MATH236 — Weeks 3&4

Vector fields

Chen notes, chapter 8

8.1 Introduction

In this chapter, we consider functions of the form

(1) $\boldsymbol{F} : A \to \mathbb{R}^n : \boldsymbol{x} \mapsto \boldsymbol{F}(\boldsymbol{x})$,

where the domain $A \subseteq \mathbb{R}^n$ is a set in the n-dimensional euclidean space, and where the codomain is also the n-dimensional euclidean space $\mathbb{R}^n$.

For each $\boldsymbol{x} \in A$, we can write $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, where $x_1, x_2, \ldots, x_n \in \mathbb{R}$. We can also write

(2) $\boldsymbol{F}(\boldsymbol{x}) = (F_1(\boldsymbol{x}), F_2(\boldsymbol{x}), \ldots, F_n(\boldsymbol{x}))$,

where $F_1(\boldsymbol{x}), F_2(\boldsymbol{x}), \ldots, F_n(\boldsymbol{x}) \in \mathbb{R}$.

Definition: vector field

A function $\boldsymbol{F}$ of the type (1) above, where $A \subseteq \mathbb{R}^n$, is called a ***vector field*** in $\mathbb{R}^n$.

The functions $F_i : A \to \mathbb{R}$, defined for $i = 1, 2, \ldots, n$ by (2), are called the *component* ***scalar fields*** of $\boldsymbol{F}$.

Remarks

- In the special cases $n = 2$ and $n = 3$, we usually write

  $\boldsymbol{F}(x, y) = (F_1(x, y), F_2(x, y))$ and $\boldsymbol{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$ respectively.

- The term *vector field* is also used more generally for functions of the type $\boldsymbol{\phi} : A \to \mathbb{R}^n$, with $A \subseteq \mathbb{R}^m$ for which $m \neq n$.

  However, here we are concerned primarily with the case of $m = n$.

Example 8.1.1 — gradient vector field, in $\mathbb{R}^n$

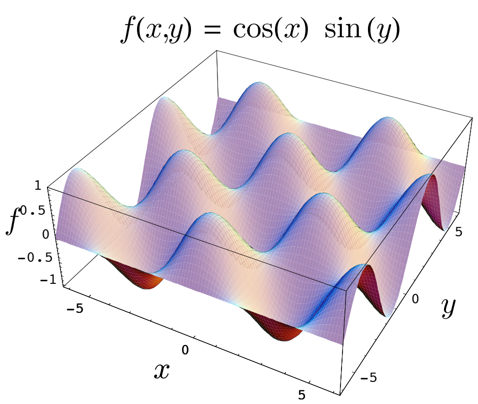
Suppose that a real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Define the function $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ by writing

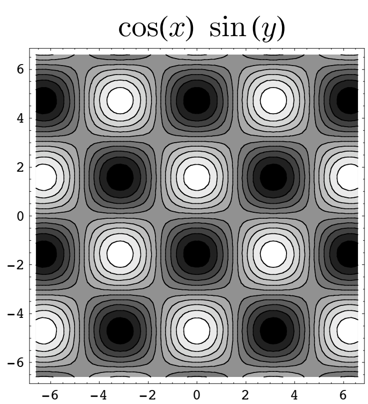
$$\boldsymbol{F}(\boldsymbol{x}) = (\nabla f)(\boldsymbol{x}) = \left(\frac{\partial f}{\partial x_1}(\boldsymbol{x}), \frac{\partial f}{\partial x_2}(\boldsymbol{x}), \ldots, \frac{\partial f}{\partial x_n}(\boldsymbol{x}) \right) \quad \text{for every } \boldsymbol{x} \in \mathbb{R}^n.$$

Recall that this is the *gradient* of f studied in Chapter 2 (MATH235).

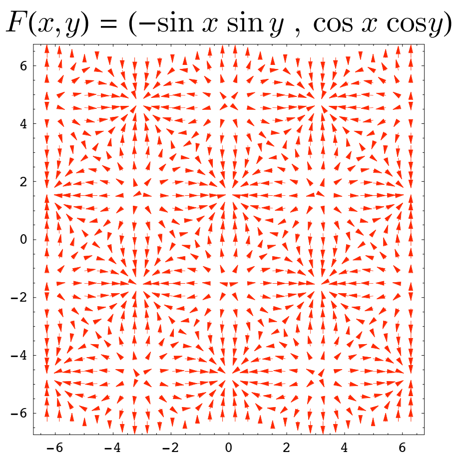
This vector field $\boldsymbol{F}$ is sometimes called a ***gradient vector field***.

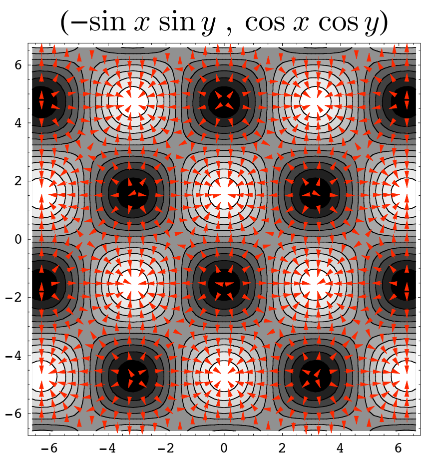
plot & contours — level curves





Vector-field plot





Example 8.1.2 — non-gradient field, in $\mathbb{R}^2$

Consider the vector field $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (-y, x)$.

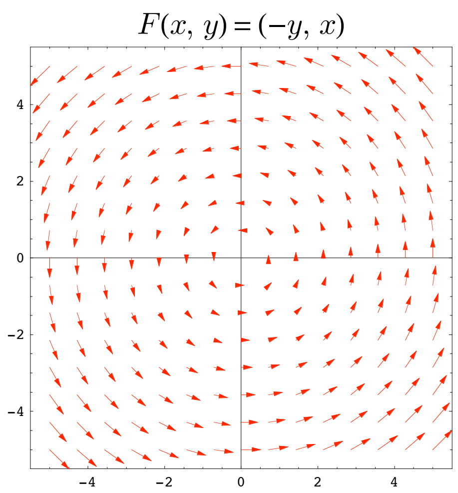
There is no continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\boldsymbol{F} = \nabla f$. To see this, note that if there were, then

$$\boldsymbol{F}(x, y) = (\nabla f)(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right),$$

so that $\frac{\partial f}{\partial x} = -y$ and $\frac{\partial f}{\partial y} = x$. It would then follow that $\frac{\partial^2 f}{\partial y\, \partial x} = -1$ and $\frac{\partial^2 f}{\partial x\, \partial y} = 1$, which is not possible.

This vector field $\boldsymbol{F}$ is an example of a ***non-gradient*** vector field.

plot



Example 8.1.3 — Newton's law of gravitation

Newton's law of gravitation states that the force acting on a point mass m at position $\boldsymbol{x} \in \mathbb{R}^3$, due to a point mass M at the origin $\boldsymbol{0}$, is given by:

$$\boldsymbol{F}(\boldsymbol{x}) = -\epsilon \frac{M\, m}{\|\boldsymbol{x}\|^3}\, \boldsymbol{x},$$

where $\epsilon > 0$ is a proportionality constant. This is an attractive force field.

Note that $\boldsymbol{F}(\boldsymbol{x}) = -\nabla f$, where $f : \mathbb{R}^3 \to \mathbb{R} : \boldsymbol{x} \mapsto -\epsilon \frac{M\, m}{\|\boldsymbol{x}\|}$,

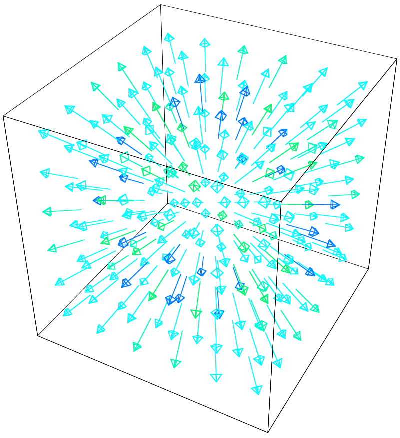
so $\boldsymbol{F}$ is a gradient vector field, the ***gravitational potential***.

