1. Using the Taylor series for \( f(x + h) \) and \( f(x - h) \) about the point \( x \), show that the central difference approximation to \( f'(x) \) has truncation error \( O(h^2) \), i.e. that
\[
f'(x) = \frac{f(x + h) - f(x - h)}{2h} + O(h^2).
\]

2. Consider the data below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

(a) Use the Vandermonde matrix method to find an interpolating polynomial for the data. [Recall that you can solve the linear system \( V a = y \) in Matlab using the command \( a = V \backslash y \).]

(b) Write down Legendre polynomials \( l_k \) (for the points \( x_k = k \) above) for \( k = 1, \ldots, 4 \) with the property that
\[
l_k(x_j) = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}
\]

(c) Using the Lagrange polynomials in 2a, write down the Lagrange interpolating polynomial for the data above. Leave your answer in terms of \( l_1, \ldots, l_4 \).

(d) Download the Matlab programs \texttt{interp\_vand.m} and \texttt{interp\_lagrange.m} from the MATH334 webpage. Read through the programs carefully and make sure you understand how they work. Use each program to compute and evaluate the interpolating polynomial at the points \( x_i = \text{linspace}(1, 5) \) and plot the result using the \texttt{plot} command.

3. Let \( p \) be the Lagrange interpolating polynomial for the function \( f(x) = \exp(x) \) at points \( x_1 = 0, x_2 = 1/2, x_3 = 1 \). Find a bound on the error \( |f(x) - p(x)| \) for \( x \in [0, 1] \).
4. Consider the polynomials

\[ p_0(x) = 3, \]
\[ p_1(x) = 2 + x, \]
\[ p_2(x) = 3x^2 + 2x - 1. \]

(a) Use the rectangle rule with \( x_1 = 1 \) to approximate

\[ \int_0^1 p_k(x) \, dx \]

for \( k = 0, 1, 2 \). In each case compute the error in the approximation.

(b) Repeat 4a using the midpoint rule \( i.e. \) the rectangle rule with \( x_1 = 1/2 \).

(c) Repeat 4a using the trapezium rule.

(d) Tabulate the errors in 4a–4c. What is the observed degree of precision of each of the three rules.

(e) Repeat 4a–4d for the integral

\[ \int_0^{1/2} p_k(x) \, dx. \]

How do the errors for each quadrature rule compare with those observed earlier? [Hint: in each case, look at the ratio between the error over \([0, 1]\) and the error over \([0, 1/2]\).]