2-nerves for bicategories

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Abstract

We describe a **Cat**-valued nerve of bicategories, which associates to every bicategory a simplicial object in **Cat**, called the 2-nerve. This becomes the object part of a 2-functor N : **NHom** $\to [\Delta^{\mathrm{op}}, \mathbf{Cat}]$, where **NHom** is a 2-category whose objects are bicategories and whose 1-cells are normal homomorphisms of bicategories. The 2-functor N is fully faithful and has a left biadjoint, and we characterize its image. The 2-nerve of a bicategory is always a weak 2-category in the sense of Tamsamani, and we show that **NHom** is biequivalent to a certain 2-category whose objects are Tamsamani weak 2-categories.

This paper concerns a notion of "2-nerve", or **Cat**-valued nerve, of bicategories.

To every category, one can associate its *nerve*; this is the simplicial set whose 0-simplices are the objects, whose 1-simplices are the morphisms, and whose n-simplices are the composable n-tuples of morphisms. The face maps encode the domains and codomains of morphisms, the composition law, and the associativity property, while the degeneracies record information about the identities.

This construction is the object part of a functor $N : \mathbf{Cat_1} \to [\Delta^{\mathrm{op}}, \mathbf{Set}]$ from the category of categories and functors, to the category of simplicial sets. This functor is fully faithful and has a left adjoint. It arises in a natural way, as the "singular functor" (see Section 1 below) of the inclusion $J : \Delta \to \mathbf{Cat_1}$ in $\mathbf{Cat_1}$ of the full subcategory Δ consisting of the non-empty finite ordinals. One can characterize the simplicial sets which lie in the image of the nerve functor as those for which certain diagrams are pullbacks.

As observed by Street, one may define the nerve of a bicategory as the simplicial set whose 0-simplices are the objects, whose 1-simplices are the morphisms, whose 2-simplices consist of a composable pair f and g and a 2-cell $gf \to h$, and so on. In this way, the category $\mathbf{Bicat_1}$ of bicategories and *normal lax functors* becomes a full subcategory of $[\Delta^{\mathrm{op}}, \mathbf{Set}]$; here a normal lax functor preserves the identities strictly, but preserves composition only up to coherent, but not necessarily invertible, comparison maps. In the important special case of invertible comparison maps, one speaks rather of *normal homomorphisms*. Once again this nerve functor $\mathbf{Bicat_1} \to [\Delta^{\mathrm{op}}, \mathbf{Set}]$ is a singular functor, this time of the inclusion $\Delta \to \mathbf{Bicat_1}$, where the non-empty finite ordinals are now seen as locally discrete (no non-identity 2-cells) bicategories. The image of this nerve functor was characterized explicitly in [4].

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In this construction, the 2-simplices are playing a double role: they encode both the 2-cells of the bicategory, and (indirectly) the composition of 1-cells. It is essentially for this reason that the normal lax functors arise. In order to obtain a tighter control over the composition of 1-cells, and in particular to extract the normal homomorphisms, one could specify, as extra structure, which 2-simplices contain not just any 2-cell but an invertible one. This gives rise to a structure called a *stratified* simplicial set, which goes back to Roberts; see [20] for a full account, and a proof of the celebrated Street-Roberts conjecture, characterizing the nerves of ω-categories.

Here we take a different approach. We consider only those 2-simplices in which the 2-cell is invertible; in order to compensate for this, however, we include arbitrary 2-cells as *morphisms* in a *category* of 1-simplices, and similarly there are categories of n-simplices for all higher n. The 2-nerve of a bicategory is then a functor $X : \Delta^{\mathrm{op}} \to \mathbf{Cat_1}$, with X_0 the discrete (no non-identity 1-cells) category of objects of the bicategory; with X_1 the category whose objects are the morphisms of the bicategory, and whose morphisms are the 2-cells; and with X_2 a category whose objects consist of composable pairs (f, g) with an *invertible* 2-cell $gf \to h$. The resulting 2-nerve construction is reminiscent of the *homotopy coherent nerve* of [3].

In order to describe this construction in terms of a singular functor, one needs to regard bicategories as objects of a *2-category*. The morphisms are the normal homomorphisms of bicategories, while the 2-cells are things we call *icons*: these are oplax natural transformations $\alpha : F \to G : \mathscr{A} \to \mathscr{B}$ between normal homomorphisms, in which the component $FA \to GA$ is an identity, for each object A of $\mathscr{A}$. We call the resulting 2-category **NHom**. Once again, there is a fully faithful inclusion $J : \Delta \to \mathbf{NHom}$, and the resulting nerve 2-functor $N : \mathbf{NHom} \to [\Delta^{\mathrm{op}}, \mathbf{Cat}]$ is fully faithful (on both 1-cells and 2-cells). This N does not have an adjoint in the ordinary strict sense, but it does have a left *biadjoint*, which is the most common situation for 2-categories. Similarly **NHom** does not have ordinary limits and colimits, but it does have the bicategorical ones of [18]. These bicategorical results are obtained using the techniques of 2-dimensional universal algebra, as in [2]. The image of N can also be characterized, although this is somewhat more complicated than in the case of ordinary categories.

There are also variants **Hom**, $\mathbf{2\text{-}Cat_{ps}}$, and $\mathbf{2\text{-}Cat_{nps}}$, of **NHom** in which one either generalizes from normal homomorphisms to homomorphisms, or specializes from bicategories to 2-categories, or both. All of these 2-categories are biequivalent. In particular, every bicategory is equivalent in **NHom** to a 2-category. (A normal homomorphism is an equivalence in **NHom** if and only if it is bijective on objects and induces equivalences of hom-categories.)

We have chosen to take **NHom** as basic, privileging normal homomorphisms over arbitrary homomorphisms. One reason for this is that it is straightforward to "normalize" an arbitrary homomorphism so as to obtain a normal one; indeed this arises in the fact, mentioned in the previous paragraph, that **NHom** is biequivalent to **Hom**. A second reason is that when we define our 2-nerves using $J : \Delta \to \mathbf{NHom}$, the resulting 2-nerves have a discrete category of 0-simplices, which is part of Tamsamani's notion of weak 2-category, mentioned below. If instead we had defined our 2-nerves using the inclusion $\Delta \to \mathbf{Hom}$, the resulting categories of 0-simplices would only have been equivalent to discrete categories, not, in general, discrete. Our results could be adapted to that setting, but we find it easier to restrict to the normal homomorphisms. Of course the composite $\mathbf{Hom} \to \mathbf{NHom} \to [\Delta^{\mathrm{op}}, \mathbf{Cat}]$ is still fully faithful in the bicategorical sense: it induces equivalences (not isomorphisms) of hom-categories.

Tamsamani [19] and Simpson [17] have each defined notions of weak 2-category as certain objects of $[\Delta^{\mathrm{op}}, \mathbf{Cat}]$; these notions then determine full sub-2-categories **Tam** and **Simpson**, both

containing the image of **NHom**. The left biadjoint of the 2-nerve construction induces left biadjoints to the fully faithful inclusions $\mathbf{NHom} \to \mathbf{Tam}$ and $\mathbf{NHom} \to \mathbf{Simpson}$, but these inclusions actually have 2-adjoints. The counits of these 2-adjunctions are invertible, while the units have components which are "pointwise equivalences": so for example if $X \in \mathbf{Tam}$, and $j : X \to NGX$ is the component at X of the unit, then for each $[n] \in \Delta$, the functor $j_n : X_n \to (NGX)_n$ is an equivalence of categories. A pointwise equivalence is not necessarily an equivalence, but it is always a "weak equivalence"; this is enough to guarantee that if one "localizes the weak equivalences", then **NHom**, **Tam**, and **Simpson** all become equivalent to the homotopy category of the Quillen model category **Bicat** of [12]. Rather than localizing, an alternative is to expand the notion of morphism of Tamsamani 2-categories: if one allows not just 2-natural transformations, but pseudonatural ones, then the resulting 2-category $\mathbf{Tam_{ps}}$ is in fact biequivalent to **NHom**.

One of the motivations for this work was to determine the precise relationship between bicategories and Tamsamani's weak 2-categories, which was only very partially sketched in [19]. In fact our construction of the 2-nerve of a bicategory differs from that of [19]: see Remark 3.3 below. On the other hand the bicategory we associate to a Tamsamani weak 2-category is the same as the one constructed in [19], however, unlike [19], we describe the functoriality of the construction. This last point was also worked out in [15], using a slightly different approach from that adopted here.

In Section 1 we describe two basic technical tools: singular functors and coskeleta. In Section 2 we recall the basic facts about nerves of categories, while in Section 3 we turn to 2-nerves. The next two sections are not needed for the rest of the paper: the first describes our basic 2-category **NHom** of bicategories using 2-monads, and deduces various useful things about it, while the second studies the biequivalence between **NHom** and various related 2-categories; in particular, we see that every bicategory is equivalent in **NHom** to a 2-category. In Section 6 we study various properties of functors $\Delta^{\mathrm{op}} \to \mathbf{Cat}$ which are 2-nerves of bicategories, which leads to a characterization theorem in Section 7, where we also establish the 2-adjunction between **NHom** and the 2-category of Tamsamani weak 2-categories, and the precise relationship between these two structures. In particular, we show that **NHom** is biequivalent to $\mathbf{Tam_{ps}}$, the 2-category of Tamsamani weak 2-categories and pseudonatural morphisms.

The basic references for bicategories and 2-categories are still [1, 10, 18].

1 Singular functors and coskeleta

In this section we briefly recall some standard material on singular functors and on coskeleta.

The results on singular functors are stated in terms of $\mathscr{V}$-categories, for a symmetric monoidal closed $\mathscr{V}$ which is complete and cocomplete. The only cases needed will be the case $\mathscr{V} = \mathbf{Set}$ of ordinary categories, and the case $\mathscr{V} = \mathbf{Cat}$ of 2-categories. With the exception of the second sentence of Proposition 1.1, everything here can be found in [8, Chapter 5].

Let $F : \mathscr{A} \to \mathscr{B}$ be a functor with small domain. There is an induced functor $\mathscr{B}(F, 1) : \mathscr{B} \to [\mathscr{A}^{\mathrm{op}}, \mathscr{V}]$ sending an object B of $\mathscr{B}$ to the functor $\mathscr{B}(F-, B) : \mathscr{A}^{\mathrm{op}} \to \mathscr{V}$, where $\mathscr{B}(F-, B)$ sends an object A to the hom-object $\mathscr{B}(FA, B)$. This functor $\mathscr{B}(F, 1)$ is sometimes called the *singular functor* of F, and it may be obtained as the composite of the Yoneda embedding $\mathscr{B} \to [\mathscr{B}^{\mathrm{op}}, \mathscr{V}]$ followed by the functor $[\mathscr{B}^{\mathrm{op}}, \mathscr{V}] \to [\mathscr{A}^{\mathrm{op}}, \mathscr{V}]$ given by restriction along F.

The basic examples of such an F will be the functor $J : \Delta \to \mathbf{Cat_1}$ (where $\mathscr{V} = \mathbf{Set}$) and the 2-functor $J : \Delta \to \mathbf{NHom}$ (where $\mathscr{V} = \mathbf{Cat}$). The resulting singular functors are then the nerve construction for categories and the 2-nerve construction for bicategories.

