \Phi(\mathscr{K})$ exhibits $\Phi(\mathscr{K})$ as the free completion of $\mathscr{K}$ under Φ-colimits. The class Φ is said to be *saturated* if, whenever $\mathscr{K}$ is small, $\Phi(\mathscr{K})$ consists exactly of the presheaves on $\mathscr{K}$ lying in Φ. (This idea goes back to [3], where the word "closed" was used rather than "saturated".) Once again the construction $\Phi(\mathscr{K})$ is pseudofunctorial in $\mathscr{K}$ and forms part of a pseudomonad Φ^* on $\mathscr{V}$-**Cat**. The union Φ^* of all the $\Phi(\mathscr{C})$ with $\mathscr{C}$ small is a new class of weights called the *saturation* of Φ.

Thus far we have spoken only of smallness of presheaves, but we shall also have cause to consider smallness of more general functors. Once again, we say that a $\mathscr{V}$-functor $S : \mathscr{K} \to \mathscr{M}$ is small if it is the left Kan extension of some $\mathscr{V}$-functor $\mathscr{C} \to \mathscr{M}$ with small domain, or equivalently, if it is the left Kan extension of its restriction to some small full subcategory of $\mathscr{K}$. This definition works best when $\mathscr{M}$ is cocomplete, so that one can form the relevant left Kan extensions, and we shall only use it in this context. An important case is where $\mathscr{M} = [\mathscr{C}, \mathscr{V}]$ for some small $\mathscr{C}$. We say that $S : \mathscr{K} \to [\mathscr{C}, \mathscr{V}]$ is *pointwise small* if the composite of S with each evaluation functor $\mathrm{ev}_C : [\mathscr{C}, \mathscr{V}] \to \mathscr{V}$ is small.

Lemma 2.3 *A functor $S : \mathscr{K} \to [\mathscr{C}, \mathscr{V}]$ is small if and only if it is pointwise small.*

Proof: Since the evaluaton functors preserve Kan extensions the "only if" part is immediate. Conversely, if S is pointwise small, then for each C there is a small full subcategory $\mathscr{B}_C$ of $\mathscr{K}$ with the property that $\mathrm{ev}_C S$ is the left Kan extension of its restriction to $\mathscr{B}_C$. Since $\mathscr{C}$ is small, the union $\mathscr{B}$ of the $\mathscr{B}_C$ is small, and now each $\mathrm{ev}_C S$ is the left Kan extension of its restriction to $\mathscr{B}$, hence the same is true of S. $\square$

In Section 8 we shall also consider the case where $\mathscr{M}$ is locally presentable.

3 Limits of small functors

As observed above, if $\mathscr{K}$ is small then $\mathscr{P}\mathscr{K}$ is the full presheaf category $[\mathscr{K}^{\mathrm{op}}, \mathbf{Set}]$ which is of course not just cocomplete but also complete. In general, however, a category of the form $\mathscr{P}\mathscr{K}$ need not be complete, as the following example, based on Example 2.2 shows:

Example 3.1 If $\mathscr{V}$ is **Set** and $\mathscr{K}$ is a large discrete category then $\mathscr{P}\mathscr{K}$ has no terminal object.

We investigate which categories $\mathscr{K}$ have the property that $\mathscr{P}\mathscr{K}$ is complete. First observe that since $\mathscr{P}\mathscr{K}$ contains the representables, any limit in $\mathscr{P}\mathscr{K}$ must be formed pointwise. Thus the question "is $\mathscr{P}\mathscr{K}$ complete?" may be rephrased as "are limits of small presheaves on $\mathscr{K}$ small?" This may appear to involve consideration of the illegitimate $[\mathscr{K}^{\mathrm{op}}, \mathscr{V}]$, but in fact this is unnecessary. Given a weight $\varphi : \mathscr{C} \to \mathscr{V}$, where $\mathscr{C}$ is small, and a diagram $S : \mathscr{C} \to \mathscr{P}\mathscr{K}$, we may regard S as a functor $\bar{S} : \mathscr{K}^{\mathrm{op}} \to [\mathscr{C}, \mathscr{V}]$, and compose $\bar{S}$ with $\{\varphi, -\} : [\mathscr{C}, \mathscr{V}] \to \mathscr{V}$, and ask whether the composite $\{\varphi, \bar{S}-\} : \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ is small.

An arbitrary $\bar{S} : \mathscr{K}^{\mathrm{op}} \to [\mathscr{C}, \mathscr{V}]$ arises in this way from some $S : \mathscr{C} \to \mathscr{P}\mathscr{K}$ if and only if $\bar{S}$ is *pointwise small*; recall from the previous section that this means that each $\mathrm{ev}_C \bar{S} : \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ is small, but that it is equivalent to $\bar{S}$ itself being small.

Proposition 3.2 *The limit of $S : \mathscr{C} \to \mathscr{P}\mathscr{K}$ weighted by $\varphi : \mathscr{C} \to \mathscr{V}$ exists if and only if $\{\varphi, \bar{S}-\}$ is small; $\mathscr{P}\mathscr{K}$ has all φ-limits if and only if $\{\varphi, R-\}$ is small for every small $R : \mathscr{K}^{\mathrm{op}} \to [\mathscr{C}, \mathscr{V}]$.*

Related to the existence of limits in $\mathscr{K}$ is the existence of a right adjoint to $\mathscr{P}F : \mathscr{P}\mathscr{K} \to \mathscr{P}\mathscr{L}$ for a functor $F : \mathscr{K} \to \mathscr{L}$. Here $\mathscr{P}F$ is given by left Kan extensions along F, so if $\mathscr{K}$ were small then $\mathscr{P}F$ would have a right adjoint given by restriction along F. In general, however, the restriction GF of a small $G : \mathscr{L}^{\mathrm{op}} \to \mathscr{V}$ need not be small; indeed the restriction $\mathscr{L}(F, L) : \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ of a representable $\mathscr{L}(-, L)$ need not be small. But if each $\mathscr{L}(F, L)$ is small, we have the right adjoint:

Proposition 3.3 *For an arbitrary functor $F : \mathscr{K} \to \mathscr{L}$, there is a right adjoint to $\mathscr{P}F : \mathscr{P}\mathscr{K} \to \mathscr{P}\mathscr{L}$ if and only if $\mathscr{L}(F, L) : \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ is small for every object L of $\mathscr{L}$, and then the right adjont is given by restriction along F.*

Proof: If $\mathscr{P}F$ has a right adjoint R, then

$$RGA \cong \mathscr{P}\mathscr{K}(YA, RG) \cong \mathscr{P}\mathscr{L}(\mathscr{P}F.YA, G) \cong \mathscr{P}\mathscr{L}(YFA, G) \cong GFA$$

for any G in $\mathscr{P}\mathscr{L}$, and so R must be given by restriction along F. Thus $RYL = \mathscr{L}(F, L)$, which must therefore be small.

Suppose conversely that each $\mathscr{L}(F, L)$ is small. Each G in $\mathscr{P}\mathscr{L}$ is a small colimit of representables. Since restricting along F preserves colimits, GF is a small colimit of functors of the form $\mathscr{L}(F, L)$, but these are small by assumption, so GF is small. $\Box$

Our first example of a large category $\mathscr{K}$ with $\mathscr{P}\mathscr{K}$ complete is the opposite of a locally presentable category.

Proposition 3.4 *$\mathscr{P}\mathscr{K}$ is complete if $\mathscr{K}^{\mathrm{op}}$ is locally presentable.*

Proof: If $\mathscr{K}^{\mathrm{op}}$ is locally presentable and $R : \mathscr{K}^{\mathrm{op}} \to [\mathscr{C}, \mathscr{V}]$ is small, then for each object C of $\mathscr{C}$ there is a regular cardinal α_C for which $\mathrm{ev}_C R$ is α_C-accessible. Since $\mathscr{C}$ is small, we may choose a regular cardinal α for which $\mathscr{K}^{\mathrm{op}}$ is an α-accessible category, R is an α-accessible functor, and φ is α-presentable in $[\mathscr{C}, \mathscr{V}]$. Then R and $\{\varphi, -\}$ preserve α-filtered colimits, hence so does $\{\varphi, R-\}$. $\Box$

Remark 3.5 The proposition remains true if $\mathscr{K}^{\mathrm{op}}$ is accessible; the comments made in Remark 2.1 still apply.

