MATH236 — Weeks 8&9

Integrals over surfaces

Chen notes, chapter 11

11.1 Integrals of scalar functions over parametrized surfaces

Suppose that the parametrized surface $\Phi : R \to \mathbb{R}^3$ whereby $(u, v) \mapsto \Phi(u, v) = (x(u, v), y(u, v), z(u, v))$ is continuously differentiable.

For any real-valued function $f(x, y, z)$ such that the composition $f \circ \Phi : R \to \mathbb{R}^3 : (u, v) \mapsto f(\Phi(u, v)) = f(x(u, v), y(u, v), z(u, v))$ is continuous, we define:

$$\int_\Phi f\, d\boldsymbol{S} = \int_\Phi f(x, y, z)\, d\boldsymbol{S} \overset{\text{def}}{=} \int\int_R f(\boldsymbol{\Phi}(u, v))\, \|\boldsymbol{t}_u \times \boldsymbol{t}_v\|\, du\, dv.$$

Remarks

- Here $dS = \|\boldsymbol{t}_u \times \boldsymbol{t}_v\|\, du\, dv$ can be considered to be the ***surface area differential*** of the parametrized surface Φ.
- Clearly $\int_\Phi f\, dS = \int\int_R f(\boldsymbol{\Phi}(u, v))\, \|\boldsymbol{t}_u \times \boldsymbol{t}_v\|\, du\, dv = \int\int_R f(\boldsymbol{\Phi}(u, v)) \left(\left| \frac{\partial(y,z)}{\partial(u,v)} \right|^2 + \left| \frac{\partial(z,x)}{\partial(u,v)} \right|^2 + \left| \frac{\partial(x,y)}{\partial(u,v)} \right|^2 \right)^{\frac{1}{2}} du\, dv.$
- Suppose that $f = 1$ identically. Then the integral simply represents the surface area of Φ.
- Note that f has only to be defined on the image surface $S = \Phi(R)$ of the parametrized surface Φ for our definition to make sense. The continuity of the composition function $f \circ \Phi$ on the elementary region R ensures the existence of the integral.
- Sometimes, Φ may only be piecewise continuously differentiable; in other words, there exists a partition of the region R into a *finite union* of elementary regions R_i, where $i = 1, \ldots, k$, such that Φ is continuously differentiable in R_i for each $i = 1, \ldots, k$. In this case, we define

$$\int_\Phi f\, dS \overset{\text{def}}{=} \sum_{i=1}^{k} \int_{\Phi_i} f(\boldsymbol{\Phi}(u, v))\, \|\boldsymbol{t}_u \times \boldsymbol{t}_v\|\, du\, dv,$$

where $\Phi_i : R_i \to \mathbb{R}^3 : (u, v) \mapsto \Phi(u, v)$. In other words, we calculate the corresponding integral for each subregion and consider the sum of the integrals.

Example 11.1.1 — parametrized cone

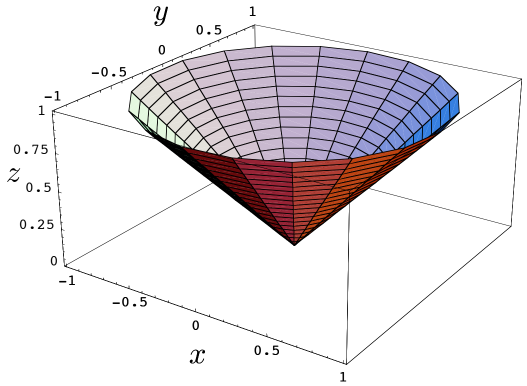
For the ***parametrized cone*** $\Phi : [0, 1] \times [0, 2\pi] \to \mathbb{R}^3$ whereby $(u, v) \mapsto (u \cos v, u \sin v, u)$, and the function $f(x, y, z) = x^2 + y^2 + z^2 + y$, we have, writing $R = [0, 1] \times [0, 2\pi]$ and noting Example 10.2.1,

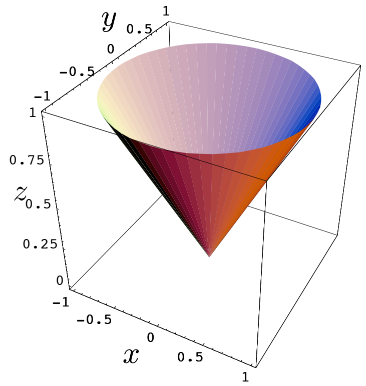
$$\int_\Phi f\, dS = \int\int_R f(u\cos v, u \sin v, u)\, \|(-u\cos v, -u\sin v, u)\|\, du\, dv = \int\int_R (u^2\cos^2 v + u^2 \sin^2 v + u^2 + u \sin v)\sqrt{2}\, u\, du\, dv$$

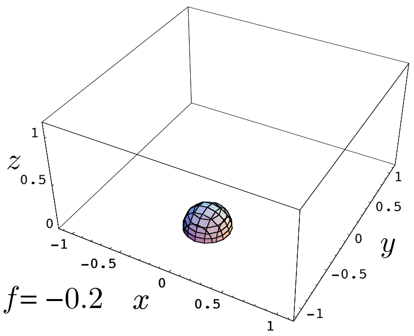
$$= \sqrt{2} \int\int_R u^2(2u + \sin v)\, du\, dv = 2\sqrt{2}\int\int_R u^3\, du\, dv + \sqrt{2}\int\int_R u^2 \sin v\, du\, dv$$

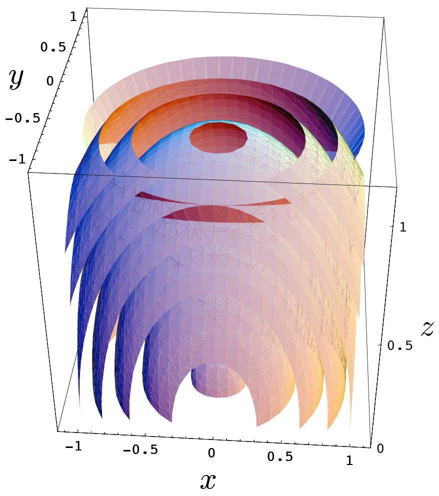
$$= 2\sqrt{2}\left(\int_0^1 u^3\, du\right)\left(\int_0^{2\pi} 1\, dv\right) + \sqrt{2}\left(\int_0^1 u^2\, du\right)\left(\int_0^{2\pi} \sin v\, dv\right) = 2\sqrt{2} \times \frac{1}{4} \times 2\pi + \sqrt{2} \times \frac{1}{3} \times 0 = \pi\sqrt{2}.$$

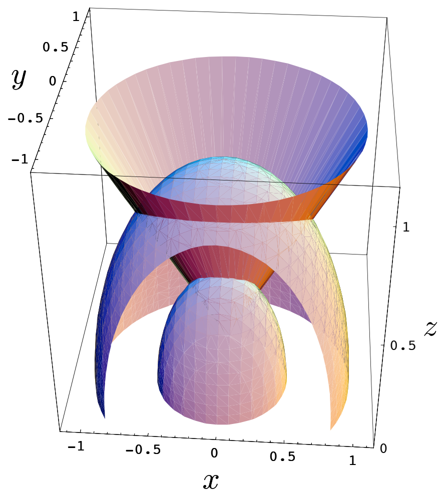
visualisation



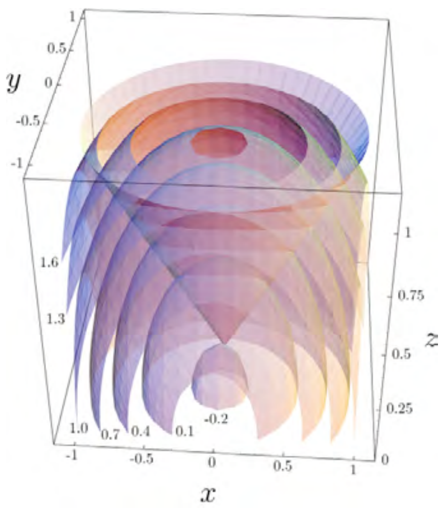




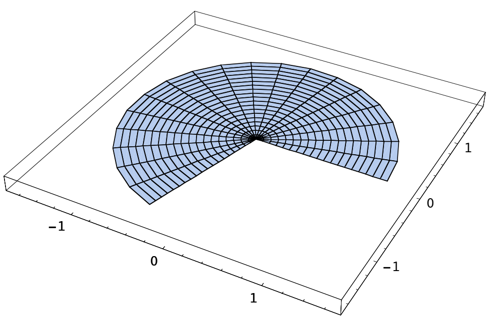


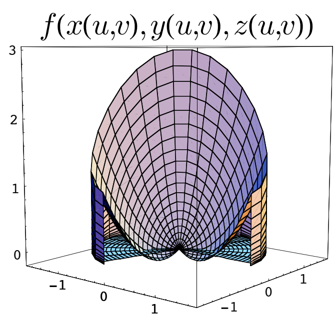


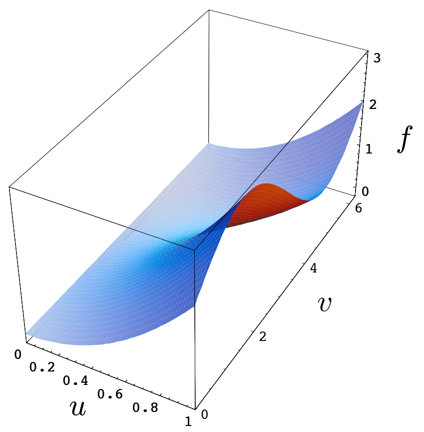
with transparency

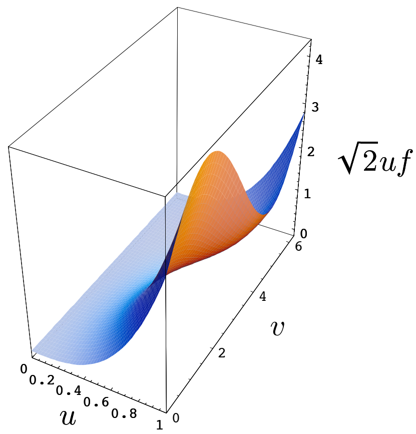


integrals









Example 11.1.2 — helicoid

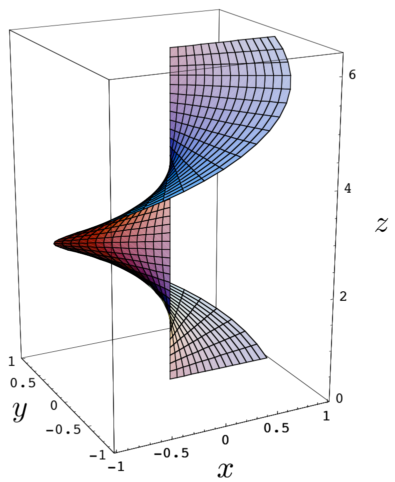
For the ***helicoid*** $\Phi : [0\,,\,1] \times [0\,,\,2\,\pi] \to \mathbb{R}^3$ whereby $(u\,,\,v) \mapsto (u\cos v\,,\,u\sin v\,,\,v)$, and the function $f(x\,,\,y\,,\,z) = \sqrt{1+x^2+y^2}+z$, we have, writing $R = [0\,,\,1] \times [0\,,\,2\,\pi]$ and noting Example 10.2 .3,