Note that $\| \boldsymbol{x} \|^{-1} = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$, so that

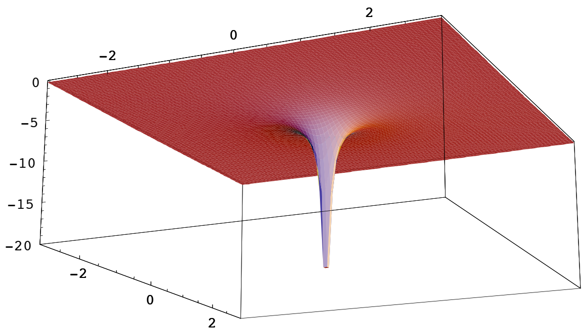
$$\nabla(- \| \boldsymbol{x} \|^{-1}) = \tfrac{1}{2}\,(x^2 + y^2 + z^2)^{-\frac{3}{2}} \times 2\,(x, y, z) \;=\; \| \boldsymbol{x} \|^{-3}\, \boldsymbol{x} \;=\; \| \boldsymbol{x} \|^{-2}\, \hat{\boldsymbol{x}}\,,$$ — inverse square law

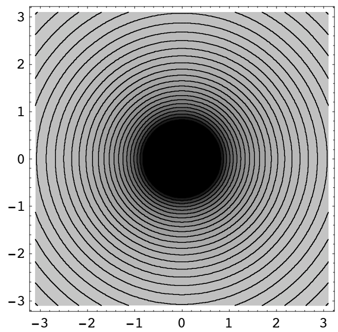
where $\hat{\boldsymbol{x}}$ denotes the unit vector in the direction of $\boldsymbol{x}$.

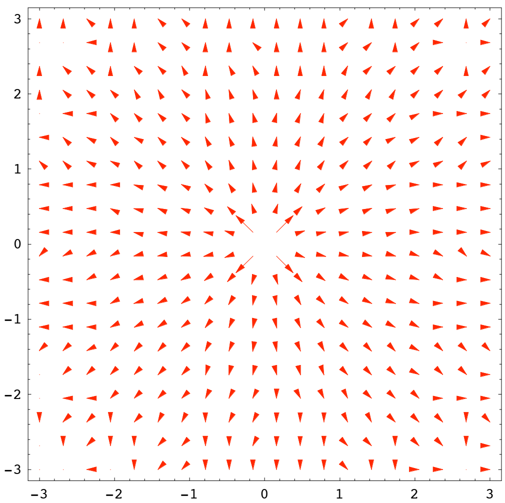
visualisation

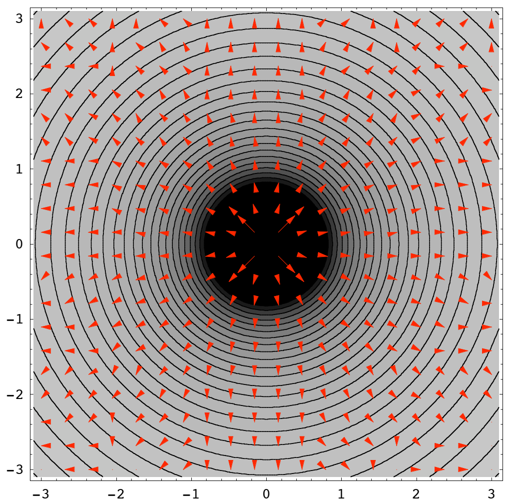


2D gradient plot









demo — Motion in a Central Field

- Mathematica demonstration

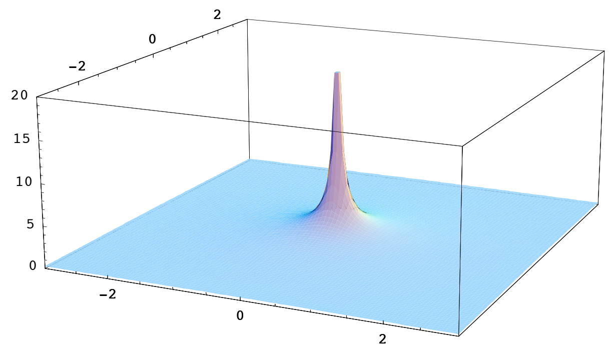
Example 8.1.4 — Coulomb's Law

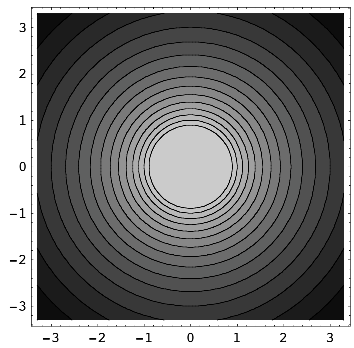
Coulomb's law in electrostatics states that the force acting on a point charge q at position $\boldsymbol{x} \in \mathbb{R}^3$ due to a point charge Q at the origin $\mathbf{0}$ is given by: $\boldsymbol{F}(\boldsymbol{x}) = \frac{\epsilon Q q}{\|\boldsymbol{x}\|^3} \boldsymbol{x}$,

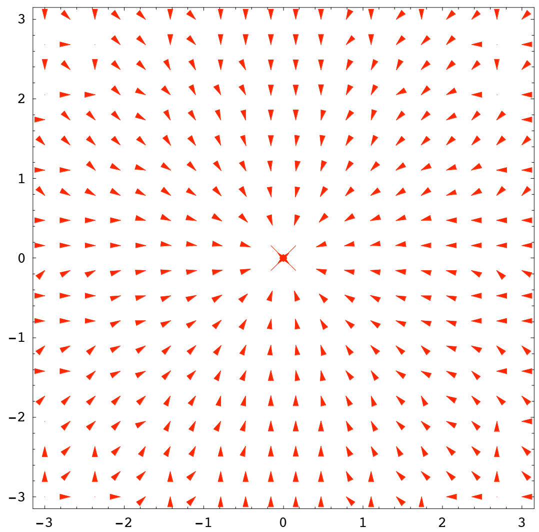
where $\epsilon > 0$ is a proportionality constant. This is a repulsive force field.

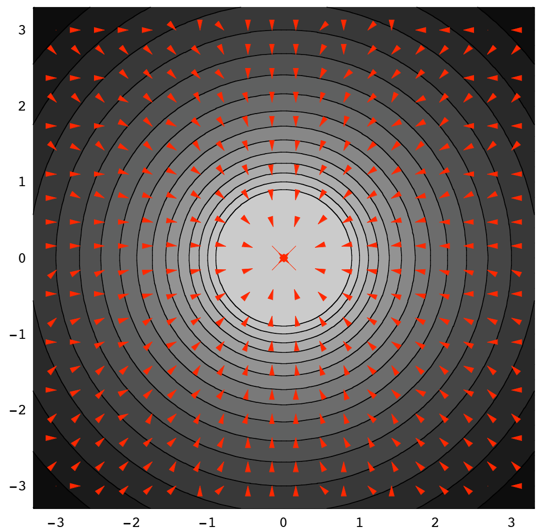
Note that $\boldsymbol{F}(\boldsymbol{x}) = -\nabla f$, where $f : \mathbb{R}^3 \to \mathbb{R} : \boldsymbol{x} \mapsto \epsilon \frac{Q q}{\|\boldsymbol{x}\|}$, so $\boldsymbol{F}$ is a gradient vector field, the ***electrostatic potential***.

visualisation









Definition: flow lines

Suppose that $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field. By a ***flow line*** of $\boldsymbol{F}$, we mean a path $\boldsymbol{\phi}(t)$ in $\mathbb{R}^n$ such that $\boldsymbol{\phi}'(t) = \boldsymbol{F}(\boldsymbol{\phi}(t))$; in other words, $\boldsymbol{F}$ yields the velocity vector of the path $\boldsymbol{\phi}(t)$.