When $\mathscr{B}(F,1)$ is fully faithful, the functor F is said to be dense. When F is itself fully faithful, there is a characterization of when F is dense, involving $\mathscr{B}$ being generated under colimits by $\mathscr{A}$, but we shall not need this characterization.

As observed by Kan, $\mathscr{B}(F,1)$ has a left adjoint provided that $\mathscr{B}$ is cocomplete; the left adjoint can then be constructed as the left Kan extension of F along the Yoneda embedding. It sends a presheaf $X : \mathscr{A}^{\mathrm{op}} \to \mathscr{V}$ to the weighted colimit $X * F$, which may be given by the coend

$$\int^{A} XA \cdot FA,$$

or, if $\mathscr{V} = \mathbf{Set}$, by the colimit of the functor

$$\mathrm{el}(X)^{\mathrm{op}} \to \mathscr{A} \to \mathscr{B}.$$

We record for future reference the following, of which the first sentence is [8, Theorem 5.13], and the second an easy consequence.

Proposition 1.1 *If $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$, with G fully faithful, then G is dense provided that GF is so, and then the identity $GF = GF$ exhibits G as the left Kan extension of GF along F. Furthermore, the singular functor $\mathscr{C}(G,1) : \mathscr{C} \to [\mathscr{B}^{\mathrm{op}}, \mathscr{V}]$ can then be obtained by first applying the singular functor $\mathscr{C}(GF,1) : \mathscr{C} \to [\mathscr{A}^{\mathrm{op}}, \mathscr{V}]$ and then right Kan extending along $F : \mathscr{A}^{\mathrm{op}} \to \mathscr{B}^{\mathrm{op}}$.*

Proof: For the second sentence observe that $\mathrm{Ran}_F \mathscr{C}(GF, C) \cong \mathscr{C}(\mathrm{Lan}_F GF, C)$. $\square$

The results on coskeleta are stated in terms of simplicial objects $[\Delta^{\mathrm{op}}, \mathbb{E}]$ in a category $\mathbb{E}$ with finite limits. Once again, the main cases will be $\mathbb{E} = \mathbf{Set}$ and $\mathbb{E} = \mathbf{Cat_1}$. Let Δ_n be the full subcategory of Δ consisting of all objects $[m]$ with $m \leq n$, and $H_n : \Delta_n \to \Delta$ the inclusion. The restriction along H_n gives a functor $[\Delta^{\mathrm{op}}, \mathbb{E}] \to [\Delta_n^{\mathrm{op}}, \mathbb{E}]$ which has a right adjoint R_n given by right Kan extension along H_n. Since H_n is fully faithful, so is R_n, and so the counit of the adjunction may be taken to be an identity. For a simplicial object X we write $\mathrm{Cosk}_n X$ for the right Kan extension of its restriction along H_n, and $c : X \to \mathrm{Cosk}_n X$ for the unit map. Then $\mathrm{Cosk}_n X$ is called the n-coskeleton of X, and X is said to be n-coskeletal if h^n is invertible. We shall be particularly interested in the maps $c_n : X_n \to (\mathrm{Cosk}_1 X)_n$.

We shall see that the nerve of a category is always 2-coskeletal, while the 2-nerve of a bicategory is always 3-coskeletal.

2 Nerves of categories

In this section we briefly recall some standard material on nerves of categories.

We write Δ for the category of finite non-empty ordinals and order-preserving maps; as usual we write $[n]$ for the ordinal $\{0 < 1 < \ldots < n\}$. Each ordinal can be seen as a category, and this provides a fully faithful inclusion functor $J : \Delta \to \mathbf{Cat}_1$ of Δ in the category $\mathbf{Cat}_1$ of categories and functors.

A simplicial set is a presheaf $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$. We follow the usual practice of writing X_n for the image under X of $[n]$.

The singular functor of the inclusion $J : \Delta \to \mathbf{Cat_1}$ is the functor $N : \mathbf{Cat}_1 \to [\Delta^{\mathrm{op}}, \mathbf{Set}]$ sending a category C to its *nerve* $NC = \mathbf{Cat}_1(J-, C)$, where $\mathbf{Cat}_1(J-, C)$ is the functor sending

an ordinal $[n]$ to the set $\mathbf{Cat}_1(J[n], C)$ of all functors from $[n]$ to C; in other words, to the set C_n of composable n-tuples of morphisms in C (with a "0-tuple" understood to mean just an object). In particular there are maps $d_0, d_1 : C_1 \to C_0$ sending a morphism to, respectively, its codomain and its domain. The maps $d_0, d_2 : C_2 \to C_1$ are the projections, while $d_1 : C_2 \to C_1$ is given by composition.

Given any simplicial object $X : \Delta^{\mathrm{op}} \to \mathbb{E}$ in a category $\mathbb{E}$ with finite limits, we can form the pullback $X_1 \times_{X_0} X_1$ of $d_0, d_1 : X_1 \to X_0$, and then since $d_0 d_2 = d_1 d_0$ there is a map $S_2 : X_2 \to X_1 \times_{X_0} X_1$ induced by d_0 and d_2. The reason for the letter "S" is that this map, and the S_n described below have often been called "Segal maps", since this approach to coherence goes back to [16]. Of course there is some ambiguity in the notation $X_1 \times_{X_0} X_1$, since the maps $X_1 \to X_0$ involved are not recorded. We exacerbate this, by abbreviating $X_1 \times_{X_0} X_1$ to X_1^2 (this will *always* denote the pullback constructed in this way). Similarly, $X_m \times_{X_0} X_n$ will denote the pullback of $d_m : X_m \to X_0$ and $d_0 : X_n \to X_0$.

A special feature of the simplicial sets which are nerves of categories is that the map $S_2 : X_2 \to X_1^2$ is invertible; this is precisely the fact that X_2 is the set of composable pairs. Similarly, for an arbitrary simplicial object X, we write X_1^n for the limit of the diagram

$$\begin{tikzcd}
X_1 \arrow[dr, "d_1"'] & & X_1 \arrow[dl, "d_0"] \arrow[dr, "d_1"'] & & \cdots & & X_1 \arrow[dl, "d_0"] \arrow[dr, "d_1"'] & & X_1 \arrow[dl, "d_0"] \\
& X_0 & & X_0 & & X_0 & & X_0 &
\end{tikzcd}$$

in which there are n copies of X_1, and $S_n : X_n \to X_1^n$ for the evident induced map. Once again, for the nerve of a category, this S_n is invertible.

It is a straightforward but important calculation that the functor $N : \mathbf{Cat}_1 \to [\Delta^{\mathrm{op}}, \mathbf{Set}]$ is fully faithful. (For example, the fact that a functor preserves composition is encoded in the fact of naturality with respect to the map $\delta_1 : [1] \to [2]$ whose image in a simplicial set is $d_1 : X_2 \to X_1$.) The fact that the nerve functor is fully faithful may alternatively be expressed by saying that $J : \Delta \to \mathbf{Cat}_1$ is *dense*; but it is probably easiest to check the fully faithfulness directly.

There are also smaller subcategories of $\mathbf{Cat_1}$ which are dense. Let Δ_c denote the subcategory Δ generated by the objects $[0]$, $[1]$, and $[2]$, and the morphisms $\delta_0, \delta_1 : [0] \to [1]$, $\sigma_0 : [1] \to [0]$, and $\delta_0, \delta_1, \delta_2 : [1] \to [2]$ (but not the degeneracy maps $[2] \to [1]$. Write J_c for the (non-full) inclusion $\Delta_c \to \mathbf{Cat_1}$. Then J_c induces a functor $N_c : \mathbf{Cat_1} \to [\Delta_c^{\mathrm{op}}, \mathbf{Set}]$ sending a category C to the restriction of its nerve NC to Δ_c^{op}, and once again this functor N_c is fully faithful. It follows, by Proposition 1.1, that the nerve of a category C is the right Kan extension along the inclusion $\Delta_c^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ of $N_c C$.

Similarly, if Δ_2 is the full subcategory of Δ containing the objects $[0]$, $[1]$, and $[2]$, then the functor $N_2 : \mathbf{Cat_1} \to [\Delta_2^{\mathrm{op}}, \mathbf{Set}]$, given by the nerve followed by restriction to Δ_2^{op}, is fully faithful. Furthermore, by Proposition 1.1 once again, N itself can be recovered as the composite

$$\mathbf{Cat_1} \xrightarrow{N_2} [\Delta_2^{\mathrm{op}}, \mathbf{Set}] \xrightarrow{\mathrm{Ran}_H} [\Delta^{\mathrm{op}}, \mathbf{Set}]$$

where Ran_H is given by right Kan extension along the inclusion $H : \Delta_2^{\mathrm{op}} \to \Delta^{\mathrm{op}}$. Thus the nerve of a category is always 2-coskeletal (see Section 1).

A simplicial set is the nerve of a category if and only if $S_n : X_n \to X_1^n$ is invertible for all $n > 1$. (Under the reasonable definition of S_1 and S_0, these last are always invertible.)

Since $\mathbf{Cat}_1$ is cocomplete, it follows for general reasons (see Section 1) that the nerve functor has a left adjoint, which sends a simplicial set X to the weighted colimit $X * J$. Explicitly, the objects of $X * J$ are the elements of X_0, while the morphisms are generated by the elements of X_1 subject to relations encoded in X_2; the higher simplices are not needed to calculate $X * J$, essentially because nerves of categories are 2-coskeletal.

3 The 2-nerve construction

We now turn to the case of bicategories. Every category may be seen as a locally discrete bicategory (that is, a bicategory in which the only 2-cells are identities). As observed in the introduction, if $\mathbf{Bicat_1}$ denotes the category of bicategories and normal lax functors, then the (fully faithful) inclusion $H : \Delta \to \mathbf{Bicat_1}$ induces a fully faithful map $\mathbf{Bicat_1}(H, 1) : \mathbf{Bicat_1} \to [\Delta^{\mathrm{op}}, \mathbf{Set}]$ (and so is dense). This construction might be called the 1-nerve of the bicategory.

Instead, we shall describe a 2-nerve construction. This requires a 2-category $\mathbf{NHom}$ of bicategories. An object of the 2-category will be a bicategory, and a morphism will be a normal homomorphism. Given normal homomorphisms $F, G : \mathscr{A} \to \mathscr{B}$, there can be a 2-cell from F to G only if F and G agree on objects; a 2-cell then consists of a 2-cell $\alpha f : Ff \to Gf$ in $\mathscr{B}$ for every 1-cell $f : A \to B$ in $\mathscr{A}$, subject to the following three conditions. First of all, the αf must be natural in f, in the sense that if $\rho : f \to g$ is a 2-cell in $\mathscr{A}$, then $\alpha g.F\rho = G\rho.\alpha f$. Secondly αf must be an identity 2-cell if f is an identity 1-cell. Thirdly, if $f : A \to B$ and $g : B \to C$ constitute a composable pair in $\mathscr{A}$, then the diagram

$$\begin{tikzcd} Fg.Ff \arrow[r, "\alpha g.\alpha f"] \arrow[d, "\varphi_{f,g}"'] & Gg.Gf \arrow[d, "\psi_{f,g}"] \\ F(gf) \arrow[r, "\alpha(gf)"'] & G(gf) \end{tikzcd}$$

of 2-cells in $\mathscr{B}$ must commute, where φ and ψ are the pseudofunctoriality isomorphisms for F and G. Such a 2-cell is called an *icon*, since it is precisely an Identity Component Oplax Natural transformation from F to G — the α are the 2-cells expressing the oplax naturality of the identity maps $FA \to GA$.