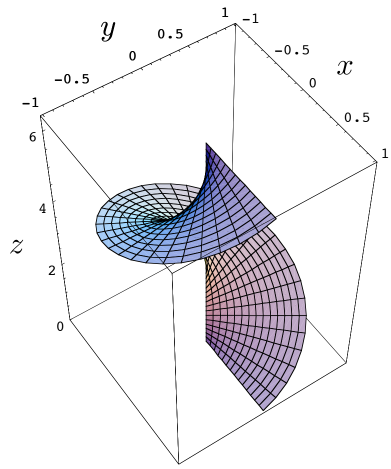
$$\int_\Phi f\,dS = \int\int_R f(u\cos v\,,\,u\sin v\,,\,v)\,\|(-\sin v\,,\,-\cos v\,,\,u)\|\,du\,dv = \int\int_R \left(\sqrt{1+u^2}+v\right)(1+u^2)^{\frac{1}{2}}\,du\,dv$$

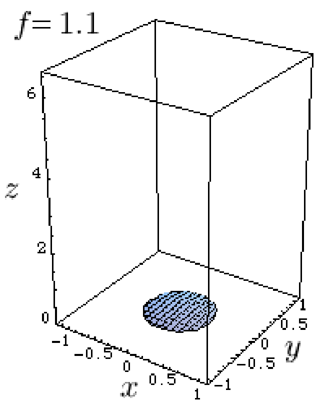
$$= \int\int_R (1+u^2)\,du\,dv + \int\int_R v\,(1+u^2)^{\frac{1}{2}}\,du\,dv = \left(\int_0^1 (1+u^2)\,du\right)\left(\int_0^{2\pi} 1\,dv\right) + \left(\int_0^1 (1+u^2)^{\frac{1}{2}}\,du\right)\left(\int_0^{2\pi} v\,dv\right)$$

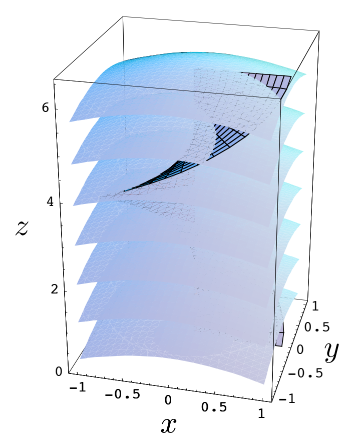
$$= \left[u + \frac{1}{3}u^3\right]_0^1 \times 2\,\pi + \frac{1}{2}\left(\sqrt{2}+\sinh^{-1}(1)\right) \times \left[\frac{1}{2}v^2\right]_0^{2\pi} = \frac{4}{3}\,\pi + \pi^2\left(\sqrt{2}+\log\left(1+\sqrt{2}\right)\right).$$

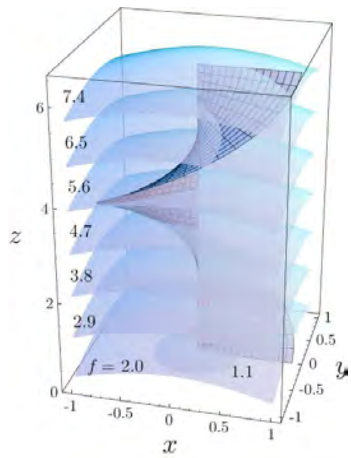
visualisation



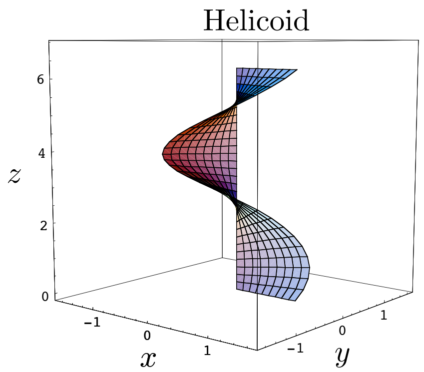


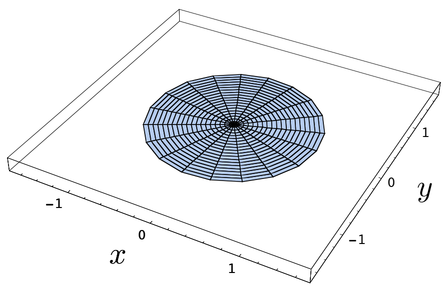


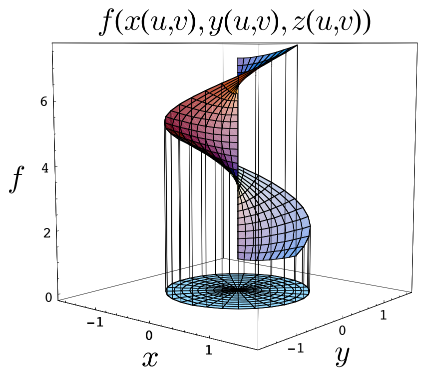

z
x
y

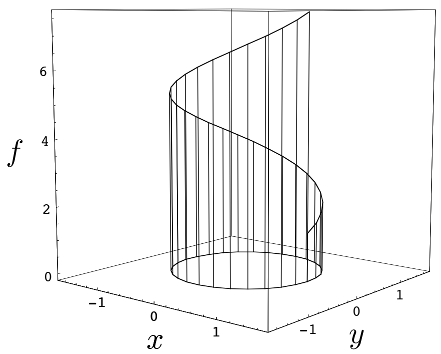

7.4
6.5
5.6
4.7
3.8
2.9
f = 2.0
1.1
z
x
y

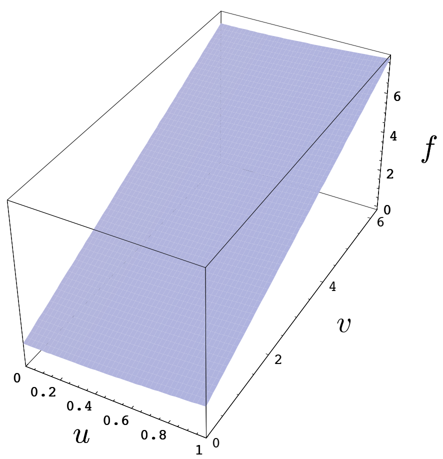
integrals

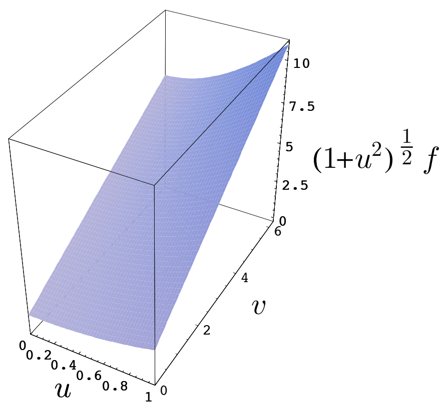

Helicoid
z
x
y


x
y


f(x(u,v), y(u,v), z(u,v))
f
x
y


f
x
y


f
u
v


(1+u^2)^(1/2) f
u
v

Example 11.1.3 — different parametrizations of the sphere

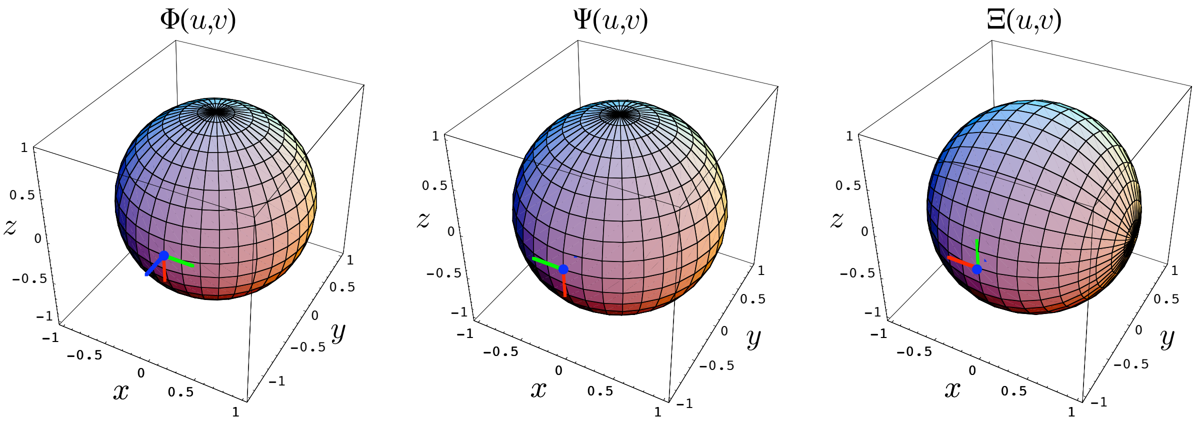
The three distinct parametrized surfaces:

$$\Phi : [0, \pi]\times[0, 2\pi] \to \mathbb{R}^3 \text{ whereby } (u, v) \mapsto (\sin u \cos v, \sin u \sin v, \cos u)$$

$$\Psi : [0, \pi]\times[0, 2\pi] \to \mathbb{R}^3 \text{ whereby } (u, v) \mapsto (\sin u \sin v, \sin u \cos v, \cos u)$$

$$\Xi : [0, \pi]\times[0, 2\pi] \to \mathbb{R}^3 \text{ whereby } (u, v) \mapsto (\cos u, \sin u \sin v, \sin u \cos v)$$

satisfy $\Phi([0, \pi]\times[0, 2\pi]) = \Psi([0, \pi]\times[0, 2\pi]) = \Xi([0, \pi]\times[0, 2\pi]) = S$, which is the unit sphere in $\mathbb{R}^3$.



