An operadic approach to internal structures

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Abstract

We study internal structures in the category of algebras for an operad, and show that these themselves admit an operadic description. The main case of interest is where the operad is on an abelian category, and the internal structures in question are those of internal category, internal n-category, or internal (cubical) n-tuple category. This allows an operadic treatment of crossed modules and other crossed structures.

1 Introduction

A basic technique in homotopical and categorical algebra is the use of internal structures in a category. A group can be defined as a set G equipped with functions $G^2 = G \times G \to G$, $G^1 = G \to G$, and $G^0 = 1 \to G$ satisfying certain equations in the form of commutativity conditions for maps $G^3 \to G$ and $G \to G$. One can now replace sets and functions by objects and morphisms of any category and this will still make sense provided that the category has finite products, so that $G \times G$ and 1 and $G \times G \times G$ make sense. Similarly, a category can be defined as a parallel pair of functions $d, c : C_1 \to C_0$ equipped with functions $i : C_0 \to C_1$ and

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$m : C_1 \times_{C_0} \times C_1 \to C_1$ satisfying various equations in the form of commutativity conditions. This time the definition makes sense in any category with pullbacks, so that for example $C_1 \times_{C_0} \times C_1$ makes sense. Thus one can speak of internal groups and internal categories in any category with finite limits. One says that groups and categories are *finite limit structures*: this really means that the *notion* of group and the *notion* of category are finite limit structures.

It is part of a general phenomenon that internal categories in the category **Grp** of groups, and internal groups in the category **Cat** of categories, are essentially the same thing: more precisely there is an equivalence of categories between these two categories of internal structures; see [10] for a discussion of this point. In fact both of these structures are further equivalent to crossed modules, and so model homotopy 2-types. Internal categories in a category $\mathscr{C}$ which is "similar to the category of groups" are often called crossed modules of $\mathscr{C}$-objects, and these have been studied in various levels of generality; see for example [10], where "similar to the category of groups" is taken to mean *semiabelian*. A typical example is crossed modules of commutative algebras: these are (equivalent to) internal categories in the category **Comm** of commutative algebras (not necessarily having a unit, defined over a fixed commutative ring R). This last case will be typical of what we study; in fact categories in **Grp** and groups in **Cat**, although simpler in many ways, will not fit into the framework developed below.

The other types of structure we shall use will be defined in terms of operads. These involve a symmetric monoidal closed category $\mathscr{V}$: we are typically interested in abelian cases, such as the category of abelian groups, or R-modules for a commutative ring R, or of differential graded R-modules (chain complexes), although the general theory will not depend on $\mathscr{V}$ being abelian. An operad T in $\mathscr{V}$ is a way of describing certain algebraic structure on an object A of $\mathscr{V}$, involving maps $A^{\otimes n} = A \otimes \ldots \otimes A \to A$, and equations between these. We write $\mathscr{V}^T$ for the category of all such "T-algebras" and their homomorphisms. For example if $\mathscr{V}$ is the category R-**Mod** of R-modules, then there is an operad T for which $\mathscr{V}^T = \mathbf{Comm}$.

The point of the paper is the interplay between finite limit structure and operad structure. A typical case is the finite limit structure of internal category and the operad structure of commutative algebras. We shall see that the symmetric monoidal structure of $\mathscr{V}$ induces a symmetric monoidal structure on the category $\mathbf{Cat}(\mathscr{V})$ of internal categories in $\mathscr{V}$, and that the operad T (on $\mathscr{V}$) for commutative algebras induces an operad T' on $\mathbf{Cat}(\mathscr{V})$ whose algebras are the internal categories in **Comm**; in other words we have the equation

$$\mathbf{Cat}(\mathscr{V}^T) = \mathbf{Cat}(\mathscr{V})^{T'}$$

which is really of course an equivalence of categories. In the case where $\mathscr{V}$ is abelian, categories in $\mathscr{V}$ are equivalent to arrows in $\mathscr{V}$, and so the right hand side becomes particularly simple.

This can be done with any operad T in $\mathscr{V}$. In the case where $\mathscr{V}$ is the category of differential graded R-modules the resulting operad T' was explicitly described in [2], where the authors worked only with crossed modules in $\mathscr{V}^T$ and arrows in $\mathscr{V}$, rather than internal categories, so that the equation above became the definition $\mathrm{CM}(\mathscr{V}^T) = \mathrm{Arr}(\mathscr{V})^{T'}$, where

CM denotes crossed modules, and Arr denotes arrows; this definition was shown to agree with existing definitions of crossed modules in certain cases.

We shall also work with finite limit structures $\mathscr{K}$ other than **Cat**, and obtain under certain conditions the corresponding equivalence

$$\mathscr{K}(\mathscr{V}^T) = \mathscr{K}(\mathscr{V})^{T'}$$

as part of our Theorem 6.2 below. The main requirement will be that the category $\mathscr{K}(\mathbf{Set})$ of all $\mathscr{K}$-structures in **Set** be cartesian closed; the construction of a suitable symmetric monoidal closed structure on $\mathscr{K}(\mathscr{V})$ in this case is due to Bastiani and Ehresmann [1].

As well as $\mathscr{K}$-structures in the category of algebras for a fixed operad T, we also consider $\mathscr{K}$-structures in the category $\mathrm{Op}(\mathscr{V})$ of operads in $\mathscr{V}$, and we show that these are equivalent to operads in the symmetric monoidal category $\mathscr{K}(\mathscr{V})$ of $\mathscr{K}$-structures in $\mathscr{V}$; thus we have an "equation"

$$\mathscr{K}(\mathrm{Op}(\mathscr{V})) = \mathrm{Op}(\mathscr{K}(\mathscr{V}))$$

which once again is really an equivalence of categories, and is part of our Theorem 5.1 below. This reduces to the *secondary operads* of [2] in the case where $\mathscr{K}$ is **Cat** and $\mathscr{V}$ is abelian. It is also possible to describe the algebras for a general operad in $\mathscr{K}(\mathscr{V})$, not just those of the form $\mathscr{K}(T)$.

The outline of the paper is as follows. In Section 2 we recall those aspects of the theory of locally finitely presentable categories that will be needed. In Section 3 we construct the tensor product on $\mathscr{K}(\mathscr{V})$, and in Section 4 we prove several some technical results which are useful for calculations involving the tensor product. On a first reading, some may choose to skip over these initial sections and to start with Section 5, where the notion of operad is recalled, and the description of operads in $\mathscr{K}(\mathscr{V})$ is given. Then in Section 6, the algebras for such operads are analyzed. Section 7 is a brief digression concerning the situation when one works with monads rather than operads. In Section 8 we consider cases where $\mathscr{K}$-structures are "categorical" — such as (internal) categories, 2-categories, double categories, and so on. Finally in Section 9, which can be regarded as the applications, we specialize further to the case where $\mathscr{V}$ is abelian, and so obtain operadic descriptions of crossed modules, crossed complexes, crossed cubes, and so on.

2 Locally finitely presentable categories

In this section we briefly review some basic facts about locally finitely presentable categories, which will be used in the description of finite limit structures. Some readers may choose to skip to Section 5 on a first reading. Gabriel and Ulmer [9] define a category $\mathscr{K}$ to be *locally finitely presentable* if it is cocomplete and has a small dense subcategory consisting of finitely presentable objects, where $G \in \mathscr{K}$ is said to be finitely presentable if the representable functor $\mathscr{K}(G, -) : \mathscr{K} \to \mathbf{Set}$ preserves filtered colimits. They then prove that $\mathscr{K}$ is locally finitely presentable if and only it is equivalent to the category $\mathrm{Lex}(\mathscr{C}, \mathbf{Set})$ of finite-limit-preserving functors from $\mathscr{C}$ to **Set**, for some small category $\mathscr{C}$ with finite limits. In fact there

is a canonical choice for $\mathscr{C}$, namely the opposite of the full subcategory $\mathscr{K}_f$ of $\mathscr{K}$ consisting of the finitely presentable objects, and this sets up a duality between locally finitely presentable categories and categories with finite limits, associating the *category of models* $\mathrm{Lex}(\mathscr{C}, \mathbf{Set})$ to a *theory* $\mathscr{C}$, and the theory $\mathscr{K}_f^{\mathrm{op}}$ to a locally finitely presentable category $\mathscr{K}$. Sometimes rather than giving the theory $\mathscr{C}$ explicitly, one gives a presentation for it in terms of a finite limit (=projective) sketch, in the sense of Ehresmann [6].