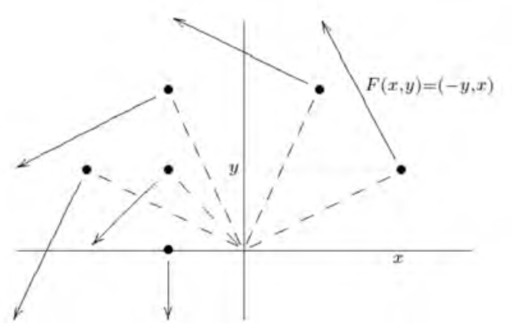
Flow lines are useful in understanding some of the properties of vector fields, as we shall see in the following examples.

Example 8.1.5 — circular flow

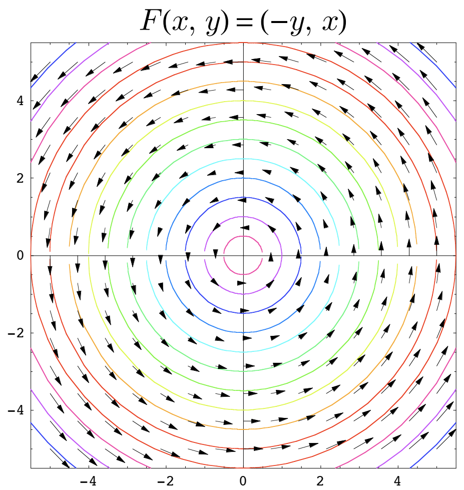
For the vector field $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n : (x, y) \mapsto (-y, x)$, the path $\boldsymbol{\phi}(t) = (\cos t, \sin t)$ is a flow line, for clearly $\boldsymbol{\phi}'(t) = \boldsymbol{\phi}(-\sin t, \cos t)$ and $\boldsymbol{F}(\boldsymbol{\phi}(t)) = \boldsymbol{F}(\cos t, \sin t) = (-\sin t, \cos t)$.

Similarly, it can be shown that for any real number $c \in \mathbb{R}$, the path $\boldsymbol{\phi}(t) = (c \cos t, c \sin t)$ is a flow line of $\boldsymbol{F}$.

Here the flow is circular, anticlockwise about the origin.



plot



Example 8.1.6 — radial flow

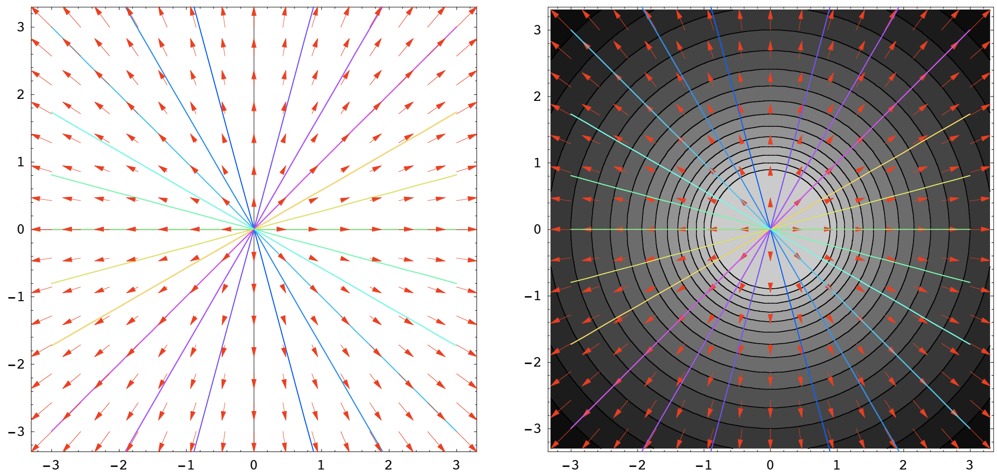
Let us return to Example 8.1.4, and consider again Coulomb's law of electrostatics, where $\boldsymbol{F}(\boldsymbol{x}) = \frac{\epsilon Q q}{\|\boldsymbol{x}\|^3} \boldsymbol{x}$.

Let $\boldsymbol{u} \in \mathbb{R}^3$ be a fixed unit vector.

The path $\boldsymbol{\phi}(t) = (3 \epsilon Q q t)^{\frac{1}{3}} \boldsymbol{u}$ is a flow line of $\boldsymbol{F}$, for clearly $\boldsymbol{\phi}'(t) = \frac{\epsilon Q q}{(3 \epsilon Q q t)^{\frac{2}{3}}} \boldsymbol{u}$ and $\boldsymbol{F}(\boldsymbol{\phi}(t)) = \frac{\epsilon Q q}{3 \epsilon Q q t \|\boldsymbol{u}\|^3} (3 \epsilon Q q t)^{\frac{1}{3}} \boldsymbol{u}$.

This shows that the flow lines are radial, away from the origin. Necessarily they are orthogonal to the level curves of the potential function.

visualisation



Example 8.1.7 — hyperbolic flow

Consider the vector field $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n : (x, y) \mapsto (x, -y)$.

For a path $\boldsymbol{\phi}(t) = (\phi_1(t), \phi_2(t))$ to be a flow line of $\boldsymbol{F}$, we must have $\boldsymbol{\phi}'(t) = \boldsymbol{F}(\boldsymbol{\phi}(t))$, so that

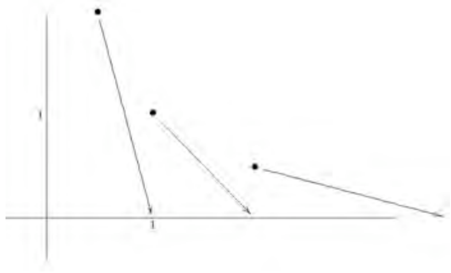
$$(\phi_1{}'(t), \phi_2{}'(t)) = \boldsymbol{F}(\phi_1(t), \phi_2(t)) = (\phi_1(t), -\phi_2(t)),$$

whence $\phi_1{}'(t) = \phi_1(t)$ and $\phi_2{}'(t) = -\phi_2(t)$. In other words, we need $\frac{d}{dt}\phi_1(t) = \phi_1(t)$ and $\frac{d}{dt}\phi_2(t) = -\phi_2(t)$.

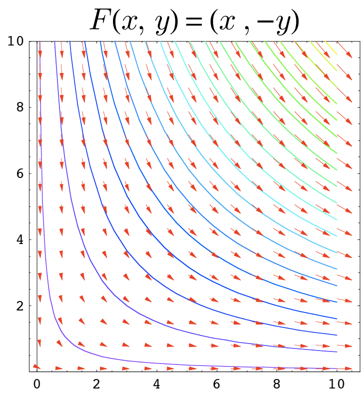
These two differential equations have solutions $\phi_1(t) = C_1 e^t$ and $\phi_2(t) = C_2 e^{-t}$, where $C_1, C_2 \in \mathbb{R}$ are constants.

It follows that flow lines of $\boldsymbol{F}$ are of the form $\boldsymbol{\phi}(t) = (C_1 e^t, C_2 e^{-t})$, where $C_1, C_2 \in \mathbb{R}$ are constants.

Note that the curve of the path $\boldsymbol{\phi}(t)$ is given by the hyperbola $x y = C_1 C_2$. The picture below shows $\boldsymbol{F}(x, y)$, at some points along the same flow line $x y = 1$.