Corollary 3.6 *$\mathscr{P}\mathscr{K}$ is complete if $\mathscr{K}$ is $[\mathscr{A},\mathscr{V}]^{\mathrm{op}}$ for a small category $\mathscr{A}$.*

In other words, $\mathscr{P}\mathscr{K}$ is complete if $\mathscr{K} = \mathscr{P}(\mathscr{A}^{\mathrm{op}})^{\mathrm{op}}$ for a small $\mathscr{A}$. We shall now show how to remove the hypothesis that $\mathscr{A}$ is small. First observe $\mathscr{P}J : \mathscr{P}\mathscr{K} \to \mathscr{P}\mathscr{L}$ is given by left Kan extension along J, so is fully faithful if J is so.

Proposition 3.7 *$\mathscr{P}\mathscr{K}$ is complete if $\mathscr{K} = \mathscr{P}(\mathscr{L}^{\mathrm{op}})^{\mathrm{op}}$.*

PROOF: Let $\mathscr{C}$ be a small category and let $R : \mathscr{K}^{\mathrm{op}} \to [\mathscr{C},\mathscr{V}]$ be small; we must show that $\{\varphi, R-\}$ is small. Now R is the left Kan extension of its restriction to a small full subcategory $\mathscr{D}$ of $\mathscr{P}(\mathscr{L}^{\mathrm{op}})$. Each $D \in \mathscr{D}$ is a small functor $\mathscr{L} \to \mathscr{V}$, so is the left Kan extension of its restriction to some small $\mathscr{B}_D$. The union $\mathscr{B}$ of the $\mathscr{B}_D$ is small, and now the full inclusion $J : \mathscr{B}^{\mathrm{op}} \to \mathscr{L}^{\mathrm{op}}$ induces a full inclusion $\mathscr{P}J : \mathscr{P}(\mathscr{B}^{\mathrm{op}}) \to \mathscr{P}(\mathscr{L}^{\mathrm{op}})$ containing $\mathscr{D}$.

Now $\mathscr{B}$ is small, so $\mathscr{P}J$ has a right adjoint J^* given by restriction along J, and thus $\mathrm{Lan}_{\mathscr{P}J}$ is itself given by restriction along J^*. Since R is the left Kan extension of its restriction S along $\mathscr{P}J$, we have

$$\{\varphi, R-\} = \{\varphi, -\}R \cong \{\varphi, -\}\mathrm{Lan}_{\mathscr{P}J}S \cong \{\varphi, -\}SJ^* \cong \mathrm{Lan}_{\mathscr{P}J}\{\varphi, -\}S = \mathrm{Lan}_{\mathscr{P}J}\{\varphi, S-\}$$

and so $\{\varphi, R-\}$ will be small if $\{\varphi, S-\}$ is so. Now $S : \mathscr{P}(\mathscr{B}^{\mathrm{op}}) \to \mathscr{V}$ is the left Kan extension of its restriction to $\mathscr{D}$, hence small, and $\mathscr{B}$ is small, so by Corollary 3.6 we conclude that $\{\varphi, S-\}$ is small. □

We are now ready to prove the main result of this section:

Theorem 3.8 *$\mathscr{P}\mathscr{K}$ is complete if and only if it has limits of representables.*

PROOF: The "only if" part is trivial, so suppose that $\mathscr{P}\mathscr{K}$ has limits of representables. Let $\mathscr{L} = \mathscr{P}(\mathscr{K}^{\mathrm{op}})^{\mathrm{op}}$, and let $Z : \mathscr{K} \to \mathscr{L}$ be the Yoneda embedding. By Proposition 3.3 the fully faithful $\mathscr{P}Z : \mathscr{P}\mathscr{K} \to \mathscr{P}\mathscr{L}$ has a right adjoint if $\mathscr{L}(Z, L)$ is small for each L. But $\mathscr{L}(Z, L) = \mathscr{P}(\mathscr{K}^{\mathrm{op}})(L, Y)$, where $L : \mathscr{K} \to \mathscr{V}$ is a small functor. Then L is the left Kan extension of its restriction to some small full subcategory $J : \mathscr{B} \to \mathscr{K}$, and now $\mathscr{P}(\mathscr{K}^{\mathrm{op}})(\mathrm{Lan}_J(LJ), Y) = \mathscr{P}(\mathscr{B}^{\mathrm{op}})(LJ, YJ)$ which is the LJ-weighted limit of a diagram of representables, thus small by assumption. This proves that $\mathscr{P}\mathscr{K}$ is a full coreflective subcategory of $\mathscr{P}\mathscr{L}$; since $\mathscr{P}\mathscr{L}$ is complete by Proposition 3.7, it follows that $\mathscr{P}\mathscr{K}$ is so. □

Corollary 3.9 *$\mathscr{P}\mathscr{K}$ is complete if $\mathscr{K}$ is so.*

4 Particular types of limit

This section gives a more refined result, dealing with particular classes of limits. It also provides an alternative proof for the main results of the previous section. It is based on the ideas of [1].

Let Φ be a class of weights. For a $\mathscr{V}$-category $\mathscr{C}$, we write $\Phi\mathscr{C}$ for the closure of the representables in $\mathscr{P}\mathscr{C}$ under Φ-weighted colimits. We suppose that the class Φ satisfies the following conditions:

(a) (smallness) If $\mathscr{C}$ is small then so is $\Phi\mathscr{C}$;

(b) (soundness) If $\mathscr{D}$ is small and Φ-complete, and $\psi : \mathscr{D} \to \mathscr{V}$ is Φ-continuous, then $\psi * - : [\mathscr{D}^{\mathrm{op}}, \mathscr{V}] \to \mathscr{V}$ is Φ-continuous.

Example 4.1

1. If $\mathscr{V}$ is **Set**, then any sound doctrine in the sense of [1] provides an example. Thus one could take Φ to be the (class of weights corresponding to the) finite limits, or the α-small limits for some regular cardinal α, or the finite products, or the finite connected limits.

2. For any locally α-presentable $\mathscr{V}$, by the results of [10, (6.11),(7.4)] one can take Φ to be the class $\mathscr{P}_\alpha$ of α-small limits.

3. If $\mathscr{V}$ is cartesian closed, then by the results of [4] (see also [12]) one can take Φ to be the class of finite products. In fact by the results of [4] this is still the case if $\mathscr{V}$ is the algebras of any commutative finitary theory over a cartesian closed category.