We have shown earlier that for the parametrized surface Φ, we have $\boldsymbol{t}_u\times\boldsymbol{t}_v = (\sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u)$.

Similarly, it can be shown that for the parametrized surfaces Ψ and Ξ, we have respectively:

$$\Psi : \ \boldsymbol{t}_u\times\boldsymbol{t}_v = (-\sin^2 u \sin v, -\sin^2 u \cos v, -\cos u \sin u),$$

$$\Xi : \ \boldsymbol{t}_u\times\boldsymbol{t}_v = (-\cos u \sin u, -\sin^2 u \sin v, -\sin^2 u \cos v).$$

Now consider the function $f(x, y, z) = z^2$. We have, writing $R = [0, \pi]\times[0, 2\pi]$ and noting Example 10.2.2,

$$\int_\Phi f\, dS = \int\int_R f(\sin u \cos v, \sin u \sin v, \cos u)\ \|(\sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u)\|\, du\, dv$$

$$= \int\int_R \cos^2 u \ |\sin u|\ du\, dv = \left(\int_0^\pi \cos^2 u \sin u\, du\right)\left(\int_0^{2\pi} 1\, dv\right) = \left(\int_{-1}^1 h^2\, dh\right)\times 2\pi = \frac{4}{3}\pi,$$

after using a substitution $h = -\cos u$. Similarly

$$\int_\Psi f\, dS = \int\int_R f(\sin u \sin v, \sin u \cos v, \cos u)\ \|(-\sin^2 u \sin v, -\sin^2 u \cos v, -\cos u \sin u)\|\, du\, dv$$

$$= \int\int_R \cos^2 u \ |\sin u|\ du\, dv = \frac{4}{3}\pi.$$

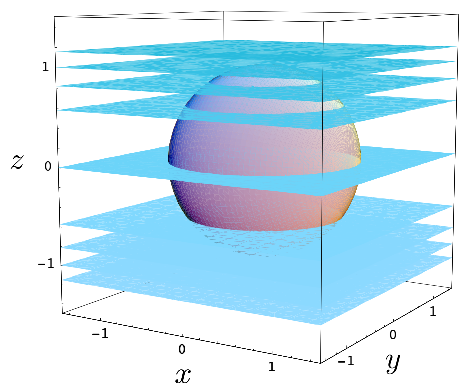
$$\int_\Xi f\, dS = \int\int_R f(\cos u, \sin u \sin v, \sin u \cos v)\ \|(-\cos u \sin u, -\sin^2 u \sin v, -\sin^2 u \cos v)\|\, du\, dv$$

$$= \int\int_R \sin^2 u \cos^2 v \ |\sin u|\ du\, dv = \left(\int_0^\pi \sin^3 u\, du\right)\left(\int_0^{2\pi} \cos^2 v\, dv\right)$$

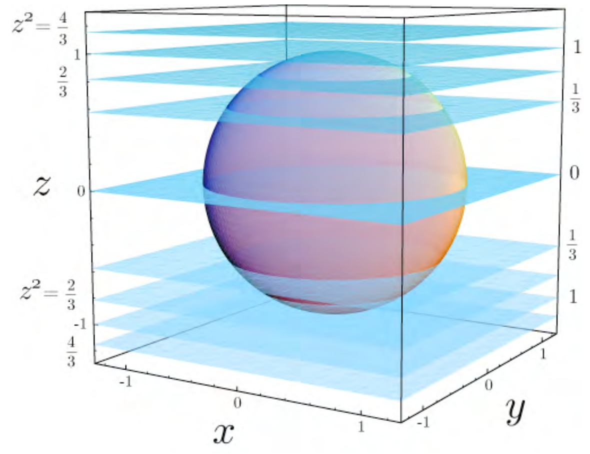
$$= \left(\int_0^\pi (1-\cos^2 u)\sin u\, du\right)\left(\int_0^{2\pi} \frac{1}{2}(1+\cos 2v)\, dv\right) = \left(\int_{-1}^1 (1-h^2)\, dh\right)\times\left(\frac{1}{2}\times 2\pi\right) = \left(2-\frac{2}{3}\right)\pi = \frac{4}{3}\pi.$$

Note that the three integrals have the same value. We shall discuss this in greater detail in Section 11.3.

visualisation



with transparency



11.2 Surface integrals, of vector fields

Suppose that the parametrized surface $\Phi : R \to \mathbb{R}^3$ whereby $(u, v) \mapsto \Phi(u, v) = (x(u, v), y(u, v), z(u, v))$ is continuously differentiable.

For any vector field $\boldsymbol{F}(x, y, z)$ such that the composition function $\boldsymbol{F} \circ \Phi : R \to \mathbb{R}^3 : (u, v) \mapsto \boldsymbol{F}(\Phi(u, v)) = \boldsymbol{F}(x(u, v), y(u, v), z(u, v))$ is continuous, we define:

$$\int_\Phi \boldsymbol{F} \cdot d\boldsymbol{S} = \int_\Phi \boldsymbol{F}(x, y, z) \cdot d\boldsymbol{S} \overset{\text{def}}{=} \int\int_R \boldsymbol{F}(\Phi(u, v)) \cdot (\boldsymbol{t}_u \times \boldsymbol{t}_v)\, du\, dv .$$

Remarks

- Here $d\boldsymbol{S} = (\boldsymbol{t}_u \times \boldsymbol{t}_v)\, du\, dv$ is the parametrized surface analogue of the velocity differential $d\boldsymbol{s} = \phi'(t)\, dt$ of a path $\boldsymbol{\phi}$.

- Clearly $\int_\Phi \boldsymbol{F} \cdot d\boldsymbol{S} = \int\int_R \boldsymbol{F}(\Phi(u, v)) \cdot (\boldsymbol{t}_u \times \boldsymbol{t}_v)\, du\, dv = \int\int_R \boldsymbol{F}(\Phi(u, v)) \cdot \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right) du\, dv.$

- Note that $\boldsymbol{F}$ has only to be defined on the image surface $S = \Phi(R)$ of the parametrized surface Φ for our definition to make sense. The continuity of the composition function $\boldsymbol{F} \circ \Phi$ on the elementary region R ensures the existence of the integral.

- Sometimes, Φ may only be piecewise continuously differentiable. As in the last section, we can calculate the corresponding integral for each subregion in a partition of the region R and consider the sum of the integrals.

- Note that if $\boldsymbol{t}_u \times \boldsymbol{t}_v \neq \boldsymbol{0}$ for every $(u, v) \in R$, then

$$\int_\Phi \boldsymbol{F} \cdot d\boldsymbol{S} = \int\int_R \boldsymbol{F}(\Phi(u, v)) \cdot (\boldsymbol{t}_u \times \boldsymbol{t}_v)\, du\, dv = \int\int_R \boldsymbol{F}(\Phi(u, v)) \cdot \frac{\boldsymbol{t}_u \times \boldsymbol{t}_v}{\|\boldsymbol{t}_u \times \boldsymbol{t}_v\|} \|\boldsymbol{t}_u \times \boldsymbol{t}_v\|\, du\, dv = \int\int_R f(\Phi(u, v))\, \|\boldsymbol{t}_u \times \boldsymbol{t}_v\|\, du\, dv$$

where $f(\boldsymbol{\Phi}(u, v)) \overset{\text{def}}{=} \boldsymbol{F}(\Phi(u, v)) \cdot \frac{\boldsymbol{t}_u \times \boldsymbol{t}_v}{\|\boldsymbol{t}_u \times \boldsymbol{t}_v\|}$.

Here $\boldsymbol{n} = \frac{\boldsymbol{t}_u \times \boldsymbol{t}_v}{\|\boldsymbol{t}_u \times \boldsymbol{t}_v\|}$ is the unit vector *normal to* the parametrized surface Φ . Then $\int_\Phi \boldsymbol{F} \cdot d\boldsymbol{S} = \int_\Phi \boldsymbol{F} \cdot \boldsymbol{n}\, dS$, so that the integral over the parametrized surface now becomes one of the type discussed in the last section.

