MATH236 — Weeks 5 & 6

Integrals over paths

Chen notes chapter 9

9.1 Integrals of scalar functions over paths

Suppose that the path $\boldsymbol{\phi} : [A\,,\, B] \to \mathbb{R}^n : t \mapsto (x_1(t),\, \ldots\,,\, x_n(t))$, is continuously differentiable. For any real valued function $f(x_1,\, \ldots\,,\, x_n)$ such that the composition function $f \circ \boldsymbol{\phi} : [A\,,\, B] \to \mathbb{R} : t \mapsto f(x_1(t),\, \ldots\,,\, x_n(t))$ is continuous, we define

$$\int_{\boldsymbol{\phi}} f\, ds = \int_{\boldsymbol{\phi}} f(x_1,\, \ldots\,,\, x_n)\, ds \overset{\text{def}}{=} \int_A^B f(\boldsymbol{\phi}(t))\ \|\boldsymbol{\phi}'(t)\|\, dt.$$

Remarks

- We are mainly interested in the special cases $n = 2$ and $n = 3$, and write respectively

$$\int_{\boldsymbol{\phi}} f\, ds = \int_{\boldsymbol{\phi}} f(x,\, y)\, ds \quad \text{and} \quad \int_{\boldsymbol{\phi}} f\, ds = \int_{\boldsymbol{\phi}} f(x,\, y,\, z)\, ds\,.$$

- Suppose that $f = 1$ identically. Then the integral simply represents the arc-length of $\boldsymbol{\phi}$.
- Note that f has only to be defined on the image curve $C = \boldsymbol{\phi}([A\,,\, B])$ of the path $\boldsymbol{\phi}$ for our definition to make sense. The continuity of the composition function $f \circ \boldsymbol{\phi}$ on the closed interval $[A\,,\, B]$ ensures the existence of the integral.
- Sometimes $\boldsymbol{\phi}$ may only be piecewise continuously differentiable; in other words, there exists a dissection $A = t_0 < t_1 < \ldots < t_k = B$ of the interval $[A\,,\, B]$ such that $\boldsymbol{\phi}$ is continuously differentiable in $[t_{i-1}\,,\, t_i]$ for each $i = 1,\, \ldots,\, k$. In this case, we define

$$\int_{\boldsymbol{\phi}} f\, ds \overset{\text{def}}{=} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} f(\boldsymbol{\phi}(t))\ \|\boldsymbol{\phi}'(t)\|\, dt.$$

  In other words, we calculate the corresponding integral for each subinterval and consider the sum of the integrals.

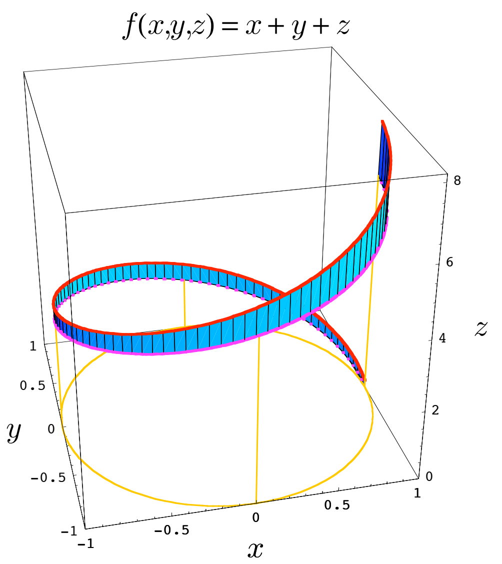
- For the special case $n = 2$ we must not confuse the integral with integrals of the type $\int_{\boldsymbol{\phi}} f(z)\, dz$, which arise frequently in complex analysis.

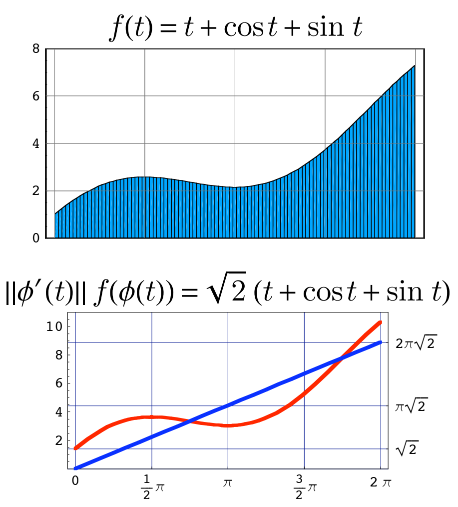
Example 9.1.1 — on a helix

Suppose that $\boldsymbol{\phi} : [0\,,\, 2\,\pi] \to \mathbb{R}^3 : t \mapsto (\cos t\,,\, \sin t\,,\, t)$ and $f(x,\, y,\, z) = x + y + z$. Then

$$\int_{\boldsymbol{\phi}} f\, ds = \int_0^{2\pi} f(\cos t\,,\, \sin t\,,\, t)\ \|(-\sin t\,,\, \cos t\,,\, 1)\|\, dt \ = \int_0^{2\pi} (\cos t + \sin t + t)\, \sqrt{2}\, dt = \tfrac{1}{2}(2\,\pi)^2\, \sqrt{2} \ = 2\,\pi^2\, \sqrt{2}\,.$$

visualisation



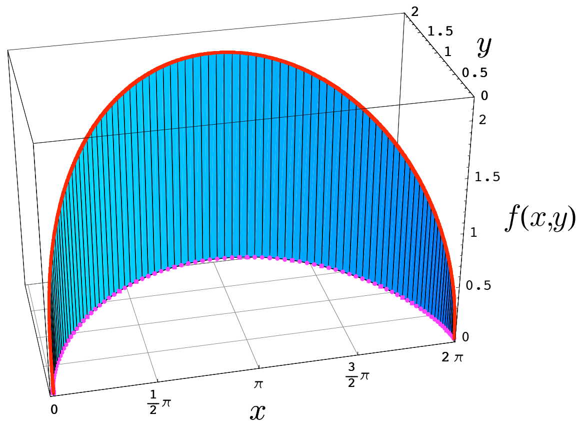


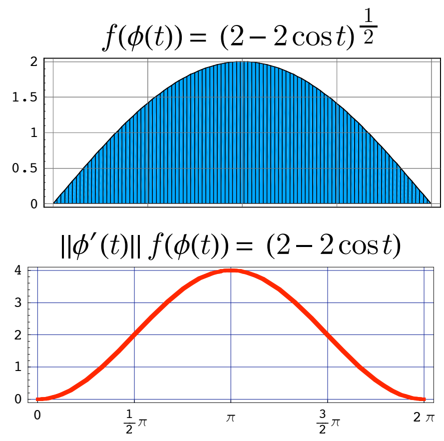
Example 9.1.2 — on a cycloid

Suppose that $\boldsymbol{\phi} : [0\,,\,2\,\pi] \to \mathbb{R}^2 : t \mapsto (t - \sin t\,,\, 1 - \cos t)$ and $f(x\,,\,y) = \sqrt{2\,y}$. Then

$$\int_{\boldsymbol{\phi}} f\,ds = \int_0^{2\pi} f(t - \sin t\,,\, 1 - \cos t)\;\|(1 - \cos t\,,\, \sin t)\|\,dt \;=\; \int_0^{2\pi} \sqrt{(2 - 2\cos t)}\;\;(2 - 2\cos t)^{\frac{1}{2}}\,dt \;=\; \int_0^{2\pi} (2 - 2\cos t)\,dt \;=\; 4\,\pi\,.$$

visualisation





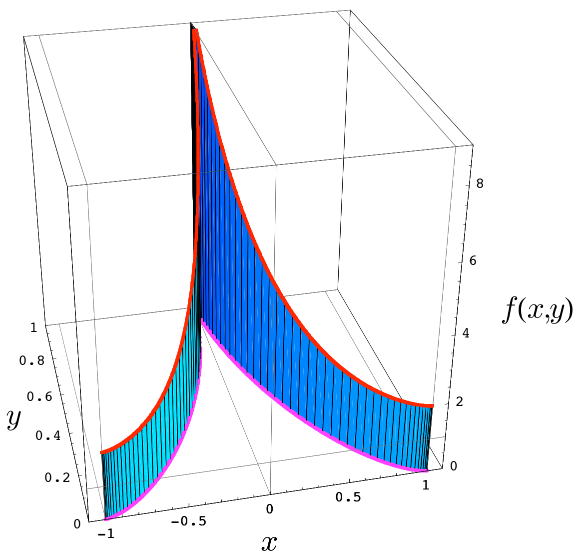
Example 9.1.3 — on a hypocycloid

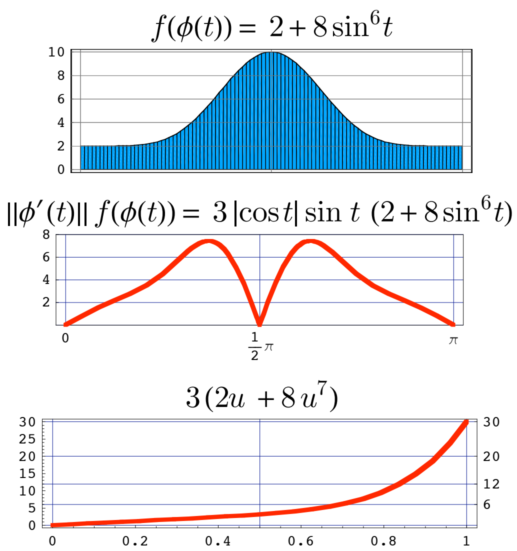
Suppose that $\boldsymbol{\phi} : [0\,,\,\pi] \to \mathbb{R}^2 : t \mapsto (\cos^3 t\,,\, \sin^3 t)$ and $f(x\,,\,y) = 2 + 8\,y^2$. Then