For a given theory $\mathscr{C}$, one can consider not just set-based models, but models in any category $\mathscr{A}$ with finite limits; that is, one can consider the category $\mathrm{Lex}(\mathscr{C}, \mathscr{A})$ of finite-limit-preserving functors from $\mathscr{C}$ to $\mathscr{A}$. For a fixed locally finitely presentable category $\mathscr{K}$, we write $\mathscr{K}(\mathscr{A})$ for the category of models in $\mathscr{A}$ of the corresponding theory; in other words $\mathscr{K}(\mathscr{A}) = \mathrm{Lex}(\mathscr{K}_f^{\mathrm{op}}, \mathscr{A})$. Of course $\mathscr{K}(\mathbf{Set})$ is just $\mathscr{K}$, while if $\mathscr{K}$ is $\mathbf{Cat}$, then $\mathscr{K}(\mathscr{A})$ will be the category of internal categories in $\mathscr{A}$, which we have previously called $\mathbf{Cat}(\mathscr{A})$, so this notation is consistent. If $\mathscr{A}$ is complete, then there is an equivalence of categories between $\mathscr{K}(\mathscr{A})$ and the category $\mathrm{Cts}(\mathscr{K}^{\mathrm{op}}, \mathscr{A})$ of continuous functors from $\mathscr{K}^{\mathrm{op}}$ to $\mathscr{A}$; the equivalence is given by right Kan extension and restriction along the inclusion $J^{\mathrm{op}} : \mathscr{K}_f^{\mathrm{op}} \to \mathscr{K}^{\mathrm{op}}$.

We always have the fully faithful inclusion of $\mathscr{K}(\mathscr{A})$ in the functor category $[\mathscr{K}_f^{\mathrm{op}}, \mathscr{A}]$. If $\mathscr{A}$ is $\mathbf{Set}$ then this has a left adjoint [14]; more generally it has a left adjoint if $\mathscr{A}$ is itself locally finitely presentable, and more generally still if $\mathscr{A}$ is locally bounded in the sense of [8]. Another class of examples where the left adjoint exists is where $\mathscr{A}$ is abelian and $\mathscr{K}$ is $\mathbf{Cat}$; this and similar examples will be treated in Section 9. In any case, if the left adjoint does exist, we shall say that $\mathscr{K}$ is *$\mathscr{A}$-reflective* (this name is non-standard).

We shall also sometimes need to consider different locally finitely presentable categories $\mathscr{K}$ and $\mathscr{L}$, with a functor $U : \mathscr{L} \to \mathscr{K}$ with left adjoint $F \dashv U$. The right adjoint U preserves filtered colimits if and only if the left adjoint F sends finitely presentable objects to finitely presentable objects; in other words, if it restricts to a map $F_f : \mathscr{K}_f \to \mathscr{L}_f$. Since the finitely presentable objects are closed under finite colimits, and the left adjoint F preserves all colimits, F_f preserves finite colimits, and so F_f^{op} preserves finite limits. Thus the functor $[F_f^{\mathrm{op}}, 1] : [\mathscr{L}_f^{\mathrm{op}}, \mathscr{A}] \to [\mathscr{K}_f^{\mathrm{op}}, \mathscr{A}]$ given by restriction along F_f^{op} itself restricts to a functor $F^* : \mathscr{L}(\mathscr{A}) \to \mathscr{K}(\mathscr{A})$.

Now the inclusion $W_{\mathscr{K}} : \mathscr{K}(\mathscr{A}) \to [\mathscr{K}_f^{\mathrm{op}}, \mathscr{A}]$ is fully faithful, so F^* will have a left adjoint provided that $W_{\mathscr{K}} F^*$ does so; but $W_{\mathscr{K}} F^*$ is also the composite

$$\mathscr{L}(\mathscr{A}) \xrightarrow[W_{\mathscr{L}}]{} [\mathscr{L}_f^{\mathrm{op}}, \mathscr{A}] \xrightarrow[[F_f^{\mathrm{op}},1]]{} [\mathscr{K}_f^{\mathrm{op}}, \mathscr{A}]$$

and $[F_f^{\mathrm{op}}, 1]$ has a left adjoint given by left Kan extension along F_f^{op} if $\mathscr{A}$ is cocomplete, while $W_{\mathscr{L}}$ has a left adjoint if $\mathscr{L}$ is $\mathscr{A}$-reflective, so we have:

Proposition 2.1 *If $U : \mathscr{L} \to \mathscr{K}$ is a filtered-colimit-preserving right adjoint between locally finitely presentable categories, and $\mathscr{L}$ is $\mathscr{A}$-reflective and $\mathscr{A}$ is cocomplete, then $F^* : \mathscr{L}(\mathscr{A}) \to \mathscr{K}(\mathscr{A})$ has a left adjoint $F_* \dashv F^*$.*

3 The symmetric monoidal structure

We now consider a symmetric monoidal closed category $\mathscr{V}$, and suppose that the locally finitely presentable $\mathscr{K}$ is $\mathscr{V}$-reflective, so that the fully faithful inclusion $W : \mathscr{K}(\mathscr{V}) \to [\mathscr{K}_f^{\mathrm{op}}, \mathscr{V}]$ has a left adjoint L. As we saw in the previous section this will be the case if $\mathscr{V}$ is locally bounded, but also in certain other cases: for example if $\mathscr{V}$ is abelian and $\mathscr{K}$ is **Cat**.

For various calculations in this section and the next, it is convenient to suppose that $\mathscr{V}$ is complete and cocomplete, and we shall write as if this is the case. This is only to simplify the proofs; the results remain true with only the assumption that $\mathscr{K}$ is $\mathscr{V}$-reflective. The completeness and cocompleteness is only used to show that certain functors, already known to exist, preserve certain limits or colimits; or that certain functors, already known to exist, are isomorphic. To see that this method of proof is valid, recall [13, Sections 3.10 and 3.11] that a symmetric monoidal closed category $\mathscr{V}$ can be embedded in a symmetric monoidal closed $\mathscr{V}'$ which is complete and cocomplete, in such a way that the symmetric monoidal closed structure as well as any existing limits and colimits are preserved.

We regard $[\mathscr{K}_f^{\mathrm{op}}, \mathscr{V}]$ as a $\mathscr{V}$-category, and $\mathscr{K}(\mathscr{V})$ as a full sub-$\mathscr{V}$-category; then the adjunction between them becomes a $\mathscr{V}$-adjunction.

There is a pointwise symmetric monoidal structure on $[\mathscr{K}_f^{\mathrm{op}}, \mathscr{V}]$, so that for functors $F, G : \mathscr{K}_f^{\mathrm{op}} \to \mathscr{V}$ the tensor product $F \otimes G$ is given by $(F \otimes G)c = Fc \otimes Gc$. This monoidal structure is also closed, with internal hom $[G, H]$ given by $[G, H]c = [\mathscr{K}_f^{\mathrm{op}}, \mathscr{V}](\mathscr{K}_f(-, c).G, H)$, where for a set n and an object G, we write $n.G$ for the coproduct of n copies of G.

Given objects F and G of $\mathscr{K}(\mathscr{V})$, seen as lying in $[\mathscr{K}_f^{\mathrm{op}}, \mathscr{V}]$, we may form their tensor product $F \otimes G$ in the latter category, and then reflect this into $\mathscr{K}(\mathscr{V})$ using the reflection L. We would like to define the resulting object $L(F \otimes G)$ as the tensor product in $\mathscr{K}(\mathscr{V})$ of F and G. The problem is that it is not clear that this is associative or closed: this is where we shall use the cartesian closedness of $\mathscr{K}$. The main result, Theorem 3.1 below, was proved by Bastiani and Ehresmann in [1]. Since [1] is a long paper, and some of its terminology may be unfamiliar to the modern reader, we present here an alternative proof, based on the reflection theorem of Day [4].

Day's reflection theorem involves the following situation. One has a symmetric monoidal closed $\mathscr{V}$-category $\mathscr{B}$, and a full reflective subcategory $\mathscr{C}$ of $\mathscr{B}$, and one wants to form a symmetric monoidal closed structure on $\mathscr{C}$ for which the reflection preserves the symmetric monoidal structure. The latter condition clearly determines what the structure must be; Day proves that this works if and only if, for every object B in $\mathscr{B}$ and every object C in $\mathscr{C}$, the internal hom $[B, C]$ in $\mathscr{B}$ actually lies in $\mathscr{C}$. Moreover, if $\mathscr{A}$ is a dense subcategory of $\mathscr{B}$, then it suffices to consider the case where B lies in $\mathscr{A}$. He further shows that if the monoidal structure on $\mathscr{B}$ is cartesian, then so will be that on $\mathscr{C}$.