Example 11.2.1 — on the cone

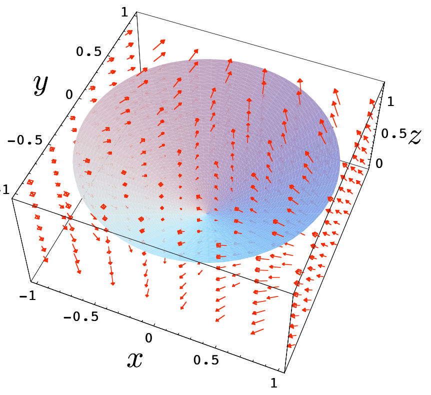
For the ***parametrized cone*** $\Phi : [0, 1] \times [0, 2\pi] \to \mathbb{R}^3$ whereby $(u, v) \mapsto (u \cos v, u \sin v, u)$, and the vector field $\boldsymbol{F}(x, y, z) = (-x, y, z)$,

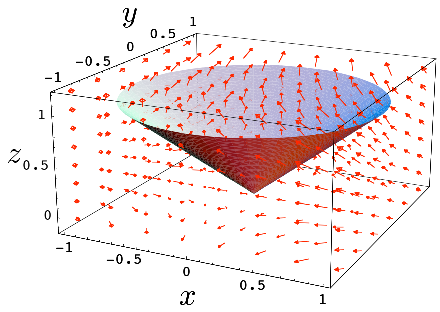
we have, writing $R = [0, 1] \times [0, 2\pi]$ and noting Example 10.2 .1,

$$\int_\Phi \boldsymbol{F} \cdot d\boldsymbol{S} = \int\int_R \boldsymbol{F}(u \cos v, u \sin v, u) \cdot (-u \cos v, -u \sin v, u)\, du\, dv = \int\int_R (-u \cos v, u \sin v, u) \cdot (-u \cos v, -u \sin v, u)\, du\, dv$$

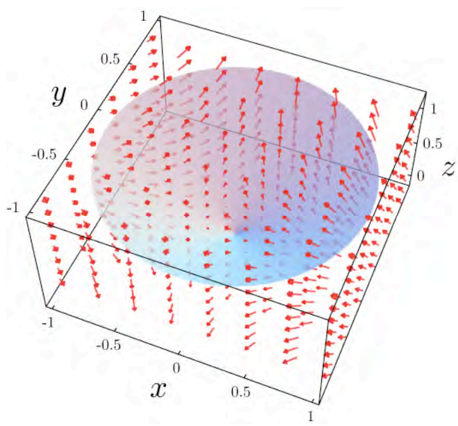
$$= \int\int_R u^2 (\cos^2 v - \sin^2 v + 1)\, du\, dv = \int\int_R 2 u^2 \cos^2 v\, du\, dv = 2 \left(\int_0^1 u^2\, du \right) \left(\int_0^{2\pi} \frac{1}{2} (1 + \cos 2 v)\, dv \right) = 2 \times \frac{1}{3} \times \left(\frac{1}{2} \times 2\pi \right) = \frac{2}{3} \pi .$$

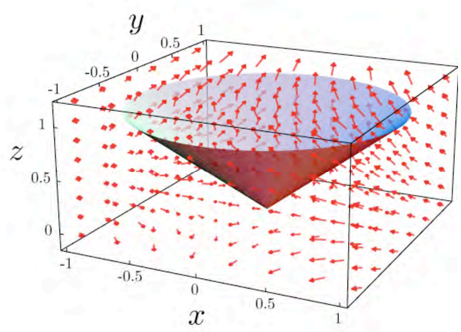
visualisation





with transparency





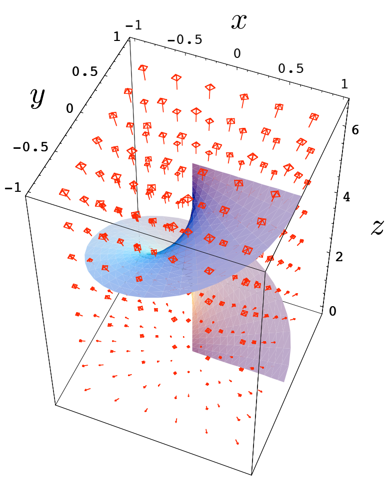
Example 11.2.2 — on the helicoid

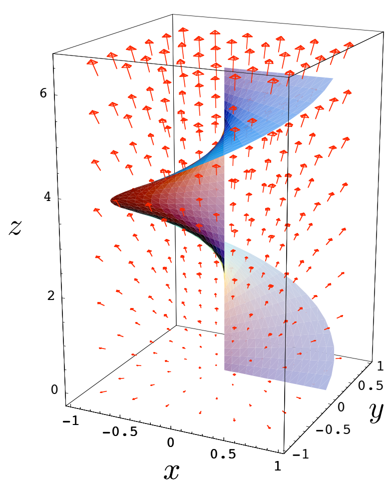
For the ***helicoid*** $\Phi : [0, 1] \times [0, 2\pi] \to \mathbb{R}^3$ whereby $(u, v) \mapsto (u \cos v, u \sin v, v)$, and the vector field $\boldsymbol{F}(x, y, z) = (x, y, z)$, we have, writing $R = [0, 1] \times [0, 2\pi]$ and noting Example 10.2 .3,

$$\int_\Phi \boldsymbol{F} \cdot d\boldsymbol{S} = \int\int_R \boldsymbol{F}(u \cos v, u \sin v, v) \cdot (\sin v, -\cos v, u)\ du\, dv = \int\int_R (u \cos v, u \sin v, v) \cdot (\sin v, -\cos v, u)\, du\, dv$$

$$= \int\int_R u\, v\, du\, dv = \left(\int_0^1 u\, du\right)\left(\int_0^{2\pi} v\, dv\right) = \frac{1}{2} \times \frac{1}{2} \times (2\pi)^2 = \pi^2 .$$

visualisation





Example 11.2.3 — different parametrizations of the sphere

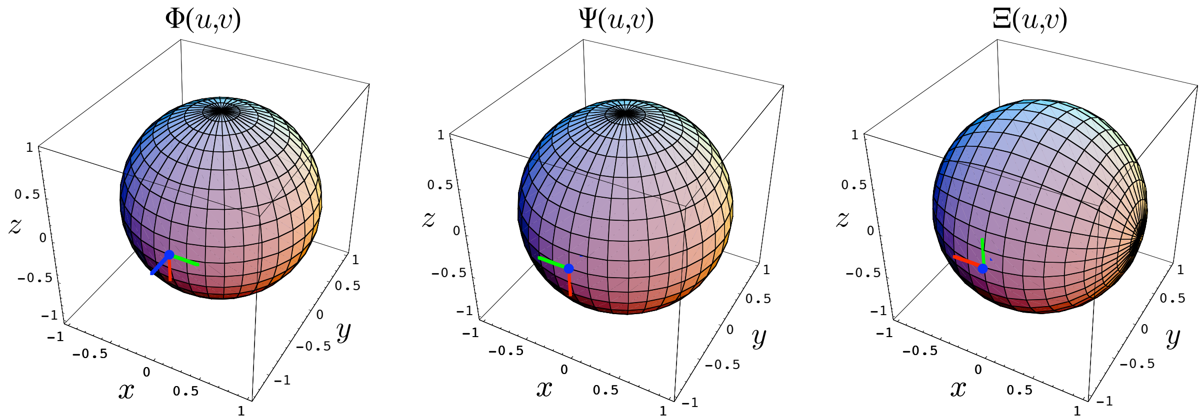
The three distinct parametrized surfaces:

$$\Phi : [0\,,\,\pi] \times [0\,,\,2\,\pi] \to \mathbb{R}^3 \text{ whereby } (u\,,\,v) \mapsto (\sin u\ \cos v\,,\,\sin u\ \sin v\,,\,\cos u)$$

$$\Psi : [0\,,\,\pi] \times [0\,,\,2\,\pi] \to \mathbb{R}^3 \text{ whereby } (u\,,\,v) \mapsto (\sin u\ \sin v\,,\,\sin u\ \cos v\,,\,\cos u)$$

$$\Xi : [0\,,\,\pi] \times [0\,,\,2\,\pi] \to \mathbb{R}^3 \text{ whereby } (u\,,\,v) \mapsto (\cos u\,,\,\sin u\ \sin v\,,\,\sin u\ \cos v)$$

satisfy $\Phi([0\,,\,\pi] \times [0\,,\,2\,\pi]) = \Psi([0\,,\,\pi] \times [0\,,\,2\,\pi]) = \Xi([0\,,\,\pi] \times [0\,,\,2\,\pi]) = S$, which is the unit sphere in $\mathbb{R}^3$.



We have respectively

$$\boldsymbol{t}_u \times \boldsymbol{t}_v = (\sin^2 u\ \cos v\,,\,\sin^2 u\ \sin v\,,\,\cos u\ \sin u) \text{ on } \Phi\,;$$

$$\boldsymbol{t}_u \times \boldsymbol{t}_v = (-\sin^2 u\ \sin v\,,\,-\sin^2 u\ \cos v\,,\,-\cos u\ \sin u) \text{ on } \Psi\,; \text{ and}$$

$$\boldsymbol{t}_u \times \boldsymbol{t}_v = (-\cos u\ \sin u\,,\,-\sin^2 u\ \sin v\,,\,-\sin^2 u\ \cos v) \text{ on } \Xi\,.$$

Now consider the vector field $\boldsymbol{F}(x,\,y,\,z) = (x\,z\,,\,y\,z\,,\,z)$. We have, writing $R = [0\,,\,\pi] \times [0\,,\,2\,\pi]$,

$$\int_\Phi \boldsymbol{F} \cdot d\boldsymbol{S} = \int\int_R \boldsymbol{F}(\sin u\ \cos v\,,\,\sin u\ \sin v\,,\,\cos u) \cdot (\sin^2 u\ \cos v\,,\,\sin^2 u\ \sin v\,,\,\cos u\ \sin u)\,d\,u\,d\,v$$

$$= \int\int_R (\sin u\ \cos u\ \cos v\,,\,\sin u\ \cos u\ \sin v\,,\,\cos u) \cdot (\sin^2 u\ \cos v\,,\,\sin^2 u\ \sin v\,,\,\cos u\ \sin u)\,d\,u\,d\,v$$

$$= \int\int_R (\sin^3 u\ \cos u\ + \cos^2 u\ \sin u)\,d\,u\,d\,v = \left(\int_0^\pi (\sin^3 u\ \cos u\ + \cos^2 u\ \sin u)\ d\,u\right)\left(\int_0^{2\pi} 1\,d\,v\right)$$

$$= \left(2\int_0^{\frac{1}{2}\pi} \cos^2 u\ \sin u\,d\,u\right) \times 2\,\pi\ =\ \tfrac{4}{3}\,\pi\,,$$

$$\int_\Psi \boldsymbol{F} \cdot d\boldsymbol{S} = \int\int_R \boldsymbol{F}(\sin u\ \sin v\,,\,\sin u\ \cos v\,,\,\cos u) \cdot (-\sin^2 u\ \sin v\,,\,-\sin^2 u\ \cos v\,,\,-\cos u\ \sin u)\,d\,u\,d\,v$$