$$\int_{\boldsymbol{\phi}} f\,ds \;=\; \int_0^{\pi} f\,(\cos^3 t\,,\, \sin^3 t)\;\|3\cos t \sin t\,(-\cos t\,,\, \sin t)\|\,dt \;=\; \int_0^{\pi} (2 + 8\sin^6 t)\;3\left|\cos t\right|\sin t\,dt = 6\int_0^{\frac{1}{2}\pi} (2\sin t + 8\sin^7 t)\cos t\,dt$$

$$= 6\int_0^1 (2\,u + 8\,u^7)\,du \;=\; 6\,[u^2 + u^8]_0^{\;1} \;=\; 6 \times 2 = 12\,.$$

visualisation





Example 9.1.4 — different paths on a circle

The three distinct paths $\boldsymbol{\phi} : [0\,,\,2\,\pi] \to \mathbb{R}^2 : t \mapsto (\cos t\,,\,-\sin t)\,,$

$\boldsymbol{\psi} : [0\,,\,1] \to \mathbb{R}^2 : t \mapsto (\cos 2\,\pi\,t\,,\,\sin 2\,\pi\,t)\,,$

$\boldsymbol{\eta} : [0\,,\,1] \to \mathbb{R}^2 : t \mapsto (\cos 2\,\pi\,t^2\,,\,\sin 2\,\pi\,t^2)$

satisfy $\boldsymbol{\phi}([0\,,\,2\,\pi]) = \boldsymbol{\psi}([0\,,\,1]) = \boldsymbol{\eta}([0\,,\,1]) = C$, the unit circle in $\mathbb{R}^2$.

Note also that the path $\boldsymbol{\phi}$ follows C in a clockwise direction, while the paths $\boldsymbol{\psi}$ and $\boldsymbol{\eta}$ follow C in an anticlockwise direction.

Now consider the function $f(x,\,y) = 1 + x + y$. Then

$$\int_{\boldsymbol{\phi}} f\,ds = \int_0^{2\pi} f\,(\cos t\,,\,-\sin t)\;\|(-\sin t,\,-\cos t\,)\|\,dt \;=\; \int_0^{2\pi} (1+\cos t - \sin t)\,1\,dt = 2\,\pi\,;$$

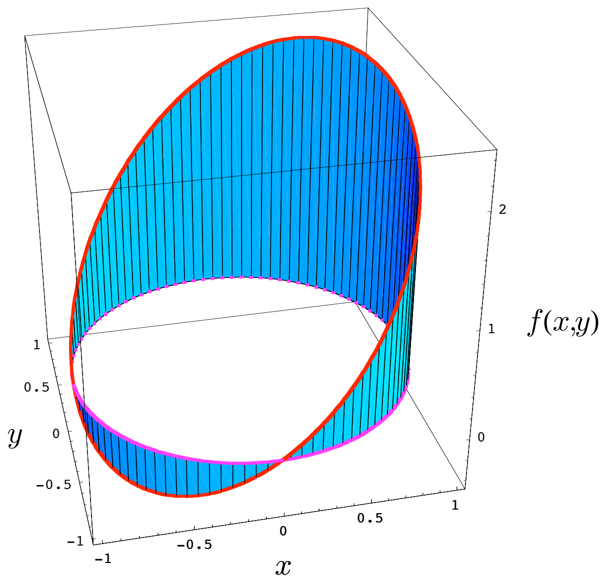
$$\int_{\boldsymbol{\psi}} f\,ds = \int_0^1 f(\cos 2\,\pi\,t\,,\,\sin 2\,\pi\,t)\;\|\,2\,\pi\,(-\sin 2\,\pi\,t,\,\cos 2\,\pi\,t)\,\|\,dt \;=\; 2\,\pi \int_0^1 (1+\cos 2\,\pi\,t + \sin 2\,\pi\,t)\,dt \;=\; 2\,\pi\,;$$

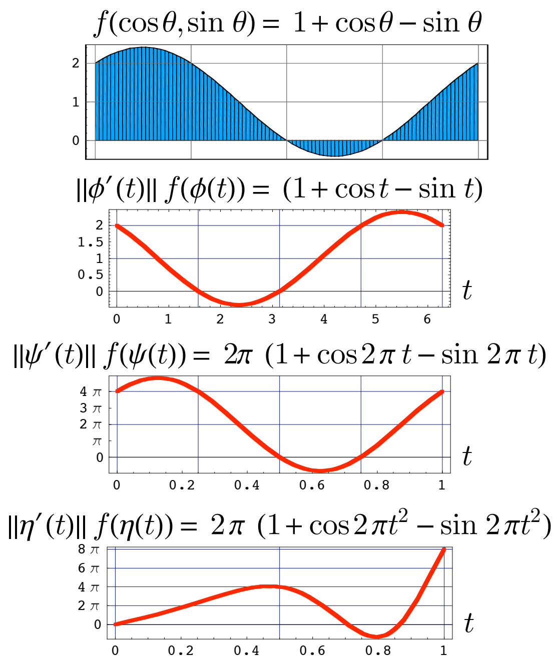
$$\int_{\boldsymbol{\eta}} f\,ds = \int_0^1 f(\cos 2\,\pi\,t^2,\,\sin 2\,\pi\,t^2)\;\|\,4\,\pi\,t(-\sin 2\,\pi\,t^2,\,\cos 2\,\pi\,t^2)\,\|\,dt \;=\; 4\,\pi \int_0^1 (1+\cos 2\,\pi\,t^2 + \sin 2\,\pi\,t^2)\,t\,dt$$

$$= 2\,\pi \int_0^1 (1+\cos 2\,\pi\,u + \sin 2\,\pi\,u)\,du = 2\,\pi.$$

Note that all three integrals have the same value. We shall show in Section 9.3 that this is not just a coincidence.

visualisation





9.2 Line integrals

Suppose that the path $\boldsymbol{\phi} : [A\,,\,B] \to \mathbb{R}^n : t \mapsto (x_1(t),\,\ldots,\,x_n(t))$, is continuously differentiable. For any vector field $\boldsymbol{F}(x_1,\,\ldots,\,x_n)$ such that the composition function $\boldsymbol{F}\circ\boldsymbol{\phi} : [A\,,\,B] \to \mathbb{R}^n : t \mapsto \boldsymbol{F}(x_1(t),\,\ldots,\,x_n(t))$ is continuous, we define

$$\int_{\boldsymbol{\phi}} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{\boldsymbol{\phi}} \boldsymbol{F}(x_1,\,\ldots,\,x_n)\cdot d\boldsymbol{s} \overset{\text{def}}{=} \int_A^B \boldsymbol{F}(\boldsymbol{\phi}(t))\cdot\boldsymbol{\phi}'(t)\,dt.$$

Remarks

- We are mainly interested in the special cases $n = 2$ and $n = 3$, and write $\int_{\boldsymbol{\phi}} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{\boldsymbol{\phi}} \boldsymbol{F}(x,\,y)\cdot d\boldsymbol{s}$ and $\int_{\boldsymbol{\phi}} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{\boldsymbol{\phi}} \boldsymbol{F}(x,\,y,\,z)\cdot d\boldsymbol{s}$.