We now apply this to the case where $\mathscr{B} = [\mathscr{K}_f^{\mathrm{op}}, \mathscr{V}]$ and $\mathscr{C} = \mathscr{K}(\mathscr{V})$, with $\mathscr{A}$ being the dense full subcategory consisting of all functors of the form $\mathscr{K}_f(-, c).I$, where I is the unit object of $\mathscr{V}$. Thus we must show that $[\mathscr{K}_f(-, c).I, G] : \mathscr{K}_f^{\mathrm{op}} \to \mathscr{V}$ preserves finite limits whenever $G : \mathscr{K}_f^{\mathrm{op}} \to \mathscr{V}$ does so. Observe, however, that if $\mathscr{V} = \mathbf{Set}$, then the pointwise monoidal structure on $[\mathscr{K}_f^{\mathrm{op}}, \mathbf{Set}]$ is the cartesian one, so that $\mathscr{K}(\mathbf{Set}) = \mathscr{K}$ will

be cartesian closed. Thus cartesian closedness is a necessary condition; we must show that it is sufficient.

Write $J : \mathscr{K}_f \to \mathscr{K}$ for the fully faithful inclusion. We have

$$\begin{aligned}
[\mathscr{K}_f(-,c).I, G]d &\cong [\mathscr{K}_f^{\mathrm{op}}, \mathscr{V}](\mathscr{K}_f(-,d).\mathscr{K}_f(-,c).I, G) \\
&\cong [\mathscr{K}_f^{\mathrm{op}}, \mathscr{V}](\mathscr{K}(J-,Jd).\mathscr{K}(J-,Jc).I, G) \\
&\cong [\mathscr{K}_f^{\mathrm{op}}, \mathscr{V}](\mathscr{K}(J-,Jd \times Jc).I, G) \\
&\cong [\mathscr{K}^{\mathrm{op}}, \mathscr{V}](\mathscr{K}(-,Jd \times Jc).I, \mathrm{Ran}_{J^{\mathrm{op}}} G) \\
&\cong \mathrm{Ran}_{J^{\mathrm{op}}} G(Jd \times Jc)
\end{aligned}$$

and so $[\mathscr{K}_f(-,c).I, G]$ is given by the composite

$$\mathscr{K}_f^{\mathrm{op}} \xrightarrow{J^{\mathrm{op}}} \mathscr{K}^{\mathrm{op}} \xrightarrow{(-\times Jc)^{\mathrm{op}}} \mathscr{K}^{\mathrm{op}} \xrightarrow{\mathrm{Ran}_{J^{\mathrm{op}}} G} \mathscr{V}.$$

In this composite J^{op} preserves finite limits since the finitely presentable objects are closed under finite colimits, and $(- \times Jc)^{\mathrm{op}}$ preserves all limits since $- \times Jc$ has a right adjoint, thus the composite will preserve finite limits provided that $\mathrm{Ran}_{J^{\mathrm{op}}} G$ does so. Since the representable functors $\mathscr{V}(X,-) : \mathscr{V} \to \mathbf{Set}$ create limits, $\mathrm{Ran}_{J^{\mathrm{op}}} G : \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ will preserve finite limits if and only if each $\mathscr{V}(X, \mathrm{Ran}_{J^{\mathrm{op}}} G) : \mathscr{K}^{\mathrm{op}} \to \mathbf{Set}$ does so; but $\mathscr{V}(X,-)$ preserves right Kan extensions, so $\mathscr{V}(X, \mathrm{Ran}_{J^{\mathrm{op}}} G) \cong \mathrm{Ran}_{J^{\mathrm{op}}} \mathscr{V}(X,G)$. Now $\mathscr{V}(X,G) : \mathscr{K}_f^{\mathrm{op}} \to \mathbf{Set}$ preserves finite limits, so by the equivalence $\mathrm{Lex}(\mathscr{K}_f^{\mathrm{op}}, \mathbf{Set}) \simeq \mathscr{K}$ can be viewed as $\mathscr{K}(J-,A) : \mathscr{K}_f^{\mathrm{op}} \to \mathbf{Set}$, for an (essentially unique) object A of $\mathscr{K}$. Now $\mathscr{K}(-,A)$ is continuous, and so is the right Kan extension $\mathrm{Ran}_{J^{\mathrm{op}}} \mathscr{K}(J-,A)$ of its restriction to the full codense subcategory $\mathscr{K}_f^{\mathrm{op}}$ of $\mathscr{K}^{\mathrm{op}}$. Thus $\mathscr{V}(X, \mathrm{Ran}_{J^{\mathrm{op}}} G) \cong \mathrm{Ran}_{J^{\mathrm{op}}} \mathscr{V}(X,G) \cong \mathrm{Ran}_{J^{\mathrm{op}}} \mathscr{K}(J-,A) \cong \mathscr{K}(-,A)$, and the latter is continuous, hence so too is $\mathrm{Ran}_{J^{\mathrm{op}}} G$, and so finally $[\mathscr{K}_f(-,c).I, G]$ preserves finite limits. Thus the conditions of Day's reflection theorem are satisfied, and we have proved:

Theorem 3.1 (Bastiani-Ehresmann) *Let $\mathscr{K}$ be a locally finitely presentable cartesian closed category, and let $\mathscr{V}$ be a symmetric monoidal closed category for which the inclusion $\mathscr{K}(\mathscr{V}) = \mathrm{Lex}(\mathscr{K}_f^{op}, \mathscr{V}) \to [\mathscr{K}_f^{op}, \mathscr{V}]$ has a left adjoint. Then this adjunction underlies a symmetric monoidal closed adjunction, where $[\mathscr{K}_f^{op}, \mathscr{V}]$ is equipped with the pointwise symmetric monoidal structure.*

4 Computing the tensor product

This section contains two techniques which will be useful in computing the tensor product. In the first, we look at what happens when $\mathscr{K}_f$ is replaced by some other dense subcategory of $\mathscr{K}$; in the second we look at what happens when $\mathscr{K}$ is itself replaced by a different cartesian closed locally finitely presentable category $\mathscr{L}$. As in the previous section, to simplify the proofs we write as if $\mathscr{V}$ were complete and cocomplete, but all results hold without this hypothesis; all we need is that $\mathscr{K}$ and $\mathscr{L}$ be $\mathscr{V}$-reflective.

Recall from Section 2 the equivalence $\mathscr{K}(\mathscr{V}) = \mathrm{Cts}(\mathscr{K}^{\mathrm{op}},\mathscr{V})$; in this section we work with $\mathrm{Cts}(\mathscr{K}^{\mathrm{op}},\mathscr{V})$ rather than $\mathscr{K}(\mathscr{V})$.

First we need an easy lemma:

Lemma 4.1 *Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be symmetric monoidal closed $\mathscr{V}$-categories, and consider adjunctions*

$$\begin{tikzcd} \mathscr{A} \arrow[r, shift right, "W"'] & \mathscr{B} \arrow[l, shift right, "L"'] \arrow[r, hook, shift right, "V"'] & \mathscr{C} \arrow[l, shift right, "K"'] \end{tikzcd}$$

with V fully faithful. If the right-hand adjunction $K \dashv V$ is symmetric monoidal closed, then the composite adjunction $LK \dashv VW$ is symmetric monoidal closed if and only if the left-hand adjunction $L \dashv W$ is so.

Proof: The "if" part is immediate from the fact that (symmetric monoidal closed) adjunctions compose. For the converse, suppose that the composite is symmetric monoidal closed. Then for objects B and B' of $\mathscr{B}$, we have $L(B \otimes B') \cong LK(VB \otimes VB') \cong LKVB \otimes LKVB' \cong LB \otimes LB'$; the details are left to the reader. $\square$

Consider an arbitrary dense full subcategory $\mathscr{H}$ of $\mathscr{K}$, with inclusion $J : \mathscr{H} \to \mathscr{K}$. Every continuous functor $G : \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ is the right Kan extension of its restriction along J^{op}: in other words $G \cong \mathrm{Ran}_{J^{\mathrm{op}}}(GJ^{\mathrm{op}})$. Suppose now that $\mathscr{G}$ is a full subcategory of $\mathscr{H}$, with inclusion $P : \mathscr{G} \to \mathscr{H}$, and that the composite JP is also dense. Then $G \cong \mathrm{Ran}_{J^{\mathrm{op}}P^{\mathrm{op}}}(GJ^{\mathrm{op}}P^{\mathrm{op}})$, and so $GJ^{\mathrm{op}} \cong \mathrm{Ran}_{J^{\mathrm{op}}P^{\mathrm{op}}}(GJ^{\mathrm{op}}P^{\mathrm{op}})J^{\mathrm{op}} \cong \mathrm{Ran}_{P^{\mathrm{op}}}(GJ^{\mathrm{op}}P^{\mathrm{op}})$. Thus we have a diagram of adjunctions

$$\begin{tikzcd} \mathrm{Cts}(\mathscr{K}^{\mathrm{op}},\mathscr{V}) \arrow[r, hook, shift right, "W"'] & {[\mathscr{G}^{\mathrm{op}},\mathscr{V}]} \arrow[l, shift right, "L"'] \arrow[r, hook, shift right, "\mathrm{Ran}_{P^{\mathrm{op}}}"'] & {[\mathscr{H}^{\mathrm{op}},\mathscr{V}]} \arrow[l, shift right, "{[P^{\mathrm{op}},\mathscr{V}]}"'] \end{tikzcd}$$

where W sends a continuous functor $G : \mathscr{K}^{\mathrm{op}} \to \mathscr{V}$ to $GJ^{\mathrm{op}}P^{\mathrm{op}}$, and the composite $\mathrm{Ran}_{P^{\mathrm{op}}}.W$ sends G to GJ^{op}. Now consider $[\mathscr{G}^{\mathrm{op}},\mathscr{V}]$ and $[\mathscr{H}^{\mathrm{op}},\mathscr{V}]$ as symmetric monoidal closed categories via the pointwise tensor product. Then the left adjoint $[P^{\mathrm{op}},\mathscr{V}]$ to $\mathrm{Ran}_{P^{\mathrm{op}}}$ is given by restriction along P, and this preserves the symmetric monoidal structure by definition, so the right hand adjunction is symmetric monoidal closed, and we are in the situation of the lemma. Thus if there is a symmetric monoidal closed structure on $\mathrm{Cts}(\mathscr{K}^{\mathrm{op}},\mathscr{V})$ for which one of the adjunctions is symmetric monoidal closed, then both of them are symmetric monoidal closed.