$$= \int\int_R (\sin u\ \sin v\,,\,\sin u\ \cos v\,,\,\cos u) \cdot (-\sin^2 u\ \sin v\,,\,-\sin^2 u\ \cos v\,,\,-\cos u\ \sin u)\,d\,u\,d\,v$$

$$= \int\int_R (-\sin^3 u\ \cos u\ - \cos^2 u\ \sin u)\,d\,u\,d\,v = \left(-\int_0^\pi (\sin^3 u\ \cos u\ + \cos^2 u\ \sin u)\ d\,u\right)\left(\int_0^{2\pi} 1\,d\,v\right)$$

$$= -\left(2\int_0^{\frac{1}{2}\pi} \cos^2 u\ \sin u\,d\,u\right) \times 2\,\pi\ =\ -\tfrac{4}{3}\,\pi\,,$$

$$\int_\Xi \boldsymbol{F} \cdot d\boldsymbol{S} = \int\int_R \boldsymbol{F}(\cos u\,,\,\sin u\ \sin v\,,\,\sin u\ \cos v) \cdot (-\cos u\ \sin u\,,\,-\sin^2 u\ \sin v\,,\,-\sin^2 u\ \cos v)\,d\,u\,d\,v$$

$$= \int\int_R (\cos u\ \sin u\ \cos v\,,\,\sin^2 u\ \cos v\ \sin v\,,\,\sin u\ \cos v) \cdot (-\cos u\ \sin u\,,\,-\sin^2 u\ \sin v\,,\,-\sin^2 u\ \cos v)\,d\,u\,d\,v$$

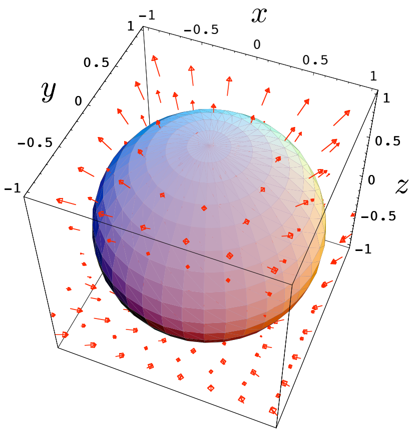
$$= \int\int_R -(\cos^2 u \sin^2 u \cos v + \sin^4 u \sin^2 v \cos v + \sin^3 u \cos^2 v)\,d\,u\,d\,v$$

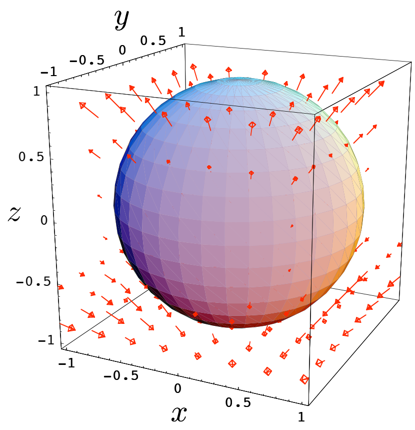
$$= -\left(\int_0^\pi \cos^2 u \sin^2 u\,d\,u\right)\left(\int_0^{2\pi} \cos v\,d\,v\right) - \left(\int_0^\pi \sin^4 u\ d\,u\right)\left(\int_0^{2\pi} \sin^2 v \cos v\,d\,v\right) - \left(\int_0^\pi \sin^3 u\,d\,u\right)\left(\int_0^{2\pi} \cos^2 v\,d\,v\right)$$

$$= (? \times 0) - (? \times 0) - \left(\int_0^\pi (1 - \cos^2 u\,) \sin u\,d\,u\right)\left(\int_0^{2\pi} \cos^2 v\,d\,v\right) = -\left(\int_{-1}^1 (1 - h^2)\,d\,h\right)\left(\int_0^{2\pi} \tfrac{1}{2}(1 + \cos 2\,v)\,d\,v\right) = -\tfrac{4}{3}\,\pi\,.$$

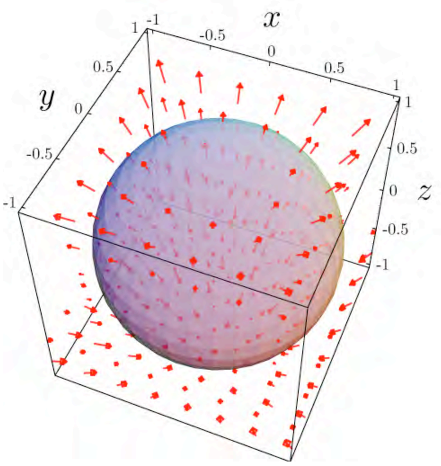
Note that the three integrals differ only in sign. We shall discuss this in greater detail in Section 11.3.

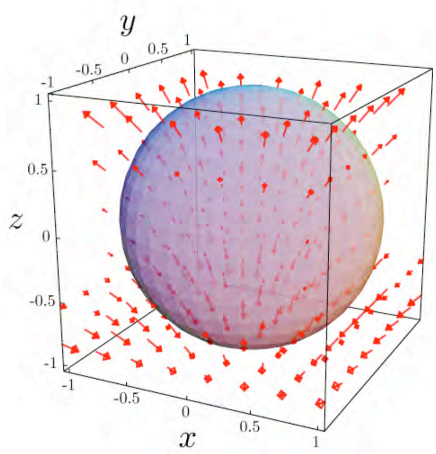
visualisation





with transparency





11.3 Equivalent parametrized surfaces

Just as most curves have two endpoints, many surfaces have two sides. For our discussion here, we shall ignore surfaces like the Möbius strip, and consider only those surfaces in $\mathbb{R}^3$ which have two sides.

Remark — Möbius strip

- The Möbius strip has only one side. To construct it, take a long rectangular strip of paper as shown below.

Hold the edge $\overline{a\,b}$ stationary and give the edge $\overline{c\,d}$ a 180° twist. Now join the edges $\overline{a\,b}$ and $\overline{c\,d}$ so that a coincides with d, and b coincides with c; then admire your artwork.

image from Wikipedia: http://en.wikipedia.org/wiki/Image:Möbius_strip.jpg

Let us return to Examples 11.1.3 and 11.2.3. Here the unit sphere S is the range of the parametrized surfaces. The surface of the unit sphere clearly has two sides, the *inside* and the *outside*.

For the parametrized surface Φ, we have $\boldsymbol{t}_u \times \boldsymbol{t}_v = (\sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u) = (\sin u)\,\Phi(u, v)$. Note that $\sin u \geq 0$ for $0 \leq u \leq \pi$, and so the vector $\boldsymbol{t}_u \times \boldsymbol{t}_v$ at $\Phi(u, v)$ points *away from* the origin in this parametrization of S. (see left image)

For the parametrized surface Ψ, we have $\boldsymbol{t}_u \times \boldsymbol{t}_v = (-\sin^2 u \sin v, -\sin^2 u \cos v, -\cos u \sin u) = (-\sin u)\,\Psi(u, v)$. Hence the vector $\boldsymbol{t}_u \times \boldsymbol{t}_v$ at $\Psi(u, v)$ points *towards* the origin. (see middle image)

Definition: equivalent surfaces

Suppose that $\Phi : R_1 \to \mathbb{R}^3$ and $\Psi : R_1 \to \mathbb{R}^3$ are continuously differentiable parametrized surfaces. Then we say that Φ and Ψ are ***equivalent*** if there exists a piecewise continuously differentiable function $h : R_1 \to R_2$ satisfying the following conditions:

(ES1) $h : R_1 \to R_2$ is essentially one-to-one and onto;

(ES2) $\Phi = \Psi \circ h$;

(ES3) writing $(s, t) = h(u, v)$, we have either $\dfrac{\partial(s,t)}{\partial(u,v)} \geq 0$ for every $(u, v) \in R_1$, $\quad(*)$

or $\dfrac{\partial(s,t)}{\partial(u,v)} \leq 0$ for every $(u, v) \in R_1$. $\quad(**)$

In this case, we say that h defines a ***change of parameters***. Furthermore, we say that the change of parameters is: ***orientation preserving*** if $(*)$ holds, and ***orientation reversing*** if $(**)$ holds.

Remark

- The condition (ES1) is essential* for integration of double integrals by change of variables. The need for condition (ES3) will become clear from the sketched proofs of Theorems 11A and 11B later in this section.

* For more details, students are referred to Chapter 6 (studied in MATH235).

Example 11.3.1 — sides of a sphere

The parametrized surfaces

$$\Phi : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3 : (u, v) \mapsto (\sin u \cos v, \sin u \sin v, \cos u),$$

$$\Psi : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3 : (s, t) \mapsto (\sin s \sin t, \sin s \cos t, \cos s)$$

are equivalent. (These are the first two parametrizations in Example 11.2.3.)

To see this, consider the function $h : [0, \pi] \times [0, 2\pi] \to [0, \pi] \times [0, 2\pi]$ where $(u, v) \mapsto (s, t)$ with $s = u$ and $t = \begin{cases} \frac{1}{2}\pi - v & \text{for } v \leq \frac{1}{2}\pi \\ \frac{5}{2}\pi - v & \text{for } v > \frac{1}{2}\pi. \end{cases}$

Clearly h is essentially one-to-one and onto. Furthermore, $\dfrac{\partial(s,t)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -1$, so that the change of parameters is *orientation reversing*.

