MATH236 — Weeks 12–13

Physical Applications & Differential Forms

Conservation Laws

Conservation of mass

Let $\boldsymbol{V}(t, x, y, z)$ be a continuously differentiable vector field on $\mathbb{R}^3$ for all times t, and let $\rho(t, x, y, z)$ be a real-valued function. Then the ***law of conservation of mass*** for $\boldsymbol{V}$ and ρ is the statement that $\frac{d}{dt}\int_\Omega \rho\, dV = -\int_{\partial\Omega} \boldsymbol{J}\cdot d\boldsymbol{S} = -\int_{\partial\Omega} \boldsymbol{J}\cdot\boldsymbol{n}\, dS$ holds for all regions Ω in $\mathbb{R}^3$, where $\boldsymbol{J} = \rho\boldsymbol{V}$.

Theorem

For $\boldsymbol{V}(t, x, y, z)$ and $\rho(t, x, y, z)$ defined on $\mathbb{R}^3$ for all times t, with $\boldsymbol{J} = \rho\boldsymbol{V}$, the law of conservation of mass is equivalent to the condition:

$$\nabla\cdot\boldsymbol{J} + \frac{\partial\rho}{\partial t} = 0 \quad \text{equivalently} \quad \rho\nabla\cdot\boldsymbol{V} + \boldsymbol{V}\cdot(\nabla\rho) + \frac{\partial\rho}{\partial t} = 0.$$

Sketch of proof

First observe that $\frac{d}{dt}\int_\Omega \rho\, dx\, dy\, dz = \int_\Omega \frac{\partial\rho}{\partial t}\, dx\, dy\, dz$, and by the divergence theorem $\int_{\partial\Omega}\boldsymbol{J}\cdot d\boldsymbol{S} = \int_\Omega \nabla\cdot\boldsymbol{J}\, dV$. Thus conservation of mass is equivalent to the condition $\int_\Omega\left(\nabla\cdot\boldsymbol{J} + \frac{\partial\rho}{\partial t}\right)dx\, dy\, dz = 0$. But this is to hold in *all* regions Ω, so it must be that $\nabla\cdot\boldsymbol{J} + \frac{\partial\rho}{\partial t} = 0$ holds everywhere.

Heat equation

The continuity equation applies also to heat transfer with some medium.

Let $T(t, x, y, z)$ be a (twice) continuously differentiable function which gives the *temperature* at all points in the medium, at each time t. Then heat flows with the vector field $\boldsymbol{F} = -\nabla T$ (from hot to cold, hence the '$-$' sign). In this context the *source* function $\rho(t, x, y, z)$ is the *energy density* (that is, energy per unit volume) which is given by $\rho = c\rho_0 T$, where ρ_0 is the *mass density* (assumed constant within a particular medium) and c is the *specific heat* of the medium. The *energy flux* vector field is $\boldsymbol{J} = k\,\boldsymbol{F} = -k\nabla T$, where k is a constant called the *conductivity* of the medium.

Thus ρ is proportional to the temperature while $\boldsymbol{J}$ follows the temerature gradient, leading to:

$$\nabla\cdot(-k\nabla T) + c\rho_0\frac{\partial T}{\partial t} = 0 \quad\Longleftrightarrow\quad \frac{\partial T}{\partial t} - \mu\nabla^2 T = 0,$$

where $\mu = k/c\rho_0$ is constant, called the *diffusivity* of the medium. This (partial) differential equation (PDE) is known as the ***Heat equation***.

It is very important in various physical applications; it governs conduction of heat, in the sense that if $T(0, x, y, z)$ is the temperature distribution at time $t = 0$ then $T(t, x, y, z)$ is fully determined for all later times $t > 0$ by a solution of this PDE. Notice that if the temperature does not change with time, then $\nabla^2 T = 0$, so that T satisfies Laplace's equation.