  Writing $\boldsymbol{F} = (F_1\,,\,F_2)$ and $d\boldsymbol{s} = (d\,x\,,\,d\,y)$ in the case $n = 2$ and $\boldsymbol{F} = (F_1\,,\,F_2\,,\,F_3)$ and $d\boldsymbol{s} = (d\,x\,,\,d\,y\,,\,d\,z)$ in the case $n = 3$, we have respectively

$$\int_\phi \boldsymbol{F}\cdot d\boldsymbol{s} = \int_\phi (F_1, F_2)\cdot(dx, dy) = \int_\phi (F_1\,dx + F_2\,dy) = \int_A^B \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt}\right) dt$$

and

$$\int_\phi \boldsymbol{F}\cdot d\boldsymbol{s} = \int_\phi (F_1, F_2, F_3)\cdot d\boldsymbol{s} = \int_\phi (F_1\,dx + F_2\,dy + F_3\,dz) = \int_A^B \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt.$$

- Note that $\boldsymbol{F}$ has only to be defined on the image curve $C = \boldsymbol{\phi}([A, B])$ of the path $\boldsymbol{\phi}$ for our definition to make sense. The continuity of the composition function $\boldsymbol{F}\circ\boldsymbol{\phi}$ on the closed interval $[A, B]$ ensures the existence of the integral.

- Sometimes $\boldsymbol{\phi}$ may only be piecewise continuously differentiable; in other words, there exists a dissection $A = t_0 < t_1 < \ldots < t_k = B$ of the interval $[A, B]$ such that $\boldsymbol{\phi}$ is continuously differentiable in $[t_{i-1}, t_i]$ for each $i = 1, \ldots, k$. In this case, we define

$$\int_\phi \boldsymbol{F}\cdot d\boldsymbol{s} \stackrel{\text{def}}{=} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \boldsymbol{F}(\boldsymbol{\phi}(t))\cdot\boldsymbol{\phi}'(t)\,dt.$$

In other words, we calculate the corresponding integral for each subinterval and consider the sum of the integrals.

- Note that if $\boldsymbol{\phi}'(t) \neq 0$ for every $t \in [A, B]$ then

$$\int_\phi \boldsymbol{F}\cdot d\boldsymbol{s} = \int_A^B \left(\boldsymbol{F}(\boldsymbol{\phi}(t))\cdot\frac{\boldsymbol{\phi}'(t)}{\|\boldsymbol{\phi}'(t)\|}\right)\|\boldsymbol{\phi}'(t)\|\,dt = \int_A^B f(\boldsymbol{\phi}(t))\,\|\boldsymbol{\phi}'(t)\|\,dt, \quad \text{where} \quad f(\boldsymbol{\phi}(t)) = \boldsymbol{F}(\boldsymbol{\phi}(t))\cdot\frac{\boldsymbol{\phi}'(t)}{\|\boldsymbol{\phi}'(t)\|}.$$

Here $\frac{\boldsymbol{\phi}'(t)}{\|\boldsymbol{\phi}'(t)\|}$ is the unit tangent vector along the path $\boldsymbol{\phi}$. The integral now becomes one of the type discussed in the last section.

- Suppose that $\boldsymbol{F}$ is a force field; e.g., a gravitational field or magnetic field. Consider a particle moving along a path $\boldsymbol{\phi}$.

  At any time t, the force on the particle will be given by $\boldsymbol{F}(\boldsymbol{\phi}(t))$.

  On the other hand, a small displacement in the time interval $[t, t+dt]$ can be described by the velocity differential $d\boldsymbol{s} = \boldsymbol{\phi}(t)\,dt$.

  It follows that the scalar product $\boldsymbol{F}(\boldsymbol{\phi}(t))\cdot\boldsymbol{\phi}'(t)\,dt$ denotes the *work* done in the time interval $[t, t+dt]$.

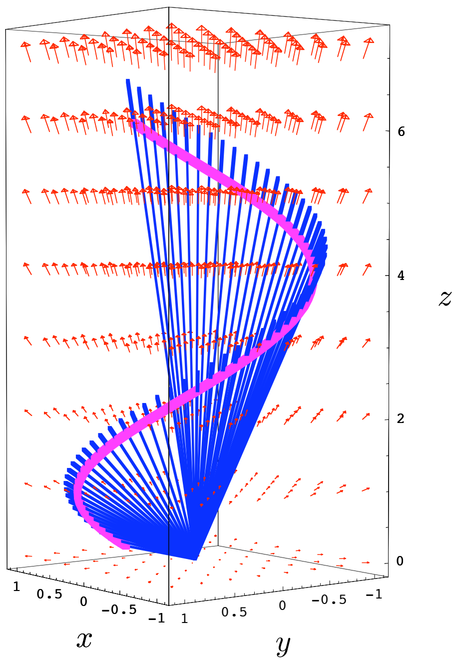
  Hence the integral describes the *total work done*.

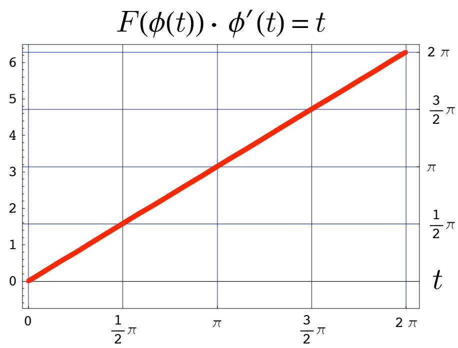
Example 9.2.1 — on a helix

Suppose that $\boldsymbol{\phi} : [0, 2\pi] \to \mathbb{R}^3 : t \mapsto (\cos t, \sin t, t)$ and $\boldsymbol{F}(x, y, z) = (x, y, z)$. Then

$$\int_\phi \boldsymbol{F}\cdot d\boldsymbol{s} = \int_0^{2\pi} \boldsymbol{F}(\cos t, \sin t, t)\cdot(-\sin t, \cos t, 1)\,dt = \int_0^{2\pi} (\cos t, \sin t, t)\cdot(-\sin t, \cos t, 1)\,dt = \int_0^{2\pi} t\,dt = 2\pi^2.$$

visualisation





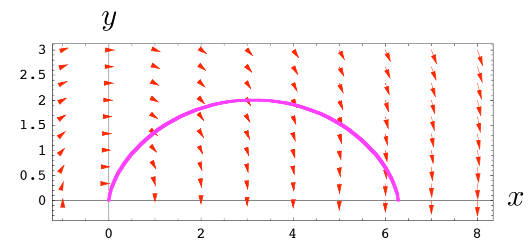
Example 9.2.2 — on a cycloid

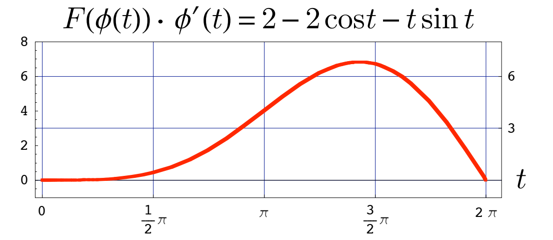
Suppose that $\boldsymbol{\phi}:[0\,,2\,\pi]\to\mathbb{R}^2: t\mapsto(t-\sin t\,,\,1-\cos t)$ and $\boldsymbol{F}(x,\,y)=(y,\,-x)$. Then

$$\int_{\boldsymbol{\phi}}\boldsymbol{F}\cdot d\boldsymbol{s}=\int_0^{2\pi}\boldsymbol{F}(t-\sin t\,,\,1-\cos t)\cdot(1-\cos t\,,\,\sin t)\,d\,t\;=\;\int_0^{2\pi}(1-\cos t\,,\,\sin t-t)\cdot(1-\cos t\,,\,\sin t)\,d\,t$$

$$=\int_0^{2\pi}(2-2\cos t-t\sin t)\,d\,t\;=\;4\,\pi-0+[t\cos t]_0^{2\pi}\;=\;6\,\pi\,.$$

visualisation





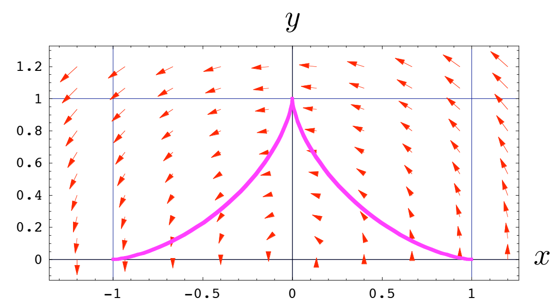
Example 9.2.3 — on a hypocycloid

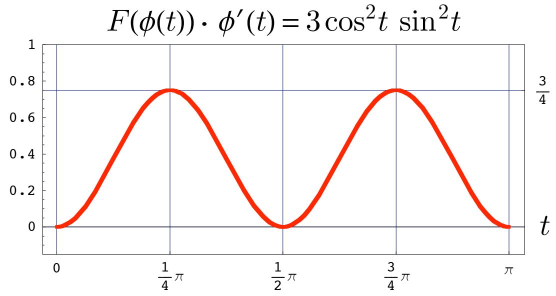
Suppose that $\boldsymbol{\phi}:[0\,,\pi]\to\mathbb{R}^2: t\mapsto(\cos^3 t\,,\,\sin^3 t)$ and $\boldsymbol{F}(x,\,y)=(-y,\,x)$. Then