Theorem 11A:

Suppose that $\Phi : R_1 \to \mathbb{R}^3$ and $\Psi : R_2 \to \mathbb{R}^3$ are two equivalent smooth continuously differentiable parametrized surfaces. Then for any real-valued function $f(x, y, z)$ such that the composition functions $f \circ \Phi : R_1 \to \mathbb{R}$ and $f \circ \Psi : R_2 \to \mathbb{R}$ are continuous, we have

$$\int_\Phi f\, dS = \int_\Psi f\, dS.$$

Sketch of proof

Since Φ and Ψ are equivalent, there exists $h : [A_1, B_1] \to [A_2, B_2]$ such that $\Phi = \Psi \circ h$. Now

$$\int_\Phi f\, dS = \iint_{R_1} f(\Phi(u, v))\, \left\| \left(\frac{\partial(\Phi_2,\Phi_3)}{\partial(u,v)}, \frac{\partial(\Phi_3,\Phi_1)}{\partial(u,v)}, \frac{\partial(\Phi_1,\Phi_2)}{\partial(u,v)} \right) \right\|\, du\, dv.$$

By the Chain rule and writing $(s, t) = h(u, v)$, we have $\dfrac{\partial(\Phi_i,\Phi_j)}{\partial(u,v)} = \dfrac{\partial(\Psi_i,\Psi_j)}{\partial(s,t)}\, \dfrac{\partial(s,t)}{\partial(u,v)}$, for each $1 \leq i \neq j \leq 3$.

It follows that
$$\int_\Phi f\, dS = \iint_{R_1} f(\Psi(s, t))\, \left\| \left(\frac{\partial(\Psi_2,\Psi_3)}{\partial(s,t)}, \frac{\partial(\Psi_3,\Psi_1)}{\partial(s,t)}, \frac{\partial(\Psi_1,\Psi_2)}{\partial(s,t)} \right) \right\|\, \left| \frac{\partial(s,t)}{\partial(u,v)} \right|\, du\, dv$$
$$= \iint_{R_2} f(\Psi(s, t))\, \left\| \left(\frac{\partial(\Psi_2,\Psi_3)}{\partial(s,t)}, \frac{\partial(\Psi_3,\Psi_1)}{\partial(s,t)}, \frac{\partial(\Psi_1,\Psi_2)}{\partial(s,t)} \right) \right\|\, ds\, dt = \int_\Psi f\, dS.$$

This completes the proof.

Theorem 11B:

Suppose that $\Phi : R_1 \to \mathbb{R}^3$ and $\Psi : R_2 \to \mathbb{R}^3$ are two equivalent smooth continuously differentiable parametrized surfaces. Then for any vector field $\boldsymbol{F}(x, y, z)$ such that the composition functions $\boldsymbol{F}\circ\Phi : R_1 \to \mathbb{R}^3$ and $\boldsymbol{F}\circ\Psi : R_2 \to \mathbb{R}^3$ are continuous, we have

$$\int_\Phi \boldsymbol{F}\cdot d\boldsymbol{S} = \pm \int_\Psi \boldsymbol{F}\cdot d\boldsymbol{S},$$

where the equality holds: (i) with the $+$sign if the change of parameters is *orientation preserving*; and (ii) with the $-$sign if the change of parameters is *orientation reversing*.

Sketch of proof

Since Φ and Ψ are equivalent, there exists $h : [A_1, B_1] \to [A_2, B_2]$ such that $\Phi = \Psi\circ h$. Now

$$\begin{aligned}\int_\Phi \boldsymbol{F}\cdot d\boldsymbol{S} &= \iint_{R_1} \boldsymbol{F}(\Phi(u, v))\cdot\left(\frac{\partial(\Phi_2,\Phi_3)}{\partial(u,v)}, \frac{\partial(\Phi_3,\Phi_1)}{\partial(u,v)}, \frac{\partial(\Phi_1,\Phi_2)}{\partial(u,v)}\right) du\, dv \\ &= \iint_{R_1} \boldsymbol{F}(\Psi(s, t))\cdot\left(\frac{\partial(\Psi_2,\Psi_3)}{\partial(s,t)}, \frac{\partial(\Psi_3,\Psi_1)}{\partial(s,t)}, \frac{\partial(\Psi_1,\Psi_2)}{\partial(s,t)}\right)\left(\frac{\partial(s,t)}{\partial(u,v)}\right) du\, dv \\ &= \iint_{R_2} \boldsymbol{F}(\Psi(s, t))\cdot\left(\frac{\partial(\Psi_2,\Psi_3)}{\partial(s,t)}, \frac{\partial(\Psi_3,\Psi_1)}{\partial(s,t)}, \frac{\partial(\Psi_1,\Psi_2)}{\partial(s,t)}\right) ds\, dt = \int_\Psi \boldsymbol{F}\cdot d\boldsymbol{S}.\end{aligned}$$

This completes the proof.

Remark

- Theorems 11A and 11B have natural extensions to the case when the paths are piecewise continuously differentiable.

  In this case, one can clearly break the paths into continuously differentiable pieces and apply Theorems 11A and 11B to each piece.

11.4 Parametrization of surfaces

As discussed at the beginning of the last section, we shall restrict our attention to surfaces in $\mathbb{R}^3$ which have two sides and are smooth, except possibly at a finite number of points. Our first task is to define an ***orientation*** for such surfaces.

Suppose that x is a point on a smooth surface S. Then if $\boldsymbol{n}$ is a unit vector normal to the surface S at x, then $-\boldsymbol{n}$ is also a unit vector normal to the surface S at x, but in the opposite direction.

We now need to make a choice as to which side of the surface we consider to be the positive side and which side we consider to be the negative side. Having made such a choice, we now take unit normal vectors $\boldsymbol{n}$ to be those that point from the negative side of the surface towards the positive side. In this case, we say that S is an ***oriented surface***.

Example 11.4.1 — sphere

Suppose that S is the unit sphere in $\mathbb{R}^3$, and we choose the outside surface to be the positive side. Then unit normal vectors point *away from the origin*. (see left image)

Example 11.4.2 — plane

Suppose that S is the $x\,y$-plane in $\mathbb{R}^3$, and we choose the bottom surface to be the positive side. Then unit normal vectors are of the form $(0, 0, -1)$.

Recall now that for any smooth parametrized surface $\Phi : R \to \mathbb{R}^3$, the unit vector $\dfrac{\boldsymbol{t}_u\times\boldsymbol{t}_v}{\|\boldsymbol{t}_u\times\boldsymbol{t}_v\|}$ is normal to the surface at the point $\Phi(u, v)$.

Definition: parametrization of a surface

Suppose that S is an oriented surface in $\mathbb{R}^3$. Then a piecewise continuously differentiable function $\Phi : R \to \mathbb{R}^3$ such that $\Phi(R) = S$ is called a ***parametrization*** of S. We say that the parametrization Φ is:

(i) ***orientation preserving*** if $\dfrac{\boldsymbol{t}_u\times\boldsymbol{t}_v}{\|\boldsymbol{t}_u\times\boldsymbol{t}_v\|} = \boldsymbol{n}$ at every point $\Phi(u, v)$ which is smooth; and

(ii) ***orientation reversing*** if $\dfrac{\boldsymbol{t}_u\times\boldsymbol{t}_v}{\|\boldsymbol{t}_u\times\boldsymbol{t}_v\|} = -\boldsymbol{n}$ at every point $\Phi(u, v)$ which is smooth.

Definition: surface integral

Suppose that S is an oriented surface in $\mathbb{R}^3$. For any real valued function $f(x, y, z)$ continuous on S, we can *define*

$$\int_S f\, d\boldsymbol{S} = \int_\Phi f\, d\boldsymbol{S},$$

where Φ is **any** parametrization of S. For any vector field $\boldsymbol{F}(x, y, z)$ continuous on S, we can *define*

$$\int_S \boldsymbol{F} \cdot d\boldsymbol{S} = \int_\Phi \boldsymbol{F} \cdot d\boldsymbol{S},$$

where Φ is **any** *orientation-preserving* parametrization of S.

Example 11.4.3 — sphere

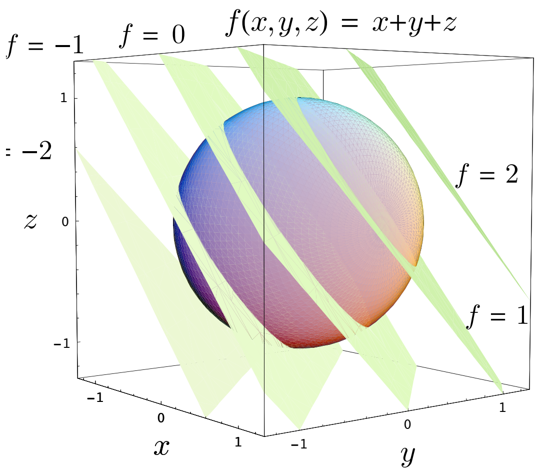
Suppose that S denotes the unit sphere $x^2 + y^2 + z^2 = 1$. Let $f(x, y, z) = x + y + z$, and consider the integral $\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$.

Now $\Phi : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$ whereby $(u, v) \mapsto (\sin u \cos v, \sin u \sin v, \cos u)$ is a parametrization of S.

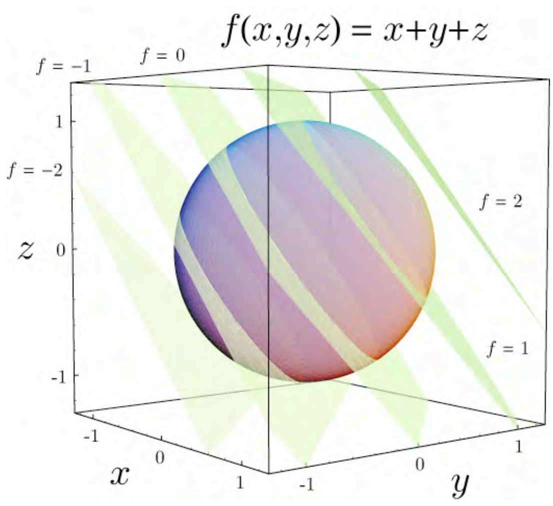
We have, writing $R = [0, \pi] \times [0, 2\pi]$,

$$\begin{aligned}
\int_S f\, d\boldsymbol{S} &= \int_\Phi f\, d\boldsymbol{S} = \iint_R f(\sin u \cos v, \sin u \sin v, \cos u)\, du\, dv \\
&= \iint_R (\sin u \cos v + \sin u \sin v + \cos u)\, \|(\sin u \cos v, \sin u \sin v, \cos u)\|\, du\, dv \\
&= \iint_R (\sin u \cos v + \sin u \sin v + \cos u)\, |\sin u|\, du\, dv \\
&= \left(\int_0^\pi \sin^2 u\, du\right)\left(\int_0^{2\pi} (\sin v + \cos v)\, dv\right) + \left(\int_0^\pi \cos u \sin u\, du\right)\left(\int_0^{2\pi} 1\, dv\right) \\
&= ? \times 0 + 0 \times 2\pi = 0.
\end{aligned}$$

visualisation

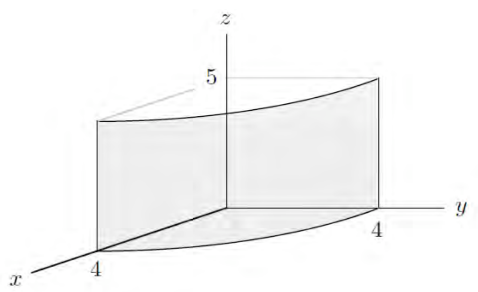


with transparency



Example 11.4.4 — portion of a cylinder

Let S denote the part of the cylinder $x^2 + y^2 = 16$ in the first octant between $z = 0$ and $z = 5$, with normal vector away from the z-axis.