Lemma 4.2 *If $\mathscr{K}$ is Φ-cocomplete and $J : \mathscr{C} \to \mathscr{K}$ is a small full subcategory, then the closure $\bar{\mathscr{C}}$ of $\mathscr{C}$ in $\mathscr{K}$ under Φ-colimits is small.*

Proof: By the smallness assumption on Φ, the free Φ-cocompletion $\Phi\mathscr{C}$ of $\mathscr{C}$ is small. Then $\bar{\mathscr{C}}$ is given, up to equivalence, by the full image of the Φ-cocontinuous extension $\bar{J} : \Phi\mathscr{C} \to \mathscr{K}$ of J; thus $\bar{\mathscr{C}}$ is small since $\Phi\mathscr{C}$ is so. □

Proposition 4.3 *If $\mathscr{K}$ is Φ-complete then so is $\mathscr{P}\mathscr{K}$.*

Proof: Let $\varphi : \mathscr{C} \to \mathscr{V}$ be in Φ, with $\mathscr{C}$ small, and let $S : \mathscr{K}^{\mathrm{op}} \to [\mathscr{C}, \mathscr{V}]$ be small. Then S is the left Kan extension of its restriction to some small full subcategory $J^{\mathrm{op}} : \mathscr{B}^{\mathrm{op}} \to \mathscr{K}^{\mathrm{op}}$. By the lemma, we may suppose $\mathscr{B}$ to be closed in $\mathscr{K}$ under Φ-limits. Thus $S = \mathrm{Lan}_{J^{\mathrm{op}}} R$, where $\mathscr{B}$ is small and Φ-complete, $J : \mathscr{B} \to \mathscr{K}$ is Φ-continuous, and $R : \mathscr{B}^{\mathrm{op}} \to [\mathscr{C}, \mathscr{V}]$. Now $\mathscr{K}(K, J-) : \mathscr{B} \to \mathscr{V}$ is Φ-continuous for all $K \in \mathscr{K}$, so $\mathscr{K}(K, J) * - : [\mathscr{B}^{\mathrm{op}}, \mathscr{V}] \to \mathscr{V}$ is Φ-continuous, so $\{\varphi, -\} : [\mathscr{C}, \mathscr{V}] \to \mathscr{V}$ preserves the left Kan extension $S = \mathrm{Lan}_{J^{\mathrm{op}}} R$. In other words

$$\{\varphi, S\} = \{\varphi, \mathrm{Lan}_{J^{\mathrm{op}}} R\} = \mathrm{Lan}_{J^{\mathrm{op}}} \{\varphi, R\}$$

and so $\{\varphi, S\}$ is small. □

Proposition 4.4 *$\mathscr{P}\mathscr{K}$ has all Φ-limits if and only if it has Φ^*-limits of representables.*

Proof: Recall from Section 2 that Φ^* is the "saturation" of Φ, so that a $\mathscr{V}$-category has Φ-limits if and only if it has Φ^*-limits, and in particular if $\mathscr{P}\mathscr{K}$ has Φ-limits then it certainly has Φ^*-limits of representables.

Let $Z : \mathscr{K} \to \mathscr{L}$ be the free Φ-completion of $\mathscr{K}$ under Φ-limits; explicitly, $\mathscr{L} = \Phi^*(\mathscr{K}^{\mathrm{op}})^{\mathrm{op}}$, and Z is the restricted Yoneda embedding. Then $\mathscr{P}\mathscr{L}$ is Φ-complete by the previous proposition. Since $\mathscr{P}Z : \mathscr{P}\mathscr{K} \to \mathscr{P}\mathscr{L}$ is fully faithful, $\mathscr{P}\mathscr{K}$ will be Φ-complete provided that $\mathscr{P}Z$ has a right adjoint. But this will happen if and only if $\mathscr{L}(Z-, F) : \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ is small for all $F \in \mathscr{L}$. Now

$$\mathscr{L}(Z-, F) = \mathscr{P}(\mathscr{K}^{\mathrm{op}})(F, Y-)$$

and the latter is an F-weighted limit of representables, with $F \in \Phi^*$. □

This provides an alternative proof of:

Corollary 4.5 *$\mathscr{P}\mathscr{K}$ is complete if $\mathscr{K}$ is.*

PROOF: If $\mathscr{K}$ is complete, then it is $\mathscr{P}_\alpha$-complete for any regular cardinal α. Thus $\mathscr{P}\mathscr{K}$ is $\mathscr{P}_\alpha$-complete for any regular cardinal α, and so is complete. □

5 The case where $\mathscr{V}_0$ is a presheaf category

For the first part of this section we suppose that $\mathscr{V} = \mathbf{Set}$, leading to Theorem 5.1. The latter should be attributed to Freyd, although it may not have been written down by him in exactly this form; it is a special case of [9, Theorem 4.8]. We include it as a warm-up for the more general case where the underlying category $\mathscr{V}_0$ of $\mathscr{V}$ is a presheaf category. This includes the case of the cartesian closed categories of directed graphs, or of simplicial sets, as well as such non-cartesian cases as the category of G-graded sets, for a group G, or the category of M-sets, for a commutative monoid M.

Suppose then that $\mathscr{V} = \mathbf{Set}$. First observe that the statement *$\mathscr{P}\mathscr{K}$ has limits if and only if it has limits of representables* remains true if by limit we mean conical limit. To say that $\mathscr{P}\mathscr{K}$ has conical limits of representables is to say that for any $S : \mathscr{C} \to \mathscr{K}$ with $\mathscr{C}$ small, the limit of YS is small. But the limit of YS is the functor $\mathrm{cone}(S) : \mathscr{K}^{\mathrm{op}} \to \mathbf{Set}$ sending an object A to the set of all cones under S with vertex A. To say that this functor is small, is to say that there is a small full subcategory $\mathscr{B}$ of $\mathscr{K}$ for which (i) any cone $\alpha : \Delta A \to S$ factorizes through a cone $\beta : \Delta B \to S$ with $B \in \mathscr{B}$, and (ii) given cones $\beta : \Delta B \to S$ and $\beta' : \Delta B' \to S$ with $B, B' \in \mathscr{B}$, and arrows $f : A \to B$ and $f' : A \to B'$ with $\beta.\Delta f = \beta'.\Delta f'$, there is a "zigzag" of cones from β to β' with vertices in $\mathscr{B}$.

The existence of a small full subcategory $\mathscr{B}$ satisfying (i) is clearly equivalent to the existence of a small set of cones through which every cone factorizes: this is the *solution set condition.* In fact, however, if this solution set condition holds for any $S : \mathscr{D} \to \mathscr{K}$ with $\mathscr{D}$ small then $\mathscr{P}\mathscr{K}$ is complete, for we shall show below that if $\mathscr{B}$ satisfies (i), then we may enlarge $\mathscr{B}$ to a new small full subcategory $\overline{\mathscr{B}}$ which satisfies (i) and (ii). This is done as follows. Let $\mathscr{B}_0 = \mathscr{B}$. We construct inductively small full subcategories $\mathscr{B}_n$ for each natural number n, and then define $\overline{\mathscr{B}}$ to be the union of the $\mathscr{B}_n$.

Let $\mathscr{D}'$ be the category obtained from $\mathscr{D}$ by freely adjoining two cones, with vertices 0 and 1, say. Let $\mathscr{B}_n$ be a small full subcategory, and consider all functors $S' : \mathscr{D}' \to \mathscr{K}$ extending $\mathscr{D}$, and sending the vertices 0 and 1 to objects of $\mathscr{B}_n$. For each such S', we may by hypothesis choose a small full subcategory $\mathscr{B}_{S'}$ of $\mathscr{K}$ which is a "solution set" for S'. Take $\mathscr{B}_{n+1}$ to be the union of $\mathscr{B}_n$ and all the $\mathscr{B}_{S'}$. This is a small union of small full subcategories, so is itself a small full subcategory. Once again, $\overline{\mathscr{B}}$ is a small union of the small full subcategories $\mathscr{B}_n$, and so is small. Clearly it satisfies (i); we check that it satisfies (ii) as well. Suppose then that $\beta : \Delta B \to S$ and $\beta' : \Delta B' \to S$ are cones over S with $B, B' \in \overline{\mathscr{B}}$, and that $f : A \to B$ and $f' : A \to B'$ are arrows with $\beta.\Delta f = \beta'.\Delta f'$. Then B, B', β, and β' together define a functor $S' : \mathscr{D}' \to \mathscr{K}$ extending S; while to give A, f, and f' is precisely to give a cone under S'. Since $B, B' \in \overline{\mathscr{B}}$, there is some $n \in \mathbb{N}$ for which $B, B' \in \mathscr{B}_n$, so there is a cone S' with vertex C in $\mathscr{B}_{S'}$ through which the cone (A, f, f') factorizes. But this C is in $\mathscr{B}_{n+1}$, and so in $\overline{\mathscr{B}}$. This proves:

Theorem 5.1 *$\mathscr{P}\mathscr{K}$ is complete if and only if for every diagram $S : \mathscr{D} \to \mathscr{K}$ with $\mathscr{D}$ small, there is a small set of cones under S through which every cone factorizes.*

We now extend this argument to the case where $\mathscr{V}_0$ is a presheaf category $[\mathscr{G}^{\mathrm{op}}, \mathbf{Set}]$. To extend the argument, we need to assume that the $\mathscr{V}$-category $\mathscr{K}$ admits tensors and cotensors by the representables; this means that for all $A, B \in \mathscr{K}$ and $G \in \mathscr{G}$, there are natural isomorphisms

$$\mathscr{K}(A, G \pitchfork B) \cong [\mathscr{G}^{\mathrm{op}}, \mathbf{Set}](\mathscr{G}(-,G), \mathscr{K}(A,B)) \cong \mathscr{K}(G \cdot A, B)$$

for objects $G \cdot A$ and $G \pitchfork B$ of $\mathscr{K}$; the first operation is called a *tensor* by G and the second a *cotensor* by G. When these exist, we say that $\mathscr{K}$ is $\mathscr{G}$-tensored and $\mathscr{G}$-cotensored. (Of course in the case $\mathscr{V} = \mathbf{Set}$ we have $\mathscr{G} = \{I\}$ and so this is automatic.)