This means that if $\mathscr{K}$ is a locally finitely presentable cartesian closed category, and $\mathscr{K}(\mathscr{V}) \to [\mathscr{K}_f^{\mathrm{op}},\mathscr{V}]$ has a left adjoint, then we may calculate the symmetric monoidal structure on $\mathscr{K}(\mathscr{V})$ using any dense full subcategory $H : \mathscr{G} \to \mathscr{K}$, as follows. Let $W : \mathscr{K}(\mathscr{V}) \to [\mathscr{G}^{\mathrm{op}},\mathscr{V}]$ be the canonical inclusion, and L its left adjoint. Then for F and G in $\mathscr{K}(\mathscr{V})$ we have $F \otimes G \cong L(WF \otimes WG)$.

It is also useful to be able to compare different finite limit structures in $\mathscr{V}$. Suppose, as in Section 2, that $\mathscr{K}$ and $\mathscr{L}$ are locally finitely presentable, and that $U : \mathscr{L} \to \mathscr{K}$ is a

filtered-colimit-preserving functor with left adjoint F, so that F restricts to $F_f : \mathscr{K}_f \to \mathscr{L}_f$, and restriction along F_f^{op} induces a functor $F^* : \mathscr{L}(\mathscr{V}) \to \mathscr{K}(\mathscr{V})$. Suppose further that $\mathscr{K}$ and $\mathscr{L}$ are cartesian closed and that $\mathscr{K}$ and $\mathscr{L}$ are $\mathscr{V}$-reflective. Then there are adjunctions

$$\mathscr{L}(\mathscr{V}) \underset{F^*}{\overset{F_*}{\leftrightarrows}} \mathscr{K}(\mathscr{V}) \underset{W_{\mathscr{K}}}{\overset{L_{\mathscr{K}}}{\hookleftarrow\!\!\!\rightarrow}} [\mathscr{K}_f^{\mathrm{op}}, \mathscr{V}] \qquad \mathscr{L}(\mathscr{V}) \underset{W_{\mathscr{L}}}{\overset{L_{\mathscr{W}}}{\hookleftarrow\!\!\!\rightarrow}} [\mathscr{L}_f^{\mathrm{op}}, \mathscr{V}] \underset{[F_f^{\mathrm{op}}, \mathscr{V}]}{\overset{\mathrm{Lan}_{F_f^{\mathrm{op}}}}{\leftrightarrows}} [\mathscr{K}_f^{\mathrm{op}}, \mathscr{V}]$$

and the composite right adjoints agree, hence so do the composite left adjoints.

Suppose finally that $F \dashv U$ is cartesian closed, so that F_f preserves finite products; then $\mathrm{Lan}_{F_f^{\mathrm{op}}}$ preserves the symmetric monoidal structure by [5, Proposition 1], and so $\mathrm{Lan}_{F_f^{\mathrm{op}}} \dashv [F_f^{\mathrm{op}}, \mathscr{V}]$ is symmetric monoidal closed. The adjunctions $L_{\mathscr{K}} \dashv W_{\mathscr{K}}$ and $L_{\mathscr{L}} \dashv W_{\mathscr{L}}$ are symmetric monoidal closed by Theorem 3.1, so finally $F_* \dashv F^*$ is symmetric monoidal closed by Lemma 4.1, and we have:

Theorem 4.2 *If $F \dashv U$ is an adjunction between locally finitely presentable cartesian closed categories $\mathscr{K}$ and $\mathscr{L}$, and if $U : \mathscr{L} \to \mathscr{K}$ preserves filtered colimits and F preserves finite products, then for any symmetric monoidal closed category $\mathscr{V}$ for which both $\mathscr{K}$ and $\mathscr{L}$ are $\mathscr{V}$-reflective, the induced adjunction $F_* \dashv F^*$ between $\mathscr{K}(\mathscr{V})$ and $\mathscr{L}(\mathscr{V})$ is symmetric monoidal closed.*

5 Operads

In this section we turn to operads. We recall below the basic ideas; for more information, see [15]. We shall be working with a symmetric monoidal adjunction

$$\mathscr{K}(\mathscr{V}) \underset{W}{\overset{L}{\hookleftarrow\!\!\!\rightarrow}} [\mathscr{G}^{\mathrm{op}}, \mathscr{V}]$$

where the symmetric monoidal structure on $[\mathscr{G}^{\mathrm{op}}, \mathscr{V}]$ is pointwise. Sections 3 and 4 give sufficient conditions for this to happen. For the main examples of interest, see Section 9; in the meantime a good example to keep in mind is that where $\mathscr{K}(\mathscr{V}) = \mathbf{Cat}(\mathscr{V})$, and $[\mathscr{G}^{\mathrm{op}}, \mathscr{V}]$ is the category of simplicial objects in $\mathscr{V}$, while W is the nerve functor.

Write $\mathbb{P}$ for the skeletal category of finite sets and bijections (the disjoint union of the symmetric groups). Then the category $[\mathbb{P}, \mathscr{V}]$ has a symmetric monoidal closed structure involving the "convolution tensor product"

$$S \otimes T = \int^{i,j \in \mathbb{P}} \mathbb{P}(i+j, -).Si \otimes Tj \quad ;$$

more explicitly, the value of $S \otimes T$ at n is given by $\Sigma_i Si \otimes T(n-i)$ where the coproduct is indexed by all i-element subsets of n. There is also a (non-symmetric) monoidal structure involving the "substitution tensor product"

$$S \circ T = \int^{n \in \mathbb{P}} Sn \otimes T^{\otimes n}$$

which can be constructed as a quotient of $\sum_n Sn \otimes T^{\otimes n}$, with the quotient being defined in terms of the actions of the symmetric group on Sn and on $T^{\otimes n}$. A monoid with respect to this tensor product is called an *operad* in $\mathscr{V}$. Since the substitution monoidal structure ultimately depends only on colimits and the tensor product in $\mathscr{V}$, if $\mathscr{W}$ is another symmetric monoidal closed category, and $F : \mathscr{V} \to \mathscr{W}$ preserves colimits and the symmetric monoidal structure, then the induced functor $[\mathbb{P}, F] : [\mathbb{P}, \mathscr{V}] \to [\mathbb{P}, \mathscr{W}]$ preserves the substitution monoidal structure.

We now turn to the adjunction $L \dashv W$ between $\mathscr{K}(\mathscr{V})$ and $[\mathscr{G}^{\mathrm{op}}, \mathscr{V}]$, and consider operads in these symmetric monoidal categories. We form the substitution monoidal structures on $[\mathbb{P}, \mathscr{K}(\mathscr{V})]$ and $[\mathbb{P}, [\mathscr{G}^{\mathrm{op}}, \mathscr{V}]]$. The left adjoint $L : [\mathscr{G}^{\mathrm{op}}, \mathscr{V}] \to \mathscr{K}(\mathscr{V})$ preserves colimits and the symmetric monoidal structure, and so the induced functor $[\mathbb{P}, L]$ preserves the substitution monoidal structure. Thus in the adjunction

$$\begin{tikzcd} {[\mathbb{P}, \mathscr{K}(\mathscr{V})]} \arrow[r, hook, shift right, "{[\mathbb{P},W]}"'] & {[\mathbb{P}, [\mathscr{G}^{\mathrm{op}}, \mathscr{V}]]} \arrow[l, shift right, "{[\mathbb{P},L]}"'] \end{tikzcd}$$

the left adjoint $[\mathbb{P}, L]$ preserves the monoidal structure. It follows ("doctrinal adjunction" [12]) that although the right adjoint does not preserve the monoidal structure, it does so up to a canonical (non-invertible) comparison map, and moreover there is an induced adjunction between the categories of internal monoids in these categories: in other words, an adjunction