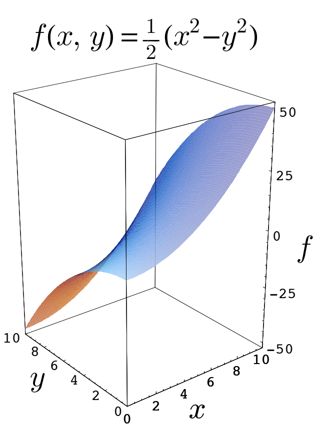


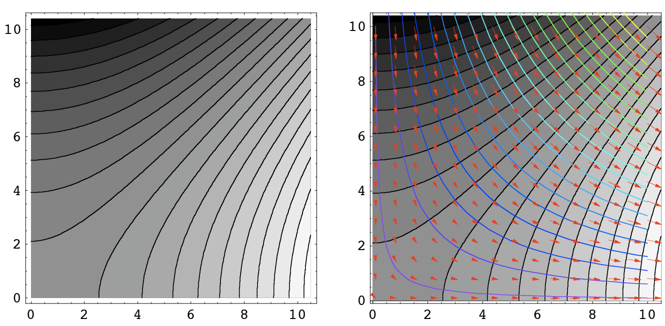
visualisation



This is a gradient field

In fact this vector field $\boldsymbol{F}$ is a *gradient* vector field for the scalar function $f(x, y) = \frac{1}{2}(x^2 - y^2)$.





Definition: gradient operator, ∇

The ***gradient operator*** in $\mathbb{R}^n$, denoted by ∇, is given by: $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)$.

Remarks

- In the special cases $n = 2$ and $n = 3$, we have respectively $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ and $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.
- Note that for any real-valued function $f(x_1, x_2, \ldots, x_n)$, the gradient vector field of f is equal to $\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)$.

8.2 Divergence of a Vector Field

Suppose that $\boldsymbol{F}$ is the vector field describing the motion of a gas or fluid. Then we may wish to discuss the rate of expansion of the volume of the fluid, under this flow.

This is a scalar valued function of a vector field.

Definition: divergence

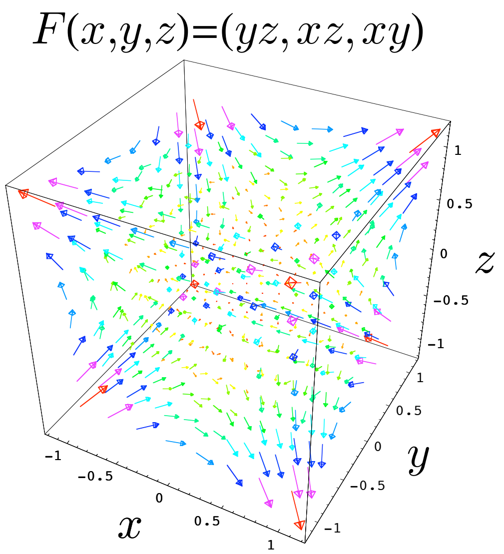
Suppose that $\boldsymbol{F} = (F_1, F_2, \ldots, F_n)$ is a vector field in $\mathbb{R}^n$. Then the ***divergence*** of $\boldsymbol{F}$ is the scalar field

$$\operatorname{div} \boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right) \cdot (F_1, F_2, \ldots, F_n) = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \ldots + \frac{\partial F_n}{\partial x_n}.$$

Example 8.2.1 — zero divergence

For the vector field $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n : (x, y, z) \mapsto (y z, x z, x y)$, we have $\operatorname{div} \boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}(y z) + \frac{\partial}{\partial y}(x z) + \frac{\partial}{\partial z}(x y) = 0$.

visualisation



Example 8.2.2 — positive divergence, expansion

For the vector field $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x, 0)$, we have $\operatorname{div} \boldsymbol{F} = 1$.

Consider next the flow lines of this vector field.

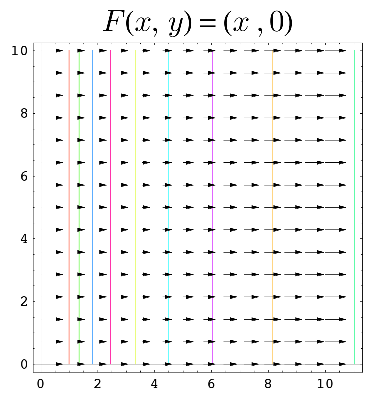
Any flow line must be a path $\boldsymbol{\phi}(t) = (\phi_1(t), \phi_2(t))$ satisfying $\boldsymbol{\phi}'(t) = \boldsymbol{F}(\boldsymbol{\phi}(t))$, so that $(\phi_1(t), \phi_2(t)) = \boldsymbol{F}(\phi_1(t), \phi_2(t)) = (\phi_1(t), 0)$.

It follows that $\phi_1(t) = C_1 e^t$ and $\phi_2(t) = C_2$, where $C_1, C_2 \in \mathbb{R}$ are constants. The flow is therefore in the x-direction.

If we think of $\boldsymbol{F}$ as a velocity field, then the speed becomes greater as we move further away from the line $x = 0$.

This corresponds to an expansion, which is consistent with $\operatorname{div} \boldsymbol{F} > 0$.

plot



Example 8.2.3 — negative divergence, contraction

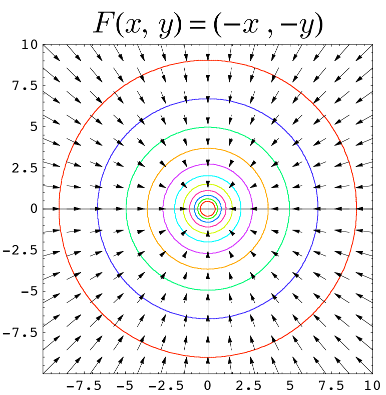
For the vector field $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (-x, -y)$, we have $\operatorname{div} \boldsymbol{F} = -2$. Consider next the flow lines of this vector field.

Any flow line must be a path $\boldsymbol{\phi}(t) = (\phi_1(t), \phi_2(t))$ satisfying $\boldsymbol{\phi}'(t) = \boldsymbol{F}(\boldsymbol{\phi}(t))$, so that $(\phi_1(t), \phi_2(t)) = \boldsymbol{F}(\phi_1(t), \phi_2(t)) = (-\phi_1(t), -\phi_2(t))$.

It follows that $\phi_1(t) = C_1 e^{-t}$ and $\phi_2(t) = C_2 e^{-t}$, where $C_1, C_2 \in \mathbb{R}$ are constants. The flow is therefore radial and towards the origin.

This corresponds to a contraction, which is consistent with $\operatorname{div} \boldsymbol{F} < 0$.

plot

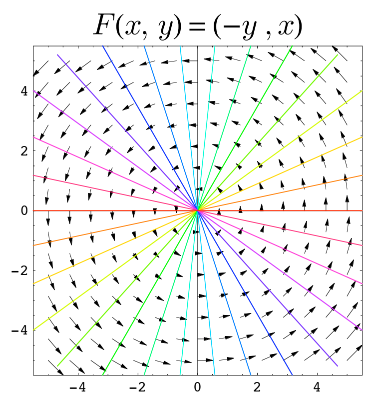


Example 8.2.4 — circular flow, no divergence

For the vector field $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (-y, x)$, we have shown in Example 8.1.5 that the paths of type $\boldsymbol{\phi}(t) = (c \cos t, c \sin t)$, where $c \in \mathbb{R}$, are flow lines of this vector field. (It can be shown that these are all the flowlines of $\boldsymbol{F}$.)

It follows that the flow is circular and anticlockwise around the origin, with no expansion or contraction. Note now that $\operatorname{div} \boldsymbol{F} = 0$.

plot



8.3 Curl of a Vector Field

While the divergence of a vector field is related to expansion or contraction, so the ***curl*** of a vector field is related to rotation — there are beaches in Sydney named after this operator!