Maxwell's equations

For the (time-dependent) *electric field* $\boldsymbol{E}(t, x, y, z)$ and *magnetic field* $\boldsymbol{H}(t, x, y, z)$, the *source distribution* $\rho(t, x, y, z)$ and *current density* $\boldsymbol{J}(t, x, y, z)$, defined on $\mathbb{R}^3$ for times t, the Maxwell's equations are the following set of (partial) differential equations:

$$\nabla\cdot\boldsymbol{E} = \rho \qquad \text{(Gauss' Law)}$$

$$\nabla\cdot\boldsymbol{H} = 0 \qquad \text{(no magnetic sources)}$$

$$\nabla\times\boldsymbol{E} + \frac{\partial\boldsymbol{H}}{\partial t} = \boldsymbol{0} \qquad \text{(Faraday's Law)}$$

and $$\nabla\times\boldsymbol{H} - \frac{\partial\boldsymbol{E}}{\partial t} = \boldsymbol{J} \qquad \text{(Ampère's Law).}$$

Gauss' Law allows measurement of charge

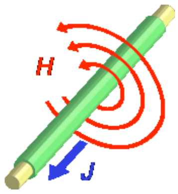
(there is no simple magnetic charge; i.e., no magnetic monopole)

Ampère's Law $\Longrightarrow$ electromagnets (Wikipedia)

Faraday's Law $\Longrightarrow$ electromagmetic induction; (Wikipedia) e.g. generators.

Ampère’s Law (time-independent)

If the physical situation is not changing with time, then Ampère’s Law means that you cannot have an electrical current without also having a magnetic field. (The term $\frac{\partial \boldsymbol{E}}{\partial t}$ then only has effect as the current is switched on or off.)



Suppose the current is passing through a surface S bounded by a closed curve C, that wraps once around the wire bearing the current.

Then: $\int_S \boldsymbol{J} \cdot d\boldsymbol{S} = \int_S (\nabla \times \boldsymbol{H}) \cdot d\boldsymbol{S} = \int_C \boldsymbol{H} \cdot d\boldsymbol{s}$.

(see *Mathematica* demos: CreationOfAMagneticFieldByAnElectricCurrent,SquareHelmholtzCoils,GalvanometerAsADCMultimeter)

Faraday’s Law

Faraday’s Law is often expressed as an integral equation.

Consider any surface S, with boundary $\partial S = C$, for which Stokes’ Theorem applies. Then we have that

$$\int_C \boldsymbol{E} \cdot d\boldsymbol{s} = \int_S (\nabla \times \boldsymbol{E}) \cdot d\boldsymbol{S} = -\int_S \frac{\partial \boldsymbol{H}}{\partial t} \cdot d\boldsymbol{S} = -\frac{\partial}{\partial t} \int_S \boldsymbol{H} \cdot d\boldsymbol{S}.$$

Now $\int_C \boldsymbol{E} \cdot d\boldsymbol{s}$ physically represents the change in voltage around the curve C, in a loop of wire, say. Also, $\int_S \boldsymbol{H} \cdot d\boldsymbol{S}$ is the *magnetic flux* passing through a surface S. Hence Faraday’s law says that: *the voltage around the loop equals the negative of the rate of change of the magnetix flux through the loop*. An extremely practical application is the generation of electricity:

spin a coil of wire within a magnetic field—an electric current will flow in the wire.

(see *Mathematica* demos: InducedEMFThroughAWire,MagneticFluxThroughALoopOfWire)

waves

Take the time derivative of Faraday’s Law :

$$\frac{\partial^2 \boldsymbol{H}}{\partial t^2} = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{E}) = -\nabla \times \frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times (\boldsymbol{J} - \nabla \times \boldsymbol{H}) = (\nabla \times \boldsymbol{J}) - \nabla \times (\nabla \times \boldsymbol{H}) = \nabla^2 \boldsymbol{H} - \nabla(\nabla \cdot \boldsymbol{H}) + (\nabla \times \boldsymbol{J}) = \nabla^2 \boldsymbol{H} + (\nabla \times \boldsymbol{J}).$$

Hence we have that $\frac{\partial^2 \boldsymbol{H}}{\partial t^2} - \nabla^2 \boldsymbol{H} = \nabla \times \boldsymbol{J}$, so that the magnetic field satisfies an inhomogeneous *wave equation*; that is, with source term $\nabla \times \boldsymbol{J}$.