Proposition 5.2 *If $\mathscr{K}$ is $\mathscr{G}$-cotensored, then $\mathscr{P}\mathscr{K}$ is complete if and only if its underlying ordinary category $(\mathscr{P}\mathscr{K})_0$ has conical limits of representables.*

Proof: Recall [11, 3.10] that every weighted limit has a canonical expression as a conical limit of cotensors. On the other hand, since every object of $\mathscr{V}_0 = [\mathscr{G}^{\mathrm{op}}, \mathbf{Set}]$ is (canonically) a conical colimit of representables, and we have $(\mathrm{colim}_i G_i) \pitchfork A \cong \lim_i (G_i \pitchfork A)$, it follows that every cotensor is canonically a conical limit of $\mathscr{G}$-cotensors, and so finally that every weighted limit is canonically a conical limit of $\mathscr{G}$-cotensors. Suppose now that we have a diagram of representables $YS : \mathscr{D} \to \mathscr{P}\mathscr{K}$ and a weight φ. We can therefore express this as a conical limit of $\mathscr{G}$-cotensors of representables. But $\mathscr{K}$ was assumed to be $\mathscr{G}$-cotensored, so a $\mathscr{G}$-cotensor of representables in $\mathscr{P}\mathscr{K}$ exists, and is representable. Thus $\mathscr{P}\mathscr{K}$ will have weighted limits of representables provided that it has conical limits of representables. Finally, $\mathscr{P}\mathscr{K}$ is cocomplete, so is certainly tensored, thus conical limits in $\mathscr{P}\mathscr{K}$ exist provided that they exist in the underlying ordinary category $(\mathscr{P}\mathscr{K})_0$ of $\mathscr{P}\mathscr{K}$, consisting of small $\mathscr{V}$-functors $\mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ and $\mathscr{V}$-natural transformations between them. □

We now adapt the argument from the $\mathscr{V} = \mathbf{Set}$ case to the $V = [\mathscr{G}^{\mathrm{op}}, \mathbf{Set}]$ case to prove:

Proposition 5.3 *If $\mathscr{K}$ is $\mathscr{G}$-tensored, then $(\mathscr{P}\mathscr{K})_0$ has conical limits of representables if and only if, for every diagram $S : \mathscr{C} \to \mathscr{K}_0$, there is a small set of cones through which every cone factorizes.*

Proof: Suppose that $(\mathscr{P}\mathscr{K})_0$ has conical limits of representables, and let $S : \mathscr{C} \to \mathscr{K}_0$ be given. Then $YS : \mathscr{C} \to (\mathscr{P}\mathscr{K})_0$ has a limit L with cone $\eta_C : L \to \mathscr{K}(-, SC)$. Also L is a small colimit of representables $\mathrm{colim}_i \mathscr{K}(-, B_i)$. For each i, there is an induced cone $\mathscr{K}(-, B_i) \to \mathscr{K}(-, SC)$, or equivalently $\beta_i C : B_i \to SC$ under S. We claim that any cone $\alpha C : A \to SC$ factorizes through one of these. Now $\mathscr{K}(-, \alpha C) : \mathscr{K}(-, A) \to \mathscr{K}(-, SC)$ must factorize through L, but $\mathscr{K}(-, A)$ is representable, so homming out of it preserves colimits, and so we get a factorization $\mathscr{K}(-, A) \to \mathscr{K}(-, B_i)$ for some i, and so the desired $A \to B_i$.

For the harder part, suppose that for each $S : \mathscr{C} \to \mathscr{K}_0$, there is a small set of cones $\beta_i C : B_i \to SC$ through which each cone factorizes. Then there is a small full subcategory $\mathscr{B}_S$ of $\mathscr{K}$ such that each cone under S factorizes through one whose vertex is in $\mathscr{B}_S$. Exactly as before, we construct the (possibly larger but still) small full subcategory $J : \overline{\mathscr{B}} \to \mathscr{K}$ with the property that if two cones with vertices in $\overline{\mathscr{B}}$ are connected, then they are connected using cones with vertices in $\overline{\mathscr{B}}$. This implies that for all $A \in \mathscr{K}$, we have $\lim_C \mathscr{K}_0(A, SC) \cong \mathrm{colim}_{B \in \overline{\mathscr{B}}} \mathscr{K}_0(A, B)$. For any $G \in \mathscr{G}$, we have

$$\begin{aligned}
\mathscr{V}_0(G, \lim_C \mathscr{K}(A, SC)) &\cong \lim_C \mathscr{V}_0(G, \mathscr{K}(A, SC)) \\
&\cong \lim_C \mathscr{K}_0(G \cdot A, SC) \\
&\cong \mathrm{colim}_{B \in \overline{\mathscr{B}}} \mathscr{K}_0(G \cdot A, B) \\
&\cong \mathrm{colim}_{B \in \overline{\mathscr{B}}} \mathscr{V}_0(G, \mathscr{K}(A, B)) \\
&\cong \mathscr{V}_0(G, \mathrm{colim}_{B \in \overline{\mathscr{B}}} \mathscr{K}(A, B))
\end{aligned}$$

where the last step uses the fact that G is representable, so $\mathscr{V}_0(G, -)$ preserves colimits. Now $\mathscr{G}$ is dense in $\mathscr{V}_0$, so we have

$$\begin{aligned}
\lim_C \mathscr{K}(A, SC) &\cong \mathrm{colim}_{B \in \overline{\mathscr{B}}} \mathscr{K}(A, B) \\
\lim_C \mathscr{K}(-, SC) &\cong \mathrm{colim}_{B \in \overline{\mathscr{B}}} \mathscr{K}(-, B)
\end{aligned}$$

but the left hand side is the presheaf $\mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ which is the pointwise limit of YS, and which we are to prove small, while the right hand side is a small colimit of representables, since $\overline{\mathscr{B}}$ is small. □

Combining the last two results, we have:

Theorem 5.4 *Suppose the underlying category $\mathscr{V}_0$ of $\mathscr{V}$ is a presheaf category $[\mathscr{G}^{\mathrm{op}}, \mathbf{Set}]$ and that $\mathscr{K}$ is a $\mathscr{G}$-tensored and $\mathscr{G}$-cotensored $\mathscr{V}$-category. Then $\mathscr{P}\mathscr{K}$ is complete if and only if the following condition is satisfied. For every small ordinary category $\mathscr{C}$ and every functor $S : \mathscr{C} \to \mathscr{K}$, there is a small set of cones $\lambda C : B \to SC$ through which every such cone factorizes.*

6 Preservation of limits

Having studied the categories $\mathscr{K}$ for which $\mathscr{P}\mathscr{K}$ is complete, we now turn to the functors $F : \mathscr{K} \to \mathscr{L}$ for which $\mathscr{P}F$ is continuous.