Indeed, a vector field with zero curl will be called ***irrotational***.

photos, from http://www.fotosearch.com/

www.fotosearch.com

http://bc.fotosearch.com/bigcomps/UPC/UPC001/cri01104.jpg
http://www.fotosearch.com/comp/PSK/PSK005/low-angle-view_~1574R-24289.jpg
http://bc.fotosearch.com/bigcomps/UPC/UPC005/shf04009.jpg
http://bc.fotosearch.com/bigcomps/BDX/BDX365/bxp125857.jpg

Definition: curl

Suppose that $\boldsymbol{F} = (F_1, F_2, F_3)$ is a vector field in $\mathbb{R}^3$.

Then the ***curl*** of $\boldsymbol{F}$ is the vector field

$$\begin{aligned}\operatorname{curl} \boldsymbol{F} = \nabla \times \boldsymbol{F} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (F_1, F_2, F_3) \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right).\end{aligned}$$

Remarks

- We can write $\operatorname{curl} \boldsymbol{F} = \nabla \times \boldsymbol{F} = \det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}$

  where $\boldsymbol{i} = (1, 0, 0)$, $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$.

- Unlike gradient and divergence which are valid in any euclidean space $\mathbb{R}^n$ for any natural number $n \in \mathbb{N}$, curl is only defined in $\mathbb{R}^3$.

- Suppose that $\boldsymbol{F}$ is a vector field in $\mathbb{R}^2$. While we cannot define $\operatorname{curl} \boldsymbol{F}$, we can nevertheless regard $\boldsymbol{F}$ as a vector field in $\mathbb{R}^3$, for which the third component is zero and the two other components are independent of the z coordinate. Then

  $$\operatorname{curl} \boldsymbol{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right) \times (F_1, F_2, 0) = \left(0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

  The function $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ is sometimes called the ***scalar curl*** of $\boldsymbol{F}$.

Example 8.3.1 — $\nabla \times (\nabla f) = 0$

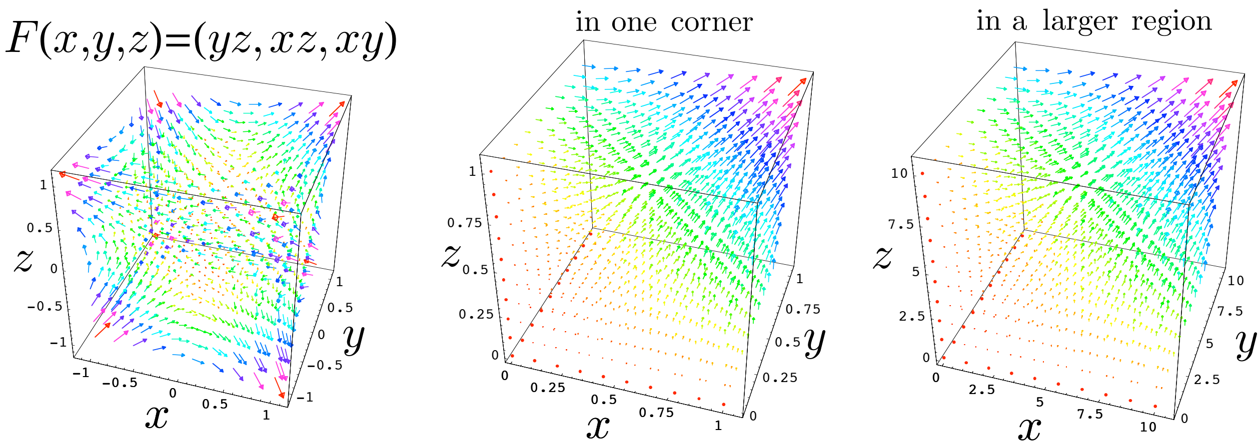
For the vector field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (y z, x z, x y)$, we have

$$\operatorname{curl} \boldsymbol{F} = \left(\frac{\partial (x y)}{\partial y} - \frac{\partial (x z)}{\partial z}, \frac{\partial (y z)}{\partial z} - \frac{\partial (x z)}{\partial x}, \frac{\partial (x z)}{\partial x} - \frac{\partial (y z)}{\partial y}\right) = \boldsymbol{0}.$$

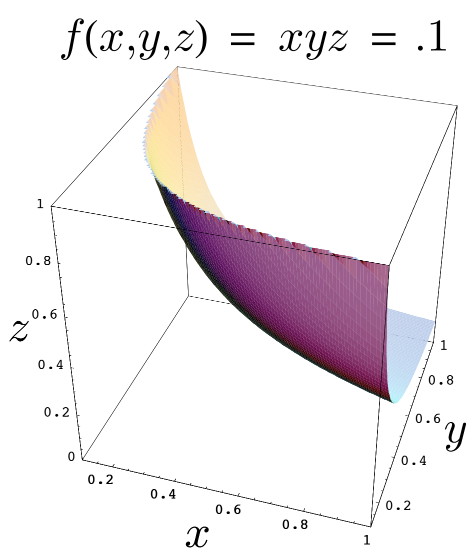
Here, note that if we consider the function $f : \mathbb{R}^3 \to \mathbb{R} : f(x, y, z) = x y z$, then $\boldsymbol{F} = \nabla f$.

We shall show later (in Theorem 8G) that $\nabla \times (\nabla f) = 0$, for any twice continuously differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$.

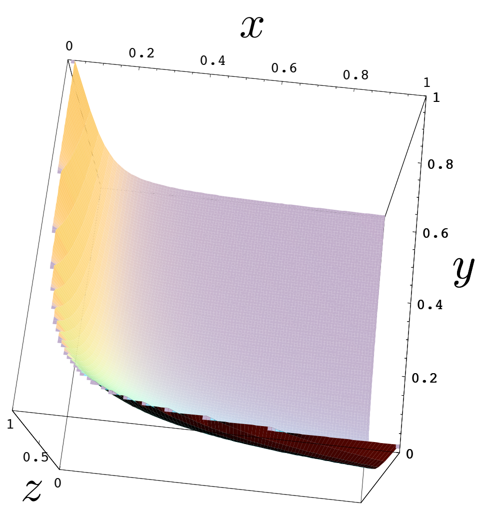
visualisation: symmetrical — one corner only & homogeneity



level surface: $f(x, y, z) = x\,y\,z = .1$



movie of level surfaces



Example 8.3.2

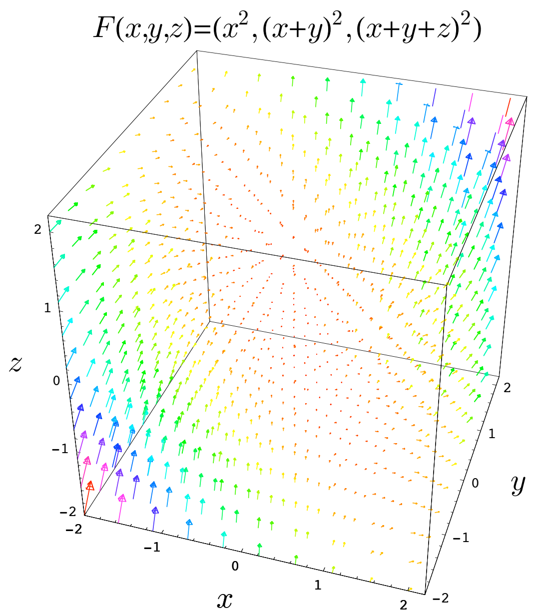
For the vector field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (x^2, (x+y)^2, (x+y+z)^2)$, we have

$$\begin{aligned}\operatorname{curl} \boldsymbol{F} &= \left(\frac{\partial\,(x+y+z)^2}{\partial y} - \frac{\partial\,(x+y)^2}{\partial z},\ \frac{\partial\,(x^2)}{\partial z} - \frac{\partial\,(x+y+z)^2}{\partial x},\ \frac{\partial\,(x+y)^2}{\partial x} - \frac{\partial\,(x^2)}{\partial y}\right)\\ &= (2\,(x+y+z) - 0,\ 0 - 2\,(x+y+z),\ 2\,(x+y) - 0)\\ &= 2\,(x+y+z,\ -x-y-z,\ x+y).\end{aligned}$$

