MATH236 — Week 7

Parametrized surfaces

Chen notes chapter 10

10.1 Introduction to parametrized surfaces

Recall that a path is essentially a parametrization of a curve. In a similar way, a surface can be parametrized. Whereas a curve can be parametrized by the use of a single real parameter, a surface can be parametrized by the use of *two* real parameters.

We shall be concerned only with surfaces in $\mathbb{R}^3$. Before we give any formal definition, let us consider two examples.

Example 10.1.1 — graph of a function

Consider a function $f : [A, B] \times [C, D] \to \mathbb{R}$. Then the graph $\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in [A, B] \times [C, D]$ and $z = f(x, y)\}$ is a surface in $\mathbb{R}^3$. Each point (x, y, z) on this surface is determined precisely by the values of the variables x and y.

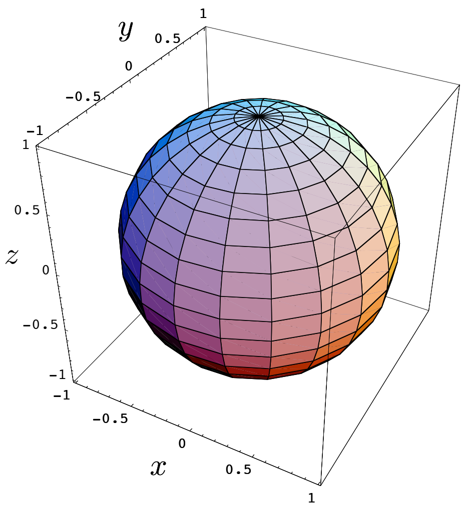
Example 10.1.2 — unit sphere

Consider the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$, with radius 1 and centre $(0, 0, 0)$. Using spherical coordinates, we can write

$$x = \sin\varphi \cos\theta, \quad y = \sin\varphi \sin\theta, \quad z = \cos\varphi,$$

where $\varphi \in [0, \pi]$ and $\theta \in [0, 2\pi]$. Each point (x, y, z) on the sphere is determined precisely by the values of the variables φ and θ.

visualisation



Definition: parametrized surface

By a ***parametrized surface*** in $\mathbb{R}^3$ we mean a continuous function of the type $\Phi : R \to \mathbb{R}^3$ where $R \subseteq \mathbb{R}^2$ is a domain.

The range $\Phi(R) = \{\Phi(u, v) : (u, v) \in R\} \subseteq \mathbb{R}^3$ of the function Φ is called a ***surface***.

Remarks

- For every $(u, v) \in R$, we can write $\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$, with components $x(u, v), y(u, v), z(u, v) \in \mathbb{R}$.
- We can think of the function Φ as twisting and bending the region R to give a surface $S = \Phi(R)$. The position of a point $(x(u, v), y(u, v), z(u, v))$ on S is determined by the values of the parameters u and v.
- Often, we refer to the parametrized surface $\Phi(u, v)$ without specifying the domain of definition of the function Φ. This is a convenient abuse of rigour.

Definition: differentiable parametrised surface

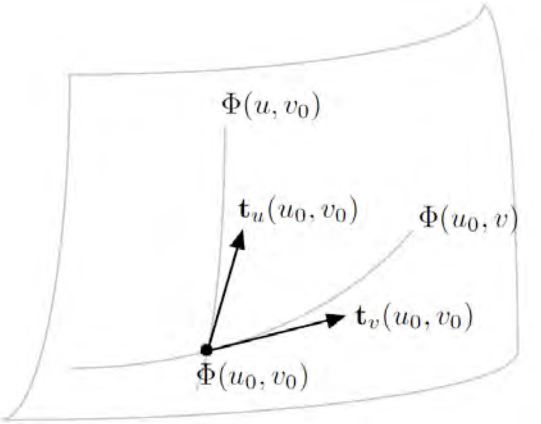
We say that a parametrized surface $\Phi : R \to \mathbb{R}^3$ is ***continuously differentiable*** if the function Φ is differentiable and the partial derivatives are continuous.