$$\begin{tikzcd} \mathrm{Op}(\mathscr{K}(\mathscr{V})) \arrow[r, hook, shift right] & \mathrm{Op}[\mathscr{G}^{\mathrm{op}}, \mathscr{V}] \arrow[l, shift right] \end{tikzcd}$$

between the categories of operads. Furthermore, one sees that an operad in $\mathscr{K}(\mathscr{V})$ is precisely an operad in $[\mathscr{G}^{\mathrm{op}}, \mathscr{V}]$ for which the underlying functor $\mathbb{P} \to [\mathscr{G}^{\mathrm{op}}, \mathscr{V}]$ actually lands in $\mathscr{K}(\mathscr{V})$. In other words, the diagram

$$\begin{tikzcd} \mathrm{Op}(\mathscr{K}(\mathscr{V})) \arrow[r, hook] \arrow[d] & \mathrm{Op}[\mathscr{G}^{\mathrm{op}}, \mathscr{V}] \arrow[d] \\ {[\mathbb{P}, \mathscr{K}(\mathscr{V})]} \arrow[r, hook] & {[\mathbb{P}, [\mathscr{G}^{\mathrm{op}}, \mathscr{V}]]} \end{tikzcd} \tag{5.1}$$

is a pullback. In particular, we may take $\mathscr{G} = \mathscr{K}_f$, and deduce that

$$\begin{tikzcd} \mathrm{Op}(\mathscr{K}(\mathscr{V})) \arrow[r, hook] \arrow[d] & \mathrm{Op}[\mathscr{K}_f^{\mathrm{op}}, \mathscr{V}] \arrow[d] \\ {[\mathbb{P}, \mathscr{K}(\mathscr{V})]} \arrow[r, hook] & {[\mathbb{P}, [\mathscr{K}_f^{\mathrm{op}}, \mathscr{V}]]} \end{tikzcd}$$

is a pullback.

On the other hand, a functor $\mathscr{K}_f^{\mathrm{op}} \to \mathrm{Op}(\mathscr{V})$ preserves finite limits if and only if its composite with the forgetful functor $\mathrm{Op}(\mathscr{V}) \to [\mathbb{P}, \mathscr{V}]$ does so, thus the square

$$\begin{tikzcd} \mathscr{K}(\mathrm{Op}(\mathscr{V})) \arrow[r, hook] \arrow[d] & {[\mathscr{K}_f^{\mathrm{op}}, \mathrm{Op}(\mathscr{V})]} \arrow[d] \\ \mathscr{K}[\mathbb{P}, \mathscr{V}] \arrow[r, hook] & {[\mathscr{K}_f^{\mathrm{op}}, [\mathbb{P}, \mathscr{V}]]} \end{tikzcd}$$

is also a pullback. There are canonical equivalences between $[\mathbb{P},[\mathscr{K}_f^{\mathrm{op}},\mathscr{V}]]$, $[\mathscr{K}_f^{\mathrm{op}}\times\mathbb{P},\mathscr{V}]$, and $[\mathscr{K}_f^{\mathrm{op}},[\mathbb{P},\mathscr{V}]]$, and since limits in $[\mathbb{P},\mathscr{V}]$ are formed pointwise, this lifts to an equivalence $[\mathbb{P},\mathscr{K}(\mathscr{V})]\simeq\mathscr{K}[\mathbb{P},\mathscr{V}]$. On the other hand, both colimits and the symmetric monoidal structure in $[\mathscr{K}_f^{\mathrm{op}},\mathscr{V}]$ are formed pointwise, so the substitution monoidal structure in $[\mathbb{P},[\mathscr{K}_f^{\mathrm{op}},\mathscr{V}]]$ is formed pointwise (with respect to objects of $\mathscr{K}_f$), and so the equivalence $[\mathbb{P},[\mathscr{K}_f^{\mathrm{op}},\mathscr{V}]]\simeq[\mathscr{K}_f^{\mathrm{op}},[\mathbb{P},\mathscr{V}]]$ also lifts to an equivalence $\mathrm{Op}[\mathscr{K}_f^{\mathrm{op}},\mathscr{V}]\simeq[\mathscr{K}_f^{\mathrm{op}},\mathrm{Op}(\mathscr{V})]$; finally since the two squares displayed are pullbacks (which are in fact pseudopullbacks [11]), we have, under the hypotheses of Theorem 3.1:

Theorem 5.1 *The equivalence $[\mathbb{P},[\mathscr{K}_f^{op},\mathscr{V}]]\simeq[\mathscr{K}_f^{op},[\mathbb{P},\mathscr{V}]]$ lifts to an equivalence*

$$\mathscr{K}(\mathrm{Op}(\mathscr{V}))\simeq\mathrm{Op}(\mathscr{K}(\mathscr{V})).$$

6 Algebras

The monoidal category $[\mathbb{P},\mathscr{V}]$, with the substitution monoidal structure, acts on $\mathscr{V}$ by the formula $T\circ A=\int^n Tn\otimes A^{\otimes n}$, where $T:\mathbb{P}\to\mathscr{V}$ and A is an object of $\mathscr{V}$. (The fact that this is an action means that there are suitably coherent isomorphisms $(S\circ T)\circ A\cong S\circ(T\circ A)$.) Thus the endofunctor $T\circ -:\mathscr{V}\to\mathscr{V}$ will become a monad when T is given an operad structure, and the category $\mathscr{V}^T$ of algebras for the operad is then defined to be the category of algebras for the monad. Once again the action $T\circ A$ is determined by colimits and the monoidal structure, so will be preserved if these are preserved.

There is also another characterization of algebras. For an object A of $\mathscr{V}$, there is an operad $\langle A,A\rangle$ in $\mathscr{V}$, with $\langle A,A\rangle n=[A^n,A]$. If T is any operad, a T-algebra structure on A is the same thing as an operad map from T to $\langle A,A\rangle$.

Once again we now consider the case where $\mathscr{V}$ is replaced by $\mathscr{K}(\mathscr{V})$ and by $[\mathscr{G}^{\mathrm{op}},\mathscr{V}]$, in the setting of the previous section, and analyze this using the adjunction $L\dashv W$ between $\mathscr{K}(\mathscr{V})$ and $[\mathscr{G}^{\mathrm{op}},\mathscr{V}]$ and the fact that L preserves colimits and the monoidal structure. Once again a basic example is where $\mathscr{K}=\mathbf{Cat}$, $\mathscr{G}$ is the simplicial category, and W is the nerve functor.

Thus if $T:\mathbb{P}\to\mathscr{K}(\mathscr{V})$ and F is an object of $\mathscr{K}(\mathscr{V})$, the action $T\circ F$ is given by $L(WT\circ WF)$. In particular there is a bijection between maps $L(WT\circ WF)\to F$ and maps $WT\circ WF\to WF$, and it is now straightforward to characterize algebras for operads in $\mathscr{K}(\mathscr{V})$ as follows, using the fully faithful map $\mathrm{Op}(\mathscr{K}(\mathscr{V}))\to\mathrm{Op}[\mathscr{G}^{\mathrm{op}},\mathscr{V}]$ of 5.1.

Since $L:[\mathbb{P},[\mathscr{G}^{\mathrm{op}},\mathscr{V}]]\to[\mathbb{P},\mathscr{K}(\mathscr{V})]$ preserves the tensor product, the right adjoint W preserves the (right-)closed structure, and so $W\langle F,F\rangle\cong\langle WF,WF\rangle$. Thus for an operad S in $[\mathscr{G}^{\mathrm{op}},\mathscr{V}]$, an LS-algebra structure on F is the same as an operad map $LS\to\langle F,F\rangle$ and so equally as an operad map $S\to W\langle F,F\rangle$ and so further as an operad map $S\to\langle WF,WF\rangle$, and so finally as an S-algebra structure on WF. Similarly a morphism of LS-algebras is the same thing as a morphism of the corresponding S-algebras. Applying this in the special case when $S=WT$, so that $T=LWT=LS$, we obtain:

Proposition 6.1 *An algebra for an operad T in $\mathscr{K}(\mathscr{V})$ is an object F of $\mathscr{K}(\mathscr{V})$ equipped with the structure of a WT-algebra, as an object of $[\mathscr{G}^{op}, \mathscr{V}]$. A morphism of T-algebras is just a morphism of the corresponding WT-algebras. In other words, the square*

$$\begin{tikzcd} \mathscr{K}(\mathscr{V})^T \arrow[r, hook] \arrow[d] & {[\mathscr{G}^{op}, \mathscr{V}]^{WT}} \arrow[d] \\ \mathscr{K}(\mathscr{V}) \arrow[r, hook] & {[\mathscr{G}^{op}, \mathscr{V}]} \end{tikzcd}$$

is a pullback.