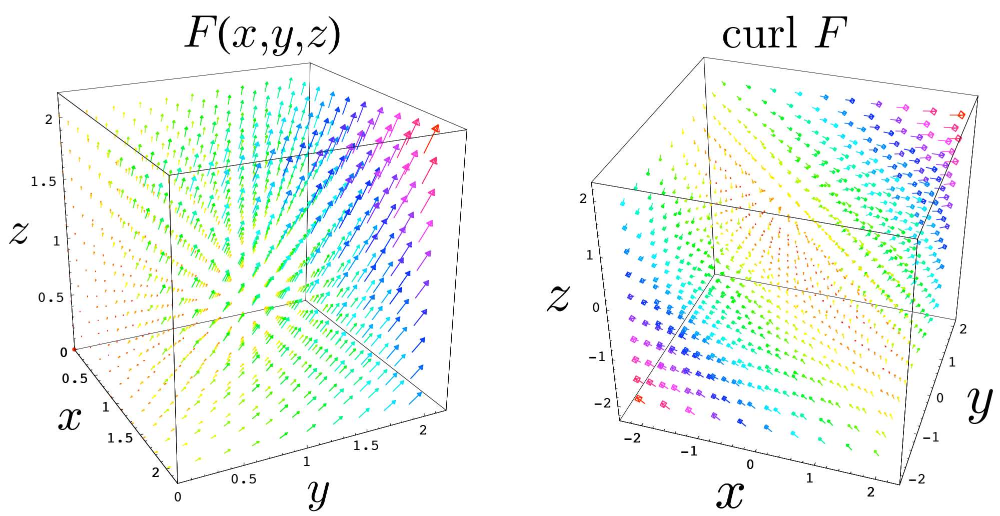
Hence $\nabla \cdot (\operatorname{curl} \boldsymbol{F}) = \frac{\partial\,(2\,(x+y+z))}{\partial x} + \frac{\partial\,(-2\,(x+y+z))}{\partial y} + \frac{\partial\,(2\,(x+y))}{\partial z} = 2 - 2 + 0 = 0$.

We shall show later in Theorem 8F that $\nabla \cdot (\nabla \times \boldsymbol{F}) = 0$, for any twice continuously differentiable function $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$.

visualisation



top corner & curl $F = 2(x+y+z\,,\,-(x+y+z)\,,\,x+y)$



Example 8.3.3

Consider again the vector field $\boldsymbol{F}: \mathbb{R}^2 \to \mathbb{R}^2 : (x\,,\,y) \mapsto (-y\,,\,x)$.

We have shown in Examples 8.1.5 and 8.2.4 that the flow is circular and anti-clockwise around the origin.

Note now that the scalar curl of $\boldsymbol{F}$ is equal to: $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1 - (-1) = 2$.

This is consistent with a positive circulation for this flow.

8.4 Basic Identities of Vector Analysis

The first three theorems do not involve curl and are therefore valid in $\mathbb{R}^n$ for any natural number $n \in \mathbb{N}$.

The first two of these theorems are easy to prove.

Theorem 8A — properties of ∇ (grad)

For any continuously differentiable functions $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$, and for any fixed real number $c \in \mathbb{R}$, we have

(a) $\nabla(f + g) = \nabla f + \nabla g$;

(b) $\nabla(c\, f) = c\, \nabla f$;

(c) $\nabla(f\, g) = f\, \nabla g + g\, \nabla f$; and

(d) $\nabla(f / g) = (g\, \nabla f - f\, \nabla g) / g^2$, at any point $x \in A$ for which $g(x) \neq 0$.

Theorem 8B — properties of $\nabla\cdot$ (div)

For any continuously differentiable functions $\boldsymbol{F} : A \to \mathbb{R}^n$ and $\boldsymbol{G} : A \to \mathbb{R}^n$, where $A \subseteq \mathbb{R}^n$, and for any fixed real number $c \in \mathbb{R}$, we have

(a) $\nabla \cdot (\boldsymbol{F} + \boldsymbol{G}) = \nabla \cdot \boldsymbol{F} + \nabla \cdot \boldsymbol{G}$;

(b) $\nabla \cdot (c\, \boldsymbol{F}) = c\, \nabla \cdot \boldsymbol{F}$.

Theorem 8C — div of product with scalar field

For any continuously differentiable functions $\boldsymbol{F} : A \to \mathbb{R}^n$ and $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$, we have $\nabla \cdot (f\, \boldsymbol{F}) = (\nabla f) \cdot \boldsymbol{F} + f\, \nabla \cdot \boldsymbol{F}$.

Proof

Let $\boldsymbol{F} = (F_1, F_2, \ldots, F_n)$, then

$$\begin{aligned}
\nabla \cdot (f\, \boldsymbol{F}) &= \frac{\partial (f\, F_1)}{\partial x_1} + \frac{\partial (f\, F_2)}{\partial x_2} + \ldots + \frac{\partial (f\, F_n)}{\partial x_n} = \left(\frac{\partial f}{\partial x_1} F_1 + f \frac{\partial F_1}{\partial x_1}\right) + \ldots + \left(\frac{\partial f}{\partial x_n} F_n + f \frac{\partial F_n}{\partial x_n}\right) \\
&= \left(\frac{\partial f}{\partial x_1} F_1 + \ldots + \frac{\partial f}{\partial x_n} F_n\right) + \left(f \frac{\partial F_1}{\partial x_1} + \ldots + f \frac{\partial F_n}{\partial x_n}\right) \\
&= \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) \cdot (F_1, \ldots, F_n) + f \left(\frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial F_n}{\partial x_n}\right) \\
&= (\nabla f) \cdot \boldsymbol{F} + f\, \nabla \cdot \boldsymbol{F}.
\end{aligned}$$

We also have the following four theorems which involve curl and are therefore restricted to $\mathbb{R}^3$.

Theorem 8D — properties of $\nabla\times$ (curl)

For any continuously differentiable functions $\boldsymbol{F} : A \to \mathbb{R}^n$ and $\boldsymbol{G} : A \to \mathbb{R}^n$, where $A \subseteq \mathbb{R}^n$, and for any fixed real number $c \in \mathbb{R}$, we have

(a) $\nabla \times (\boldsymbol{F} + \boldsymbol{G}) = \nabla \times \boldsymbol{F} + \nabla \times \boldsymbol{G}$;

(b) $\nabla \times (c\, \boldsymbol{F}) = c\, \nabla \times \boldsymbol{F}$; and

(c) $\nabla \cdot (\boldsymbol{F} \times \boldsymbol{G}) = (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{G} - \boldsymbol{F} \cdot (\nabla \times \boldsymbol{G})$.

Proof

Parts (a) and (b) are easy to check.

To prove (c), let $\boldsymbol{F} = (F_1, F_2, F_3)$ and $\boldsymbol{G} = (G_1, G_2, G_3)$. Then

$$\begin{aligned}\nabla\cdot(\boldsymbol{F}\times\boldsymbol{G}) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(F_2 G_3 - F_3 G_2,\ F_3 G_1 - F_1 G_3,\ F_1 G_2 - F_2 G_1)\\ &= \frac{\partial(F_2 G_3 - F_3 G_2)}{\partial x} + \frac{\partial(F_3 G_1 - F_1 G_3)}{\partial y} + \frac{\partial(F_1 G_2 - F_2 G_1)}{\partial z}\\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)G_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)G_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)G_3 - F_1\left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}\right) - F_2\left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}\right) - F_3\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)\\ &= (\nabla\times\boldsymbol{F})\cdot\boldsymbol{G} - \boldsymbol{F}\cdot(\nabla\times\boldsymbol{G}),\end{aligned}$$

using the sum and product rules for differentiation, and rearranging terms.