Remark 3.1 In the important special case where the bicategories $\mathscr{A}$ and $\mathscr{B}$ have only one object, so that they may be regarded as monoidal categories, with F and G then becoming strong monoidal functors, an icon is precisely a monoidal natural transformation.

Remark 3.2 As pointed out to us by Bob Paré, the 2-category $\mathbf{NHom}$ can be seen as living within the 2-category $\mathbf{LxDbl}$ of pseudo double categories, lax double functors, and horizontal transformations, studied by Grandis and Paré in [6]. From this point of view, it is the restriction from pseudo double categories to bicategories (seen as pseudo double categories in which all horizontal arrows are identities) that leads to the restriction to transformations whose components are identities. See the last paragraph of [6, Section 2.2].

Every category can be seen as a locally discrete bicategory — that is, a bicategory with no non-identity 2-cells. Seen in this way, $\mathbf{Cat_1}$ becomes a full sub-2-category of $\mathbf{NHom}$— a normal

homomorphism between locally discrete bicategories is just a functor between the corresponding categories, and there are no non-identity icons between such normal homomorphisms.

Thus we can in turn regard Δ as a full sub-2-category of $\mathbf{NHom}$, once again there are no non-identity 2-cells. It is this fully faithful inclusion $J : \Delta \to \mathbf{NHom}$ whose singular 2-functor $N = \mathbf{NHom}(J, 1) : \mathbf{NHom} \to [\Delta^{\mathrm{op}}, \mathbf{Cat}]$ gives our 2-nerve construction.

Remark 3.3 This is not the same as the construction described by Tamsamani on page 54 of [19]: his construction has nothing corresponding to the coherence condition for normal homomorphisms $[n] \to \mathscr{B}$.

One of the main results of the paper will be Theorem 3.7 below, which states that the 2-nerve 2-functor $N : \mathbf{NHom} \to [\Delta^{\mathrm{op}}, \mathbf{Cat}]$ is fully faithful.

Just as for categories, it is not necessary to use all of Δ. Write Δ_b for the sub-category of Δ generated by the objects [0], [1], and [2], and all morphisms between them, as well as the object [3] and the four maps $\delta_i : [2] \to [3]$. In fact we shall see that the inclusion $H : \Delta_b \to \mathbf{NHom}$ is also dense, so that the induced $N_b = \mathbf{NHom}(H, 1) : \mathbf{NHom} \to [\Delta_b^{\mathrm{op}}, \mathbf{Cat}]$ is fully faithful; and by Proposition 1.1 this will imply that $N : \mathbf{NHom} \to [\Delta^{\mathrm{op}}, \mathbf{Cat}]$ is fully faithful; it will likewise imply that the corresponding 2-functor $\mathbf{NHom} \to [\Delta_3^{\mathrm{op}}, \mathbf{Cat}]$ is fully faithful, where now Δ_3^{op} is the full subcategory of Δ^{op} consisting of all objects $[n]$ with $n \leq 3$.

As a first step to proving that N_b is fully faithful, we describe a little more explicitly the 2-nerve 2-functor. We write $\mathscr{B}_n$ for the category $\mathbf{NHom}([n], \mathscr{B})$ of n-simplices of the 2-nerve $N\mathscr{B}$ of $\mathscr{B}$.

For an ordinal $[n]$ and a bicategory $\mathscr{B}$, a normal homomorphism $[n] \to \mathscr{B}$ consists of the following data in $\mathscr{B}$

- an object B_i for each $i \in [n]$
- a morphism $b_{ij} : B_i \to B_j$ for each $i, j \in [n]$ with $i < j$
- an invertible 2-cell $\beta_{ijk} : b_{jk}b_{ij} \cong b_{ik}$ for each $i, j, k \in [n]$ with $i < j < k$

subject to the condition that the diagram

$$\begin{tikzcd}
b_{kl}(b_{jk}b_{ij}) \arrow[r, "b_{kl}\beta_{ijk}"] \arrow[dd, "\alpha"'] & b_{kl}b_{ik} \arrow[dr, "\beta ikl"] & \\
 & & b_{il} \\
(b_{kl}b_{jk})b_{ij} \arrow[r, "\beta_{jkl}b_{ij}"'] & b_{jl}b_{ij} \arrow[ur, "\beta_{ijl}"'] &
\end{tikzcd}$$

commutes for all $i, j, k, l \in [n]$ with $i < j < k < l$.

Given another such normal homomorphism $(C, c, \gamma) : [n] \to \mathscr{B}$, an icon $(B, b, \beta) \to (C, c, \gamma)$ consists of

- satisfaction of the equation $B_i = C_i$ for each $i \in [n]$
- a 2-cell $\varphi_{ij} : b_{ij} \to c_{ij}$ for each $i, j \in [n]$, $i < j$

such that the diagram

$$\begin{tikzcd}
b_{jk}b_{ij} \arrow[r, "\beta_{ijk}"] \arrow[d, "\varphi_{jk}\varphi_{ij}"'] & b_{ik} \arrow[d, "\varphi_{ik}"] \\
c_{jk}c_{ij} \arrow[r, "\gamma_{ijk}"'] & c_{ik}
\end{tikzcd}$$

commutes for all $i, j, k \in [n]$ with $i < j < k$.

Suppose now that $X : \Delta_b^{\mathrm{op}} \to \mathbf{Cat}$ has X_0 discrete and that $\mathscr{B}$ is a bicategory. We consider what it is to give a morphism $F : X \to N_b\mathscr{B}$ in $[\Delta_b^{\mathrm{op}}, \mathbf{Cat}]$. If x and y are in X_0, we write $X(x, y)$ for the fibre over (x, y) of the map $X_1 \to X_0 \times X_0$ induced by d_1 and d_0.

The category $\mathscr{B}_0$ is discrete; its objects are the objects of $\mathscr{B}$ and it has no non-identity morphisms, thus F_0 simply assigns to each $x \in X_0$ an object Fx of $\mathscr{B}$.

An object of $\mathscr{B}_1$ is a morphism of $\mathscr{B}$, while a morphism of $\mathscr{B}_1$ is a 2-cell in $\mathscr{B}$. Thus $\mathscr{B}_1$ is the coproduct of the hom-categories $\mathscr{B}(A, B)$ as A and B range over all the objects of $\mathscr{B}$. The face maps $d_0, d_1 : \mathscr{B}_1 \to \mathscr{B}_0$ give the codomain and the domain objects of a 1-cell or 2-cell. The degeneracy map $s_0 : \mathscr{B}_0 \to \mathscr{B}_1$ sends an object to the identity 1-cell on the object. Thus to give $F_1 : X_1 \to \mathscr{B}_1$, compatible with the face maps d_0 and d_1, is to give a functor $F : X(x, y) \to \mathscr{B}(Fx, Fy)$ for all $x, y \in X_0$.

An object of $\mathscr{B}_2$ consists of morphisms $b_{01} : B_0 \to B_1$, $b_{12} : B_1 \to B_2$, and $b_{02} : B_0 \to B_2$, and an invertible 2-cell $\beta : b_{12}b_{01} \to b_{02}$. A morphism of $\mathscr{B}_2$ consists of three 2-cells of $\mathscr{B}$, satisfying the coherence condition given above. The face maps $d_0, d_1, d_2 : \mathscr{B}_2 \to \mathscr{B}_1$ pick out the three sides of the 2-simplex (d_i picks out the map b_{jk}, where i, j, and k are distinct). The degeneracy maps $s_0, s_1 : \mathscr{B}_1 \to \mathscr{B}_2$ send a 1-cell $b : B \to B'$ to the 2-simplices defined using the identity isomorphisms $1_{B'}f \cong f$ and $f1_B \cong f$. Thus to give $F_2 : X_2 \to \mathscr{B}_2$ compatible with the face maps is to give, for each object ξ of X_2, an invertible 2-cell $F_2\xi : Fd_0\xi.Fd_2\xi \cong Fd_1\xi$ in $\mathscr{B}$, as in

$$\begin{tikzcd}
 & Fd_0d_2\xi \arrow[dr, "Fd_0\xi"] \arrow[d, phantom, "\Downarrow F_2\xi"] & \\
Fd_1d_2\xi \arrow[ur, "Fd_2\xi"] \arrow[rr, "Fd_1\xi"'] & {} & Fd_0d_0\xi
\end{tikzcd} \tag{1}$$

natural with respect to the 1-cells in X_2. Compatibility with respect to the degeneracy maps asserts that F_2s_0f and F_2s_1f are the 2-simplices arising from the identity isomorphisms $1.Ff \cong Ff \cong Ff.1$ in $\mathscr{B}$.

Finally to give $F_3 : X_3 \to \mathscr{B}_3$, compatible with the degeneracy maps, is to assert that for each object Ξ of X_3, the diagram

$$\begin{tikzcd}
Fx_{23}(Fx_{12}.Fx_{01}) \arrow[r, "Fx_{23}.F_2\xi_{012}"] \arrow[dd, "\alpha"'] & Fx_{23}.Fx_{02} \arrow[dr, "F_2\xi_{023}"] & \\
 & & Fx_{03} \\
(Fx_{23}.Fx_{12})Fx_{01} \arrow[r, "F_2\xi_{123}.Fx_{01}"'] & Fx_{13}.Fx_{01} \arrow[ur, "F_2\xi_{013}"'] &
\end{tikzcd} \tag{2}$$

in $\mathscr{B}$ commutes, where x_{ij} is $d_k d_l \Xi$ for a suitable choice of k and l, and ξ_{ijk} is $d_l \Xi$ for a suitable l, while α is the associativity isomorphism.

We record this as:

Proposition 3.4 *If $X : \Delta_b^{op} \to \mathbf{Cat}$ has X_0 discrete, and $\mathscr{B}$ is a bicategory, then to give a morphism $F : X \to N_b\mathscr{B}$ in $[\Delta_b^{op}, \mathbf{Cat}]$ is to give (i) an object Fx of $\mathscr{B}$ for each $x \in X_0$, (ii) a functor $F : X(x, y) \to \mathscr{B}(Fx, Fy)$ for each $x, y \in X_0$, with $Fs_0x = 1_{Fx}$, (iii) an invertible 2-cell $F_2\xi : Fd_0\xi.Fd_2\xi \cong Fd_1\xi$, as in* (1) *above, for each object $\xi \in X_2$, such that (iv) $F_2\xi$ is natural in ξ, (v) the $F_2 s_i \xi$ are the identity isomorphisms, and (vi)* (2) *holds for all $\Xi \in X_3$.*

Suppose now that $F, G : X \to N_b\mathscr{B}$ are two morphisms in $[\Delta_b^{\mathrm{op}}, \mathbf{Cat}]$. What is it to give a 2-cell (modification) between them? A similar analysis to that above gives:

Proposition 3.5 *If $X : \Delta_b^{op} \to \mathbf{Cat}$ with X_0 discrete, $\mathscr{B}$ is a bicategory, and $F, G : X \to N_b\mathscr{B}$, then to give a 2-cell $F \to G$ is (i) to assert that $Fx = Gx$ for all $x \in X_0$, (ii) to give a 2-cell $\varphi f : Ff \to Gf$ in $\mathscr{B}$, for every $f \in X_1$, such that (iii) φf is natural in f, (iv) $\varphi s_0 x$ is an identity 2-cell for each $x \in X_0$, and the diagram*

$$\begin{tikzcd}
Fd_0\xi.Fd_2\xi \arrow[r, "\varphi d_0\xi.\varphi d_2\xi"] \arrow[d, "F_2\xi"'] & Gd_0\xi.Gd_2\xi \arrow[d, "G_2\xi"] \\
Fd_1\xi \arrow[r, "\varphi d_1\xi"'] & Gd_1\xi
\end{tikzcd}$$

of 2-cells in $\mathscr{B}$ commutes for all objects ξ of X_2.