$$\int_{\boldsymbol{\phi}}\boldsymbol{F}\cdot d\boldsymbol{s}=\int_0^{\pi}\boldsymbol{F}(\cos^3 t\,,\,\sin^3 t)\cdot(-\cos t\,,\,\sin t)\,3\cos t\sin t\,d\,t\;=\;\int_0^{\pi}3\cos t\sin t\,(-\sin^3 t\,,\,\cos^3 t)\cdot(-\cos t\,,\,\sin t)\,d\,t$$

$$=3\int_0^{\pi}\cos^2 t\,\sin^2 t\,d\,t=\tfrac{3}{4}\int_0^{\pi}\sin^2 2\,t\,d\,t\;=\;\tfrac{3}{8}\int_0^{\pi}(1-\cos 4\,t)\,d\,t\;=\;\tfrac{3}{8}\,\pi\,.$$

visualisation





Example 9.2.4 — different paths on a circle

The three distinct paths

$$\boldsymbol{\phi}:[0\,,2\,\pi]\to\mathbb{R}^2: t\mapsto(\cos t\,,\,-\sin t)\,,$$

$$\boldsymbol{\psi}:[0\,,1]\to\mathbb{R}^2: t\mapsto(\cos 2\,\pi\,t\,,\,\sin 2\,\pi\,t)\,,$$

$$\boldsymbol{\eta}:[0\,,1]\to\mathbb{R}^2: t\mapsto(\cos 2\,\pi\,t^2\,,\,\sin 2\,\pi\,t^2)$$

satisfy $\boldsymbol{\phi}([0\,,2\,\pi])=\boldsymbol{\psi}([0\,,1])=\boldsymbol{\eta}([0\,,1])=C$, the unit circle in $\mathbb{R}^2$. Note also that the path $\boldsymbol{\phi}$ follows C in a clockwise direction, while the paths $\boldsymbol{\psi}$ and $\boldsymbol{\eta}$ follow C in an anticlockwise direction. Now consider the function $\boldsymbol{F}(x,\,y)=(-y\,,\,x)$. Then

$$\int_{\boldsymbol{\phi}}\boldsymbol{F}\cdot d\boldsymbol{s}=\int_0^{2\pi}\boldsymbol{F}(\cos t\,,\,-\sin t)\cdot(-\sin t,\,-\cos t\,)\,d\,t\;=\;\int_0^{2\pi}(\sin t\,,\,\cos t)\cdot(-\sin t,\,-\cos t\,)\,d\,t\;=\;\int_0^{2\pi}(-1)\,d\,t=-2\,\pi\,;$$

$$\int_{\boldsymbol{\psi}}\boldsymbol{F}\cdot d\boldsymbol{s}\;=\;\int_0^1\boldsymbol{F}(\cos 2\,\pi\,t\,,\,\sin 2\,\pi\,t)\cdot 2\,\pi(-\sin 2\,\pi\,t,\,\cos 2\,\pi\,t)\,d\,t\;=\;2\,\pi\int_0^1(-\sin 2\,\pi\,t\,,\,\cos 2\,\pi\,t)\cdot(-\sin 2\,\pi\,t,\,\cos 2\,\pi\,t)\,d\,t$$

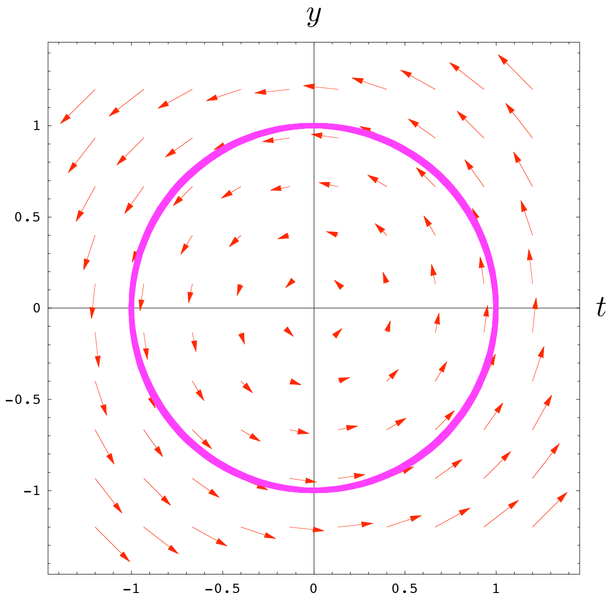
$$=2\,\pi\int_0^1 1\,d\,t=2\,\pi\,;$$

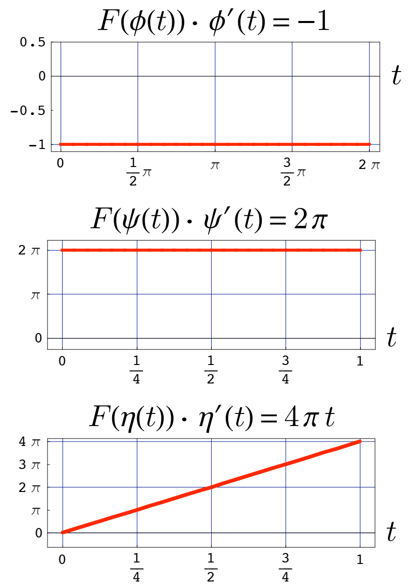
$$\int_{\boldsymbol{\eta}}\boldsymbol{F}\cdot d\boldsymbol{s}\;=\;\int_0^1\boldsymbol{F}(\cos 2\,\pi\,t^2,\,\sin 2\,\pi\,t^2)\cdot 4\,\pi\,t(-\sin 2\,\pi\,t^2,\,\cos 2\,\pi\,t^2)\,d\,t\;=\;4\,\pi\int_0^1(-\sin 2\,\pi\,t^2,\,\cos 2\,\pi\,t^2)\cdot(-\sin 2\,\pi\,t^2,\,\cos 2\,\pi\,t^2)\,t\,d\,t$$

$$=2\,\pi\int_0^1 2\,t\,d\,t\;=2\,\pi\,[t^2]_0^1=2\,\pi.$$

Note that $-\int_{\phi} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{\psi} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{\eta} \boldsymbol{F}\cdot d\boldsymbol{s}$, where $\boldsymbol{\psi}$ and $\boldsymbol{\eta}$ follow the unit circle C in the same direction, while $\boldsymbol{\phi}$ follows C in the opposite direction. The three integrals have the same absolute value, differing only in sign. We shall show in Section 9.3 that this is not just a coincidence.

visualisation





Example 9.2.5 — different paths with the same endpoints

The three distinct paths

$$\boldsymbol{\phi} : [0\,,\,1] \to \mathbb{R}^2 : t \mapsto (t\,,\,t)\,,$$

$$\boldsymbol{\psi} : [0\,,\,1] \to \mathbb{R}^2 : t \mapsto (t\,,\,t^2)\,,$$

$$\boldsymbol{\eta} : [0\,,\,\tfrac{1}{2}\,\pi] \to \mathbb{R}^2 : t \mapsto (1 - \cos t\,,\,\sin t)$$

all have the same initial point $(0\,,\,0)$ and the same terminal point $(1\,,\,1)$. The curve $\boldsymbol{\phi}([0\,,\,1])$ is part of the straight line $y = x$, the curve $\boldsymbol{\psi}([0\,,\,1])$ is part of the parabola $y = x^2$, while the curve $\boldsymbol{\eta}([0\,,\,1])$ is part of the circle $(x-1)^2 + y^2 = 1$. Hence the three paths have different curves. Consider now the vector field $\boldsymbol{F}(x\,,\,y) = (y\,,\,x)$. Then

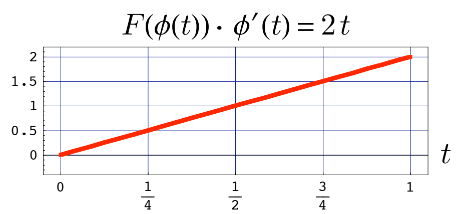
$$\int_{\phi} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_0^1 \boldsymbol{F}(t\,,\,t)\cdot(1\,,\,1)\,dt = \int_0^1 2\,t\,dt = 1$$

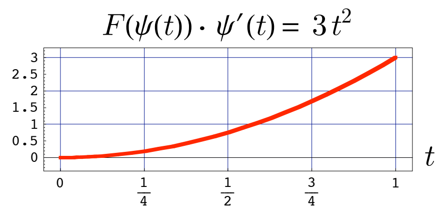
$$\int_{\psi} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_0^1 \boldsymbol{F}(t\,,\,t^2)\cdot(1\,,\,2\,t)\,dt = \int_0^1 (t^2\,,\,t)\cdot(1\,,\,2\,t)\,dt = \int_0^1 3\,t^2\,dt = [t^3]_0^1 = 1$$