Suppose that a parametrized surface $\Phi : R \to \mathbb{R}^3$ is differentiable at a point $(u_0, v_0) \in R$. Keeping the first parameter u fixed at u_0, we consider the function $v \mapsto \Phi(u_0, v)$ in a neighbourhood of v_0. The image of this function is a curve on the surface $S = \Phi(R)$, and a tangent vector to this curve at the point $\Phi(u_0, v_0)$ is given by

$$\boldsymbol{t}_v(u_0, v_0) = \left(\frac{\partial x}{\partial v}(u_0, v_0), \frac{\partial y}{\partial v}(u_0, v_0), \frac{\partial z}{\partial v}(u_0, v_0)\right).$$

On the other hand, keeping the second parameter v fixed at v_0, we consider the function $u \mapsto \Phi(u, v_0)$ in a neighbourhood of u_0. The image of this function is a curve on the surface $S = \Phi(R)$, and a tangent vector to this curve at the point $\Phi(u_0, v_0)$ is given by

$$\boldsymbol{t}_u(u_0, v_0) = \left(\frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0)\right).$$



It follows that $\boldsymbol{t}_u(u_0, v_0)$ and $\boldsymbol{t}_v(u_0, v_0)$ are two vectors *tangent to the surface* $S = \Phi(R)$, at the point $\Phi(u_0, v_0)$. Unless they are parallel or opposite, these two vectors determine the ***tangent plane*** to the surface $S = \Phi(R)$, at the point $\Phi(u_0, v_0)$.

In this case, the vector $\boldsymbol{t}_u(u_0, v_0) \times \boldsymbol{t}_v(u_0, v_0)$ is a vector *normal to the surface* $S = \Phi(R)$, at the point $\Phi(u_0, v_0)$.

Definition: smooth parametrised surface

We say that a parametrized surface $\Phi : R \to \mathbb{R}^3$ is ***smooth*** at a point $\Phi(u_0, v_0)$ if $\boldsymbol{t}_u(u_0, v_0) \times \boldsymbol{t}_v(u_0, v_0) \neq \mathbf{0}$. We say that a parametrized surface $\Phi : R \to \mathbb{R}^3$ is ***smooth*** if it is smooth at *every* point $\Phi(u_0, v_0)$ where $(u_0, v_0) \in R$; in other words, if $\boldsymbol{t}_u \times \boldsymbol{t}_v \neq \mathbf{0}$ in R.

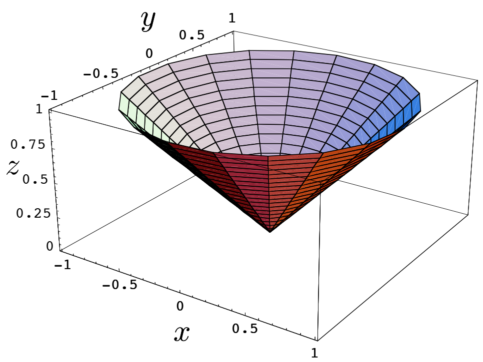
Example 10.1.3 — cone

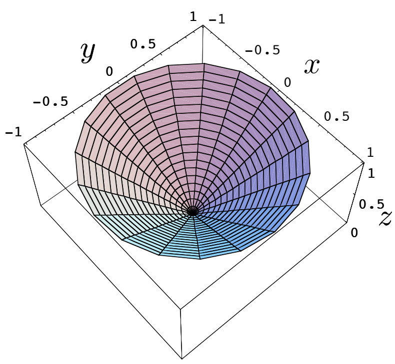
For the parametrized cone $\Phi : [0, 1] \times [0, 2\pi] \to \mathbb{R}^3$ whereby $(u, v) \mapsto (u \cos v, u \sin v, u)$, we have that

$$\boldsymbol{t}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (\cos v, \sin v, 1) \quad \text{and} \quad \boldsymbol{t}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (-u \sin v, u \cos v, 0).$$