We now specialize the last two propositions to the case where X too has the form $N_b\mathscr{A}$ for a bicategory $\mathscr{A}$. In the resulting description of a morphism $N_b\mathscr{A} \to N_b\mathscr{B}$, we see that (i) amounts to the assignment of an object FA of $\mathscr{B}$ for each object A of $\mathscr{A}$, and (ii) amounts to a functor $F : \mathscr{A}(A, B) \to \mathscr{B}(FA, FB)$ for all objects A and B, with $F1_A = 1_{FA}$ for all A. If $f : A \to B$ and $g : B \to C$ are 1-cells in $\mathscr{A}$, then the identity 1-cell on gf determines a 2-simplex $\xi \in N_b\mathscr{A}_2$ with faces f, g, and gf, and then $F_2\xi$, for this ξ, is an invertible 2-cell $\varphi_{f,g} : Fg.Ff \cong F(gf)$. Essentially by the Yoneda lemma, to give the $F_2\xi$ as in (iii) satisfying naturality as in (iv) is just to give such $\varphi_{f,g}$, natural in f and g; then a general 2-simplex $(f, g, \xi : gf \to h)$ must be sent to the 2-simplex $(Ff, Fg, F_2\xi : Fg.Ff \to Fh)$, where $F_2\xi$ is now the composite

$$Fg.Ff \xrightarrow{\varphi_{f,g}} F(gf) \xrightarrow{F\xi} Fh.$$

Finally (v) and (vi) assert precisely that these φ satisfy the normalization and 3-cocycle conditions to make F into a normal homomorphisms from $\mathscr{A}$ to $\mathscr{B}$. Similarly, if $G : N_b\mathscr{A} \to N_b\mathscr{B}$ is another morphism in $[\Delta_b^{\mathrm{op}}, \mathbf{Cat}]$, then to give a 2-cell $F \to G$ is precisely to give an icon between the corresponding normal homomorphisms. This proves:

Theorem 3.6 *The 2-functor $N_b : \mathbf{NHom} \to [\Delta_b^{op}, \mathbf{Cat}]$ is fully faithful, or equivalently the inclusion $\Delta_b \to \mathbf{NHom}$ is dense.*

As an immediate consequence we have, by Proposition 1.1:

Theorem 3.7 *The 2-nerve 2-functor $N : \mathbf{NHom} \to [\Delta^{op}, \mathbf{Cat}]$ is fully faithful, or equivalently the inclusion $\Delta \to \mathbf{NHom}$ is dense.*

4 The 2-dimensional universal algebra point of view

As we said in the introduction, the following two sections are not needed in the rest of the paper, and can be omitted on a first reading.

A **Cat**-graph [21] has objects $X, Y, Z, \ldots$, with "hom-categories" $\mathscr{G}(X,Y)$ for each pair of objects X and Y. With the obvious notion of morphism, this defines a category, which is in fact locally finitely presentable. Here, however, we want to make it into a 2-category. Given **Cat**-graph morphisms $M, N : \mathscr{G} \to \mathscr{H}$, a 2-cell $M \to N$ exists only if M and N agree on objects, in which case it consists of a natural transformation

$$\begin{tikzcd} \mathscr{G}(X,Y) \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\Downarrow"] & \mathscr{H}(MX,MY) \end{tikzcd}$$

for each pair of objects X and Y. These objects, morphisms, and 2-cells now form a 2-category **CG**, which is itself locally finitely presentable, in the sense of [7].

There is an evident forgetful 2-functor $U : \mathbf{Hom} \to \mathbf{CG}$, and it is a routine exercise to give a presentation, in the sense of [9], for a finitary 2-monad T on **CG** for which **Hom** is the 2-category T-Alg of (strict) T-algebras, (pseudo) T-morphisms, and T-transformations. To see this, let $\mathbf{n}$ denote the **Cat**-graph with objects $0, 1, \ldots, n$, with $\mathbf{n}(i,j) = 1$ if $i < j$ and all other hom-categories empty, and let $i : \mathbf{0} \to \mathbf{n}$ denote the map sending 0 to i. Finally let I denote the **Cat**-graph with objects 0 and 1, with $I(0,1)$ the "free-living isomorphism", and all other hom-categories empty; thus there are two isomorphic maps from 0 to 1. Then to make a **Cat**-graph into a bicategory, one must equip it with operations

$$\mathbf{CG}(\mathbf{2},\mathscr{G}) \xrightarrow{M} \mathbf{CG}(\mathbf{1},\mathscr{G})$$

$$\mathbf{CG}(\mathbf{0},\mathscr{G}) \xrightarrow{j} \mathbf{CG}(\mathbf{1},\mathscr{G})$$

$$\mathbf{CG}(\mathbf{3},\mathscr{G}) \xrightarrow{\alpha} \mathbf{CG}(I,\mathscr{G})$$

$$\mathbf{CG}(\mathbf{1},\mathscr{G}) \xrightarrow{\lambda} \mathbf{CG}(I,\mathscr{G})$$

$$\mathbf{CG}(\mathbf{1},\mathscr{G}) \xrightarrow{\rho} \mathbf{CG}(I,\mathscr{G})$$

specifying composition, identities, associativity isomorphisms, and left and right identity isomorphisms, subject to equations between derived operations, which specify such things as the domain and codomain of composites, and the coherence condition for the associativity isomorphism. For example the domains and codomains of composites are specified by commutativity of the diagrams

$$\begin{tikzcd} \mathbf{CG}(\mathbf{2},\mathscr{G}) \arrow[r, "M"] \arrow[dr, "\mathbf{CG}(0,\mathscr{G})"'] & \mathbf{CG}(\mathbf{1},\mathscr{G}) \arrow[d, "\mathbf{CG}(0,\mathscr{G})"] \\ & \mathbf{CG}(\mathbf{0},\mathscr{G}) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{CG}(\mathbf{2},\mathscr{G}) \arrow[r, "M"] \arrow[dr, "\mathbf{CG}(2,\mathscr{G})"'] & \mathbf{CG}(\mathbf{1},\mathscr{G}) \arrow[d, "\mathbf{CG}(1,\mathscr{G})"] \\ & \mathbf{CG}(\mathbf{0},\mathscr{G}) \end{tikzcd}$$

while the domain of the associativity isomorphism is specified by commutativity of

$$\begin{tikzcd} \mathbf{CG}(\mathbf{3},\mathscr{G}) \arrow[r, "\alpha"] \arrow[d, "M_1"'] & \mathbf{CG}(I,\mathscr{G}) \arrow[d, "\mathbf{CG}(d,\mathscr{G})"] \\ \mathbf{CG}(\mathbf{2},\mathscr{G}) \arrow[r, "M"'] & \mathbf{CG}(\mathbf{1},\mathscr{G}) \end{tikzcd}$$

wherein $d : \mathbf{1} \to I$ is one of the identity-on-object inclusions, and $M_1 : \mathbf{CG}(\mathbf{3}, \mathscr{G}) \to \mathbf{CG}(\mathbf{2}, \mathscr{G})$ is the map representing "composing the first two maps of a composable triple". The latter is uniquely determined by commutativity of

$$\begin{tikzcd} \mathbf{CG}(\mathbf{3},\mathscr{G}) \arrow[r, "M_1"] \arrow[d, "\mathbf{CG}(p',\mathscr{G})"'] & \mathbf{CG}(\mathbf{2},\mathscr{G}) \arrow[d, "\mathbf{CG}(p,\mathscr{G})"] \\ \mathbf{CG}(\mathbf{2},\mathscr{G}) \arrow[r, "M"'] & \mathbf{CG}(\mathbf{1},\mathscr{G}) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{CG}(\mathbf{3},\mathscr{G}) \arrow[r, "M_1"] \arrow[dr, "\mathbf{CG}(r,\mathscr{G})"'] & \mathbf{CG}(\mathbf{2},\mathscr{G}) \arrow[d, "\mathbf{CG}(q,\mathscr{G})"] \\ & \mathbf{CG}(\mathbf{1},\mathscr{G}) \end{tikzcd}$$

wherein $p : \mathbf{1} \to \mathbf{2}$ and $p' : \mathbf{2} \to \mathbf{3}$ are inclusions, $q : \mathbf{1} \to \mathbf{2}$ sends an object $i \in \mathbf{1}$ to $i + 1 \in \mathbf{2}$, and $r : \mathbf{1} \to \mathbf{3}$ sends i to $i + 2$.

Although our main interest is in (normal) homomorphisms, and so as usual it is the pseudo morphisms of T-algebras which are most important, the strict morphisms of T-algebras are also of considerable theoretical importance, and they are precisely the strict homomorphisms of bicategories.

As a consequence of the fact that **Hom** has the form T-Alg, we may deduce, thanks to [2]:

Theorem 4.1 *The 2-category* **Hom** *has products, inserters, and equifiers, and therefore has all bicategorical limits. It also has bicategorical colimits. A homomorphism of bicategories is an equivalence if and only if the underlying morphism of* **Cat***-graphs is an equivalence. An icon is invertible if and only if the underlying 2-cell in* **CG** *is invertible. If $\mathscr{B}$ is a bicategory, and $M : \mathscr{G} \to U\mathscr{B}$ an equivalence in* **CG***, we may "transport" the bicategory structure to obtain a bicategory $\mathscr{A}$ and an equivalence $F : \mathscr{A} \to \mathscr{B}$ with $U\mathscr{A} = \mathscr{G}$ and $UF = M$. If $G : \mathscr{C} \to \mathscr{B}$ is a homomorphism, and $\rho : N \to UG$ is an invertible 2-cell in* **CG***, we may "transport" the homomorphism structure to obtain a homomorphism $H : \mathscr{C} \to \mathscr{B}$ and an invertible icon $\sigma : H \to G$ with $UH = N$ and $U\sigma = \rho$.*

5 Some other 2-categories of bicategories

The 2-category **NHom** has a full sub-2-category $\mathbf{2\text{-}Cat_{nps}}$ consisting of the 2-categories. (Here "nps" is short for "normal pseudofunctor": this is the name often given to normal homomorphisms between 2-categories.) Thus the inclusion $\mathbf{2\text{-}Cat_{nps}} \to \mathbf{NHom}$ is fully faithful; we shall see that it is also biessentially surjective on objects, and so a biequivalence. There is also a larger 2-category **Hom**, whose objects are the bicategories, but with arbitrary homomorphisms (not necessarily normal) as 1-cells. It is fairly straightforward to extend the definition of icons, so as to allow icons between arbitrary homomorphisms; the only slight subtlety is that rather than asking $\alpha 1_A$ be an identity 2-cell when 1_A is an identity 1-cell, one rather asks for $\alpha 1_A$ to be suitably compatible with the identity constraints $F1_A \cong 1_{FA}$ and $G1_A \cong 1_{GA}$. The inclusion of **NHom** in **Hom** is bijective on objects and locally fully faithful; we shall see that it is also locally an equivalence, and so a biequivalence. Finally there is a full sub-2-category $\mathbf{2\text{-}Cat_{ps}}$ of **Hom** consisting of the 2-categories, and once again the inclusions $\mathbf{2\text{-}Cat_{ps}} \to \mathbf{Hom}$ and $\mathbf{2\text{-}Cat_{nps}} \to \mathbf{2\text{-}Cat_{ps}}$ are biequivalences.