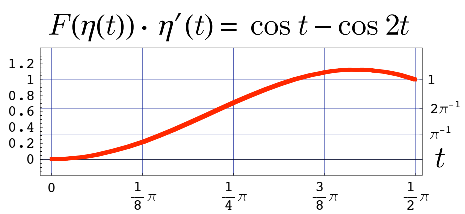
$$\int_{\eta} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_0^{\frac{1}{2}\pi} \boldsymbol{F}(1-\cos t\,,\,\sin t)\cdot(\sin t\,,\,\cos t)\,dt = \int_0^{\frac{1}{2}\pi} (\sin t\,,\,1-\cos t)\cdot(\sin t\,,\,\cos t)\,dt = \int_0^{\frac{1}{2}\pi} (\sin^2 t - \cos^2 t + \cos t)\,dt$$

$$= \int_0^{\frac{1}{2}\pi} (\cos t - \cos 2\,t)\,dt = \left[\sin t - \tfrac{1}{2}\sin 2\,t\right]_0^{\frac{1}{2}\pi} = 1.$$

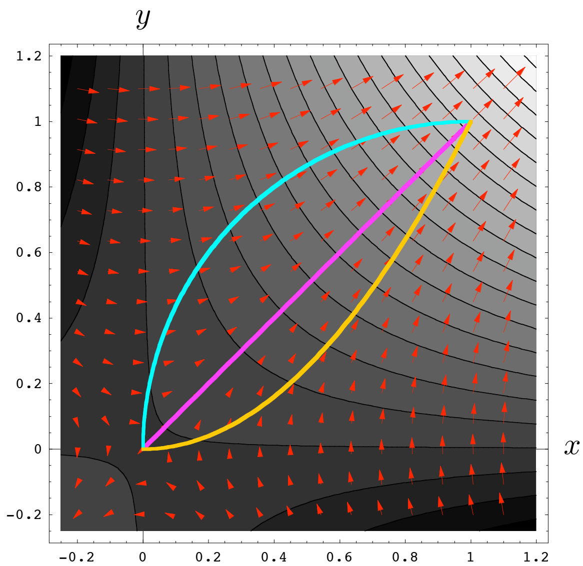
plots

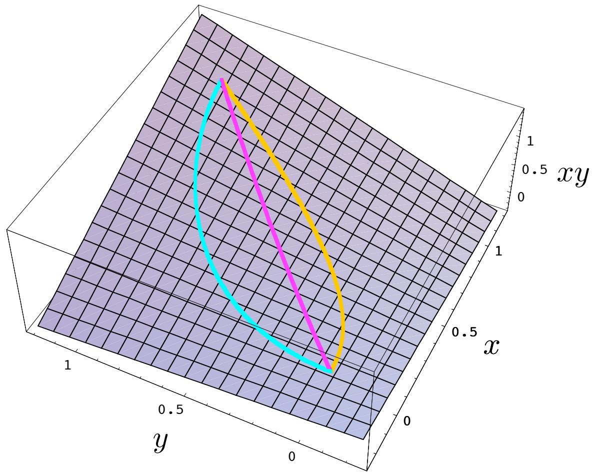






visualisation





this is a gradient field

Next note that $\boldsymbol{F} = \nabla f$, where $f(x, y) = x\,y$. Hence we have that

$$\int_{\phi} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\phi} \nabla f \cdot d\boldsymbol{s}, \quad \int_{\psi} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\psi} \nabla f \cdot d\boldsymbol{s}, \quad \int_{\eta} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\eta} \nabla f \cdot d\boldsymbol{s}.$$

Observe that $f(1, 1) - f(0, 0) = 1 - 0 = 1$, so is it a coincidence that

$$\int_{\phi} \nabla f \cdot d\boldsymbol{s} = \int_{\psi} \nabla f \cdot d\boldsymbol{s} = \int_{\eta} \nabla f \cdot d\boldsymbol{s} = f(1, 1) - f(0, 0),$$

so that the integrals depend only on the endpoints of the paths? On the other hand, note that $\boldsymbol{F}$ is the total derivative of f, so this is really just a statement like the Fundamental Theorem of Calculus. The images above show the line integrals to be just giving different routes for climbing the *potential hill* defined by the function $f(x, y)$; the total height climbed is the same in each case.

Let us investigate this problem in general. Suppose that $\boldsymbol{F}$ is a gradient vector field in $\mathbb{R}^n$, so that there exists a continuously differentiable function $f(x_1, \ldots, x_n)$ such that $\boldsymbol{F} = \nabla f$. Suppose that $\boldsymbol{\phi} : [A, B] \to \mathbb{R}^n$ is a continuously differentiable path.

Consider the composite function $g = f \circ \boldsymbol{\phi} : [A, B] \to \mathbb{R}$. By the Chain rule, we have

$$g'(t) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\boldsymbol{\phi}(t)) & \ldots & \frac{\partial f}{\partial x_n}(\boldsymbol{\phi}(t)) \end{pmatrix} \begin{pmatrix} \boldsymbol{\phi}_1'(t) \\ \vdots \\ \boldsymbol{\phi}_n'(t) \end{pmatrix}$$

where the right hand side is the matrix product of the total derivatives $(\mathbb{D} f)(\boldsymbol{\phi}(t))$ and $(\mathbb{D}\boldsymbol{\phi})(t)$.

It follows that $g'(t) = \nabla f(\boldsymbol{\phi}(t)) \cdot \boldsymbol{\phi}'(t) = \boldsymbol{F}(\boldsymbol{\phi}(t)) \cdot \boldsymbol{\phi}'(t)$, and so

$$\int_{\phi} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_A^B \boldsymbol{F}(\boldsymbol{\phi}(t)) \cdot \boldsymbol{\phi}'(t)\, dt = \int_A^B g'(t)\, dt = g(B) - g(A) = f(\boldsymbol{\phi}(B)) - f(\boldsymbol{\phi}(A))$$

by the Fundamental Theorem of Calculus applied to the function g.

We have proved the following result.

Theorem 9A — line integral of ∇ *f*

Suppose that $\boldsymbol{F} = \nabla f$ is a gradient vector field in $\mathbb{R}^n$. Then for any continuously differentiable path $\boldsymbol{\phi} : [A, B] \to \mathbb{R}^n$ such that the composition function $\boldsymbol{F} \circ \boldsymbol{\phi} : [A, B] \to \mathbb{R}^n$ is continuous, we have $\int_{\phi} \boldsymbol{F} \cdot d\boldsymbol{s} = f(\boldsymbol{\phi}(B)) - f(\boldsymbol{\phi}(A))$.

9.3 Equivalent paths

We return to the questions posed by Examples 9.1.4 and 9.2.4.

Definition: change of parameter

Suppose that $\boldsymbol{\phi} : [A_1 , B_1] \to \mathbb{R}^n$ and $\boldsymbol{\psi} : [A_2 , B_2] \to \mathbb{R}^n$ are two continuously differentiable paths. Then we say that $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are ***equivalent*** if there exists a continuously differentiable and *strictly monotonic* function $h : [A_1 , B_1] \to [A_2 , B_2]$ such that $h([A_1 , B_1]) = [A_2 , B_2]$ and $\boldsymbol{\phi} = \boldsymbol{\psi} \circ h$. In this case, we say that the function h defines a ***change of parameter***.

Furthermore, we say that the change of parameter is: (i) ***orientation preserving*** if h is strictly increasing; and (ii) ***orientation reversing*** if h is strictly decreasing.

Remarks

- It is easy to see that if two paths are equivalent, then they have the same curve. If the change of parameter is *orientation preserving*, then the curve is followed in the *same direction*. If the change of parameter is *orientation reversing*, then the curve is followed in *different directions*.
- Note that the change of parameter is: *orientation preserving* if and only if $h'(t) \geq 0$ for every $t \in [A_1 , B_1]$; and *orientation reversing* if and only if $h'(t) \leq 0$ for every $t \in [A_1 , B_1]$.
- Since $h : [A_1 , B_1] \to [A_2 , B_2]$ is strictly monotonic and onto, it follows that it has an inverse function $h^{-1} : [A_2 , B_2] \to [A_1 , B_1]$. Clearly $\boldsymbol{\psi} = \boldsymbol{\phi} \circ h^{-1}$. Furthermore, the inverse function is also continuously differentiable.

Example 9.3.1

Recall the three distinct paths

$$\boldsymbol{\phi} : [0 , 2\pi] \to \mathbb{R}^2 : t \mapsto (\cos t , -\sin t),$$
$$\boldsymbol{\psi} : [0 , 1] \to \mathbb{R}^2 : t \mapsto (\cos 2\pi t , \sin 2\pi t),$$
$$\boldsymbol{\eta} : [0 , 1] \to \mathbb{R}^2 : t \mapsto (\cos 2\pi t^2 , \sin 2\pi t^2)$$

considered in Examples 9.1.4 and 9.2.4.