Hence $\boldsymbol{t}_u \times \boldsymbol{t}_v = (\cos v, \sin v, 1) \times (-u \sin v, u \cos v, 0) = (-u \cos v, -u \sin v, u)$. It follows that the parametrized cone is smooth everywhere except at $(0, 0, 0)$.

visualisation





Example 10.1.4 — different parametrizations of the sphere

For the parametrized sphere $\Phi : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$ whereby $(u, v) \mapsto (\sin u \cos v, \sin u \sin v, \cos u)$, we have that

$$\boldsymbol{t}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (\cos u \cos v, \cos u \sin v, -\sin u) \quad \text{and} \quad \boldsymbol{t}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (-\sin u \sin v, \sin u \cos v, 0).$$

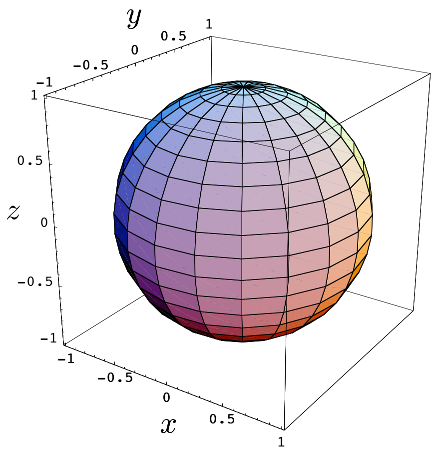
Hence $\boldsymbol{t}_u \times \boldsymbol{t}_v = (\cos u \cos v, \cos u \sin v, -\sin u) \times (-\sin u \sin v, \sin u \cos v, 0) = (\sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u)$.

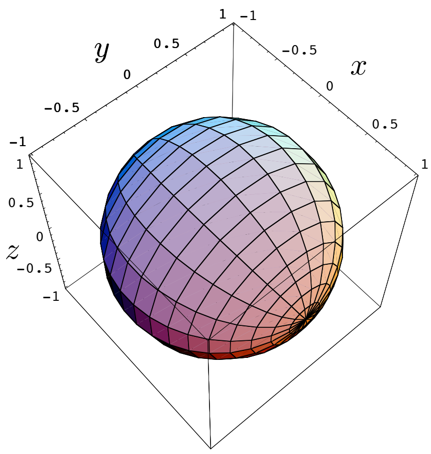
It follows that the parametrized sphere is smooth everywhere except at $(0, 0, \pm 1)$.

A similar argument shows that the parametrized sphere $\Psi : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$ whereby $(u, v) \mapsto (\cos u, \sin u \cos v, \sin u \sin v)$ is smooth everywhere except at $(\pm 1, 0, 0)$.

Note that both Φ and Ψ are parametrizations of the same unit sphere $x^2 + y^2 + z^2 = 1$.

visualisation





Example 10.1.5 — different parametrisations of the square

For the parametrized surface $\Phi : [-1, 1] \times [-1, 1] \to \mathbb{R}^3$ whereby $(u, v) \mapsto (u, v, 0)$, we have that

$$\boldsymbol{t}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (1, 0, 0) \quad \text{and} \quad \boldsymbol{t}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (0, 1, 0) \quad \text{so that} \quad \boldsymbol{t}_u \times \boldsymbol{t}_v = (0, 0, 1).$$

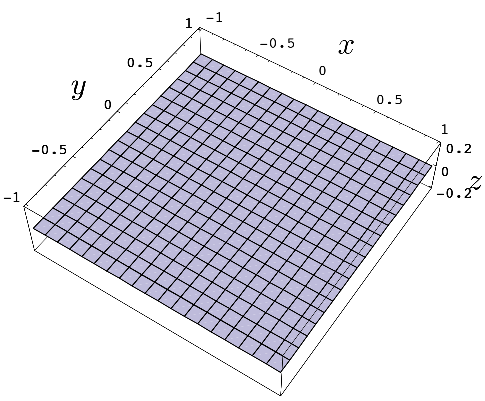
It follows that the parametrized square is smooth everywhere. Note that the surface Φ is the square with vertices at $(\pm 1, \pm 1, 0)$.