Similarly take the time derivative of Ampère’s Law :

$$\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \frac{\partial}{\partial t}(\nabla \times \boldsymbol{H}) - \frac{\partial \boldsymbol{J}}{\partial t} = \nabla \times \frac{\partial \boldsymbol{H}}{\partial t} - \frac{\partial \boldsymbol{J}}{\partial t} = -\nabla \times (\nabla \times \boldsymbol{E}) - \frac{\partial \boldsymbol{J}}{\partial t} = \nabla^2 \boldsymbol{E} - \nabla(\nabla \cdot \boldsymbol{E}) - \frac{\partial \boldsymbol{J}}{\partial t} = \nabla^2 \boldsymbol{E} - \left(\frac{\partial \boldsymbol{J}}{\partial t} + \nabla \rho\right).$$

Hence we have that $\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \nabla^2 \boldsymbol{E} = \frac{\partial \boldsymbol{J}}{\partial t} + \nabla \rho$, so that the electric field also satisfies an inhomogeneous *wave equation*, with source term $\frac{\partial \boldsymbol{J}}{\partial t} + \nabla \rho$.

gauge field and guage potential

Starting from $\nabla \cdot \boldsymbol{H} = 0$ there must be a ***gauge field*** $\boldsymbol{A}$, such that $\boldsymbol{H} = \nabla \times \boldsymbol{A}$; note that this determines $\boldsymbol{A}$ only up to the ***gauge freedom*** $\boldsymbol{A} \mapsto \boldsymbol{A} + f$, for any scalar function f. Now Faraday’s Law becomes $\boldsymbol{0} = \nabla \times \boldsymbol{E} + \frac{\partial}{\partial t}(\nabla \times \boldsymbol{A}) = \nabla \times \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right)$, so that $\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\nabla \phi$, for some ***guage potential*** function ϕ (defined up to addition of a constant). Thus the electric and magnetic fields are fully determined by the guage pair $(\boldsymbol{A}, \phi)$, via the expressions: $\boldsymbol{H} = \nabla \times \boldsymbol{A}$ and $\boldsymbol{E} = -\nabla \phi - \frac{\partial \boldsymbol{A}}{\partial t}$. The gauge freedom $\boldsymbol{A} \mapsto \boldsymbol{A} + \nabla f$ means that ϕ is determined up to $\phi \mapsto \phi - \frac{\partial f}{\partial t}$.

Using these, Gauss’ Law gives $\rho = \nabla \cdot \left(-\nabla \phi - \frac{\partial \boldsymbol{A}}{\partial t}\right) = -\nabla^2 \phi - \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A})$, while Ampère’s Law gives that

$$\boldsymbol{J} = \nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times (\nabla \times \boldsymbol{A}) - \frac{\partial}{\partial t}\left(-\nabla \phi - \frac{\partial \boldsymbol{A}}{\partial t}\right) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} + \frac{\partial^2 \boldsymbol{A}}{\partial t^2} + \frac{\partial}{\partial t}(\nabla \phi) = \frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} + \nabla\left(\nabla \cdot \boldsymbol{A} + \frac{\partial \phi}{\partial t}\right).$$

Now if it were true that necessarily $\nabla \cdot \boldsymbol{A} + \frac{\partial \phi}{\partial t} = 0$ everywhere, then these expressions would reduce to: $\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} = \boldsymbol{J}$ and $\nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = \rho$. That is, both A and ϕ satisfy inhomogeneous *wave equations*, with the specified current and charge distributions.

To show that this is indeed always achievable, consider next the gauge freedom: $\boldsymbol{A} \mapsto \boldsymbol{A} + \nabla f$. That is, suppose that the pair $(\boldsymbol{A}_0, \phi_0)$ determines the fields $\boldsymbol{E}$ and $\boldsymbol{H}$. Then for $\boldsymbol{A} = \boldsymbol{A}_0 + \nabla f$ and $\phi \mapsto \phi_0 - \frac{\partial f}{\partial t}$ we first verify that:

$$\rho = -\nabla^2 \phi_0 - \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A}_0) = -\nabla^2\left(\phi - \frac{\partial f}{\partial t}\right) - \frac{\partial}{\partial t}(\nabla \cdot (\boldsymbol{A} - \nabla f)) = -\nabla^2 \phi - \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A}) + \left(\nabla^2 \frac{\partial f}{\partial t} - \frac{\partial}{\partial t}(\nabla^2 f)\right) = -\nabla^2 \phi - \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A}), \text{ and}$$

$$\boldsymbol{J} = \frac{\partial^2 \boldsymbol{A}_0}{\partial t^2} - \nabla^2 \boldsymbol{A}_0 + \nabla\left(\nabla \cdot \boldsymbol{A}_0 + \frac{\partial \phi_0}{\partial t}\right) = \frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} - \left(\frac{\partial^2}{\partial t^2} \nabla f - \nabla^2(\nabla f)\right) + \nabla\left(\nabla \cdot \boldsymbol{A}_0 + \frac{\partial \phi_0}{\partial t}\right) = \frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} + \nabla\left(-\frac{\partial^2 f}{\partial t^2} + \nabla^2 f + \nabla \cdot \boldsymbol{A}_0 + \frac{\partial \phi_0}{\partial t}\right).$$

This latter equation means that, *if* we can solve the (inhomogeneous) *wave equation*: $\frac{\partial^2 f}{\partial t^2} - \nabla^2 f = \nabla \cdot \boldsymbol{A}_0 + \frac{\partial \phi_0}{\partial t}$, *then* we can choose the gauge field and potential $(\boldsymbol{A}, \phi)$ such that: $\nabla \cdot \boldsymbol{A} + \frac{\partial \phi}{\partial t} = 0$, $\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} = \boldsymbol{J}$ and $\nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = \rho$. Indeed it *is* possible to solve for f under very general conditions on the form of $\boldsymbol{A}_0$ and ϕ_0. Thus the study of solutions for Maxwell’s equations *reduces to studying wave equations*.

Note that *any* funtion of the form $f(x, y, z, t) = g(x - t)$ is a solution of the homogeneous wave equation $\nabla^2 f - \frac{\partial^2 f}{\partial t^2} = 0$. This solution just propagates the graph of g along the x-direction, like a wave. Thus, particularly in regions without sources or currents, one might conjecture that solutions of Maxwell's equations are wavelike in nature. Historically it was Maxwell's great achievement to postulate the existence of electromagnetic waves. This soon led to the discovery of radio waves. (Much later came the realisation that light also is an example of electromagnetic radiation.)

The study of waves in general contexts is the content of the unit MATH331, for next year's study.

Differential Forms

integration of differential forms on $\mathbb{R}^3$

f		$\eta = F_1\,dx + F_2\,dy + F_3\,dz$		$\omega = F_1\,dy \wedge dz + F_2\,dz \wedge dx + F_3\,dx \wedge dy$		$\kappa = \rho\,dx \wedge dy \wedge dz$
0 – forms	$\xrightarrow[f \mapsto \nabla f]{d}$	1 – forms	$\xrightarrow[F \mapsto \nabla \times F]{d}$	2 – forms	$\xrightarrow[F \mapsto \nabla \cdot F]{d}$	3 – forms
$\Downarrow$		$\Downarrow$		$\Downarrow$		$\Downarrow$
$f(a)$	*Fund. Thm. Calc.*	$\int_C \eta$	*Stokes ' Thm.*	$\int_S \omega$	*Gauss ' Thm.*	$\int_V \kappa$
$\Uparrow$		$\Uparrow$		$\Uparrow$		$\Uparrow$
signed points	$\xleftarrow[\partial]{}$	**oriented curves**	$\xleftarrow[\partial]{}$	**oriented surfaces**	$\xleftarrow[\partial]{}$	**elementary regions**

integration of differential forms on $\mathbb{R}^2$

f		$\eta = F_1\,dx + F_2\,dy$		$\omega = \rho\,dx \wedge dy$
0 – forms	$\xrightarrow[f \mapsto \nabla f]{d}$	1 – forms	$\xrightarrow[F \mapsto \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}]{d}$	2 – forms
$\Downarrow$		$\Downarrow$		$\Downarrow$
$f(a)$	*Fund. Thm. Calc.*	$\int_C \eta$	*Green ' s Thm.*	$\int_R \omega$
$\Uparrow$		$\Uparrow$		$\Uparrow$
signed points	$\xleftarrow[\partial]{}$	**oriented curves**	$\xleftarrow[\partial]{}$	**elementary regions**

wedge product of forms

$\psi \wedge \chi = (-1)^{\deg \psi \times \deg \chi}\; \chi \wedge \psi$ for any forms ψ and χ.