Let us examine first of all $\boldsymbol{\psi}$ and $\boldsymbol{\eta}$. The function $h : [0 , 1] \to [0 , 1] : t \mapsto \sqrt{t}$ is strictly increasing. It defines an orientation-preserving change of parameter with $\boldsymbol{\psi} = \boldsymbol{\eta} \circ h$. Note that the inverse function $h^{-1} : [0 , 1] \to [0 , 1] : t \mapsto t^2$ is also strictly increasing, and $\boldsymbol{\eta} = \boldsymbol{\psi} \circ h^{-1}$. Clearly $\boldsymbol{\psi}$ and $\boldsymbol{\eta}$ follow the unit circle in the same direction.

Consider next $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$. The function $h_2 : [0 , 1] \to [0 , 2\pi] : t \mapsto 1 - \frac{t}{2\pi}$ is strictly decreasing. It defines an orientation-reversing change of parameter with $\boldsymbol{\phi} = \boldsymbol{\psi} \circ h_2$. Note that the inverse function $h_2^{-1} : [0 , 2\pi] \to [0 , 1] : t \mapsto 2\pi(1 - t)$ is also strictly decreasing, and $\boldsymbol{\psi} = \boldsymbol{\phi} \circ h_2^{-1}$. Clearly $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ follow the unit circle in opposite directions.

Theorem 9B

Suppose that $\boldsymbol{\phi} : [A_1 , B_1] \to \mathbb{R}^n$ and $\boldsymbol{\psi} : [A_2 , B_2] \to \mathbb{R}^n$ are two equivalent continuously differentiable paths. Then for any real-valued function $f(x_1 , \ldots , x_n)$ such that the composition functions $f \circ \boldsymbol{\phi} : [A_1 , B_1] \to \mathbb{R}$ and $f \circ \boldsymbol{\psi} : [A_2 , B_2] \to \mathbb{R}$ are continuous, we have $\int_{\boldsymbol{\phi}} f \, ds = \int_{\boldsymbol{\psi}} f \, ds$.

Proof

Since $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are equivalent, there exists $h : [A_1 , B_1] \to [A_2 , B_2]$ such that $\boldsymbol{\phi} = \boldsymbol{\psi} \circ h$. It follows from the Chain Rule that $\boldsymbol{\phi}'(t) = \boldsymbol{\psi}'(h(t)) h'(t)$, and so

$$\int_{\boldsymbol{\phi}} f \, ds = \int_{A_1}^{B_1} f(\boldsymbol{\phi}(t)) \, \| \boldsymbol{\phi}'(t) \| \, dt = \int_{A_1}^{B_1} f(\boldsymbol{\psi}(h(t))) \, \| \boldsymbol{\psi}'(h(t)) \, h'(t) \| \, dt.$$

In the orientation-preserving case, we have $h'(t) \geq 0$ always. So with a change of variables $u = h(t)$, we have

$$\int_{\boldsymbol{\phi}} f \, ds = \int_{A_1}^{B_1} f(\boldsymbol{\psi}(h(t))) \, \| \boldsymbol{\psi}'(h(t)) \| \, h'(t) \, dt = \int_{A_2}^{B_2} f(\boldsymbol{\psi}(u)) \, \| \boldsymbol{\psi}'(u) \| \, du = \int_{\boldsymbol{\psi}} f \, ds.$$

In the orientation-reversing case, we have $h'(t) \leq 0$ always. So with a change of variables $u = h(t)$, we have

$$\int_{\boldsymbol{\phi}} f \, ds = -\int_{A_1}^{B_1} f(\boldsymbol{\psi}(h(t))) \, \| \boldsymbol{\psi}'(h(t)) \| \, h'(t) \, dt = -\int_{B_2}^{A_2} f(\boldsymbol{\psi}(u)) \, \| \boldsymbol{\psi}'(u) \| \, du = \int_{A_2}^{B_2} f(\boldsymbol{\psi}(u)) \, \| \boldsymbol{\psi}'(u) \| \, du = \int_{\boldsymbol{\psi}} f \, ds.$$

All cases are covered, so this completes the proof.

Theorem 9C

Suppose that $\phi : [A_1, B_1] \to \mathbb{R}^n$ and $\psi : [A_2, B_2] \to \mathbb{R}^n$ are two equivalent continuously differentiable paths. Then for any vector field $F(x_1, \ldots, x_n)$ such that the composition functions $F \circ \phi : [A_1, B_1] \to \mathbb{R}^n$ and $F \circ \psi : [A_2, B_2] \to \mathbb{R}^n$ are continuous, we have $\int_\phi F \cdot ds = \pm \int_\psi F \cdot ds$, where the equality holds: (i) with +sign if the change of parameter is orientation preserving; and (ii) with the −sign if the change of parameter is orientation reversing.

Proof

Since ϕ and ψ are equivalent, there exists $h : [A_1, B_1] \to [A_2, B_2]$ such that $\phi = \psi \circ h$. It follows from the Chain Rule that $\phi'(t) = \psi'(h(t))\, h'(t)$, and so

$$\int_\phi F \cdot ds = \int_{A_1}^{B_1} F(\phi(t)) \cdot \phi'(t)\, dt = \int_{A_1}^{B_1} F(\psi(h(t))) \cdot \psi'(h(t))\, h'(t)\, dt.$$

With a change of variables $u = h(t)$, we have in the orientation-preserving case:

$$\int_\phi F \cdot ds = \int_{A_2}^{B_2} F(\psi(u)) \cdot \psi'(u)\, du = \int_\psi F \cdot ds,$$

and in the orientation-reversing case,

$$\int_\phi F \cdot ds = \int_{B_2}^{A_2} F(\psi(u)) \cdot \psi'(u)\, du = -\int_{A_2}^{B_2} F(\psi(u)) \cdot \psi'(u)\, du = -\int_\psi F \cdot ds.$$

All cases are covered, so this completes the proof.

Remark

- Theorems 9B and 9C have natural extensions to the case when the paths are piecewise continuously differentiable. In this case, one can clearly break the paths into continuously differentiable pieces and apply Theorems 9B and 9C to each piece.

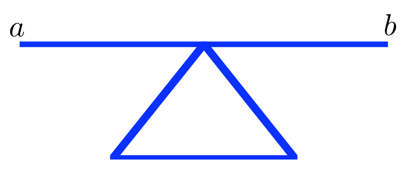
9.4 Simple curves

Theorems 9B and 9C demonstrate that integrals over differentiable paths depend only on the (oriented) curves of these paths. It therefore seems natural to try to express the theory in terms of these curves instead of the paths. The purpose of this section is to consider this problem.

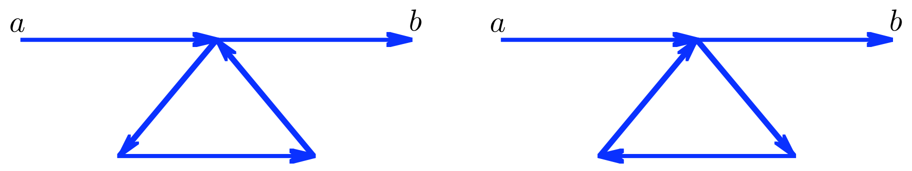
Before we start, we examine the example below which suggests that some care is required.

Example 9.4.1

Consider the curve below with endpoints indicated.



Clearly it is not enough to say that a path has initial point a and terminal point b, since any two paths that trace the curve in the two different ways indicated below are clearly not equivalent.



To temporarily avoid situations like this, we make the following definition.

Definition: simple curve

By a ***simple curve*** C in $\mathbb{R}^n$, we mean the image $C = \boldsymbol{\phi}([A, B])$ of a piecewise continuously differentiable path $\boldsymbol{\phi} : [A, B] \to \mathbb{R}^n$ with the property that $\boldsymbol{\phi}(t_1) \neq \boldsymbol{\phi}(t_2)$ whenever $A \leq t_1 < t_2 \leq B$, with the possible exception that $\boldsymbol{\phi}(A) = \boldsymbol{\phi}(B)$ may hold.

A simple curve together with a direction is called an ***oriented simple curve***. The function $\boldsymbol{\phi}$ is called a ***parametrization*** of the oriented simple curve C, and the parametrization is said to be: (i) *orientation preserving* if $\boldsymbol{\phi}$ follows the direction of C; and (ii) *orientation reversing* if $\boldsymbol{\phi}$ follows the opposite direction of C.