We saw $\mathscr{P}\mathscr{K}$ is *always* complete if $\mathscr{K}$ is small; the situation for functors is totally different:

Example 6.1 Let $\mathscr{V}$ =**Set**, let $\mathscr{K}$ be the terminal category 1, let $\mathscr{L}$ be the discrete category 2, and let $F : \mathscr{K} \to \mathscr{L}$ be the first injection. Then $\mathscr{P}\mathscr{K} = \mathbf{Set}$ and $\mathscr{P}\mathscr{L} = \mathbf{Set}^2$, which are of course complete; but $\mathscr{P}F : \mathbf{Set} \to \mathbf{Set}^2$ is the functor sending a set X to $(X, 0)$, which clearly fails to preserve the terminal object.

Consider a functor $F : \mathscr{K} \to \mathscr{L}$, where $\mathscr{P}\mathscr{K}$ and $\mathscr{P}\mathscr{L}$ are complete, a weight $\varphi : \mathscr{C} \to \mathscr{V}$ and a diagram $S : \mathscr{C} \to \mathscr{P}\mathscr{K}$. Let $R : \mathscr{K}^{\mathrm{op}} \to [\mathscr{C}, \mathscr{V}]$ be the corresponding pointwise small functor. To say that $\mathscr{P}F$ preserves the limit $\{\varphi, S\}$ is to say that $\{\varphi, -\} : [\mathscr{C}, \mathscr{V}] \to \mathscr{V}$ preserves the left Kan extension $\mathrm{Lan}_F R$; that is, the colimit $\mathscr{L}(L, F-) * R$ for each object L of $\mathscr{L}$.

Proposition 6.2 *If $F : \mathscr{K} \to \mathscr{L}$ is a right adjoint then $\mathscr{P}F$ is continuous.*

PROOF: If F has a left adjoint G, then

$$\{\varphi, \mathscr{L}(L, F-) * R\} \cong \{\varphi, \mathscr{K}(GL, -) * R\} \cong \{\varphi, RGL\}$$

while

$$\mathscr{L}(L,F-)*\{\varphi,R\}\cong\mathscr{L}(GL,-)*\{\varphi,R\}\cong\{\varphi,RGL\}.$$

□

Along the same lines, observe that $\mathscr{P}F.Y\cong YF$, so that if $\mathscr{P}F$ is continuous then F must preserve any limits which exist.

Suppose that $F:\mathscr{K}\to\mathscr{L}$ is given, with $\mathscr{P}\mathscr{K}$ and $\mathscr{P}\mathscr{L}$ complete. Then $\mathscr{P}F$ is continuous if and only if each $\mathrm{ev}_L.\mathscr{P}F$ is so; but $\mathrm{ev}_L.\mathscr{P}F$ is just $\mathscr{L}(L,F)*-$. If $\mathscr{K}$ is small, then $\mathscr{L}(L,F)*-$ is continuous if and only if it is α-continuous for every regular cardinal α; in other words, if $\mathscr{L}(L,F)$ is α-flat for every α.

More generally, if $\mathscr{P}\mathscr{K}$ is α-complete, we say that a functor $G:\mathscr{K}\to\mathscr{V}$ is α-flat if $G*-:\mathscr{P}\mathscr{K}\to\mathscr{V}$ is α-continuous, and ∞-flat if $G*-$ is continuous; that is, if G is α-flat for every α. Thus $\mathscr{P}F$ will be continuous if and only if each $\mathscr{L}(L,F)$ is ∞-flat. Similarly, if Φ is a class of weights satisfying the conditions in Section 4 and $\mathscr{P}\mathscr{K}$ is Φ-complete, we say that $G:\mathscr{K}\to\mathscr{V}$ is Φ-flat if $G*-$ is Φ-continuous.

Lemma 6.3 *If $\mathscr{K}$ is complete and $G:\mathscr{K}\to\mathscr{V}$ continuous then $\mathrm{Lan}_Y G:\mathscr{P}\mathscr{K}\to\mathscr{V}$ is continuous.*

Proof: First observe that if $\mathscr{K}$ is complete then $\mathscr{P}\mathscr{K}$ is so. Let $\varphi:\mathscr{D}\to\mathscr{V}$ be a weight, and let $S:\mathscr{D}\to\mathscr{P}\mathscr{K}$ correspond to the pointwise small functor $R:\mathscr{K}^{\mathrm{op}}\to[\mathscr{D},\mathscr{V}]$. For $X\in\mathscr{P}\mathscr{K}$, we have the formula $(\mathrm{Lan}_Y G)X=X*G$, thus to say that $\mathrm{Lan}_Y G$ preserves the limit $\{\varphi,S\}$ is to say that $\{\varphi,-\}$ preserves the colimit $G*R$. Since R is pointwise small, it is the left Kan extension of its restriction to some full subcategory $\mathscr{B}^{\mathrm{op}}$ of $\mathscr{K}^{\mathrm{op}}$. Let α be a regular cardinal for which φ is α-small. We may choose $\mathscr{B}$ to be closed in $\mathscr{K}$ under α-limits, then the inclusion $J:\mathscr{B}\to\mathscr{K}$ preserves α-limits. Then $G*R\cong G*(\mathrm{Lan}_J(RJ))\cong GJ*RJ$ by [11, 4.1], and $GJ:\mathscr{B}\to\mathscr{V}$ preserves α-limits, hence so does $GJ*-$ by [10, (6.11),(7.4)], and now

$$\{\varphi,G*R\}\cong\{\varphi,GJ*RJ\}\cong GJ*\{\varphi,RJ\}\cong G*\mathrm{Lan}_J\{\varphi,RJ\}.$$

On the other hand Lan_J preserves α-limits since J does so, thus

$$G*\mathrm{Lan}_J\{\varphi,RJ\}\cong G*\{\varphi,\mathrm{Lan}_J(RJ)\}\cong G*\{\varphi,R\}.$$

This proves that $G*-$ preserves the limit $\{\varphi,R\}$, and so that $\mathrm{Lan}_Y G$ preserves $\{\varphi,S\}$. □

Theorem 6.4 *Let $\mathscr{K}$ and $\mathscr{L}$ be complete. Then $F:\mathscr{K}\to\mathscr{L}$ is continuous if and only if $\mathscr{P}F:\mathscr{P}\mathscr{K}\to\mathscr{P}\mathscr{L}$ is so.*

Proof: The "if part" was observed above. Suppose then that F is continuous. Then each $\mathscr{L}(L,F)$ is continuous, so $\mathrm{Lan}_Y\mathscr{L}(L,F)$ is continuous, but $\mathrm{Lan}_Y\mathscr{L}(L,F)\cong\mathrm{ev}_L.\mathscr{P}F$, and so $\mathscr{P}F$ is continuous, since limits in $\mathscr{P}\mathscr{L}$ are constructed pointwise. □

Remark 6.5 The Yoneda embedding $Y:\mathscr{K}\to\mathscr{P}\mathscr{K}$ preserves any existing limits, and is continuous if $\mathscr{K}$ is complete. The pseudomonad $\mathscr{P}$ is of the Kock-Zöberlein type, and so the multiplication $\mathscr{P}\mathscr{P}\mathscr{K}\to\mathscr{P}\mathscr{K}$ has both adjoints so also preserves any existing limits (or colimits). Thus the pseudomonad $\mathscr{P}$ lifts from $\mathscr{V}$**-Cat** to the 2-category of complete $\mathscr{V}$-categories, continuous $\mathscr{V}$-functors, and $\mathscr{V}$-natural transformations.

Remark 6.6 Suppose once again that Φ is a class of weights satisfying the conditions of Section 4. Suppose that $\mathscr{K}$ and $\mathscr{L}$ are Φ-complete and $F : \mathscr{K} \to \mathscr{L}$ is Φ-continuous. Then each $\mathscr{L}(L, F)$ is Φ-continuous, so each $\mathscr{L}(L, F) * -$ is Φ-continuous; that is, each $\mathrm{ev}_L \mathscr{P}(F)$ is Φ-continuous, and so finally $\mathscr{P}F : \mathscr{P}K \to \mathscr{P}L$ is Φ-continuous. Thus the pseudomonad Φ^* lifts from $\mathscr{V}$**-Cat** to the 2-category of Φ-complete $\mathscr{V}$-categories, Φ-continuous $\mathscr{V}$-functors, and $\mathscr{V}$-natural transformations.