Since the monoidal structure of $[\mathscr{G}^{\mathrm{op}}, \mathscr{V}]$ is pointwise, algebras for operads in $[\mathscr{G}^{\mathrm{op}}, \mathscr{V}]$ are easy to describe. An operad in $[\mathscr{G}^{\mathrm{op}}, \mathscr{V}]$ is the same thing as a functor $T : \mathscr{G}^{\mathrm{op}} \to \mathrm{Op}(\mathscr{V})$. An algebra is then a functor $F : \mathscr{G}^{\mathrm{op}} \to \mathscr{V}$, with a Tc-algebra structure $\varphi c : Tc \circ Fc \to Fc$ for each object c of $\mathscr{G}$, with the property that for every morphism $\gamma : c \to d$ in $\mathscr{G}$, the map $F\gamma : Fd \to Fc$ is a morphism of Td-algebras, where the Td-algebra structure on Fc is given by the composite

$$Td \circ Fc \xrightarrow{T\gamma \circ Fc} Tc \circ Fc \xrightarrow{\varphi c} Fc.$$

The diagonal map $\Delta : \mathscr{V} \to [\mathscr{G}^{\mathrm{op}}, \mathscr{V}]$, which sends an object A of $\mathscr{V}$ to the functor $\mathscr{G}^{\mathrm{op}} \to \mathscr{V}$ which is constant at $\mathscr{V}$, preserves colimits and the symmetric monoidal structure, and so sends an operad T in $\mathscr{V}$ to an operad ΔT in $[\mathscr{G}^{\mathrm{op}}, \mathscr{V}]$. By the preceding discussion, an algebra for ΔT is a functor $F : \mathscr{G}^{\mathrm{op}} \to \mathscr{V}$ with a T-algebra structure on each Fc, for which the $F\gamma : Fd \to Fc$ are T-morphisms; but this is precisely a functor $\mathscr{G}^{\mathrm{op}} \to \mathscr{V}^T$. Thus we have an equivalence of categories $[\mathscr{G}^{\mathrm{op}}, \mathscr{V}^T] \simeq [\mathscr{G}^{\mathrm{op}}, \mathscr{V}]^{\Delta T}$.

Finally, we may compose ΔT with the reflection $L : [\mathscr{K}_f^{\mathrm{op}}, \mathscr{V}] \to \mathscr{K}(\mathscr{V})$ to obtain an operad $L\Delta T$ in $\mathscr{K}(\mathscr{V})$ and an equivalence $\mathscr{K}(\mathscr{V}^T) \simeq \mathscr{K}(\mathscr{V})^{L\Delta T}$, giving:

Theorem 6.2 *If $\mathscr{K}$ is a locally finitely presentable cartesian closed category, and $\mathscr{G}$ is a dense subcategory of $\mathscr{K}$, and $\mathscr{V}$ is a symmetric monoidal closed category for which the fully faithful functor $\mathscr{K}(\mathscr{V}) \to [\mathscr{G}^{op}, \mathscr{V}]$ has a left adjoint, then for any operad T in $\mathscr{V}$, there is an equivalence of categories $\mathscr{K}(\mathscr{V}^T) \simeq \mathscr{K}(\mathscr{V})^{L\Delta T}$, where $\Delta : \mathscr{V} \to [\mathscr{G}^{op}, \mathscr{V}]$ is the diagonal map.*

7 Monads

In the rest of the paper we have been working with an operad T, but there is an analogous theory in which one considers the more general situation of a monad T. In some ways the situation is then easier: one now replaces the symmetric monoidal closed $\mathscr{V}$ by a mere category $\mathscr{A}$, and considers the (mere) category $\mathscr{K}(\mathscr{A})$ of $\mathscr{K}$-structures in $\mathscr{A}$. Thus there is no need to construct or calculate the tensor product. Under suitable conditions, given a monad T on $\mathscr{A}$ there is a monad T' on $\mathscr{K}(\mathscr{A})$ for which $\mathscr{K}(\mathscr{A})^{T'}$ is equivalent to $\mathscr{K}(\mathscr{A}^T)$.

Although operads can be seen as a special case of monads, our main results are not a special case of those in this section, since one still needs to show that if T is (that is, comes from) an operad then T' is one.

Suppose then that T is a monad on $\mathscr{A}$, with category of algebras $\mathscr{A}^T$ and forgetful functor $U^T : \mathscr{A}^T \to \mathscr{A}$. Then T induces a "pointwise" monad $[\mathscr{K}_f^{\mathrm{op}}, T]$ on $[\mathscr{K}_f^{\mathrm{op}}, \mathscr{A}]$ whose category of algebras is just $[\mathscr{K}_f^{\mathrm{op}}, \mathscr{A}^T]$, and whose forgetful functor is $[\mathscr{K}_f^{\mathrm{op}}, U]$: $[\mathscr{K}_f^{\mathrm{op}}, \mathscr{A}^T] \to [\mathscr{K}_f^{\mathrm{op}}, \mathscr{A}]$. Since limits in $\mathscr{A}^T$ are formed as in $\mathscr{A}$, we have a pullback

$$\begin{tikzcd} \mathscr{K}(\mathscr{A}^T) \arrow[r] \arrow[d, "V"'] & {[\mathscr{K}_f^{\mathrm{op}}, \mathscr{A}^T]} \arrow[d, "{[\mathscr{K}_f^{\mathrm{op}}, U^T]}"] \\ \mathscr{K}(\mathscr{A}) \arrow[r] & {[\mathscr{K}_f^{\mathrm{op}}, \mathscr{A}]} \end{tikzcd}$$

with horizontal arrows the fully faithful inclusions, and we wish to show that $V : \mathscr{K}(\mathscr{A}^T) \to \mathscr{K}(\mathscr{A})$ exhibits $\mathscr{K}(\mathscr{A}^T)$ as the category of algebras for a monad T' on $\mathscr{K}(\mathscr{A})$; in other words, that V is monadic. It is well-known that V reflects isomorphisms and satisfies the Beck condition, since $[\mathscr{K}_f^{\mathrm{op}}, U^T]$ does so, thus by Beck's theorem [16, VI.7], V will be monadic provided that it has a left adjoint. This is quite a mild condition; it holds, for example, if $\mathscr{A}$ is cocomplete and the endofunctor $T : \mathscr{A} \to \mathscr{A}$ has a rank (preserves α-filtered colimits for some regular cardinal α). This in turn will be the case, at least for locally presentable $\mathscr{A}$, if T-algebra structure can be described in terms of operations and equations of arity less than α; in particular, if it can be done with finite arities.

8 The case of internal categorical structures

A paradigmatic case is that where $\mathscr{K}$ is the category **Cat** of categories, which is well-known to be locally finitely presentable and cartesian closed. This is a case where it is convenient not to use $\mathbf{Cat}_f$, but rather various other small dense subcategories of **Cat**.

One possibility is the category $\mathbf{\Delta}$ of finite non-empty ordinals. We may regard these as partially ordered sets, and so as categories, and thus obtain a fully faithful inclusion $J : \mathbf{\Delta} \to \mathbf{Cat}$. The induced functor $W = \mathbf{Cat}(J, -) : \mathbf{Cat} \to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ is the nerve functor, which is well-known to be fully faithful, so that J is dense. Thus the tensor product in $\mathbf{Cat}(\mathscr{V})$ of internal categories C and D is given by taking the pointwise tensor product $WC \otimes WD$ of their nerves, and then reflecting this into $\mathbf{Cat}(\mathscr{V})$ to obtain the tensor product $L(WC \otimes WD)$. This example was also given in [1].

By the general theory, we see by (5.1) that an operad in $\mathbf{Cat}(\mathscr{V})$ is just a simplicial operad in $\mathscr{V}$ for which the underlying map $\mathbb{P} \to [\mathbf{\Delta}^{\mathrm{op}}, \mathscr{V}]$ lands in $\mathbf{Cat}(\mathscr{V})$. Similarly, an algebra for such an operad is a (simplicial) algebra for the simplicial operad, for which the underlying simplicial object is actually the nerve of a category.

For many purposes this is all that one needs to know, but in the following section, where we shall take $\mathscr{V}$ to be abelian, it is useful to have a more explicit description of the tensor

product in $\mathbf{Cat}(\mathscr{V})$. To obtain this, we use the full subcategory $\mathbf{\Delta}'$ of $\mathbf{\Delta}$ determined by the objects 1, 2, and 3. The inclusion $H : \mathbf{\Delta}' \to \mathbf{Cat}$ is still dense, and so the adjunction

$$\begin{tikzcd} \mathbf{Cat}(\mathscr{V}) \arrow[r, hook, shift right, "V"'] & {[(\mathbf{\Delta}')^{\mathrm{op}}, \mathscr{V}]} \arrow[l, shift right, "K"'] \end{tikzcd}$$

is still symmetric monoidal closed, and the tensor product can be calculated as $C \otimes D \cong K(VC \otimes VD)$.