Thus $dx \wedge dx = 0 = dy \wedge dy = dz \wedge dz$ also $dy \wedge dx = -dx \wedge dy$ and $dx \wedge dz = -dz \wedge dx$ and $dz \wedge dy = -dy \wedge dz$.

differential d

$d(f\,\chi) = df \wedge \chi + f\,d\chi$ for any function f and form χ.

$d(\psi \wedge \chi) = d\psi \wedge \chi + (-1)^{\deg \psi}\; \psi \wedge d\chi$ for any forms ψ and χ — think of d as being (an operator) 1-form.

$d(dx) = 0 = d(dy) = d(dz)$ — in general, this means that $d^2 = 0$.

differential of a 0-form (in $\mathbb{R}^3$):

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = \nabla f \cdot (dx, dy, dz).$$

$$\begin{aligned} d(df) &= d\left(\frac{\partial f}{\partial x}\right) \wedge dx + d\left(\frac{\partial f}{\partial y}\right) \wedge dy + d\left(\frac{\partial f}{\partial z}\right) \wedge dz \\ &= \left(\frac{\partial^2 f}{\partial y \partial x}\,dy + \frac{\partial^2 f}{\partial z \partial x}\,dz\right) \wedge dx + \left(\frac{\partial^2 f}{\partial x \partial y}\,dx + \frac{\partial^2 f}{\partial z \partial y}\,dz\right) \wedge dy + \left(\frac{\partial^2 f}{\partial x \partial z}\,dx + \frac{\partial^2 f}{\partial y \partial z}\,dy\right) \wedge dz \\ &= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) dy \wedge dz + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right) dz \wedge dx + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) dx \wedge dy = 0\,. \end{aligned}$$

differential of a 1-form (in $\mathbb{R}^3$):

$$\begin{aligned} d\eta &= d(\mathbf{F} \cdot (dx, dy, dz)) = dF_1 \wedge dx + dF_2 \wedge dy + dF_3 \wedge dz \\ &= \left(\frac{\partial F_1}{\partial y}\,dy + \frac{\partial F_1}{\partial z}\,dz\right) \wedge dx + \left(\frac{\partial F_2}{\partial x}\,dx + \frac{\partial F_2}{\partial z}\,dz\right) \wedge dy + \left(\frac{\partial F_3}{\partial x}\,dx + \frac{\partial F_3}{\partial y}\,dy\right) \wedge dz \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy \wedge dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dz \wedge dx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx \wedge dy \\ &= \nabla \times \mathbf{F} \cdot (dy \wedge dz, dz \wedge dx, dx \wedge dy)\,. \end{aligned}$$

$$d(d\eta) = d\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\wedge dy\wedge dz + d\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\wedge dz\wedge dx + d\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\wedge dx\wedge dy$$
$$= \left(\frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_2}{\partial x\partial z} + \frac{\partial^2 F_1}{\partial y\partial z} - \frac{\partial^2 F_3}{\partial y\partial x} + \frac{\partial^2 F_2}{\partial z\partial x} - \frac{\partial^2 F_1}{\partial z\partial y}\right) dx\wedge dy\wedge dz = 0\,.$$

differential of a 2-form (in $\mathbb{R}^3$):

$$d\omega = d(\boldsymbol{F}\cdot(dy\wedge dz, dz\wedge dx, dx\wedge dy)) = dF_1\wedge dy\wedge dz + dF_2\wedge dz\wedge dx + dF_3\wedge dx\wedge dy$$
$$= \left(\frac{\partial F_1}{\partial x}\, dx\wedge dy\wedge dz\right) + \left(\frac{\partial F_2}{\partial y}\, dy\wedge dz\wedge dx\right) + \left(\frac{\partial F_3}{\partial z}\, dz\wedge dx\wedge dy\right) = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\wedge dy\wedge dz$$
$$= (\nabla\cdot\boldsymbol{F})\, dx\wedge dy\wedge dz\,.$$