To see these facts, we first describe the equivalences and the invertible 2-cells in **Hom**.

Lemma 5.1 *Let $F, G : \mathscr{A} \to \mathscr{B}$ be homomorphisms of bicategories. An icon $\alpha : F \to G$ is invertible if and only if each $\alpha f : Ff \to Gf$ is invertible (in other words, if the oplax natural transformation is actually pseudonatural). A 2-cell in* **NHom**, $\mathbf{2\text{-}Cat_{ps}}$, *or* $\mathbf{2\text{-}Cat_{nps}}$ *is an isomorphism if and only if it is one in* **Hom**.

Proof: By Theorem 4.1, the icon α is invertible if and only if the underlying 2-cell $U\alpha$ in **CG** is invertible, but clearly this says precisely that each αf is invertible. The results for **NHom**, **2-Cat**$_{\mathbf{ps}}$, and **2-Cat**$_{\mathbf{nps}}$ are immediate since these are all locally full sub-2-categories of **Hom**. $\square$

As a consequence we have:

Proposition 5.2 *Every homomorphism $F : \mathscr{A} \to \mathscr{B}$ of bicategories is isomorphic in* **Hom** *to a normal homomorphism.*

Proof: Consider the underlying morphism $UF : U\mathscr{A} \to U\mathscr{B}$ in **CG**. There is an evident morphism $M : U\mathscr{A} \to U\mathscr{B}$ defined like UF except on the identity 1-cells, which are sent to the corresponding identity 1-cells in $\mathscr{B}$. There is an invertible 2-cell $\rho : M \to UF$ in **CG** which is the identity except on identity 1-cells, where it is the canonical isomorphism $F1_A \cong 1_{FA}$. We may transport the homomorphism structure to obtain a homomorphism $G : \mathscr{A} \to \mathscr{B}$ and an isomorphism $\sigma : G \to F$ with $UG = M$ and $U\sigma = \rho$. Clearly G is in fact a normal homomorphism. $\square$

Lemma 5.3 *Let $\mathscr{A}$ and $\mathscr{B}$ be bicategories. A homomorphism $F : \mathscr{A} \to \mathscr{B}$ is an equivalence in* **Hom** *if and only if it is bijective on objects and induces an equivalence of hom-categories. A 1-cell in* **NHom**, **2-Cat**$_{\mathbf{ps}}$, *or* **2-Cat**$_{\mathbf{nps}}$ *is an equivalence if and only if it is one in* **Hom**.

Proof: The statement about **Hom** is more or less immediate from the fact that F is an equivalence if and only if the underlying morphism UF of **Cat**-graphs is an equivalence. The main point of interest is that if G is equivalence inverse to F, then $GF \cong 1$ and $FG \cong 1$ in **Hom**, which forces GF and FG to act as the identity on objects, and so for F to be bijective on objects. The case of **2-Cat**$_{\mathbf{ps}}$ follows immediately from that of **Hom**; while those of **NHom** and **2-Cat**$_{\mathbf{nps}}$ now follow using the proposition. $\square$

It is well-known that every bicategory is biequivalent to a 2-category, and that this biequivalence may be chosen to be bijective on objects [14]. But a biequivalence which is bijective on objects is precisely an equivalence in **Hom**. This now proves:

Theorem 5.4 *Each of the inclusions* **NHom** $\to$ **Hom**, **2-Cat**$_{\mathbf{ps}}$ $\to$ **Hom**, *and* **2-Cat**$_{\mathbf{nps}}$ $\to$ **Hom** *is a biequivalence of 2-categories.*

The 2-monad of the previous section can be modified so that T-Alg becomes not **Hom** but **2-Cat**$_{\mathbf{ps}}$. On the other hand, we can modify the base 2-category **CG** by asking each hom-category $\mathscr{G}(X, X)$ to have a chosen object, preserved by the morphisms and 2-cells, and the resulting 2-category **RCG** is once again locally finitely presentable. There are suitable finitary 2-monads on **RCG** for which the resulting 2-categories T-Alg are respectively **NHom** and **2-Cat**$_{\mathbf{nps}}$.

We leave to the reader the resulting modifications of Theorem 4.1 dealing with **NHom**, **2-Cat**$_{\mathbf{ps}}$, and **2-Cat**$_{\mathbf{nps}}$. As a further consequence of [2] we have:

Theorem 5.5 *The singular functor $N = \mathbf{NHom}(J, 1) : \mathbf{NHom} \to [\Delta^{op}, \mathbf{Cat}]$ has a left biadjoint.*

Proof: Let T be the finitary 2-monad on **RCG** for which T-Alg is **NHom**. Just as in the case of **Hom**, a *strict* morphism of T-algebras is precisely a strict homomorphism of bicategories. We write **SHom** or T-$\mathrm{Alg}_{\mathrm{s}}$ for the sub-2-category consisting of the strict homomorphisms. The

inclusion $I : \mathbf{SHom} \to \mathbf{NHom}$ has a left adjoint L; this follows once again from [2], or can be proved directly. Now the composite $NI : \mathbf{SHom} \to [\Delta^{\mathrm{op}}, \mathbf{Cat}]$ can be written as $\mathbf{NHom}(J, I)$, which by the adjunction $L \dashv I$ is just $\mathbf{SHom}(LJ, 1)$; that is, the singular functor of $LJ : \Delta \to \mathbf{SHom}$. Now $\mathbf{SHom}$ is cocomplete *as a 2-category*, and so $NI = \mathbf{SHom}(LJ, 1)$ has a 2-adjoint F; thus by [2, Theorem 5.1], the composite IF is biadjoint to N.

The left biadjoint IF is in fact a 2-functor, and the unit $1 \to NIF$ is 2-natural; neither of these facts is true for a general biadjunction. □

We shall also see below that if we restrict the codomain of N we can obtain a very special left 2-adjoint, which is almost a 2-equivalence.

6 Properties of 2-nerves

We have already seen that the 2-nerve $N\mathscr{B}$ of a bicategory is the right Kan extension of a 2-functor $N_b : \Delta_b^{\mathrm{op}} \to \mathbf{Cat}$ along the inclusion $\Delta_b^{\mathrm{op}} \to \Delta^{\mathrm{op}}$, and so in particular that it is 3-coskeletal. We have also seen that $N\mathscr{B}_0$ is discrete.

For each $n > 1$, the Segal map $S_n : \mathscr{B}_n \to \mathscr{B}_1^n$ is a surjective equivalence. When $n = 2$, for example, this says (i) for a composable pair $f : A \to B$ and $g : B \to C$, there exist a morphism $h : A \to C$ and an invertible 2-cell $\varphi : gf \cong h$, and (ii) given data as above, and also $f' : A \to B$, $g' : B \to C$, $h' : A \to C$, and $\varphi' : g'f' \cong h'$, then for each pair $\alpha : f \to f'$ and $\beta : g \to g'$ of 2-cells, there is a unique 2-cell $\gamma : h \to h'$ for which the evident pasting diagram (involved in the definition of morphisms of 2-simplices) commutes. Of course for (i), we may take $h = gf$ and φ to be the identity, while for (ii), we may (and must!) take γ to be the composite $\varphi'.\beta\alpha.\varphi^{-1}$.

The fact that these Segal maps are surjective equivalences will be important in the following section, where we turn to the notions of weak 2-category due to Tamsamani and to Simpson.

We say that a functor $p : E \to B$ is a *discrete isofibration* or *dif*, if for each $e \in E$ and each isomorphism $\beta : b \to pe$ in B, there exists a unique isomorphism $\varepsilon : e' \to e$ in E with $p\varepsilon = \beta$ (and so also $pe' = b$). (This property might equally be called "unique transport of structure" or "unique invertible-path lifting".)

Recall that we write c_n for the n-component of the canonical map $X \to \mathrm{Cosk}_{n-1}X$ from a simplicial object X to its $n-1$-coskeleton. Our next observation is that c_2 is a dif for $X = N\mathscr{B}$. A 2-simplex in the 1-coskeleton of $N\mathscr{B}$ consists of three maps $f : A \to B$, $g : B \to C$, and $h : A \to C$ in $\mathscr{B}$; thus the fact that c_2 is a dif amounts to the (evident) fact that if $\varphi : gf \cong h$, and we are given isomorphisms $\alpha : f' \to f$, $\beta : g' \to g$, and $\gamma : h' \to h$, then there is a unique way to paste these together to obtain an isomorphism $\varphi' : g'f' \to h'$.

Similarly c_3 is a dif: this amounts to the (equally evident) fact that for an n-simplex consisting of morphisms $x_{ij} : X_i \to X_j$ and invertible 2-cells $\xi_{ijk} : x_{jk}x_{ij} \to x_{ik}$, given invertible 2-cells $\zeta_{ij} : x'_{ij} \to x_{ij}$ for each $i < j$, when one constructs the unique induced ξ'_{ijk} guaranteed by the fact that c_2 is a dif, these ξ'_{ijk} fit together to form an 3-simplex.

We shall see below that a simplicial object $X : \Delta^{\mathrm{op}} \to \mathbf{Cat}$ is the 2-nerve of a bicategory if and only if (i) X_0 is discrete, (ii) $S_n : X_n \to X_1^n$ is an equivalence for all $n > 1$, (iii) c_2 and c_3 are difs, and (iv) X is 3-coskeletal.

In the remainder of this section we establish two results which will be used in the comparison between bicategories and Tamsamani weak 2-categories to which we turn in the following section. Up until now, the only sort of morphisms between functors $\Delta^{\mathrm{op}} \to \mathbf{Cat}$ have been the

(2-)natural transformations. But since **Cat** is a 2-category, it is possible, and indeed reasonable, to consider also pseudonatural transformations. There is a 2-category $\mathbf{Ps}(\Delta^{\mathrm{op}}, \mathbf{Cat})$ of (2-)functors, pseudonatural transformations, and modifications, but we shall also be interested in the sub 2-category $\mathbf{NPs}(\Delta^{\mathrm{op}}, \mathbf{Cat})$ containing only those pseudonatural transformations $X \to Y$ for which the pseudonaturality isomorphism with respect to each epimorphism in Δ is an identity. We call such a pseudonatural transformation *normal pseudonatural*, since the strict naturality with respect to epimorphisms is closely related to normality of pseudofunctors/homomorphisms. Notice that "normal pseudonatural transformation" is not ascribed any meaning in general, only in this special case of pseudonatural transformations between functors from Δ^{op} to **Cat**.

The first of these two results says that we can "normalize" pseudonatural transformations whose domain is a 2-nerve.