7 Monoidal structure on $\mathscr{P}\mathscr{K}$

In this section we suppose that $\mathscr{K}$ is a $\mathscr{V}$-category for which $\mathscr{P}\mathscr{K}$ is complete. If $\mathscr{K}$ is small, so that $\mathscr{P}\mathscr{K}$ is $[\mathscr{K}^{\mathrm{op}}, \mathscr{V}]$, monoidal closed structures on $\mathscr{P}\mathscr{K}$ correspond to promonoidal structures on $\mathscr{K}^{\mathrm{op}}$ [6]. These consist of $\mathscr{V}$-functors $P : \mathscr{K}^{\mathrm{op}} \otimes \mathscr{K} \otimes \mathscr{K} \to \mathscr{V}$ and $J : \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ equipped with coherent associativity and unit isomorphisms.

If $\mathscr{K}$ is large, we shall insist that $P(-; A, B) : \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ and $J : \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ be small, and we write $P : \mathscr{K} \otimes \mathscr{K} \to \mathscr{P}\mathscr{K} : (A, B) \mapsto P(-; A, B)$ and $J \in \mathscr{P}\mathscr{K}$. If $F, G \in \mathscr{P}\mathscr{K}$ are given, we define $F \otimes G$ using the usual convolution formula:

$$F \otimes G = \int^{A,B} P(-; A, B) \otimes FA \otimes GB.$$

This is small, since each $P(-; A, B)$ is small by assumption, so $\int^A P(-; A, B) \otimes FA$ is a small (F-weighted) colimit of small presheaves for each B, and so $\int^{A,B} P(-; A, B) \otimes FA \otimes GB$ is itself a small colimit of small presheaves, hence small.

In the usual case, where $\mathscr{K}$ is small, this monoidal structure is closed, with (right) internal hom given by

$$\begin{aligned}[G, H] &\cong \int_{B,C} [P(C; -, B) \otimes GB, HC] \\ &\cong \int_{B,C} [GB, [P(C; -, B), HC]]\end{aligned}$$

If $\mathscr{K}$ is large, this need not lie in $\mathscr{P}\mathscr{K}$, but if it does so, then it will still provide the internal hom. Now G is small, and the expression above for $[G, H]$ is precisely the G-weighted limit of the functor sending B to $\int_C [P(C; -, B), HC]$. Since $\mathscr{P}\mathscr{K}$ is complete this limit will exist provided that this functor actually lands in $\mathscr{P}\mathscr{K}$; that is, provided that

$$\int_C [P(C; -, B), HC] : \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$$

is small for all $B \in \mathscr{K}$.

The case of the other internal hom is similar, and we have:

Proposition 7.1 *The convolution monoidal category $\mathscr{P}\mathscr{K}$ is closed if and only if the presheaves $\int_C [P(C; -, B), HC]$ and $\int_C [P(C; B, -), HC]$ are small for all $B \in \mathscr{K}$.*

An important special case is where the promonoidal structure P is a filtered colimit $P = \mathrm{colim}_i P_i$ of promonoidal structures P_i which are in fact monoidal, as in

$$P_i(C; A, B) = \mathscr{K}(C, A \otimes_i B).$$

We call such a promonoidal structure P *approximately monoidal*; of course every monoidal structure is approximately monoidal. (We are using the fact that the colimit is filtered to obtain the associativity and unit isomorphisms; a general colimit of promonoidal structures need not be promonoidal.)

In the approximately monoidal case a simplification is possible, since

$$\begin{aligned}\int_C [P(C;-,B),HC] &\cong \int_C [\mathrm{colim}_i P_i(C;-,B),HC] \\ &\cong \lim_i \int_C [P_i(C;-,B),HC] \\ &\cong \lim_i \int_C [\mathscr{K}(C,-\otimes_i B),HC] \\ &\cong \lim_i H(-\otimes_i B)\end{aligned}$$

which is small provided each $H(-\otimes_i B)$ is so. But H is small, so has the form $\mathrm{Lan}_J(HJ)$ for some $J:\mathscr{D}\to\mathscr{K}^{\mathrm{op}}$ with $\mathscr{D}$ small. Then

$$\begin{aligned}H(-\otimes_i B) &\cong \int^D \mathscr{K}^{\mathrm{op}}(D,-\otimes_i B)\cdot HJD \\ &\cong \int^D \mathscr{K}(-\otimes_i B,D)\cdot HJD\end{aligned}$$

which is a small (HJ-weighted) colimit of presheaves $\mathscr{K}(-\otimes_i B,D)$ with $D\in\mathscr{D}$, so will be small provided that the $\mathscr{K}(-\otimes_i B,D)$ are so. Once again the case of the other internal hom is similar, and we have:

Proposition 7.2 *The convolution monoidal category $\mathscr{PK}$ arising from an approximately monoidal structure on $\mathscr{K}$ is closed if and only if the presheaves $\mathscr{K}(-\otimes_i B,D)$ and $\mathscr{K}(B\otimes_i -,D)$ are small for all B and D in $\mathscr{K}$, and for each monoidal structure $\otimes_i$.*

In particular we have:

Proposition 7.3 *The convolution monoidal category $\mathscr{PK}$ arising from a monoidal structure on $\mathscr{K}$ is closed if and only if the presheaves $\mathscr{K}(-\otimes B,D)$ and $\mathscr{K}(B\otimes -,D)$ are small for all B and D in $\mathscr{K}$.*

Example 7.4

1. The special case where $\mathscr{V}=\mathbf{Set}$ and the monoidal structure is cartesian was proved in [15].
2. If $\mathscr{K}$ is not just monoidal but closed then the $\mathscr{K}(-\otimes B,D)$ and $\mathscr{K}(B\otimes -,D)$ are not just small but representable, and so $\mathscr{PK}$ is monoidal closed.
3. If $\mathscr{V}$ is cartesian monoidal (so that $\otimes=\times$), and $\mathscr{K}=\mathscr{E}^{\mathrm{op}}$ where $\mathscr{E}$ is also cartesian monoidal, then $\mathscr{K}(-\otimes B,D)=\mathscr{E}(D,-\times B)=\mathscr{E}(D,B)\times\mathscr{E}(D,-)$ which is given by tensoring the representable $\mathscr{E}(D,-)$ by the $\mathscr{V}$-object $\mathscr{E}(D,B)$, and so is small. Thus once again $\mathscr{PK}$ is monoidal closed.

8 Functors with codomain other than $\mathscr{V}$

In this section we consider small functors $\mathscr{K}^{\mathrm{op}} \to \mathscr{M}$ where $\mathscr{M}$ is cocomplete, building on our earlier work on the case $\mathscr{M} = \mathscr{V}$ and $\mathscr{M} = [\mathscr{C}, \mathscr{V}]$.

In that earlier work, we considered when, for a small functor $S : \mathscr{K}^{\mathrm{op}} \to [\mathscr{C}, \mathscr{V}]$, each $\{\varphi, S\}$ was small. But $\{\varphi, -\}$ is just the representable functor $[\mathscr{C}, \mathscr{V}](\varphi, -)$, which motivates the following definition: a functor $S : \mathscr{K} \to \mathscr{M}$ is *representably small* if each $\mathscr{M}(M, S) : \mathscr{K} \to \mathscr{V}$ is small. Thus Corollary 3.9 asserts that if $\mathscr{K}$ is complete then every small functor $\mathscr{K}^{\mathrm{op}} \to [\mathscr{C}, \mathscr{V}]$ is representably small.