$d(d\omega) = d(\nabla\cdot\boldsymbol{F})\wedge dx\wedge dy\wedge dz = 0$, as there are no 4-forms on $\mathbb{R}^3$.

differential of a 3-form (in $\mathbb{R}^3$):

$$d\kappa = d(f\, dx\wedge dy\wedge dz) = \left(\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz\right)\wedge(dx\wedge dy\wedge dz) = 0\,.$$

Integration theorems, on $\mathbb{R}^3$

Fundamental theorem of calculus:

$\int_C df = \int_{\partial C} f = [f]_a^b$ for a curve C from a to b.

Stokes' theorem:

$\int_S d\eta = \int_{\partial S}\eta$ for a surface S with (oriented) boundary ∂S.

Gauss' divergence theorem:

$\int_V d\omega = \int_{\partial V}\omega$ for a region V with outward-pointing normals on the boundary ∂V.

Integration theorems, on $\mathbb{R}^2$

Fundamental theorem of calculus:

$\int_C df = \int_{\partial C} f = [f]_a^b$ for a curve C from a to b.

Greens' theorem:

$\int_R d\eta = \int_{\partial R}\eta$ for a region R with (oriented) boundary ∂R. With $\eta = P\,dx + Q\,dy$ then $d\eta = \frac{\partial P}{\partial y}\, dy\wedge dx + \frac{\partial Q}{\partial x}\, dx\wedge dy = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\wedge dy$.

Exact sequence of forms, on $\mathbb{R}^3$

Let Ω^k denote the space of k-forms on $\mathbb{R}^3$. Then we have an exact sequence:

$$0\longrightarrow\mathbb{R}\xrightarrow{i}\Omega^0\xrightarrow{d}\Omega^1\xrightarrow{d}\Omega^2\xrightarrow{d}\Omega^3\longrightarrow 0 \qquad \text{corresponding to:} \qquad 0\longrightarrow\mathbb{R}\xrightarrow{i}\Omega^0\xrightarrow{\nabla}\Omega^1\xrightarrow{\nabla\times}\Omega^2\xrightarrow{\nabla\cdot}\Omega^3\longrightarrow 0.$$

That is:

$df = 0 \Rightarrow f$ is constant; that is, $\nabla f = 0$ for constant functions only;

$d\eta = 0 \Rightarrow \eta = df$ for some function f, with η a 1-form; equivalently, $\nabla\times\boldsymbol{F} = 0 \Rightarrow \boldsymbol{F} = \nabla f$;

$d\omega = 0 \Rightarrow \omega = d\eta$ for some 1-form η, with ω a 2-form; equivalently, $\nabla\cdot\boldsymbol{H} = 0 \Rightarrow \boldsymbol{H} = \nabla\times\boldsymbol{F}$;

any 3-form κ can be written as $\kappa = d\omega$, for some 2-form ω.

Exact sequence of forms, on $\mathbb{R}^2$

Let Ω^k denote the space of k-forms on $\mathbb{R}^3$. Then we have an exact sequence:

$$0\longrightarrow\mathbb{R}\xrightarrow{i}\Omega^0\xrightarrow{d}\Omega^1\xrightarrow{d}\Omega^2\longrightarrow 0 \qquad \text{corresponding to:} \qquad 0\longrightarrow\mathbb{R}\xrightarrow{i}\Omega^0\xrightarrow{\nabla}\Omega^1\xrightarrow{\text{curl}}\Omega^2\longrightarrow 0.$$

That is:

$df = 0 \Rightarrow f$ is constant; that is, $\nabla f = 0$ for constant functions only;

$d\eta = 0 \Rightarrow \eta = df$ for some function f, with η a 1-form; equivalently, $\text{curl}\,\boldsymbol{F} = 0 \Rightarrow \boldsymbol{F} = \nabla f$ — using the *scalar* curl;

any 2-form ω can be written as $\omega = d\eta$, for some 1-form η.