Definition: integral along a curve

Suppose that C is an oriented simple curve in $\mathbb{R}^n$. For any real-valued function $f(x_1, \ldots, x_n)$ continuous on C, we can define $\int_C f\, ds = \int_{\boldsymbol{\phi}} f\, ds$, where $\boldsymbol{\phi}$ is *any* parametrization of C. For any vector field $\boldsymbol{F}(x_1, \ldots, x_n)$ continuous on C, we can define $\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\boldsymbol{\phi}} \boldsymbol{F} \cdot d\boldsymbol{s}$, where $\boldsymbol{\phi}$ is *any orientation-preserving* parametrization of C.

The integrals along the curve $\int_C f\, ds$ and $\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$ are well defined in view of Theorems 9B and 9C respectively.

Remarks

- Suppose that the oriented simple curve C^- is obtained from the oriented simple curve C by taking the opposite orientation. Then
  $\int_{C^-} f\, ds = \int_C f\, ds$ and $\int_{C^-} \boldsymbol{F} \cdot d\boldsymbol{s} = -\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$.

- The theory can be extended to curves that are not simple, provided that we indicate very carefully how these curves are to be followed, and take note where some parts may be followed more than once. In particular, it is often convenient to break up an oriented curve into several components, each of which is simple.

For example, if $C = C_1 + \ldots + C_k$, where the sum denotes that the oriented curve C is obtained by following the oriented (simple) curves $C_1, \ldots, C_k$ one after another, then we have

$$\int_C f\, ds = \sum_{i=1}^{k} \int_{C_i} f\, ds \quad \text{and} \quad \int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \sum_{i=1}^{k} \int_{C_i} \boldsymbol{F} \cdot d\boldsymbol{s}.$$

In this case, each of $C_1, \ldots, C_k$ can be parametrized separately, so it doesn't matter if two or more of the curves have points in common.

Example 9.4.2

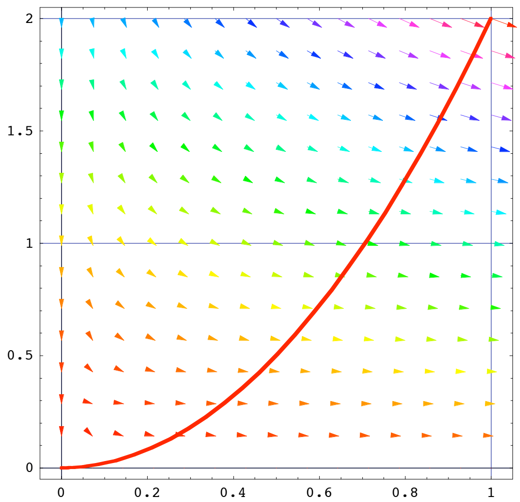
Let $\boldsymbol{F}(x, y) = (3xy, -y^2)$ and let C denote the path of the parabola $y = 2x^2$ from $(1, 2)$ to $(0, 0)$.

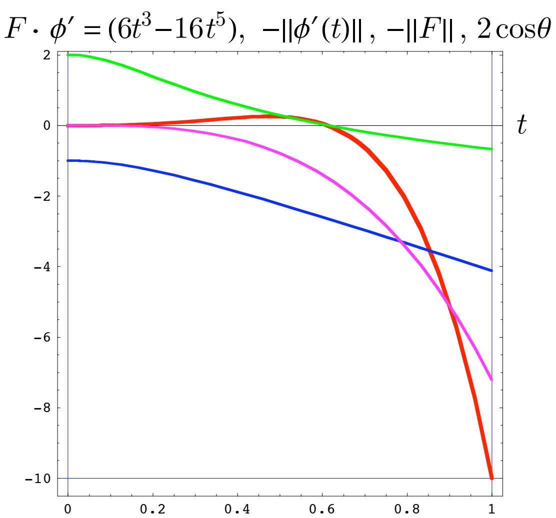
Clearly $\boldsymbol{\phi} : [0, 1] \to \mathbb{R}^2 : t \mapsto (t, 2t^2)$ is an orientation preserving parametrization of C^-, and so

$$\int_{C^-} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\boldsymbol{\phi}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_0^1 \boldsymbol{F}(\boldsymbol{\phi}(t)) \cdot \boldsymbol{\phi}'(t)\, dt = \int_0^1 \boldsymbol{F}(t, 2t^2) \cdot (1, 4t)\, dt = \int_0^1 (6t^3, -4t^4) \cdot (1, 4t)\, dt = \int_0^1 (6t^3 - 16t^5)\, dt = \left[\tfrac{3}{2}t^4 - \tfrac{16}{6}t^6\right]_0^1 = \tfrac{3}{2} - \tfrac{16}{6} = -\tfrac{7}{6}.$$

Hence $\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \frac{7}{6}$.

visualisation





Example 9.4.3

Let $\boldsymbol{F}(x, y, z) = (2x - y + z, x + y - z^2, 3x - 2y + 4z)$, and let C denote the circle on the xy-plane with centre at $(0, 0)$ and radius 3, followed in the anti-clockwise direction on the xy-plane.

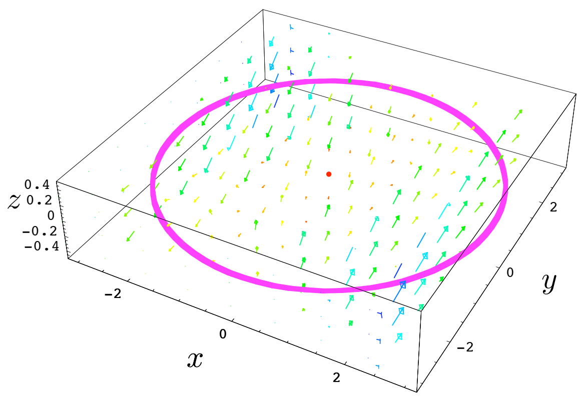
Clearly $\boldsymbol{\phi} : [0, 2\pi] \to \mathbb{R}^3 : t \mapsto (3\cos t, 3\sin t, 0)$ is an orientation-preserving parametrization of C, and so

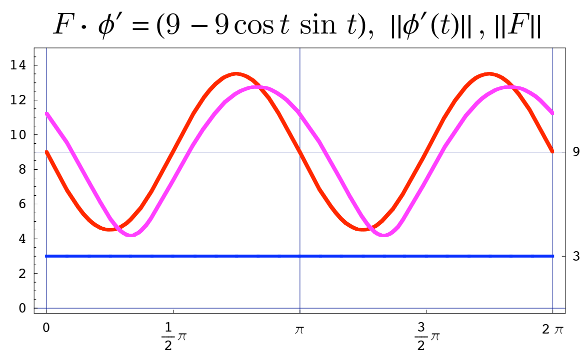
$$\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\boldsymbol{\phi}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_0^{2\pi} \boldsymbol{F}(\boldsymbol{\phi}(t)) \cdot \boldsymbol{\phi}'(t)\, dt = \int_0^{2\pi} \boldsymbol{F}(3\cos t, 3\sin t, 0) \cdot (-3\sin t, 3\cos t, 0)\, dt$$

$$= \int_0^{2\pi} 3^2 (2\cos t - \sin t, \cos t + \sin t, 3\cos t - 2\sin t) \cdot (-\sin t, \cos t, 0)\, dt = 9 \int_0^{2\pi} (\sin^2 t - 2\cos t \sin t + \cos^2 t + \sin t \cos t)\, dt$$

$$= 9 \int_0^{2\pi} (1 - \cos t \sin t)\, dt = 18\pi.$$

visualisation





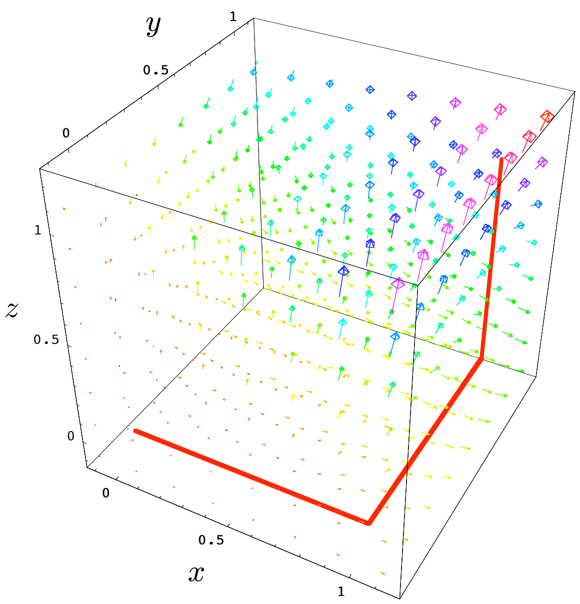
Example 9.4.4

Let $\boldsymbol{F}(x, y, z) = (3x^2 + 6y, -14yz, 20xz^2)$, and let C denote a succession of the straight line segments from $(0, 0, 0)$ to $(1, 0, 0)$ to $(1, 1, 0)$ to $(1, 1, 1)$.