Theorem 8E — curl of product with scalar field

For any continuously differentiablefunctions $\boldsymbol{F} : A \to \mathbb{R}^n$ and $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$, we have $\nabla\times(f\,\boldsymbol{F}) = (\nabla f)\times\boldsymbol{F} + f\,\nabla\times\boldsymbol{F}$.

Proof

Let $\boldsymbol{F} = (F_1, F_2, F_3)$. Then

$$\begin{aligned}\nabla\times(f\,\boldsymbol{F}) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\times(f F_1, f F_2, f F_3)\\ &= \left(\frac{\partial(f F_3)}{\partial y} - \frac{\partial(f F_2)}{\partial z}, \frac{\partial(f F_1)}{\partial z} - \frac{\partial(f F_3)}{\partial x}, \frac{\partial(f F_2)}{\partial x} - \frac{\partial(f F_1)}{\partial y}\right)\\ &= \left(\frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2, \frac{\partial f}{\partial z}F_1 - \frac{\partial f}{\partial x}F_3, \frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1\right) + f\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\\ &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\times(F_1, F_2, F_3) + f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\times(F_1, F_2, F_3)\\ &= (\nabla f)\times\boldsymbol{F} + f\,\nabla\times\boldsymbol{F}\end{aligned}$$

using the sum and product rules for differentiation, and rearranging terms.

Theorem 8F — div of a curl vanishes, $\nabla\cdot(\nabla\times\boldsymbol{F}) \equiv 0$

For any twice continuously differentiablefunction $\boldsymbol{F} : A \to \mathbb{R}^3$, where $A \subseteq \mathbb{R}^n$, we have $\nabla\cdot(\nabla\times\boldsymbol{F}) = 0$.

Proof

Let $\boldsymbol{F} = (F_1, F_2, F_3)$. Then

$$\begin{aligned}\nabla\cdot(\nabla\times\boldsymbol{F}) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\\ &= \frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_2}{\partial x\partial z} + \frac{\partial^2 F_1}{\partial y\partial z} - \frac{\partial^2 F_3}{\partial y\partial x} + \frac{\partial^2 F_2}{\partial z\partial x} - \frac{\partial^2 F_1}{\partial z\partial y} = \left(\frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_3}{\partial y\partial x}\right) + \left(\frac{\partial^2 F_2}{\partial z\partial x} - \frac{\partial^2 F_2}{\partial x\partial z}\right) + \left(\frac{\partial^2 F_1}{\partial y\partial z} - \frac{\partial^2 F_1}{\partial z\partial y}\right)\\ &= 0 + 0 + 0 = 0,\end{aligned}$$

in view of the fact that for a twice continuously differentiable function g, the 2nd partial derivatives satisfy $\frac{\partial^2 g}{\partial x\partial y} = \frac{\partial^2 g}{\partial y\partial x}$, etc. .

Theorem 8G — curl of a grad vanishes, $\nabla\times(\nabla f) \equiv 0$

For any twice continuously differentiable function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^3$, we have $\nabla\times(\nabla f) = \boldsymbol{0}$.

Proof

We have that

$$\begin{aligned}\nabla\times(\nabla f) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\times\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y}, \frac{\partial^2 f}{\partial z\partial x} - \frac{\partial^2 f}{\partial x\partial z}, \frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y\partial x}\right)\\ &= (0, 0, 0) = \boldsymbol{0},\end{aligned}$$

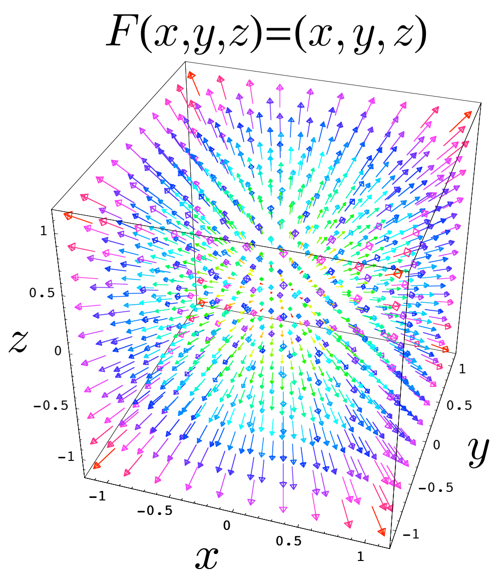
in a similar way to the previous result.

Example 8.4.1

Consider the vector field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (x, y, z)$. It is easily checked that $\nabla \cdot \boldsymbol{F} = 3$.

It follows that there is *no* function $\boldsymbol{G} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\boldsymbol{F} = \nabla \times \boldsymbol{G}$, for otherwise $\nabla \cdot \boldsymbol{F} = 0$ by Theorem 8F.

visualisation

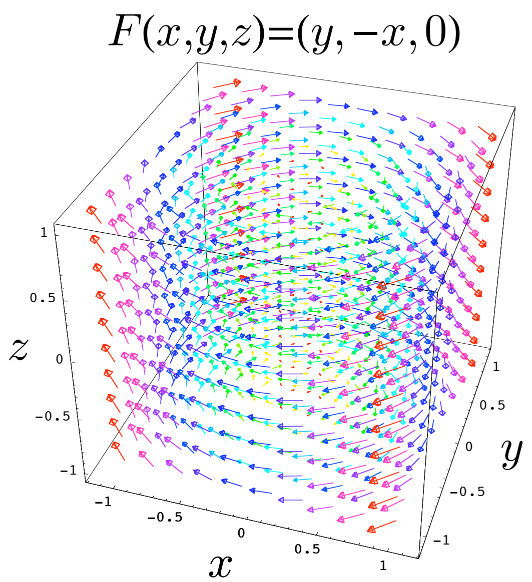


Example 8.4.2

Consider the vector field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (y, -x, 0)$. It is easily checked that $\nabla \times \boldsymbol{F} = (0, 0, -2)$.

It follows that there is *no* function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\boldsymbol{F} = \nabla f$, for otherwise $\nabla \times \boldsymbol{F} = 0$ by Theorem 8G.

visualisation



For any twice continuously differentiable function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$, we have

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right) \cdot \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right) = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2}.$$

Definition: Laplace operator, ∇^2

The ***Laplace operator*** ∇^2 in $\mathbb{R}^n$, also known as the ***Laplacian***, is defined to be the divergence of the gradient, so that for any twice continuously differentiable function $f : A \to \mathbb{R}$, we have:

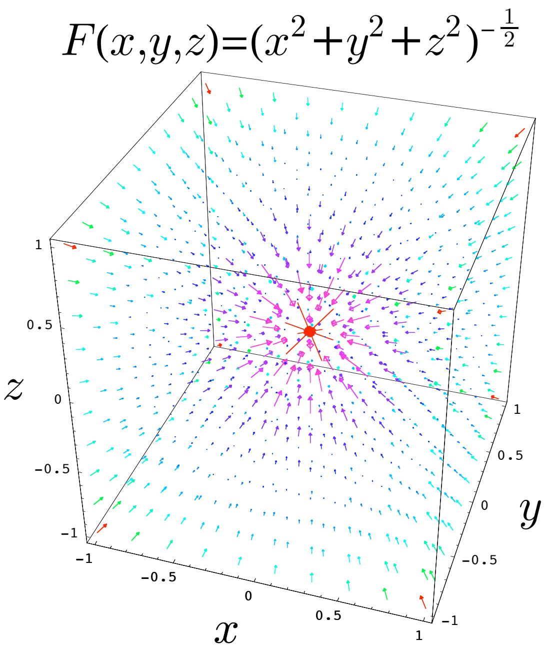
$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2}.$$

Example 8.4.3

A function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$, is said to *satisfy Laplace's equation* if $\nabla^2 f = 0$.