Proposition 6.1 *Any morphism $f : N\mathscr{A} \to X$ in $\mathbf{Ps}(\Delta^{op}, \mathbf{Cat})$, with $N\mathscr{A}$ the 2-nerve of a bicategory $\mathscr{A}$, is isomorphic to a normal pseudonatural transformation.*

Proof: We shall define inductively functors $g_n : (N\mathscr{A})_n \to X_n$ equipped with natural isomorphisms $\psi_n : f_n \cong g_n$ such that the composite

$$g_{n+1}s_i \xrightarrow{\psi_{n+1}^{-1}s_i} f_{n+1}s_i \xrightarrow{f_{s_i}} s_i f_n \xrightarrow{\psi_n s_i} s_i g_n$$

is an identity for each i; here the isomorphism $f_{s_i} : f_{n+1}s_i \cong s_i f_n$ is the pseudonaturality isomorphism. Then the pseudonaturality isomorphisms for f can be transported across the isomorphisms ψ_n to give pseudonaturality isomorphisms for the g_n, and the composite displayed above will be precisely the induced pseudonaturality isomorphism $g_{n+1}s_i \cong s_i g_n$; thus g will become normal pseudonatural, in such a way that ψ is an invertible modification.

We take g_0 to be f_0, and ψ_0 to be the identity. To define g_{n+1} and ψ_{n+1}, it suffices to choose, for each object $x \in (N\mathscr{A})_{n+1}$, an object $g_{n+1}x \in X_{n+1}$ and an isomorphism $\psi_{n+1}x : f_{n+1}x \cong g_{n+1}x$, such that the composite

$$g_{n+1}s_i y \xrightarrow{\psi_{n+1}^{-1}s_i y} f_{n+1}s_i y \xrightarrow{f_{s_i y}} s_i f_n y \xrightarrow{\psi_n s_i y} s_i g_n y$$

is an identity for each i; then g_{n+1} becomes a functor and ψ_{n+1} a natural isomorphism.

If x is non-degenerate, we take $g_{n+1}x$ to be $f_{n+1}x$ and $\psi_{n+1}x$ to be the identity. If $x = s_i y$, we take $g_{n+1}x = s_i g_n y$, and $\psi_{n+1}x$ to be the composite $\psi_n s_i y . f_{s_i y} : f_{n+1}s_i y \to s_i g_n y = g_{n+1}s_i y$. The only thing to check is that this is well-defined. Now s_i is a section, so there can be at most one y with $s_i y = x$; but it is possible that $x = s_i y = s_j z$, with $j < i$. It is at this stage that we use the fact that the domain of f is the 2-nerve of a bicategory. For in this case we necessarily have $y = s_j w$ and $z = s_{i-1}w$ for some $w \in (N\mathscr{A})_{n-1}$: this boils down to the fact that in a bicategory the left and right identity isomorphisms $1_A 1_A \cong 1_A$ must agree. Now $s_i g_n y = s_i g_n s_j w = s_i s_j g_{n-1} w = s_j s_{i-1} g_{n-1} w = s_j g_n s_{i-1} w = s_j g_n z$, and so $g_{n+1}x$ is indeed well-defined, and the well-definedness of $\psi_{n+1}x$ is similar. $\square$

Remark 6.2 It is clear from the proof that one could relax the assumption that the domain is the 2-nerve of a bicategory. What is really used is that certain commutative squares of degeneracy maps are actually pullbacks.

The second result asserts a kind of "fibrancy" or "coflexibility" property of simplicial objects $X : \Delta^{\rm op} \to \mathbf{Cat}$ of the form $N\mathscr{B}$ for a bicategory $\mathscr{B}$. (See [11] for more about the relationship between flexibility and cofibrancy.) It shows that if we have a *normal* pseudonatural transformation whose *codomain* is the 2-nerve of a bicategory, then we can replace it by an isomorphic 2-natural transformation. Combined with the previous result, this will imply that any pseudonatural transformation between 2-nerves of bicategories is isomorphic to a 2-natural transformation.

It was shown in [2] that the inclusion $[\mathscr{A},\mathscr{B}] \to \mathbf{Ps}(\mathscr{A},\mathscr{B})$ admits a left adjoint whenever $\mathscr{A}$ is a small 2-category and $\mathscr{B}$ a cocomplete one. For suitable choices of $\mathscr{A}$ and $\mathscr{B}$ it follows that $[\Delta^{\rm op}, \mathbf{Cat}] \to \mathbf{Ps}(\Delta^{\rm op}, \mathbf{Cat})$ admits both a left and a right adjoint. A straightforward variant of this (which is still in fact a special case of the main theorem of [2]) shows that likewise the inclusion $[\Delta^{\rm op}, \mathbf{Cat}] \to \mathbf{NPs}(\Delta^{\rm op}, \mathbf{Cat})$ admits both adjoints. It is the right adjoint of the latter inclusion which concerns us here; it sends a functor $X : \Delta^{\rm op} \to \mathbf{Cat}$ to another such functor X^+; and by the universal property of the adjunction combined with the Yoneda lemma we see that

$$\begin{aligned} X^+_n &\cong [\Delta^{\rm op}, \mathbf{Cat}](\Delta(-,n), X^+) \\ &\cong \mathbf{NPs}(\Delta^{\rm op}, \mathbf{Cat})(\Delta(-,n), X) \end{aligned}$$

and now the component at X of the unit of the adjunction is the map $j : X \to X^+$ whose n-component is the inclusion of $[\Delta^{\rm op}, \mathbf{Cat}](\Delta(-,n), X)$ in $\mathbf{NPs}(\Delta^{\rm op}, \mathbf{Cat})(\Delta(-,n), X)$.

The *counit* is a normal pseudonatural $p : X^+ \to X$ with $pj = 1$, and the arguments of [2] show that $jp \cong 1$, and so that j is an equivalence in $\mathbf{NPs}(\Delta^{\rm op}, \mathbf{Cat})$. We shall say that X is *coflexible* (but *fibrant* would also be a good name) if $j : X \to X^+$ has a retraction r in $[\Delta^{\rm op}, \mathbf{Cat}]$; it then follows, as in [2], that $r \cong rjp = p$, and so $jr \cong jp \cong 1$, and so finally that j is an equivalence in $[\Delta^{\rm op}, \mathbf{Cat}]$.

Alternatively, rather than relying on [2], one could *define* X^+ by $X^+_n = \mathbf{NPs}(\Delta^{\rm op}, \mathbf{Cat})(\Delta(-,n), X)$ and j as the inclusion, and then prove directly that it has the properties stated above.

Theorem 6.3 *The 2-nerve $N\mathscr{B}$ of any bicategory $\mathscr{B}$ is coflexible.*

PROOF: First we make slightly more explicit the description of X^+, when $X = N\mathscr{B}$.

A normal pseudonatural from $\Delta(-,n)$ to X consists of an object $\xi \in X_n$ equipped with, for each non-identity monomorphism $\delta_i : [m] \to [n]$ in Δ, an object ξ_δ and an isomorphism $u_{\delta_i} : d_i\xi \cong \xi_\delta$ in X_m. A morphism between two such objects is just a morphism between their underlying objects in X_n.

In particular, since $[0]$ has no subobjects, X^+_0 is just X_0. On the other hand, while $[1]$ does have two subobjects (the two maps $\delta_0, \delta_1 : [0] \to [1]$), the category X_0 has no non-identity isomorphisms, and so once again X^+_1 is just X_1.

When it comes to X^+_2 things become more interesting. Once again, there are no non-identity isomorphisms in X_0, but there are non-identity isomorphisms in X_1, and three non-identity monomorphisms $[1] \to [2]$. Thus an object of X^+_2 consists of an object $\xi \in X_2$, equipped with an isomorphism $u_i : d_i\xi \cong \xi_i$ in X_1, for $i = 0,1,2$. A morphism in X^+_2 is just a morphism between the underlying objects in X_2. The inclusion $j : X_2 \to X^+_2$ equips $\xi \in X_2$ with identity isomorphisms. The degeneracies $X^+_1 \to X^+_2$ are induced by j from the degeneracies $X_1 \to X_2$.

An object in X^+_3 consists of an object $\Xi \in X_3$, equipped with isomorphisms $v_i : d_i\Xi \cong \Xi_i$ in X_2 for $i = 0,1,2,3$; as well as isomorphisms $w_d : d\Xi \cong \Xi_d$ in X_1, where d runs though each of the six

monomorphisms $[1] \to [3]$ in Δ. The face map $d_i : X_3^+ \to X_2^+$ sends such an object $(\Xi, (v_i), (w_d))$ to the 2-simplex in X^+ consisting of Ξ_i, equipped with the isomorphism $x_{ij} : d_j \Xi_i \cong \Xi_{ij}$ given by

$$d_j \Xi_i \xrightarrow{d_j v_i^{-1}} d_j d_i \Xi = d\Xi \xrightarrow{w_d} \Xi_d$$

where $d = d_j d_i$; we shall sometimes write $\Xi_d = \Xi_{ij}$.

We shall now construct the desired retraction $r : X^+ \to X$, using the fact that $X = N\mathscr{B}$, and the description of morphisms into such objects of $[\Delta^{\mathrm{op}}, \mathbf{Cat}]$.

For $n = 0$ and $n = 1$, the map $j : X_n \to X_n^+$ is the identity, and so we take r to be the identity; clearly $r : X_1^+ \to X_1$ is compatible with degeneracies. We define $r_2 : X_2^+ \to X_2$ on objects to take $(\xi, (u_i))$ to the 2-simplex in X_2 given by

$$\xi_2.\xi_0 \xrightarrow{u_2^{-1}.u_0^{-1}} d_2\xi.d_0\xi \xrightarrow{\xi} d_1\xi \xrightarrow{u_1} \xi_1.$$

This is compatible with the face maps by construction, and compatible with the degeneracies by definition of degeneracies in X_2^+. The functoriality of r_2 is clear.

It remains to check the compatibility condition codified by a 3-simplex. But this asserts the commutativity in the bicategory $\mathscr{B}$ of diagram (2) in Section 3, which asserts the equality, for each object of X_3^+, of a parallel pair of arrows in X_1. But the construction of this parallel pair is natural in the objects of X_3^+, and holds for objects in the image of $j : X_3 \to X_3^+$, so holds for all objects, since j is an equivalence. □

7 The Tamsamani and Simpson notions of weak 2-category

Let **Tam** denote the full sub-2-category of $[\Delta^{\mathrm{op}}, \mathbf{Cat}]$ consisting of those X for which X_0 is discrete and each $S_n : X_n \to X_1^n$ is an equivalence, and **Simpson** the smaller full sub-2-category of those X for which moreover each S_n is surjective. We speak of *Tamsamani 2-categories* and *Simpson 2-categories*, but in fact Tamsamani used the name *2-nerve*, while Simpson used the name *easy 2-category*.

We saw in the previous section that if X is the 2-nerve of a bicategory, then each $S_n : X_n \to X_1^n$ is a surjective equivalence and X_0 is discrete. Thus the 2-nerve of a bicategory is a Simpson 2-category, and so necessarily a Tamsamani 2-category.

The left biadjoint of Theorem 5.5 also gives biadjoints to the inclusions of **NHom** in each of **Simpson** and **Tam**, but in fact there exist 2-adjoints. For a Tamsamani 2-category X, there is a bicategory GX, defined in [19], whose objects are the elements of X_0, and whose 1-cells and 2-cells are the objects and morphisms of X_1, with vertical composition of 2-cells given by the composition law in X_1. Since $S_2 : X_2 \to X_1 \times_{X_0} X_1$ is an equivalence, we can choose a functor $M : X_1 \times_{X_0} X_1 \to X_1$ and an isomorphism $\sigma : d_1 \cong MS_2$, as in

$$\begin{tikzcd} X_2 \arrow[r, "S_2"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\Swarrow\sigma"] & X_1^2 \arrow[dl, "M"] \\ X_1 & \end{tikzcd}$$

and this M gives the composition of 1-cells and the horizontal composition of 2-cells.