Note that S can be parametrized by $\Phi : R \to \mathbb{R}^3 : (u, v) \mapsto (4 \sin u, 4 \cos u, v)$ where $R = [0, \frac{1}{2}\pi] \times [0, 5]$.

Let $F(x, y, z) = (z, x, -3y^2 z)$, and consider the integral $\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$. Note that

$$\boldsymbol{t}_u \times \boldsymbol{t}_v = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \times \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (4\cos u, -4\sin u, 0) \times (0, 0, 1) = -4(\sin u, \cos u, 0).$$

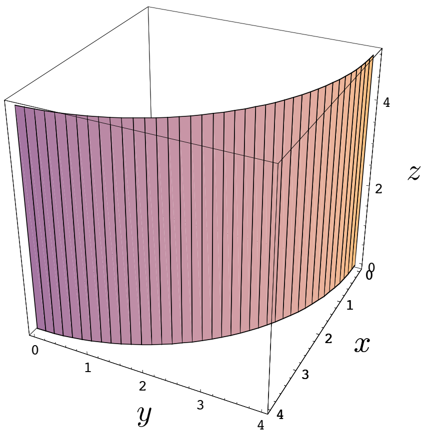
so Φ is an *orientation-reversing* parametrization of S.

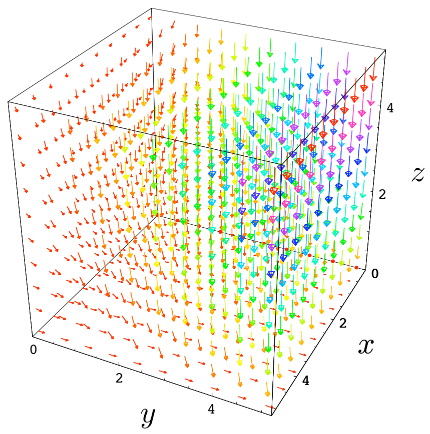
We have that $\int_S \boldsymbol{F} \cdot d\boldsymbol{S} = -\int_\Phi \boldsymbol{F} \cdot d\boldsymbol{S} = -\int\int_R \boldsymbol{F}(4\sin u, 4\cos u, v) \cdot (-4\sin u, -4\cos u, 0)\, du\, dv$

$$= -\int\int_R (v, 4\sin u, -48 v\cos^2 u) \cdot (-4\sin u, -4\cos u, 0)\, du\, dv = 4\int\int_R (v\sin u + 4\sin u\cos u)\, du\, dv$$

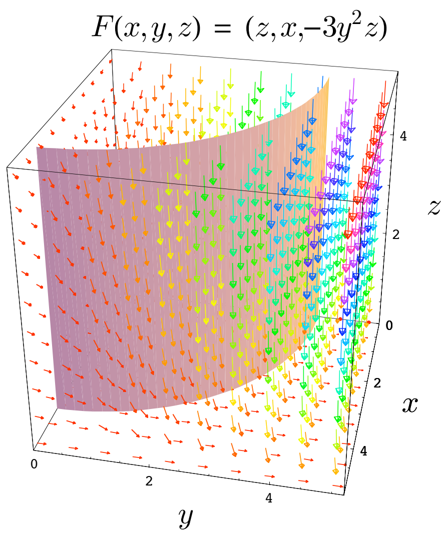
$$= 4\left(\int_0^{\frac{1}{2}\pi} \sin u\, du\right)\left(\int_0^5 v\, dv\right) + 8\left(\int_0^{\frac{1}{2}\pi} \sin 2u\, du\right)\left(\int_0^5 1\, dv\right) = (4 \times \tfrac{1}{2} \times 25) + (8 \times 1 \times 5) = 90.$$

visualisation

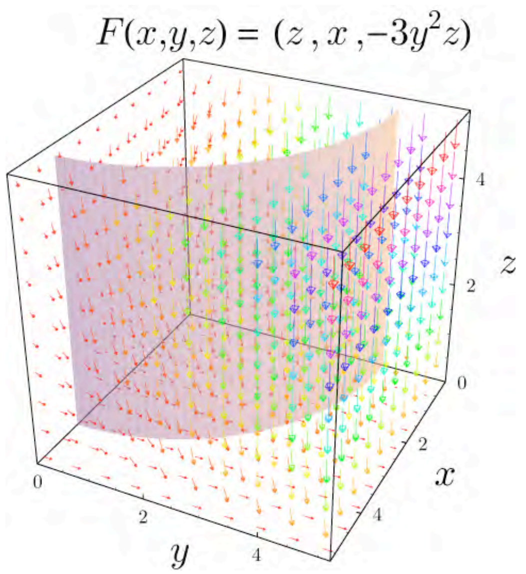


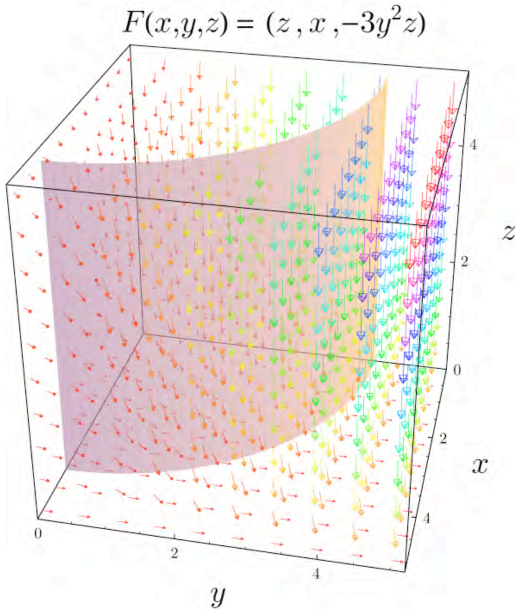


The vector-lengths have been log-scaled in the image at above-right.



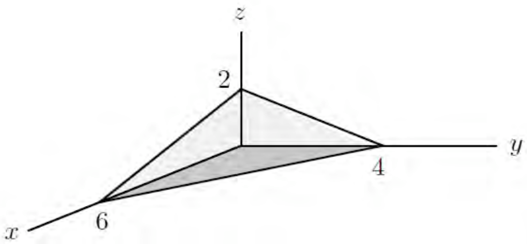
with transparency





Example 11.4.5 — triangular region

Let S denote the part of the plane $2x+3y+6z=12$ in the first octant, with normal vector away from the origin.



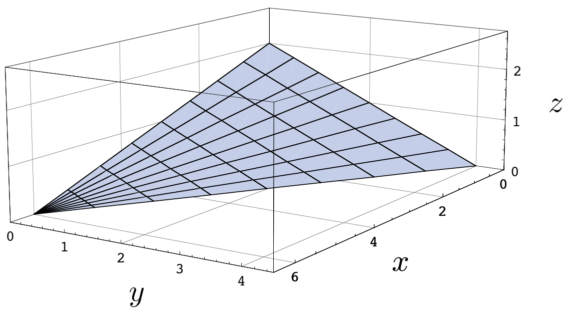
Note that S is a triangle with vertices $(6,0,0)$, $(0,4,0)$ and $(0,0,2)$, and can be parametrized by $\Phi : R \to \mathbb{R}^3$ whereby $(u,v) \mapsto (u, v, \frac{1}{6}(12-2u-3v))$, where R is the triangular region $R = \{(u,v) \in R^2 : u, v \geq 0$ and $2u+3v \leq 12\}$, so Φ is an *orientation-preserving* parametrization of S.

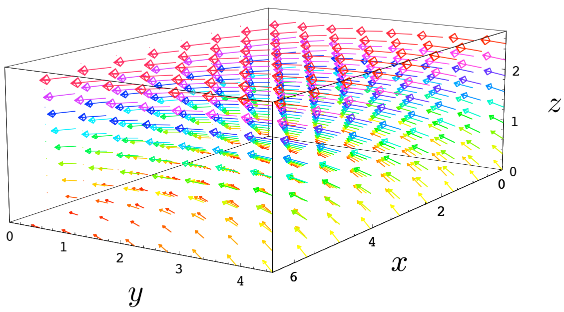
Let $F(x,y,z) = (18z, -12, 3y)$, and consider the integral $\int_S \boldsymbol{F}\cdot d\boldsymbol{S}$. Note that

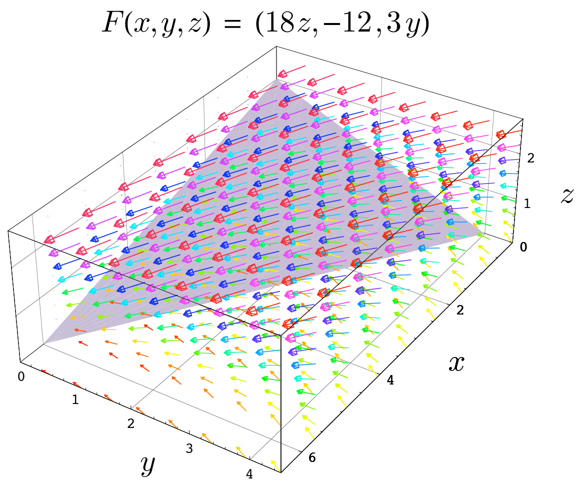
$$\boldsymbol{t}_u \times \boldsymbol{t}_v = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \times \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = \left(1, 0, -\tfrac{1}{3}\right) \times \left(0, 1, -\tfrac{1}{2}\right) = \left(\tfrac{1}{3}, \tfrac{1}{2}, 1\right).$$