In fact the subcategory of $\mathbf{\Delta}'$ generated by the maps

$$\begin{tikzcd} 1 \arrow[r, shift left=2, "\delta_0"] \arrow[r, shift right=2, "\delta_1"'] & 2 \arrow[l, "\sigma_0"'] \arrow[r, shift left=3, "\delta_0"] \arrow[r, "\delta_1" description] \arrow[r, shift right=3, "\delta_2"'] & 3 \end{tikzcd}$$

is final in $\mathbf{\Delta}$, so that we can calculate the colimit involved in the reflection K using just this smaller subcategory.

An internal double category in $\mathscr{V}$ is an internal category in $\mathbf{Cat}(\mathscr{V})$, so one could treat internal double categories in $\mathscr{V}$ by iterating this process; alternatively, one can simply take $\mathscr{K}$ to be the cartesian closed locally finitely presentable category of double categories. The case of n-tuple categories (multiple/cubical categories) for $n > 2$ is similar.

On the other hand, one could take $\mathscr{K}$ to be the cartesian closed locally finitely presentable category **2-Cat** of 2-categories, and so treat internal 2-categories. A 2-category can be regarded as a double category with no non-identity vertical arrows (in other words, the category of objects is discrete). But there is also another possibility, which is to regard a 2-category as a double category with no non-identity horizontal arrows, and this will turn out to be useful in certain calculations. More formally, let $U : \mathbf{Cat} \to \mathbf{Set}$ be the functor sending a category to its set of objects, and let $\mathbf{Cat}(U) : \mathbf{Cat}(\mathbf{Cat}) \to \mathbf{Cat}(\mathbf{Set}) = \mathbf{Cat}$ be the induced functor sending a double category to its category of horizontal arrows. Finally let $D : \mathbf{Set} \to \mathbf{Cat}$ be the functor sending a set to the discrete category on that set (no non-identity arrows). Then there is a pullback

$$\begin{tikzcd} \mathbf{2\text{-}Cat} \arrow[r, "E"] \arrow[d] & \mathbf{Cat}(\mathbf{Cat}) \arrow[d, "\mathbf{Cat}(U)"] \\ \mathbf{Set} \arrow[r, "D"'] & \mathbf{Cat} \end{tikzcd}$$

in which the horizontal arrows are fully faithful. Moreover E (like D) preserves finite products and has a right adjoint, given by discarding non-identity horizontal arrows. This right adjoint preserves filtered colimits, and so the conditions of Theorem 4.2 are satisfied. This means that the adjunction between $\mathbf{2\text{-}Cat}(\mathscr{V})$ and $\mathbf{Cat}^2(\mathscr{V})$ will be symmetric monoidal closed; in other words, the full subcategory of $\mathbf{Cat}^2(\mathscr{V})$ consisting of the internal 2-categories (seen as double categories with no non-identity horizontal arrows) is closed under the tensor product.

The case of higher categories can be treated in a similar fashion. Let U : $_{(n-1)}$-$\mathbf{Cat} \to$ $_{(n-2)}$-$\mathbf{Cat}$ now be the functor which discards all $(n-1)$-cells, $\mathbf{Cat}(U) : \mathbf{Cat}({}_{(n-1)}\text{-}\mathbf{Cat}) \to$

$\mathbf{Cat}({}_{(n-2)}\text{-}\mathbf{Cat})$ the induced functor, and $D : {}_{(n-2)}\text{-}\mathbf{Cat} \to \mathbf{Cat}({}_{(n-2)}\text{-}\mathbf{Cat})$ the functor which regards each object of ${}_{(n-2)}\text{-}\mathbf{Cat}$ as a discrete category. Then once again we have a pullback

$$\begin{tikzcd} n\text{-}\mathbf{Cat} \arrow[r, "E"] \arrow[d] & \mathbf{Cat}({}_{(n-1)}\text{-}\mathbf{Cat}) \arrow[d, "\mathbf{Cat}(U)"] \\ {}_{(n-2)}\text{-}\mathbf{Cat} \arrow[r, "D"'] & \mathbf{Cat}({}_{(n-2)}\text{-}\mathbf{Cat}) \end{tikzcd}$$

with E a fully faithful finite-product-preserving functor with a filtered-colimit-preserving right adjoint, so that the resulting adjunction satisfies the conditions of Theorem 4.2, and the induced adjunction between $n\text{-}\mathbf{Cat}(\mathscr{V})$ and $\mathbf{Cat}({}_{(n-1)}\text{-}\mathbf{Cat}(\mathscr{V}))$ is symmetric monoidal closed. Explicitly, we are regarding an internal n-category

$$C_n \rightrightarrows C_{n-1} \rightrightarrows \cdots \rightrightarrows C_1 \rightrightarrows C_0$$

in $\mathscr{V}$ as the internal category

$$\begin{tikzcd} C_n \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & C_{n-2} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left, "1"] \arrow[d, shift right, "1"'] & \cdots \arrow[r, shift left] \arrow[r, shift right] & C_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left, "1"] \arrow[d, shift right, "1"'] & C_0 \arrow[d, shift left, "1"] \arrow[d, shift right, "1"'] \\ C_{n-1} \arrow[r, shift left] \arrow[r, shift right] & C_{n-2} \arrow[r, shift left] \arrow[r, shift right] & \cdots \arrow[r, shift left] \arrow[r, shift right] & C_1 \arrow[r, shift left] \arrow[r, shift right] & C_0 \end{tikzcd}$$

in ${}_{(n-1)}\text{-}\mathbf{Cat}(\mathscr{V})$. In particular, Theorem 4.2 tells us that the image of $n\text{-}\mathbf{Cat}(\mathscr{V})$ is closed in $\mathbf{Cat}({}_{(n-1)}\text{-}\mathbf{Cat}(\mathscr{V}))$ under the tensor product.

9 The abelian case

As an application, we consider the case where $\mathscr{V}$ is abelian, which is important in homological and homotopical algebra. For certain $\mathscr{K}$, the category $\mathscr{K}(\mathscr{V})$ is very simple. The basic fact is the Dold-Kan correspondence: this gives an equivalence between the category $[\Delta^{\mathrm{op}}, \mathscr{V}]$ of simplicial objects in $\mathscr{V}$ and the category $\mathbf{Chain}(\mathscr{V})$ of chain complexes in $\mathscr{V}$ (of non-negative degree). The equivalence sends a simplicial object X to the chain complex NX with $(NX)_n$ equal to the intersection of the $d_i : X_n \to X_{n-1}$ for $0 < i \leq n$, and with $\partial : (NX)_n \to (NX)_{n-1}$ given by (the restriction of) $d_0 : X_n \to X_{n-1}$.

9.1 Operadic crossed modules

If $\mathscr{K}$ is $\mathbf{Cat}$, then $\mathscr{K}(\mathscr{V})$ is a full subcategory of $[\Delta^{\mathrm{op}}, \mathscr{V}]$, and so may be regarded as a full subcategory of $\mathbf{Chain}(\mathscr{V})$; explicitly, as the full subcategory of chain complexes which vanish above degree one. In other words, $\mathbf{Cat}(\mathscr{V})$ is equivalent to the category $\mathscr{V}^{\mathbf{2}}$ of arrows in $\mathscr{V}$. The equivalence sends a category $d_0, d_1 : C_1 \to C_0$ to the arrow $\partial : \ker(d_1) \to C_0$, where ∂ is the restriction of $d_0 : C_1 \to C_0$ to the kernel of d_1. The inverse equivalence

sends an arrow $\partial : A \to M$ to the evident category C with object-of-objects $C_0 = M$ and object-of-morphisms $C_1 = A \oplus M$.

Thus $\mathscr{V}^{\mathbf{2}}$ becomes a full subcategory of $[\Delta^{\mathrm{op}}, \mathscr{V}]$ via the nerve functor, sending $\partial : A \to M$ to the evident simplicial object X with $X_n = A^n \oplus M$. This full subcategory is reflective; an easy calculation shows that the reflection sends an arbitrary simplicial object X to $\partial : X_1' \to X_0$, where X_1' is the quotient of X_1 by (the image of) $d_0 - d_1 + d_2 : X_2 \to X_1$, and where $\partial : X_1' \to X_0$ is the map induced by $d_0 - d_1 : X_1 \to X_0$.