In this section we investigate the relationship between smallness and representable smallness for more general $\mathscr{M}$. We have already seen that smallness does not in general imply representable smallness. For an explicit counterexample in the case $\mathscr{M} = \mathscr{V}$ we have:

Example 8.1 As in Example 3.1 let $\mathscr{V}$ be **Set**, and let $\mathscr{K}$ be any large set X, seen as a discrete category. Then a presheaf on $\mathscr{K}$ is an X-indexed set $A \to X$, and it is small if and only if A is so. Certainly $x : 1 \to X$ is small, for any $x \in X$; this corresponds to the representable presheaf $X(-, x) : X \to \mathbf{Set}$ sending x to 1, and all other elements to 0. Now $\mathbf{Set}(0, X(-, x))$ is the terminal presheaf, which as we have seen is not small. Thus $X(-, x)$ is small but not representably small.

To see that a representably small functor need not be small, we have:

Example 8.2 If $\mathscr{K}$ is a large $\mathscr{V}$-category for which $\mathscr{P}\mathscr{K}$ is complete (for example if $\mathscr{K}$ is complete), then the Yoneda embedding $Y_{\mathscr{K}^{\mathrm{op}}} : \mathscr{K}^{\mathrm{op}} \to \mathscr{P}(\mathscr{K}^{\mathrm{op}})$ is representably small. For if $F \in \mathscr{P}(\mathscr{K}^{\mathrm{op}})$, the composite $\mathscr{P}(\mathscr{K}^{\mathrm{op}})(F, Y) : \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ is the F-weighted limit of $Y : \mathscr{K} \to \mathscr{P}\mathscr{K}$, so is small since F is small and $\mathscr{P}\mathscr{K}$ is complete. But $Y_{\mathscr{K}^{\mathrm{op}}} : \mathscr{K}^{\mathrm{op}} \to \mathscr{P}(\mathscr{K}^{\mathrm{op}})$ is not small unless $\mathscr{K}$ is so. For if $Y_{\mathscr{K}^{\mathrm{op}}}$ were small, $\mathscr{K}^{\mathrm{op}}$ would have a small full subcategory $J : \mathscr{C}^{\mathrm{op}} \to \mathscr{K}^{\mathrm{op}}$ for which $Y = \mathrm{Lan}_J(YJ)$, so

$$\mathscr{K}(-, A) = \int^{C \in \mathscr{C}} \mathscr{K}(JC, A) \cdot \mathscr{K}(-, JC)$$

for all A, and in particular

$$\mathscr{K}(A, A) = \int^{C} \mathscr{K}(JC, A) \cdot \mathscr{K}(A, JC)$$

and so the identity $1 : A \to A$ must factorize through some JC; in other words, each $A \in \mathscr{K}$ is a retract of some object in $\mathscr{C}$. But this clearly implies that $\mathscr{K}$ is small.

As a first positive result we have:

Proposition 8.3 *If $\mathscr{K}$ is a $\mathscr{V}$-category for which $\mathscr{P}\mathscr{K}$ admits cotensors, a presheaf $F : \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ is small if and only if it is representably small. In particular this will be the case if $\mathscr{P}\mathscr{K}$ is complete.*

Proof: Representably small presheaves are always small, since $\mathscr{V}(I, F)$ is just F, for any presheaf F. It remains to show that any small presheaf $F : \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ is representably small. Suppose then that $X \in \mathscr{V}$. Then $\mathscr{V}(X, F)$ is the cotensor $X \pitchfork F$ of F by X, which is small by assumption. $\square$

For the remainder of the section we suppose that $\mathscr{K}$ is a $\mathscr{V}$-category for which $\mathscr{P}\mathscr{K}$ is complete, and that $\mathscr{M}$ is a locally presentable $\mathscr{V}$-category. If β is a regular cardinal for which $\mathscr{M}$ is locally β-presentable, write $\mathscr{M}_\beta$ for the full subcategory of $\mathscr{M}$ consisting of the β-presentable objects, and $W : \mathscr{M} \to [\mathscr{M}_\beta^{\mathrm{op}}, \mathscr{V}]$ for the canonical (fully faithful) inclusion.

Lemma 8.4 *For a $\mathscr{V}$-functor $S : \mathscr{K}^{\mathrm{op}} \to \mathscr{M}$, the following are equivalent:*

(a) S is representably small;

(b) WS is small;

(c) S is small.

Proof: $(a) \Rightarrow (b)$. To say that S is representably small is to say that $\mathscr{M}(M, S)$ is small for all $M \in \mathscr{M}$; to say that WS is small is to say that this is so for all $M \in \mathscr{M}_\beta$, so this is immediate.

$(b) \Rightarrow (c)$. For each $M \in \mathscr{M}_\beta$ we have $\mathscr{M}(M, S)$ small, so it is the left Kan extension of its restriction to some full subcategory $\mathscr{D}_M$ of $\mathscr{K}^{\mathrm{op}}$. Since $\mathscr{M}_\beta$ is small, the union $\mathscr{D}$ of the $\mathscr{D}_M$ is small, and each $\mathscr{M}(M, S)$ is the left Kan extension of its restriction to $\mathscr{D}$. Thus WS is the left Kan extension of its restriction to $\mathscr{D}$. But W is fully faithful, and so reflects Kan extensions; thus also S is the left Kan extension of its restriction to $\mathscr{D}$.

$(c) \Rightarrow (a)$. This is by far the hardest implication; we prove it in several steps, analogous to the main steps used in preparation for the proof of Theorem 3.8. Suppose then that S is small and $M \in \mathscr{M}$; we must show that $\mathscr{M}(M, S)$ is small.

Case 1: $\mathscr{K}^{\mathrm{op}}$ is locally presentable. Since S is small, it is the left Kan extension $\mathrm{Lan}_{J^{\mathrm{op}}} R$ along $J^{\mathrm{op}} : \mathscr{C}^{\mathrm{op}} \to \mathscr{K}^{\mathrm{op}}$ of some $R : \mathscr{C}^{\mathrm{op}} \to \mathscr{M}$ with $\mathscr{C}$ small. Since $\mathscr{C}$ is small and $\mathscr{K}^{\mathrm{op}}$ and $\mathscr{M}$ are locally presentable, there exists a regular cardinal $\gamma \geq \beta$ for which each JC is γ-presentable in $\mathscr{K}^{\mathrm{op}}$ and M is γ-presentable in $\mathscr{M}$. Now (a) $\mathscr{K}^{\mathrm{op}}$ is the free completion under γ-filtered colimits of the full subcategory $(\mathscr{K}^{\mathrm{op}})_\gamma$ of $\mathscr{K}^{\mathrm{op}}$ consisting of the γ-presentable objects, (b) S preserves γ-filtered colimits, and (c) $\mathscr{M}(M, -)$ preserves γ-filtered colimits. Thus $\mathscr{M}(M, S)$ preserves γ-filtered colimits, so is the left Kan extension of its restriction to $(\mathscr{K}^{\mathrm{op}})_\gamma$. This proves that $\mathscr{M}(M, S)$ is small, and so that S is representably small.

Case 2: $\mathscr{K}^{\mathrm{op}} = \mathscr{P}(\mathscr{L}^{\mathrm{op}})$. Then S is the left Kan extension of its restriction to some small full subcategory $\mathscr{D}$ of $\mathscr{P}(\mathscr{L}^{\mathrm{op}})$. Each $D \in \mathscr{D}$ is a small functor $\mathscr{L} \to \mathscr{V}$, so is the left Kan extension of its restriction to some small $\mathscr{B}_D$. The union $\mathscr{B}$ of the $\mathscr{B}_D$ is small, and now the full inclusion $J : \mathscr{B}^{\mathrm{op}} \to \mathscr{L}^{\mathrm{op}}$ induces a full inclusion $\mathscr{P}J : \mathscr{P}(\mathscr{B}^{\mathrm{op}}) \to \mathscr{P}(\mathscr{L}^{\mathrm{op}})$ whose image contains $\mathscr{D}$.