Let C_1 denote the straight line segment from $(0, 0, 0)$ to $(1, 0, 0)$; C_2 denote the straight line segment from $(1, 0, 0)$ to $(1, 1, 0)$; and C_3 denote the straight line segment from $(1, 1, 0)$ to $(1, 1, 1)$.

Clearly $\boldsymbol{\phi} : [0, 1] \to \mathbb{R}^3 : t \mapsto (t, 0, 0)$, $\boldsymbol{\psi} : [0, 1] \to \mathbb{R}^3 : t \mapsto (1, t, 0)$ and $\boldsymbol{\eta} : [0, 1] \to \mathbb{R}^3 : t \mapsto (1, 1, t)$ are orientation-preserving parametrizations of C_1, C_2, C_3 respectively.

visualisation



integrate the vector field

Hence

$$\int_{C_1} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{\phi} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_0^1 \boldsymbol{F}(\boldsymbol{\phi}(t))\cdot\boldsymbol{\phi}'(t)\,dt = \int_0^1 \boldsymbol{F}(t,0,0)\cdot(1,0,0)\,dt = \int_0^1 (3t^2,0,0)\cdot(1,0,0)\,dt = \int_0^1 3t^2\,dt = 1;$$

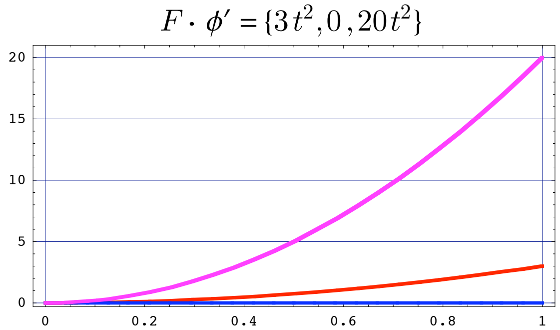
$$\int_{C_2} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{\psi} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_0^1 \boldsymbol{F}(\boldsymbol{\psi}(t))\cdot\boldsymbol{\psi}'(t)\,dt = \int_0^1 \boldsymbol{F}(1,t,0)\cdot(0,1,0)\,dt = \int_0^1 (3+6t,0,0)\cdot(0,1,0)\,dt = \int_0^1 0\,dt = 0;$$

$$\int_{C_3} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{\eta} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_0^1 \boldsymbol{F}(\boldsymbol{\eta}(t))\cdot\boldsymbol{\eta}'(t)\,dt = \int_0^1 \boldsymbol{F}(1,1,t)\cdot(0,0,1)\,dt = \int_0^1 (9,-14t,20t^2)\cdot(0,0,1)\,dt$$

$$= \int_0^1 20t^2\,dt = \frac{20}{3},$$

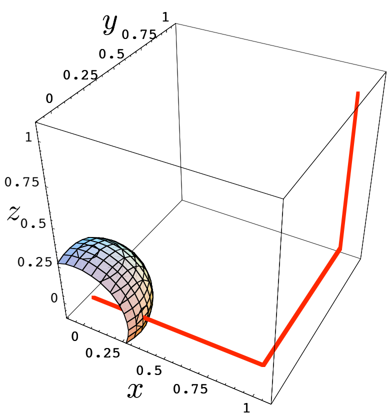
so that $$\int_C \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{C_1} \boldsymbol{F}\cdot d\boldsymbol{s} + \int_{C_2} \boldsymbol{F}\cdot d\boldsymbol{s} + \int_{C_3} \boldsymbol{F}\cdot d\boldsymbol{s} = 1 + 0 + \frac{20}{3} = \frac{23}{3}.$$

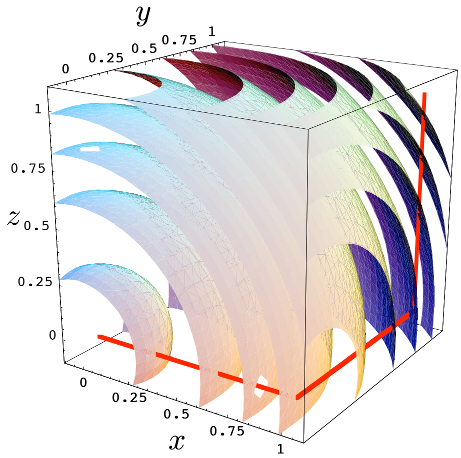
graphs



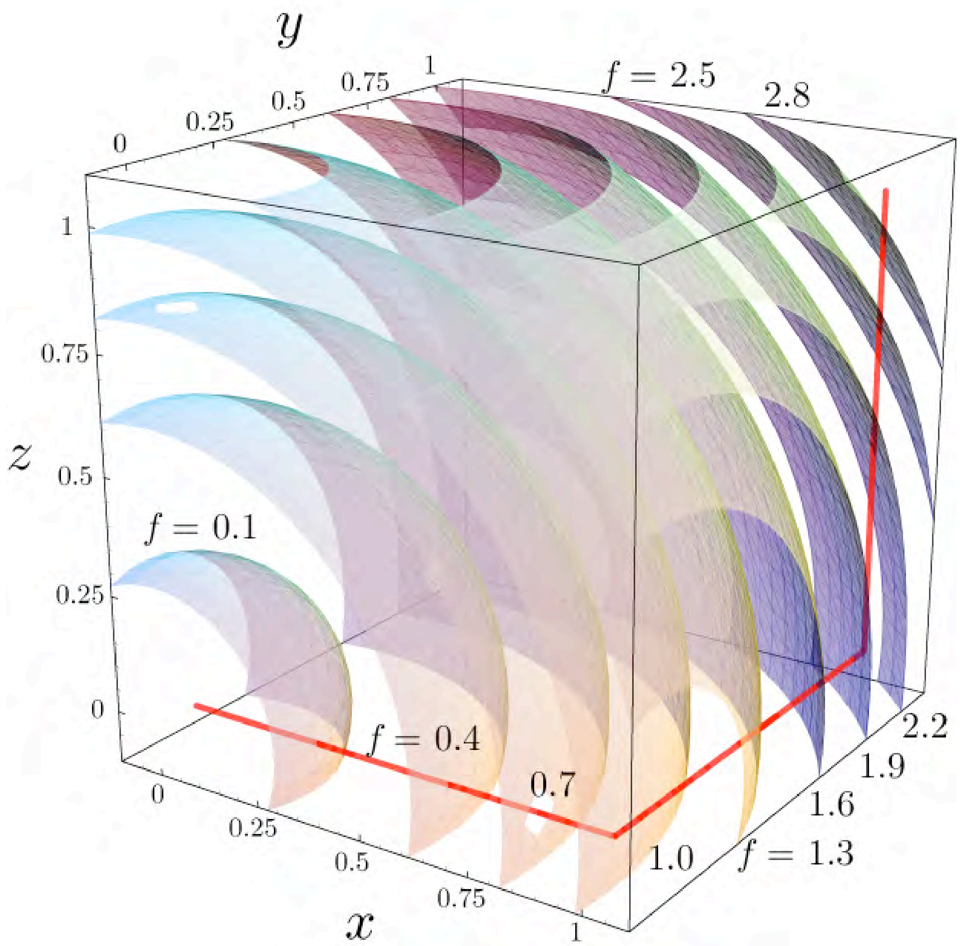
level surfaces of a function—nested spherical shells (movie)

Next let $f(x, y, z) = x^2 + y^2 + z^2$.





... using transparency



integrate the function

Then

$$\int_{C_1} f\,ds = \int_{\phi} f\,ds = \int_0^1 f(\phi(t))\ \|\phi'(t)\|\,dt\ = \int_0^1 f(t\,,0\,,0)\ \|(1\,,0\,,0)\|\,dt = \int_0^1 t^2\,dt = \frac{1}{3}\,;$$

$$\int_{C_2} f\,ds = \int_{\psi} f\,ds = \int_0^1 f(\psi(t))\ \|\psi'(t)\|\,dt\ = \int_0^1 f(1\,,t\,,0)\ \|(0\,,1\,,0)\|\,dt\ = \int_0^1 (1+t^2)\,dt = \frac{4}{3}\,;$$

$$\int_{C_3} f\,ds = \int_{\eta} f\,ds = \int_0^1 f(\eta(t))\ \|\eta'(t)\|\,dt\ = \int_0^1 f(1\,,1\,,t)\ \|(0\,,0\,,1)\|\,dt = \int_0^1 (2+t^2)\,dt = \frac{7}{3}\,;$$

so that

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \int_{C_3} f\,ds = \frac{1}{3} + \frac{4}{3} + \frac{7}{3} = \frac{12}{3} = 4\,.$$

graphs