The symmetric monoidal structure on $\mathbf{Cat}(\mathscr{V})$ can be transported across the equivalence to give a symmetric monoidal structure on $\mathscr{V}^{\mathbf{2}}$. Since the tensor product of categories is calculated as the reflection into $\mathbf{Cat}(\mathscr{V})$ of the pointwise tensor product of the nerves, the tensor product of $\partial : A \to M$ and $\partial : A' \to M'$ is constructed by first forming the corresponding simplicial objects, then tensoring these pointwise (term-by-term), and finally reflecting the resulting simplicial object into $\mathscr{V}^{\mathbf{2}}$, as in the last paragraph. To simplify the notation we omit the symbol $\otimes$, and write CD for the tensor product C and D. Then the tensor product of $\partial : A \to M$ and $\partial : A' \to M'$ is $\partial : AM' +_{AA'} A'M \to MM'$, where the domain is the pushout of $A\partial : AA' \to AM'$ and $\partial A' : AA' \to A'M$, and where $\partial : AM' +_{AA'} MA' \to MM'$ is the map induced by the universal property of the pushout applied to $\partial M' : AM' \to MM'$ and $M\partial : MA' \to MM'$.

It follows that for an operad T in $\mathscr{V}$ there is an operad $T' = (0 \to T)$ on $\mathscr{V}^{\mathbf{2}}$ whose algebras are the internal categories in $\mathscr{V}^T$; the symmetric monoidal structure on $\mathscr{V}^{\mathbf{2}}$ and the operad T' were described by Baues, Minian, and Richter [2], who called T'-algebras "crossed modules over T", and observed that this gives the usual notion of crossed module when T is the operad for commutative or associative algebras.

9.2 Operadic crossed n-cubes

If $\mathscr{K}$ is the category $\mathbf{Cat}^n$ of n-tuple categories, we may iterate the analysis of the previous section to obtain an equivalence $\mathbf{Cat}^n(\mathscr{V}) \simeq \mathscr{V}^{\mathbf{2}^n}$, and a corresponding operadic description of n-fold categories in $\mathscr{V}^T$ for an operad T in $\mathscr{V}$. Since $\mathscr{V}^{\mathbf{2}^n} \simeq (\mathscr{V}^{\mathbf{2}^{n-1}})^{\mathbf{2}}$, the induced operad $T^{(n)}$ on $\mathscr{V}^{\mathbf{2}^n}$ may be seen as $0 \to T^{(n-1)}$, where $T^{(n-1)}$ is the operad on $\mathscr{V}^{\mathbf{2}^{n-1}}$ whose algebras are internal $(n-1)$-tuple categories in $\mathscr{V}^T$.

In the case where $\mathscr{V}$ is R-**Mod** and T is the operad for commutative or associative algebras, $\mathscr{K}(\mathscr{V}^T)$ is the category of cat^n-algebras, which can also be described as crossed n-cubes of algebras [7].

9.3 Operadic crossed n-complexes

This is the case where $\mathscr{K}$ is the cartesian closed category n-**Cat** of (strict) n-categories. We shall calculate the induced tensor product on n-$\mathbf{Cat}(\mathscr{V})$. As we saw in Section 8, there

is a pullback

$$\begin{tikzcd} n\text{-}\mathbf{Cat}(\mathscr{V}) \arrow[r, "E"] \arrow[d] & \mathbf{Cat}((n-1)\text{-}\mathbf{Cat}(\mathscr{V})) \arrow[d, "\mathbf{Cat}(U)"] \\ (n-2)\text{-}\mathbf{Cat}(\mathscr{V}) \arrow[r, "D"'] & \mathbf{Cat}((n-2)\text{-}\mathbf{Cat}(\mathscr{V})) \end{tikzcd}$$

Since $\mathbf{Cat}(\mathscr{W}) \simeq \mathscr{W}^{\mathbf{2}}$ for an abelian category $\mathscr{W}$, we recover by induction the well-known equivalences $n\text{-}\mathbf{Cat}(\mathscr{V}) \simeq \mathbf{Chain}_n(\mathscr{V})$ between n-categories in $\mathscr{V}$ and chain complexes in $\mathscr{V}$ which vanish above degree n. As observed in Section 8, the map $E : n\text{-}\mathbf{Cat}(\mathscr{V}) \to \mathbf{Cat}((n-1)\text{-}\mathbf{Cat}(\mathscr{V}))$ is fully faithful, and its image is closed under the tensor product, allowing us to construct inductively the tensor product on $n\text{-}\mathbf{Cat}(\mathscr{V})$ for all n. As in the previous section, we omit all tensor products, writing AB for $A \otimes B$, and we write $A'B +_{A'B'} AB'$ for the pushout of $A'\partial : A'B' \to AB$ and $\partial B' : A'B' \to AB'$, when the maps $\partial : A' \to A$ and $\partial : B' \to B$ are understood.

Proposition 9.1 *The tensor product $C \otimes D$ of chain complexes C and D in $\mathbf{Chain}_n(\mathscr{V})$ has*

$$(CD)_i = \operatorname{coker}\big(C_iD_1 \xrightarrow{C_i\partial} C_iD_0 \big) \oplus \operatorname{coker}\big(C_1D_i \xrightarrow{\partial D_i} C_0D_i \big)$$

for $1 < i \leq n$, with $(CD)_0 = C_0D_0$ and $(CD)_1 = C_1D_0 +_{C_1D_1} C_0D_1$. For $0 \leq i < n$, the differential $\partial : (CD)_{i+1} \to (CD)_i$ is the map induced by $\partial D_0 : C_{i+1}D_0 \to C_iD_0$ and $C_0\partial : C_0D_{i+1} \to C_0D_i$.

For computational purposes it is useful to observe that the functors $\mathbf{Chain}_n(\mathscr{V}) \to \mathbf{Chain}_{n-1}(\mathscr{V})$, given by truncation, preserve the tensor product, since the corresponding functors $U : n\text{-}\mathbf{Cat} \to (n-1)\text{-}\mathbf{Cat}$ do so. From the homotopy point of view, the truncation functor is less interesting than the one that quotients out the image of the things in the top dimension; nonetheless, the compatibility of the tensor product with the truncation functors will be used in the following section where we consider chains of arbitrary length.

Suppose now that T is an operad in $\mathscr{V}$. Then $L\Delta T$ is the operad $0 \to 0 \to \ldots \to 0 \to T$ in $\mathbf{Chain}_n(\mathscr{V})$. The category of algebras for $L\Delta T$ is equivalent to $n\text{-}\mathbf{Cat}(\mathscr{V}^T)$. When $\mathscr{V}$ is $R\text{-}\mathbf{Mod}$ and T is the operad for commutative or associative algebras, then $n\text{-}\mathbf{Cat}(\mathscr{V}^T)$ is equivalent to the category of crossed n-complexes of algebras (see [3] for the corresponding statement for n-categories in $\mathbf{Grp}$). Thus we have obtained in these cases an operadic description of crossed n-complexes.

9.4 Operadic crossed complexes

This is the case where $\mathscr{K}$ is the cartesian closed category $\omega\text{-}\mathbf{Cat}$ of (strict) ω-categories. This can be seen as the limit of the diagram

$$\cdots \longrightarrow n\text{-}\mathbf{Cat} \longrightarrow (n-1)\text{-}\mathbf{Cat} \longrightarrow \cdots \longrightarrow (1\text{-}\mathbf{Cat} = \mathbf{Cat}) \longrightarrow 0\text{-}\mathbf{Cat} = \mathbf{Set}$$

while $\mathbf{Chain}(\mathscr{V})$ is the limit of the diagram

$$\cdots \longrightarrow \mathbf{Chain}_n(\mathscr{V}) \longrightarrow \mathbf{Chain}_{n-1}(\mathscr{V}) \longrightarrow \cdots \longrightarrow (\mathbf{Chain}_1(\mathscr{V}) = \mathscr{V}^{\mathbf{2}}) \longrightarrow \mathbf{Chain}_0(\mathscr{V}) = \mathscr{V}$$

and so we recover the well-known equivalence $\omega\text{-}\mathbf{Cat}(\mathscr{V}) \simeq \mathbf{Chain}(\mathscr{V})$. Furthermore, the symmetric monoidal closed structures pass to the limits, giving (equivalent) symmetric monoidal structures on $\omega\text{-}\mathbf{Cat}(\mathscr{V})$ and on $\mathbf{Chain}(\mathscr{V})$. The description of the tensor product on $\mathbf{Chain}(\mathscr{V})$ is as in Proposition 9.1 except there is no upper bound n.

This tensor product is of course different to the "usual" tensor product on $\mathbf{Chain}(\mathscr{V})$, but there is a canonical map from the usual tensor product to this one.

As in the previous example, if T is an operad in $\mathscr{V}$, then $\overline{T} = (\ldots \to 0 \to T)$ is an operad in $\mathbf{Chain}(\mathscr{V})$ whose category of algebras is equivalent to $\omega\text{-}\mathbf{Cat}(\mathscr{V}^T)$. If $\mathscr{V}$ is R-$\mathbf{Mod}$ and T is the operad for commutative or associative algebras, then $\omega\text{-}\mathbf{Cat}(\mathscr{V}^T)$ is equivalent to the category of crossed complexes of algebras [3]. Thus we have an operadic description of crossed complexes of algebras in these cases.

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