An example of such a function is given in the case $n = 3$ by $f(\boldsymbol{x}) = f(x, y, z) = \frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}} = \frac{1}{\|\boldsymbol{x}\|}$.

visualisation



Theorem 8H — Laplacian of a product

For any twice continuously differentiable functions $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$, we have

(a) $\nabla^2 (f\, g) = (\nabla^2 f)\, g + 2\, (\nabla f \cdot \nabla g) + f\, \nabla^2 g$;

(b) $\nabla \cdot (f\, \nabla g - g\, \nabla f) = f\, \nabla^2 g - g\, \nabla^2 f$.

Proof

Note that

$$\begin{aligned}\nabla^2 (f\, g) &= \frac{\partial^2 (f\, g)}{\partial x_1^2} + \ldots + \frac{\partial^2 (f\, g)}{\partial x_n^2} \\ &= \left(\frac{\partial^2 f}{\partial x_1^2}\, g + 2\, \frac{\partial f}{\partial x_1}\, \frac{\partial g}{\partial x_1} + f\, \frac{\partial^2 g}{\partial x_1^2}\right) + \ldots + \left(\frac{\partial^2 f}{\partial x_n^2}\, g + 2\, \frac{\partial f}{\partial x_n}\, \frac{\partial g}{\partial x_n} + f\, \frac{\partial^2 g}{\partial x_n^2}\right) \\ &= \left(\frac{\partial^2 f}{\partial x_1^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2}\right) g + f \left(\frac{\partial^2 g}{\partial x_1^2} + \ldots + \frac{\partial^2 g}{\partial x_n^2}\right) + 2 \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) \cdot \left(\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n}\right) \\ &= (\nabla^2 f)\, g + 2\, (\nabla f \cdot \nabla g) + f\, \nabla^2 g.\end{aligned}$$

This gives (a).

On the other hand, $\nabla\cdot(f\nabla g - g\nabla f) = \nabla\cdot\left(f\left(\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n}\right) - \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) g\right)$

$$= \nabla\cdot\left(f\frac{\partial g}{\partial x_1} - \frac{\partial f}{\partial x_1} g, \ldots, f\frac{\partial g}{\partial x_n} - \frac{\partial f}{\partial x_n} g\right)$$

$$= \left(f\frac{\partial^2 g}{\partial x_1^2} + \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_1} - \frac{\partial^2 f}{\partial x_1^2} g - \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_1}\right) + \ldots + \left(f\frac{\partial^2 g}{\partial x_n^2} + \frac{\partial f}{\partial x_n}\frac{\partial g}{\partial x_n} - \frac{\partial^2 f}{\partial x_n^2} g - \frac{\partial f}{\partial x_n}\frac{\partial g}{\partial x_n}\right)$$

$$= \left(f\frac{\partial^2 g}{\partial x_1^2} - \frac{\partial^2 f}{\partial x_1^2} g\right) + \ldots + \left(f\frac{\partial^2 g}{\partial x_n^2} - \frac{\partial^2 f}{\partial x_n^2} g\right)$$

$$= f\left(\frac{\partial^2 g}{\partial x_1^2} + \ldots + \frac{\partial^2 g}{\partial x_n^2}\right) - \left(\frac{\partial^2 f}{\partial x_1^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2}\right) g$$

$$= f\nabla^2 g - g\nabla^2 f.$$

Theorem 8J — cross product of gradients

For any twice continuously differentiable functions $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^3$, we have $\nabla\cdot(\nabla f \times \nabla g) = 0$.

Proof

Exercise for the reader; do it in a similar way to the others!

Definition: Total Derivative, $\mathbb{D}F$

Let in $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field, then the ***total derivative*** $\mathbb{D}\boldsymbol{F}$ is the 3×3 matrix of partial derivatives:

$$\mathbb{D}\boldsymbol{F} = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{pmatrix}.$$

$$\mathbb{D}^+\boldsymbol{F} = \tfrac{1}{2}\left(\mathbb{D}\boldsymbol{F} + (\mathbb{D}\boldsymbol{F})^{\mathrm{T}}\right) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{1}{2}\left(\frac{\partial F_1}{\partial y} + \frac{\partial F_2}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial F_1}{\partial z} + \frac{\partial F_3}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial F_2}{\partial x} + \frac{\partial F_1}{\partial y}\right) & \frac{\partial F_2}{\partial y} & \frac{1}{2}\left(\frac{\partial F_2}{\partial z} + \frac{\partial F_3}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial F_3}{\partial y} + \frac{\partial F_2}{\partial z}\right) & \frac{\partial F_3}{\partial z} \end{pmatrix}$$

is the ***symmetric part*** of $\mathbb{D}\boldsymbol{F}$. Notice that $(\mathbb{D}^+\boldsymbol{F})^{\mathrm{T}} = \mathbb{D}^+\boldsymbol{F}$.

$$\mathbb{D}^-\boldsymbol{F} = \tfrac{1}{2}\left(\mathbb{D}\boldsymbol{F} - (\mathbb{D}\boldsymbol{F})^{\mathrm{T}}\right) = \frac{1}{2}\begin{pmatrix} 0 & \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}\right) & \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) \\ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) & 0 & \left(\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y}\right) \\ \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) & \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) & 0 \end{pmatrix}$$

is the ***anti-symmetric part*** of $\mathbb{D}\boldsymbol{F}$. Notice that $(\mathbb{D}^-\boldsymbol{F})^{\mathrm{T}} = -\mathbb{D}^-\boldsymbol{F}$.

Remarks

- $\mathbb{D}\boldsymbol{F} = \mathbb{D}^+\boldsymbol{F} + \mathbb{D}^-\boldsymbol{F}$, so the information contained within the full derivative $\mathbb{D}\boldsymbol{F}$ is precisely that contained within its symmetric and anti-symmetric parts $\mathbb{D}^+\boldsymbol{F}$ and $\mathbb{D}^-\boldsymbol{F}$.
- The components of $\mathbb{D}^-\boldsymbol{F}$ are the same (up to $\pm$sign) as the components of $\nabla\times\boldsymbol{F}$; namely,
  $$\operatorname{curl}\boldsymbol{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}.$$
- For a gradient vector field $\boldsymbol{F} = \nabla f$ in $\mathbb{R}^3$ we have that $\mathbb{D}(\nabla f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y\,\partial x} & \frac{\partial^2 f}{\partial z\,\partial x} \\ \frac{\partial^2 f}{\partial x\,\partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z\,\partial y} \\ \frac{\partial^2 f}{\partial x\,\partial z} & \frac{\partial^2 f}{\partial y\,\partial z} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$,

which is necessarily a symmetric matrix; thus $\mathbb{D}(\nabla f) = \mathbb{D}^{+}(\nabla f)$ with $\mathbb{D}^{-}(\nabla f) = \mathbf{0}$.

Together with the previous remark this shows that necessarily have that $\nabla \times (\nabla f) = \mathbf{0}$, for any continuous scalar function f on $\mathbb{R}^3$.

In particular, for the general (homogeneous) quadratic function $f(x, y, z) = a x^2 + 2 b x y + 2 c x z + d y^2 + 2 g y z + h z^2$

we have that $\frac{1}{2} \mathbb{D}(\nabla f) = \begin{pmatrix} a & b & c \\ b & d & g \\ c & g & h \end{pmatrix}$, which is the matrix of the quadratic form for $f(x, y, z)$.

- For any vector field $\boldsymbol{F}$ we have that:

$$\nabla \cdot \boldsymbol{F} = \text{trace}\, \mathbb{D}\boldsymbol{F} = \frac{\partial \boldsymbol{F}_1}{\partial x} + \frac{\partial \boldsymbol{F}_2}{\partial y} + \frac{\partial \boldsymbol{F}_3}{\partial z},$$

where the ***trace*** of a square matrix is the sum of the elements on the major diagonal.

In particular, for a gradient field we have the ***Laplacian***: $\nabla \cdot (\nabla f) = \text{trace}\, \mathbb{D}(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$.