Composing σ with the degeneracy maps $s_0, s_1 : X_1 \to X_2$ gives

$$1 = d_1 s_0 \xrightarrow{\sigma s_0} MS_2 s_0 = M\binom{1}{s_0 d_1}$$

$$1 = d_1 s_1 \xrightarrow{\sigma s_1} MS_2 s_1 = M\binom{s_0 d_0}{1}$$

which are the identity isomorphisms for the bicategory GX. For the associativity isomorphism $M(M \times 1) \cong M(1 \times M)$, consider the pasting diagrams

$$\begin{tikzcd}
X_3 \arrow[r, "\binom{d_0}{d_2 d_2}"] \arrow[d, "d_2"'] & X_2 \times_{X_0} X_1 \arrow[r, "S_2 \times 1"] \arrow[d, "d_1 \times 1"'] \arrow[dr, phantom, "\Downarrow\sigma\times 1"] & X_1^3 \arrow[dl, "M \times 1"] \\
X_2 \arrow[r, "S_2"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\Downarrow\sigma"] & X_1^2 \arrow[dl, "M"] & \\
X_1 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
X_3 \arrow[r, "\binom{d_0 d_1}{d_3}"] \arrow[d, "d_1"'] & X_1 \times_{X_0} X_2 \arrow[r, "1 \times S_2"] \arrow[d, "1 \times d_1"'] \arrow[dr, phantom, "\Downarrow 1\times\sigma"] & X_1^3 \arrow[dl, "1 \times M"] \\
X_2 \arrow[r, "S_2"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\Downarrow\sigma"] & X_1^2 \arrow[dl, "M"] & \\
X_1 & &
\end{tikzcd}$$

The left hand composite of the two diagrams are equal, and the top composite of each diagram is just the equivalence $S_3 : X_3 \to X_1^3$; thus there is a unique invertible $\alpha : M(M \times 1) \cong M(1 \times M)$ which when pasted onto the left diagram gives the right diagram, and we take this α to be the associativity isomorphism. The coherence condition for associativity is checked using the fact that S_4 is an equivalence: the two pasting composites which are to be proven equal are each pasted onto the right hand side of the following diagram (in which the operation " $\times_{X_0}$ " has been written as juxtaposition)

$$\begin{tikzcd}
X_4 \arrow[r, "\binom{d_0}{d_2 d_2 d_2}"] \arrow[d, "d_3"'] & X_3 X_1 \arrow[r, "\binom{d_0}{d_2 d_2}\times 1"] \arrow[d, "d_2 1"'] & X_2 X_1^2 \arrow[r, "S_2 11"] \arrow[d, "d_1 11"'] \arrow[dr, phantom, "\Downarrow\sigma 11"] & X_1^4 \arrow[dl, "M11"] \\
X_3 \arrow[r, "\binom{d_0}{d_2 d_2}"] \arrow[d, "d_2"'] & X_2 X_1 \arrow[r, "S_2 1"] \arrow[d, "d_1 1"'] \arrow[dr, phantom, "\Downarrow\sigma 1"] & X_1^3 \arrow[dl, "M1"] & \\
X_2 \arrow[r, "S_2"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\Downarrow\sigma"] & X_1^2 \arrow[dl, "M"] & & \\
X_1 & & &
\end{tikzcd}$$

and in each case, after three steps, one gets the same result, namely

$$\begin{tikzcd}
X_4 \arrow[r, "\binom{d_0d_0d_0}{d_2}"] \arrow[d, "d_1"'] & X_1X_3 \arrow[r, "1\binom{d_0d_1}{d_3}"] \arrow[d, "1d_1"'] & X_1^2X_2 \arrow[r, "11S_2"] \arrow[d, "11d_1"'] \arrow[dr, phantom, "\Downarrow 11\sigma"] & X_1^4 \arrow[dl, "11M"] \\
X_3 \arrow[r, "\binom{d_0d_1}{d_3}"] \arrow[d, "d_1"'] & X_1X_2 \arrow[r, "1S_2"] \arrow[d, "1d_1"'] \arrow[dr, phantom, "\Downarrow 1\sigma"] & X_1^3 \arrow[dl, "1M"] & \\
X_2 \arrow[r, "S_2"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\Downarrow \sigma"] & X_1^2 \arrow[dl, "M"] & & \\
X_1 & & &
\end{tikzcd}$$

Now both these pasting composites are invertible, and the arrow across the top is the equivalence $S_4 : X_4 \to X_1^4$, so the desired result follows. A similar but easier argument establishes the coherence for the identities.

Having constructed the bicategory GX, we can now of course take its 2-nerve NGX, which we temporarily name X'. We are going to construct a morphism $u : X \to NGX = X'$. Clearly $X'_0 = X_0$ and $X'_1 = X_1$, and so we may take u_0 and u_1 to be the identities. Now X'_2 may be constructed as the *pseudolimit* of the map $M : X_1^2 \to X_1$; that is, the universal category equipped with functors $S'_2 : X'_2 \to X_1^2$ and $d'_1 : X'_2 \to X_1$, and an isomorphism $\rho : MS'_2 \cong d'_1$. The components d'_0 and d'_2 of S'_2, along with d'_1, are the face maps $X'_2 \to X'_1 = X_1$. Thus a map into X'_2 is determined by its composite with the face maps $X'_2 \to X_1$ and with ρ. By a general property of such pseudolimits S'_2 must be a surjective equivalence.

We construct $u_2 : X_2 \to X'_2$ as the unique map compatible with the face maps $X'_2 \to X'_1$, and with $\rho u_2 = \sigma$. Notice that $S'_2 u_2 = \binom{d'_0}{d'_2} u_2 = \binom{d_0}{d_2} = S_2$, and both S'_2 and S_2 are equivalences, thus also u_2 is an equivalence. The degeneracy map $s'_0 : X'_1 = X_1 \to X'_2$ is the unique map satisfying $d'_0 s'_0 = d'_1 s'_0 = 1$, $d'_2 s'_0 = s_0 d'_1$, and $\rho s'_0 = \sigma s_0$; similarly $s'_1 : X_1 \to X'_2$ is the unique map satisfying $d'_0 s'_1 = s'_0 d'_0$, $d'_1 s'_1 = d'_2 s'_1 = 1$, and $\rho s'_1 = \sigma s_1$. It is now straightforward to verify that u_2 is compatible with the degeneracy maps.

The category X'_3 is the pseudolimit of the diagram

$$\begin{tikzcd}
X_1^3 \arrow[r, "M\times 1"] \arrow[d, "1\times M"'] \arrow[dr, phantom, "\Downarrow \alpha"] & X_1^2 \arrow[d, "M"] \\
X_1^2 \arrow[r, "M"'] & X_1
\end{tikzcd}$$

in other words, the universal category X'_3 equipped with morphisms $S'_3 : X'_3 \to X_1^3$ and $K_1, K_2 : X'_3 \to X_1^2$ and $L : X'_3 \to X_1$, and with invertible 2-cells $\kappa_1 : K_1 \cong (1 \times M)S'_3$, $\kappa_2 : (M \times 1)S'_3 \cong K_2$, and $\lambda_1 : L \cong MK_1$. (This may appear asymmetric but it is not; there is a uniquely induced isomorphism $\lambda_2 : L \cong MK_2$ suitably compatible with the associativity isomorphism α.) Once again, it is an immediate consequence that S'_3 is a surjective equivalence.

There is a unique map $u_3 : X_3 \to X'_3$ with $S'_3 u_3 = S_3$, $K_1 u_3 = S_2 d_1$, $K_2 u_3 = S_2 d_2$, $L u_3 = d_1 d_2$, $\kappa_1 u_3 = d_0 d_0 \times \sigma d_3$, $\kappa_2 u_3 = \sigma d_0 \times d_2 d_2$, while $\lambda_1 u_3 = \sigma d_1$. Since $S'_3 u_3 = S_3$ and S'_3 and S_3 are equivalences, u_3 is an equivalence too.

Next we describe the face maps $X'_3 \to X'_2$. First of all d'_1 is the unique map induced by K_1, L, and λ_1, while similarly d'_2 is that induced by K_2, L, and λ_2. The other two are only slightly more complicated: S'_3, K_1, and κ_1 are composed with the projection $X_1^2 \to X_1$ onto the first factor, to obtain the data inducing d'_3, while S'_3, K_2, and κ_2 are composed with the projection $X_1^2 \to X_1$ onto the second factor to obtain the data inducing d'_0. One now verifies that u_3 is compatible with these face maps: for example, $d'_1 u_3$ is the map $X_3 \to X'_2$ induced by $K_1 u_3 = S_2 d_3$, $L u_3 = d_1 d_3$, and $\lambda_1 u_3 = \sigma d_1$, but these are precisely the data that describe $u_2 d_1$. The other face maps are treated similarly, and so we have now defined a map u between the restrictions of X and X' to Δ_b^{op}, but since X' is the right Kan extension of its restriction to Δ_b^{op}, it follows that u extends uniquely to give a map $X \to X'$.

Since X is a Tamsamani 2-category and X' is the 2-nerve of a bicategory, the Segal maps $S_n : X_n \to X_1^n$ and $S'_n : X'_n \to X_1^n$ are equivalences; on the other hand, $u_n : X_n \to X'_n$ is clearly compatible with the Segal maps, and so it follows that each u_n is an equivalence. A morphism $f : X \to Y$ of simplicial object is said to be a *pointwise equivalence* if, as here, each f_n is an equivalence.

Thus for every Tamsamani 2-category X, we have constructed a pointwise equivalence $u : X \to NGX$ to the 2-nerve of a bicategory GX.

Suppose now that X is 3-coskeletal and c_2 and c_3 are difs. We shall show that u_2 and u_3 are isomorphisms, so that u is an isomorphism, and X is the 2-nerve of a bicategory (namely GX).

We already know that u_2 is an equivalence; we must show that it is bijective on objects. Suppose then that $\varphi : gf \cong h$ is an object of X'_2. Since $S_2 : X_2 \to X_1^2$ is an equivalence, there exists a $\xi \in X_2$ with isomorphisms $\alpha : f \cong d_2\xi$ and $\beta : g \cong d_0\xi$. The isomorphism $\sigma : MS_2\xi \cong d_1\xi$ involved in the definition of composition in GX goes from $d_0\xi.d_2\xi$ to $d_1\xi$. Combining it with α, β, and φ^{-1} provides an isomorphism $\gamma : h \to d_1\xi$ as in

$$h \xrightarrow{\varphi^{-1}} gf \xrightarrow{\beta.\alpha} d_0\xi.d_2\xi \xrightarrow{\sigma} d_1\xi,$$

and now α, β, and γ together constitute an isomorphism $\varphi' : (f, g, h) \cong c_2\xi$ in $(\mathrm{Cosk}_1 X)_2$, and so since c_2 is a dif, there is a unique $\varphi'' : \xi' \cong \xi$ in X_2 with $c_2\xi'' = (f, g, h)$ and $c_2\varphi'' = \varphi'$; in other words, with $u_2\xi'' = (f, g, \varphi : gf \cong h)$. This proves that u_2 is bijective on objects, and so an isomorphism.