Now $\mathscr{B}$ is small, so $\mathscr{P}J$ has a right adjoint J^* given by restriction along J, and thus $\mathrm{Lan}_{\mathscr{P}J}$ is itself given by restriction along J^*. Since S is the left Kan extension of its restriction Q along $\mathscr{P}J$, we have

$$\mathscr{M}(M, S) = \mathscr{M}(M, \mathrm{Lan}_{\mathscr{P}J} Q) = \mathscr{M}(M, QJ^*) = \mathscr{M}(M, Q)J^* = \mathrm{Lan}_{\mathscr{P}J}\mathscr{M}(M, Q)$$

and so $\mathscr{M}(M, S)$ will be small if $\mathscr{M}(M, Q)$ is so. Now Q is the left Kan extension of its restriction to $\mathscr{D}$, hence small, so $\mathscr{M}(M, Q)$ is small by Case 1. This proves that $\mathscr{M}(M, S)$ is small, and so that S is representably small.

Case 3: $\mathscr{P}\mathscr{K}$ is complete. The left Kan extension $\mathrm{Lan}_Y(S) : \mathscr{P}(\mathscr{K}^{\mathrm{op}}) \to \mathscr{M}$ of S along the Yoneda embedding is small, so by Case 2 is representably small. Thus each $\mathscr{M}(M, \mathrm{Lan}_Y(S)) : \mathscr{P}(\mathscr{K}^{\mathrm{op}})^{\mathrm{op}} \to \mathscr{V}$ is small; that is, a small colimit of representables. Now restriction along the

Yoneda embedding preserves colimits, so it will send small presheaves to small presheaves provided that it sends representables to small presheaves; but the latter is equivalent to completeness of $\mathscr{P}\mathscr{K}$. Thus each $\mathscr{M}(M,S)$ is small, and S is representably small. $\square$

Write $[\mathscr{K}^{\mathrm{op}},\mathscr{M}]_s$ for the $\mathscr{V}$-category of all small $\mathscr{V}$-functors from $\mathscr{K}^{\mathrm{op}}$ to $\mathscr{M}$.

Theorem 8.5 *Let $\mathscr{M}$ be a locally presentable $\mathscr{V}$-category, and $\mathscr{K}$ a $\mathscr{V}$-category for which $\mathscr{P}\mathscr{K}$ is complete. Then $[\mathscr{K}^{\mathrm{op}},\mathscr{M}]_s$ is complete.*

PROOF: Let $\varphi:\mathscr{D}\to\mathscr{V}$ and $S:\mathscr{D}\to[\mathscr{K}^{\mathrm{op}},\mathscr{M}]_s$ be given, where $\mathscr{D}$ is small. Since $\mathscr{D}$ is small, the functor $\bar{S}:\mathscr{K}^{\mathrm{op}}\to[\mathscr{D},\mathscr{M}]$ corresponding to S is small. The "pointwise limit" is the composite

$$\mathscr{K}^{\mathrm{op}} \xrightarrow{\bar{S}} [\mathscr{D},\mathscr{M}] \xrightarrow{\{\varphi,-\}} \mathscr{M}$$

and provided that this is small, and so lies in $[\mathscr{K}^{\mathrm{op}},\mathscr{M}]_s$, it will be the limit. Since $\mathscr{M}$ is locally presentable, by the lemma it will suffice to show that each composite with $\mathscr{M}(M,-)$ is small. But for any $X:\mathscr{D}\to\mathscr{M}$ we have

$$\begin{aligned}\mathscr{M}(M,\{\varphi,X\}) &\cong \{\varphi,\mathscr{M}(M,X)\}\\ &\cong \int_D [\varphi D,\mathscr{M}(M,XD)]\\ &\cong \int_D \mathscr{M}(\varphi D\cdot M, XD)\\ &\cong [\mathscr{D},\mathscr{M}](\varphi_M, X)\end{aligned}$$

where $\varphi_M:\mathscr{D}\to\mathscr{M}$ is the functor sending D to $\varphi D\cdot M$, so now $\mathscr{M}(M,\{\varphi,-\})$ is representable as $[\mathscr{D},\mathscr{M}](\varphi_M,-):[\mathscr{D},\mathscr{M}]\to\mathscr{V}$.

Now $[\mathscr{D},\mathscr{M}]$ is locally presentable, so by the lemma once again the small $\bar{S}$ is representably small, and so $[\mathscr{D},\mathscr{M}](\varphi_M,\bar{S}):\mathscr{K}^{\mathrm{op}}\to\mathscr{V}$ is small; but we have just seen that this is the composite of $\bar{S}$ with $\mathscr{M}(M,\{\varphi,-\})$. This now proves that $\{\varphi,-\}\circ\bar{S}$ is representably small, and so small, and it therefore provides the desired limit $\{\varphi,S\}$. $\square$

9 Isbell conjugacy

If $\mathscr{C}$ is a small category then as well as the Yoneda embedding $Y:\mathscr{C}\to[\mathscr{C}^{\mathrm{op}},\mathscr{V}]$ there is also the "dual" Yoneda embedding $Z:\mathscr{C}\to[\mathscr{C},\mathscr{V}]^{\mathrm{op}}$, and this induces an adjunction between $[\mathscr{C}^{\mathrm{op}},\mathscr{V}]$ and $[\mathscr{C},\mathscr{V}]^{\mathrm{op}}$ called "Isbell conjugacy". The left adjoint $L:[\mathscr{C}^{\mathrm{op}},\mathscr{V}]\to[\mathscr{C},\mathscr{V}]^{\mathrm{op}}$ is given by $\mathrm{Lan}_Y Z$.

What happens if we replace $\mathscr{C}$ be an arbitrary category $\mathscr{K}$? Then we have $Y:\mathscr{K}\to\mathscr{P}\mathscr{K}$ and $Z:\mathscr{K}\to\mathscr{P}(\mathscr{K}^{\mathrm{op}})^{\mathrm{op}}$, but do we still have the adjunction between them? A sufficient condition for the left adjoint $L:\mathscr{P}\mathscr{K}\to\mathscr{P}(\mathscr{K}^{\mathrm{op}})^{\mathrm{op}}$ to exist is that $\mathscr{P}(\mathscr{K}^{\mathrm{op}})^{\mathrm{op}}$ be cocomplete, or equivalently $\mathscr{P}(\mathscr{K}^{\mathrm{op}})$ complete, but in fact this is also necessary. For if $\mathrm{Lan}_Y Z$ does exist, then for each small $F:\mathscr{C}^{\mathrm{op}}\to\mathscr{V}$ the colimit $F*Z$ in $\mathscr{P}(\mathscr{K}^{\mathrm{op}})^{\mathrm{op}}$ exists. But then for any $\varphi:\mathscr{C}^{\mathrm{op}}\to\mathscr{V}$ and $S:\mathscr{C}\to\mathscr{K}$, we have $\mathrm{Lan}_S\varphi$ small, and $(\mathrm{Lan}_S\varphi)*Z=\varphi ZS$, and so $\mathscr{P}(\mathscr{K}^{\mathrm{op}})^{\mathrm{op}}$ has arbitrary colimits of representables, $\mathscr{P}(\mathscr{K}^{\mathrm{op}})$ has arbitrary limits of representables, and so $\mathscr{P}(\mathscr{K}^{\mathrm{op}})$ is in fact complete.

Thus $\mathrm{Lan}_Y Z : \mathscr{P}\mathscr{K} \to \mathscr{P}(\mathscr{K}^{\mathrm{op}})^{\mathrm{op}}$ exists if and only if $\mathscr{P}(\mathscr{K}^{\mathrm{op}})$ is complete, and dually the putative right adjoint $\mathscr{P}(\mathscr{K}^{\mathrm{op}})^{\mathrm{op}} \to \mathscr{P}\mathscr{K}$ exists if and only if $\mathscr{P}\mathscr{K}$ is complete.

In particular, both will exist if $\mathscr{K}$ is complete and cocomplete.

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