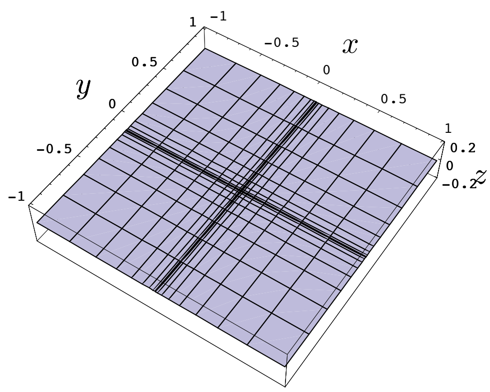
For the parametrized surface $\Psi : [-1, 1] \times [-1, 1] \to \mathbb{R}^3$ whereby $(u, v) \mapsto (u^3, v^3, 0)$, we have that

$$\boldsymbol{t}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (3u^2, 0, 0) \quad \text{and} \quad \boldsymbol{t}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (0, 3v^2, 0) \quad \text{so that} \quad \boldsymbol{t}_u \times \boldsymbol{t}_v = (0, 0, 9u^2 v^2).$$

It follows that this parametrized square is smooth everywhere except at points $\Psi(u, v)$ where $u = 0$ or $v = 0$. Note that the surface Ψ is also the square with vertices at $(\pm 1, \pm 1, 0)$.

visualisation





Remarks

- Examples 10.1.4 and 10.1.5 show that smoothness depends on the parametrization and not just the surface. Indeed, we say that a surface S is smooth if *there exists* a parametrization $\Phi : R \to \mathbb{R}^3$ which is smooth and such that $\Phi(R) = S$.

- Suppose that a parametrized surface $\Phi : R \to \mathbb{R}^3$ is smooth at a point $(x_0, y_0, z_0) = \Phi(u_0, v_0)$. Then $\boldsymbol{n} = \boldsymbol{t}_u \times \boldsymbol{t}_v$, evaluated at (u_0, v_0), is a normal vector to the surface $S = \Phi(R)$ at (x_0, y_0, z_0). It follows that the equation of the ***tangent plane*** to S at (x_0, y_0, z_0) is given by $(x - x_0, y - y_0, z - z_0) \cdot \boldsymbol{n} = 0$, where $\boldsymbol{n}$ is evaluated at (u_0, v_0).

Example 10.1.6 — smooth graph

Suppose that $f : [A, B] \times [C, D] \to \mathbb{R}$ is a differentiable function. Then its graph $\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in [A, B] \times [C, D] \text{ and } z = f(x, y)\}$ is the range of the function $\Phi : [A, B] \times [C, D] \to \mathbb{R}^3$ whereby $(u, v) \mapsto (u, v, f(u, v))$. We have

$$\boldsymbol{t}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = \left(1, 0, \frac{\partial f}{\partial u}\right) \text{ and } \boldsymbol{t}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = \left(0, 1, \frac{\partial f}{\partial v}\right) \text{ so that } \boldsymbol{t}_u \times \boldsymbol{t}_v = \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right) \neq \boldsymbol{0}.$$