Once again, we already know that u_3 is an equivalence, and must show that it is bijective on objects. Injectivity is easy: if Ξ and Ξ' are objects of X_3 with $u_3\Xi = u_3\Xi'$, then there is a unique isomorphism $\Xi \cong \Xi'$ sent by u_3 to the identity; but it then easily follows that it is sent by $c_3 : X_3 \to (\mathrm{Cosk}_1 X)_3$ to the identity, and so that it must itself be an identity. As for surjectivity, suppose now that Ξ' is an object of X'_3. We know that there is an isomorphism $\varphi : \Xi' \cong u_3\Xi$ for some $\Xi \in X_3$. Applying the map $c'_3 : X'_3 \to (\mathrm{Cosk}_1 X')_3$ gives an isomorphism $c'_3\varphi : c'_3\Xi' \cong c'_3 u_3 \Xi$ in $\mathrm{Cosk}_1 X')_3$. But X' and X have the same 1-coskeleton, so $(\mathrm{Cosk}_1 X')_3 = (\mathrm{Cosk}_1 X)_3$ and $c'_3 u_3 = c_3$; and now the previous isomorphism may be seen as an isomorphism $c'_3\varphi : c'_3\Xi' \cong c_3\Xi$ in $(\mathrm{Cosk}_1 X)_3$. Since c_3 is a discrete isofibration, there is a unique isomorphism $\varphi_1 : \Xi_1 \cong \Xi$ in X_3 with $c_3\varphi_1 = c'_3\varphi$ (and so also $c_3\Xi_1 = c'_3\Xi'$). But now $u_3\varphi_1 : u_3\Xi_1 \cong u_3\Xi$ is an isomorphism in X'_3 with the property that $c'_3 u_3 \varphi_1 = c_3\varphi_1 = c'_3\varphi$, and so since c'_3 is also a discrete fibration, $u_3\varphi_1 = \varphi$ and $u_3\Xi_1 = \Xi'$. This proves:

Theorem 7.1 *A 2-functor $X : \Delta^{op} \to \mathbf{Cat}$ is the 2-nerve of a bicategory if and only if (i) it is 3-coskeletal, (ii) X_0 is discrete, (iii) the Segal maps $S_n : X_n \to X_1^n$ are equivalences, and (iv) $c_2 : X_2 \to (\mathrm{Cosk}_1 X)_2$ and $c_3 : X_3 \to (\mathrm{Cosk}_1 X)_3$ are discrete isofibrations.*

We now show that $u : X \to NGX$ is the unit of a 2-adjunction between **NHom** and **Tam**. To do this, let $\mathscr{B}$ be a bicategory and $F : X \to N\mathscr{B}$ a morphism in **Tam** (equivalently, in $[\Delta^{\mathrm{op}}, \mathbf{Cat}]$). We must show that there is a unique morphism $F' : NGX \to N\mathscr{B}$ with $F'u = F$; the two dimensional aspect of the universal property will then follow, since **NHom** has cotensors, preserved by $N : \mathbf{NHom} \to \mathbf{Tam}$. Now to give $F' : NGX \to N\mathscr{B}$ is equivalently to give a normal homomorphism $F'' : GX \to \mathscr{B}$. Since u_0 and u_1 are the identities, the action of F'' on objects, 1-cells, and 2-cells is already determined; it will remain only to give the pseudofunctoriality isomorphisms.

To give a map $F : X \to N\mathscr{B}$ is to give a function $F = F_0 : X_0 \to \mathscr{B}_0$, functors $F : X(x,y) \to \mathscr{B}(Fx, Fy)$, and an invertible 2-cell $\chi_\xi : Fd_2\xi.Fd_0\xi \to Fd_1\xi$ for each object $\xi \in X_2$ such that (i) χ_ξ is natural in ξ, (ii) χ_ξ is the relevant identity isomorphism whenever ξ is degenerate, and (iii) if Ξ is an object of X_3, then the diagram

$$\begin{tikzcd}
(Fd_0d_0\Xi.Fd_2d_0\Xi).Fd_2d_3\Xi \arrow[r, "\chi_{d_0\Xi}.1"] \arrow[d, equal] & Fd_1d_0\Xi.Fd_2d_3\Xi \arrow[r, equal] & Fd_0d_2\Xi.Fd_2d_2\Xi \arrow[r, "\chi_{d_2\Xi}"] & Fd_1d_2\Xi \arrow[dd, equal] \\
(Fd_0d_1\Xi.Fd_0d_3\Xi).Fd_2d_3\Xi \arrow[d, "\alpha"'] & & & \\
Fd_0d_1\Xi.(Fd_0d_3\Xi.Fd_2d_3\Xi) \arrow[r, "1.\chi_{d_3\Xi}"'] & Fd_0d_1\Xi.Fd_1d_3\Xi \arrow[r, equal] & Fd_0d_1\Xi.Fd_2d_1\Xi \arrow[r, "\chi_{d_1\Xi}"'] & Fd_1d_1\Xi
\end{tikzcd}$$

in $\mathscr{B}$ commutes.

How can we extend this to a morphism $NGX \to N\mathscr{B}$? On 0-simplices and 1-simplices there is no change: we still use the same function $F : X_0 \to \mathscr{B}_0$ and the same functors $F : X(x,y) \to \mathscr{B}(Fx, Fy)$. When it comes to 2-simplices, we know that every object $(f, g, \varphi : gf \cong h)$ of NGX is isomorphic to an object of the form $u_2\xi$ for a $\xi \in X_2$, so must be sent to a 2-simplex in $N\mathscr{B}$ isomorphic to the image of ξ under F. But compatibility with the face maps tells where the faces f, g, and h must go — namely to Ff, Fg, and Fh, and now everything else is uniquely determined. Explicitly, fix $\xi \in X_2$, and an isomorphism $u_2\xi \cong (f, g, \varphi)$, given by $\alpha : d_2\xi \cong f$, $\beta : d_0\xi \cong g$, and $\gamma : d_1\xi \cong h$. Then (f, g, φ) must be sent to the 2-simplex of $N\mathscr{B}$ made up of Ff, Fg, and

$$Fg.Ff \xrightarrow{F\beta.F\alpha} Fd_0\xi.Fd_2\xi \xrightarrow{F_2(\xi)} Fd_1\xi \xrightarrow{F\gamma^{-1}} Fh$$

where $F_2(\xi)$ is the image of the 2-simplex $\xi \in X_2$ under the map $F : X \to N\mathscr{B}$. (Notice that the final result does not depend on the choice of ξ or the isomorphism $u_2\xi \cong (f, g, \varphi)$.

This proves:

Theorem 7.2 *The 2-nerve 2-functor $N : \mathbf{NHom} \to \mathbf{Tam}$, seen as landing in the 2-category* **Tam** *of Tamsamani 2-categories, has a left 2-adjoint given by G. Since N is fully faithful, the counit $GN \to 1$ is invertible. Each component $u : X \to NGX$ of the unit is a pointwise equivalence, and u_0 and u_1 are identities.*

If u were in fact an equivalence in **Tam**, then N would be fully faithful and biessentially surjective, and so a biequivalence. Since each $u_n : X_n \to NGX_n$ is an equivalence, we can choose inverse equivalences $v_n : NGX_n \to X_n$, and these will automatically become the components of a *pseudonatural* transformation $v : NGX \to X$, but there is no reason in general why they should be natural, and so there is no reason in general why $u : X \to NGX$ should be an equivalence in **Tam**. One response would be to "localize" **NHom** and **Tam** by inverting certain morphisms (and throwing away the 2-cells).

As is always the case with a full reflective subcategory, if one inverts the components of the unit — in this case the $u : X \to NGX$ — then one recovers the subcategory. One could, however, consider inverting larger classes of maps, for instance the pointwise equivalences, or more generally the *weak equivalences* (or *external equivalences* in [19]): a morphism $f : X \to Y$ is a weak equivalence if and only if Gf is a biequivalence of bicategories. One can show that inverting these maps in **NHom** and **Tam** gives equivalent categories. In fact, in the case where one uses the weak equivalences, the resulting categories are also equivalent to the homotopy categories of the Quillen model categories **2-Cat** and $\mathbf{Bicat_s}$ of [11, 12]; here **2-Cat** is the category of 2-categories and 2-functors, and $\mathbf{Bicat_s}$ the category of bicategories and strict homomorphisms.

A less violent approach than inverting these morphisms is to use a simplicial localization, as in [5]; this time, using [5, Corollary 3.6] , one obtains weakly equivalent simplicial categories after localization.

Here we adopt a more precise and explicit approach, in which we expand our notion of morphism in **Tam** to allow not just natural transformations, but pseudonatural ones. Let $\mathbf{Tam_{ps}}$ be the full sub-2-category of $\mathbf{Ps}(\Delta^{\mathrm{op}}, \mathbf{Cat})$ consisting of the Tamsamani 2-categories.

Theorem 7.3 *The 2-nerve 2-functor* $\mathbf{NHom} \to \mathbf{Tam_{ps}}$ *is a biequivalence of 2-categories. An object of* $\mathbf{Tam_{nps}}$ *is in the image of the functor if and only if it satisfies the conditions of Theorem 7.1; a morphism between 2-nerves of bicategories is in the image if and only if it is (not just pseudonatural but 2-) natural.*

Proof: The statements about the image have already been proven.

We already know that the 2-nerve 2-functor is locally fully faithful. To say that it is locally essentially surjective on objects is to say that any pseudonatural transformation $f : N\mathscr{A} \to N\mathscr{B}$ is isomorphic to a 2-natural transformation. By Proposition 6.1, f is isomorphic to a normal pseudonatural transformation g. By the universal property of $p : N\mathscr{B}^+ \to N\mathscr{B}$, there is a unique 2-natural $h : N\mathscr{A} \to N\mathscr{B}^+$ for which $g = ph$, and now by Theorem 6.3 there is a 2-natural r isomorphic to p, and so $g = ph \cong rh$ with rh 2-natural. Thus the nerve 2-functor is locally an equivalence. It remains to show that it is biessentially surjective on objects; that is, that every Tamsamani 2-category is equivalent in $\mathbf{Tam_{ps}}$ to the 2-nerve of a bicategory. But if X is a Tamsamani 2-category then we have the 2-nerve NGX, and the pointwise equivalence $u : X \to NGX$, and every pointwise equivalence is an equivalence in $\mathbf{Tam_{ps}}$. □

Remark 7.4 One could also consider the sub-2-category $\mathbf{Tam_{nps}}$ of $\mathbf{Tam_{ps}}$ containing only the normal pseudonatural maps, and this would once again be biequivalent: this time local essential surjectivity uses only Theorem 6.3, but one now needs to use Proposition 6.1 to prove biessential surjectivity.

Finally we consider what happens in the one-object case. The full sub-2-category of **NHom** consisting of the one-object bicategories is precisely the 2-category of monoidal categories, normal

strong monoidal functors, and monoidal natural transformations. On the other hand, a one-object Tamsamani weak 2-category is what has sometimes been called a homotopy monoidal category [13]. The 2-adjunction between **NHom** and **Tam** restricts to the one-object case, and so once again, one obtains a weak equivalence between the simplicial localizations of the category of monoidal categories and strong monoidal functors, and the category of homotopy monoidal categories and morphisms thereof. Likewise, Theorem 7.3 restricts to the one-object case.

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