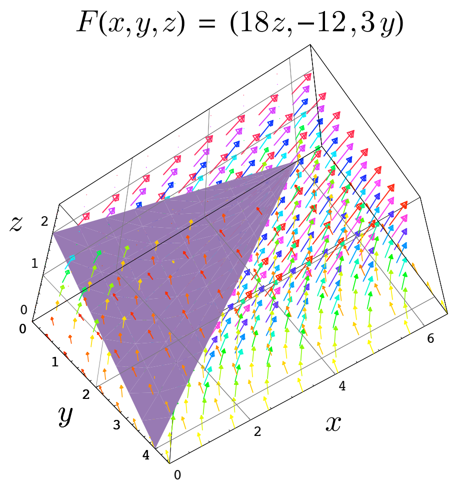
We have that
$$\int_S \boldsymbol{F}\cdot d\boldsymbol{S} = \int_\Phi \boldsymbol{F}\cdot d\boldsymbol{S} = \int\!\!\int_R \boldsymbol{F}\left(u, v, \tfrac{1}{6}(12-2u-3v)\right)\cdot\left(\tfrac{1}{3}, \tfrac{1}{2}, 1\right) du\, dv$$
$$= \int\!\!\int_R \left(3(12-2u-3v), -12, 3v\right)\cdot\left(\tfrac{1}{3}, \tfrac{1}{2}, 1\right) du\, dv = \int\!\!\int_R \left((12-2u-3v)+6-3v\right) du\, dv$$
$$= \int\!\!\int_R (6-2u)\, du\, dv = \int_0^4 \left(\int_0^{6-\frac{3}{2}v} (6-2u)\, du\right) dv = \int_0^4 \left[6u - u^2\right]_0^{6-\frac{3}{2}v} dv = \int_0^4 \left(9v - \tfrac{9}{4}v^2\right) dv$$
$$= 3\left[\tfrac{3}{2}v^2 - \tfrac{1}{4}v^3\right]_0^4 = 3(24-16) = 24.$$

visualisation



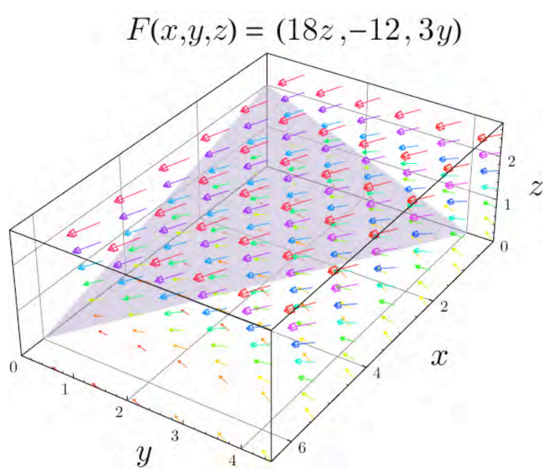


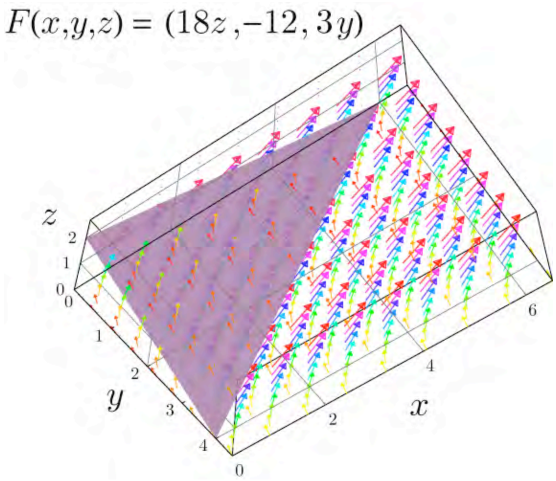




Viewed from above, it's hard to see how much flux is through the surface. The view from underheath shows this better.

with transparency





Example 11.4.6 — surface of a cube

Let S denote the surface of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, with outward normal vector. Let $F(x, y, z) = (x, y, z)$, and consider the integral $\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$.

To evaluate this integral, consider first of all the face S_1 with vertices $(\pm 1, \pm 1, 1)$. The function $\Phi_1 : [-1, 1] \times [-1, 1] \to \mathbb{R}^3$ whereby $(u, v) \mapsto (u, v, 1)$ is a parametrization of S_1. Note that $\boldsymbol{t}_u \times \boldsymbol{t}_v = (1, 0, 0) \times (0, 1, 0) = (0, 0, 1)$, so Φ is an *orientation-preserving* parametrization of S_1.

We have, writing $R = [-1, 1] \times [-1, 1]$, that

$$\int_{S_1} \boldsymbol{F} \cdot d\boldsymbol{S} = \int_{\Phi_1} \boldsymbol{F} \cdot d\boldsymbol{S} = \int\int_R \boldsymbol{F}(u, v, 1) \cdot (0, 0, 1)\, du\, dv = \int\int_R (u, v, 1) \cdot (0, 0, 1)\, du\, dv = \int\int_R 1\, du\, dv$$

$$= \left(\int_{-1}^{1} 1\, du\right)\left(\int_{-1}^{1} 1\, dv\right) = 2 \times 2 = 4.$$

Consider next the face S_2 with vertices $(\pm 1, \pm 1, -1)$. The function $\Psi_1 : [-1, 1] \times [-1, 1] \to \mathbb{R}^3 : (u, v) \mapsto (u, v, -1)$ is a parametrization of S_2. Note that $\boldsymbol{t}_u \times \boldsymbol{t}_v = (1, 0, 0) \times (0, 1, 0) = (0, 0, 1)$, so Ψ_1 is an *orientation-reversing* parametrization of S_2. We have, writing $R = [-1, 1] \times [-1, 1]$, that

$$\int_{S_2} \boldsymbol{F} \cdot d\boldsymbol{S} = -\int_{\Psi_1} \boldsymbol{F} \cdot d\boldsymbol{S} = -\int\int_R \boldsymbol{F}(u, v, 1) \cdot (0, 0, 1)\, du\, dv = -\int\int_R (u, v, -1) \cdot (0, 0, 1)\, du\, dv = -\int\int_R (-1)\, du\, dv$$

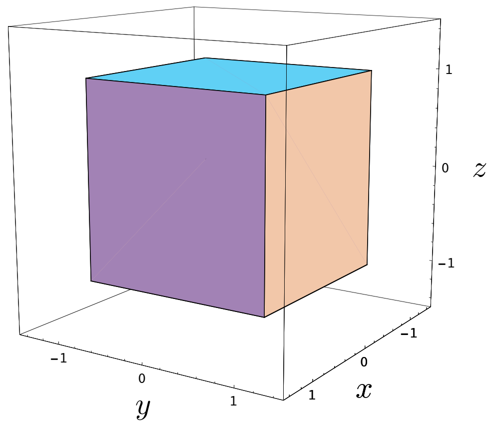
$$= +\left(\int_{-1}^{1} 1\, du\right)\left(\int_{-1}^{1} 1\, dv\right) = 2 \times 2 = 4.$$

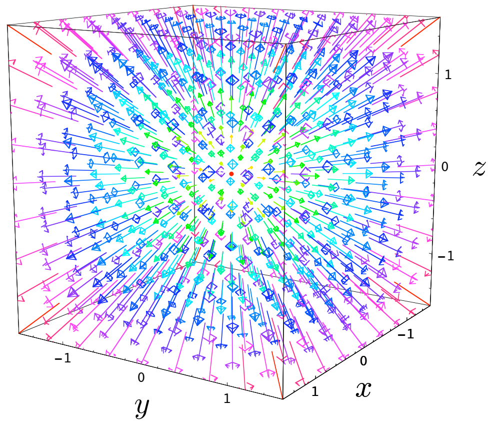
Thus $\int_{S_1} \boldsymbol{F} \cdot d\boldsymbol{S} + \int_{S_2} \boldsymbol{F} \cdot d\boldsymbol{S} = 8$.

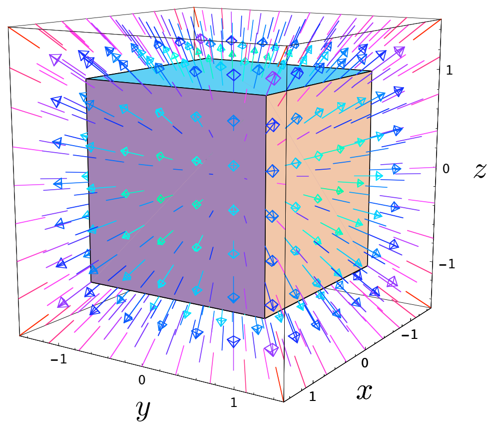
It follows from symmetry arguments, concerning the other two pairs of opposite faces, that the full surface integral gives

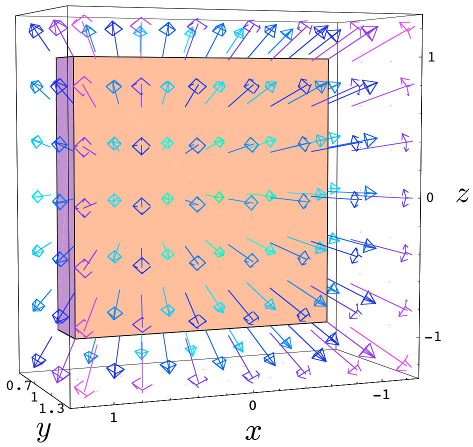
$$\int_S \boldsymbol{F} \cdot d\boldsymbol{S} = 3 \times 8 = 24.$$

visualisation









with transparency