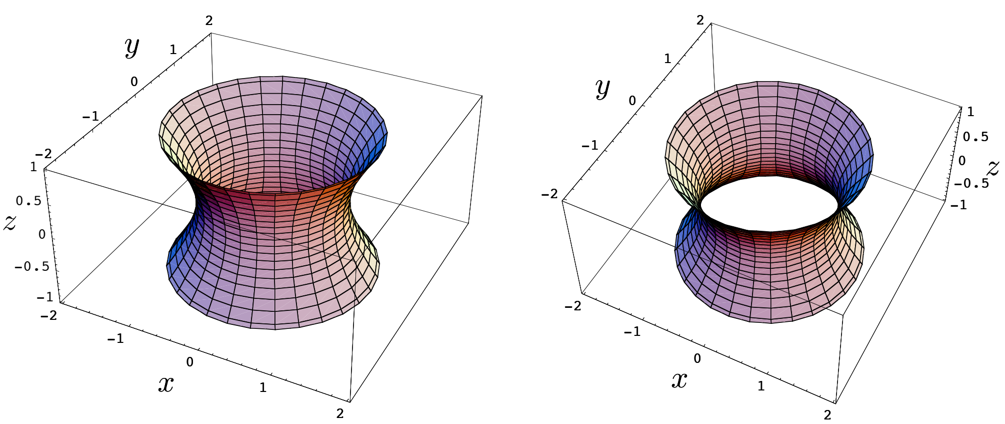
It follows that the surface is smooth.

Example 10.1.7 — hyperboloid of 1 sheet

The *hyperboloid of one sheet* is given by the equation $x^2 + y^2 - z^2 = 1$. We can write $x = r\cos u$ and $y = r\sin u$, so that $r^2 - z^2 = 1$. We can then write $r = \cosh v$ and $z = \sinh v$. Hence we have the parametrization: $x = \cos u \cosh v$, $y = \sin u \cosh v$, $z = \sinh v$.

Consider now the function $\Phi(u, v) = (\cos u \cosh v, \sin u \cosh v, \sinh v)$. We have $\boldsymbol{t}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (-\sin u \cosh v, \cos u \cosh v, 0)$ and $\boldsymbol{t}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (\cos u \sinh v, \sin u \sinh v, \cosh v)$ so that $\boldsymbol{t}_u \times \boldsymbol{t}_v = (\cos u \cosh^2 v, \sin u \cosh^2 v, -\sinh v \cosh v) \neq \boldsymbol{0}$. It follows that this surface is smooth.

visualisation



10.2 Surface Area

For the remainder of these notes, we restrict our attention to piecewise smooth surfaces. These are finite unions of the ranges of parametrized surfaces of the type $\Phi_i : R_i \to \mathbb{R}^3$, where R_i is an elementary region* in $\mathbb{R}^2$, Φ_i is continuously differentiable and one-to-one, except possibly on the boundary of R_i, and $S_i = \Phi(R_i)$ is smooth, except possibly at a finite number of points.

* For a discussion of elementary regions in $\mathbb{R}^2$, students are referred to Section 5.4 (studied in MATH235).

Definition: surface area

Suppose that $\Phi : R \to \mathbb{R}^3$ is a continuously differentiable parametrized surface. Then the quantity $\mathcal{A} = \int\int_R \| \boldsymbol{t}_u \times \boldsymbol{t}_v \| \, du \, dv$ is called the ***surface area*** of the parametrized surface Φ.

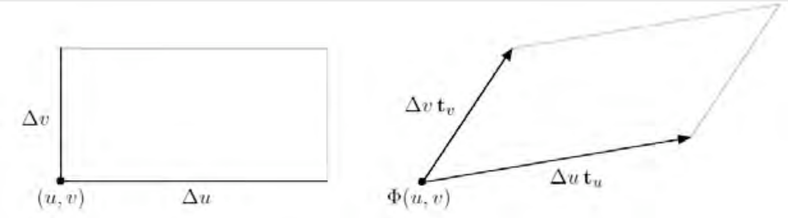
Remarks

- Note that $\boldsymbol{t}_u \times \boldsymbol{t}_v = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \times \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}, \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}, \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right) = \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)}\right)$,

  so that $\|\boldsymbol{t}_u \times \boldsymbol{t}_v\| = \left(\left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(z,x)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2\right)^{\frac{1}{2}} = \left(\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2\right)^{\frac{1}{2}}$.

  Hence $\mathcal{A} = \int\int_R \left(\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2\right)^{\frac{1}{2}} du\, dv$.

- To justify the definition, consider a small rectangle in R with bottom left vertex (u, v) and top right vertex $(u + \Delta u, v + \Delta v)$.



  The image under Φ of this rectangle can be approximated by a parallelogram in $\mathbb{R}^3$, with area $\|\Delta u\, \boldsymbol{t}_u \times \Delta v\, \boldsymbol{t}_v\| = \|\boldsymbol{t}_u \times \boldsymbol{t}_v\|\, \Delta u\, \Delta v$.

Example 10.2.1 — cone

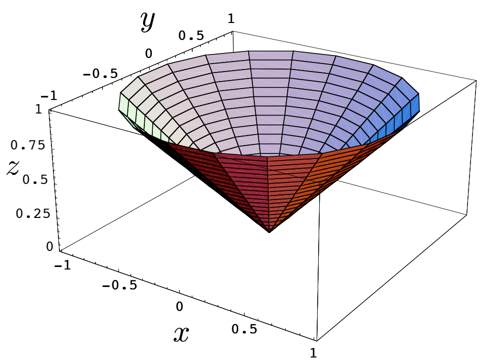
For the parametrized cone $\Phi : [0, 1] \times [0, 2\pi] \to \mathbb{R}^3$ whereby $(u, v) \mapsto (u \cos v, u \sin v, u)$, we have $\boldsymbol{t}_u \times \boldsymbol{t}_v = (-u \cos v, -u \sin v, u)$, so that $\|\boldsymbol{t}_u \times \boldsymbol{t}_v\| = (u^2 \cos^2 v + u^2 \sin^2 v + u^2)^{\frac{1}{2}} = \sqrt{2}\, u$.

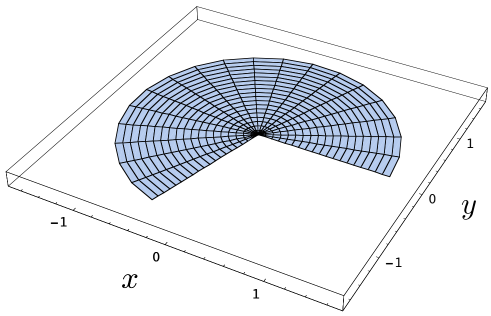
Alternatively, we have $\|\boldsymbol{t}_u \times \boldsymbol{t}_v\| = \left(\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2\right)^{\frac{1}{2}} = (u^2 + u^2 \sin^2 v + u^2 \cos^2 v)^{\frac{1}{2}} = \sqrt{2}\, u$.

Hence the surface area is given by:

$$\mathcal{A} = \int_0^{2\pi}\left(\int_0^1 \sqrt{2}\, u\, du\right) dv = \sqrt{2}\, \pi.$$

visualisation





Example 10.2.2 — sphere

For the parametrized sphere $\Phi : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$ whereby $(u, v) \mapsto (\sin u \cos v, \sin u \sin v, \cos u)$, we have $\boldsymbol{t}_u \times \boldsymbol{t}_v = (\sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u)$, so that $\|\boldsymbol{t}_u \times \boldsymbol{t}_v\| = (\sin^4 u + \cos^2 u \sin^2 u)^{\frac{1}{2}} = |\sin u|$.

Alternatively, we have $\|\boldsymbol{t}_u \times \boldsymbol{t}_v\| = \left(\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2\right)^{\frac{1}{2}} = (\sin^2 u \cos^2 u + \sin^4 u \sin^2 v + \sin^4 u \cos^2 v)^{\frac{1}{2}} = |\sin u|$.

Hence the surface area is given by:

$$\mathcal{A} = \int_0^{2\pi}\left(\int_0^{\pi} |\sin u|\, du\right) dv = \int_0^{2\pi} 2\left(\int_0^{\frac{1}{2}\pi} \sin u\, du\right) dv = 4\pi.$$

Example 10.2.3 — on a helicoid

For the helicoid $\Phi : [0\,,\,1] \times [0\,,\,2\,\pi] \to \mathbb{R}^3$ whereby $(u\,,\,v) \mapsto (u\cos v\,,\,u\sin v\,,\,v)$, we have

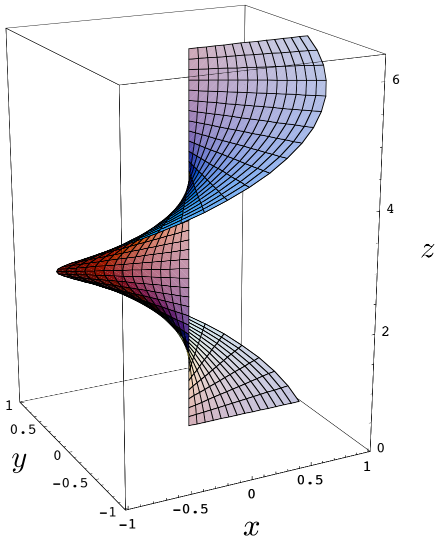
$\boldsymbol{t}_u = (\cos v\,,\,\sin v\,,\,0)$ and $\boldsymbol{t}_v = (-u\sin v\,,\,u\cos v\,,\,1)$ so that $\boldsymbol{t}_u \times \boldsymbol{t}_v = (\sin v\,,\,-\cos v\,,\,u)$ so that $\|\,\boldsymbol{t}_u \times \boldsymbol{t}_v\,\| = (1+u^2)^{\frac{1}{2}}$.

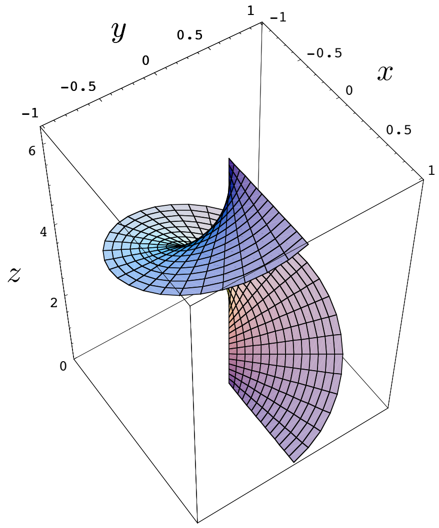
Alternatively, we have $\|\,\boldsymbol{t}_u \times \boldsymbol{t}_v\,\| = \left(\left| \frac{\partial(x,y)}{\partial(u,v)} \right|^2 + \left| \frac{\partial(x,z)}{\partial(u,v)} \right|^2 + \left| \frac{\partial(y,z)}{\partial(u,v)} \right|^2 \right)^{\frac{1}{2}} = (u^2 + \cos^2 v + \sin^2 v)^{\frac{1}{2}} = (1+u^2)^{\frac{1}{2}}$.

Hence the surface area is given by:

$$\mathcal{A} = \int_0^{2\pi} \left(\int_0^1 (1+u^2)^{\frac{1}{2}}\, du \right) dv = 2\,\pi \times \frac{1}{2}\left(\sqrt{2} + \sinh^{-1}(1)\right) = \pi\left(\sqrt{2} + \log\left(1+\sqrt{2}\right)\right) \approx 7.2118\,.$$

visualisation





photos of helicoids

Example 10.2.4 — surface area of a graph

Suppose that $f : R \to \mathbb{R}$ is a continuously differentiable function, where $R \subseteq \mathbb{R}^2$ is an elementary region. Then its graph

$\{(x\,,\,y\,,\,z) \in \mathbb{R}^3 : (x\,,\,y) \in R$ and $z = f(x\,,\,y)\}$ is the range of the function $\Phi : R \to \mathbb{R}^3$ whereby $(u\,,\,v) \mapsto (u\,,\,v\,,\,f(u\,,\,v))$ and $\boldsymbol{t}_u \times \boldsymbol{t}_v = \left(-\frac{\partial f}{\partial u}\,,\,-\frac{\partial f}{\partial v}\,,\,1\right)$. Hence the surface area of the graph is

$$\mathcal{A} = \int\int_R \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 + 1}\;du\,dv\,.$$