MATH236 — Weeks 10–12

Integration Theorems

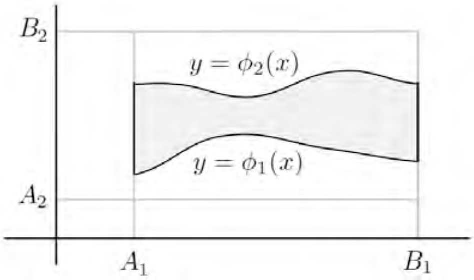
Chen notes, chapter 12

12.1 Green's Theorem

Recall from Section 5.4 (studied in MATH235) that a region of the type:

(1) $\quad R = \{(x, y) \subseteq \mathbb{R}^2 \ : \ x \in [A_1, B_1] \text{ and } \phi_1(x) \le y \le \phi_2(x)\},$

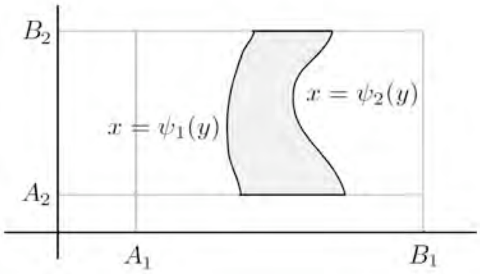
where the functions $\phi_1 : [A_1, B_1] \to \mathbb{R}$ and $\phi_2 : [A_1, B_1] \to \mathbb{R}$ are continuous in the interval $[A_1, B_1]$ and where $\phi_1(x) \le \phi_2(x)$ for every $x \in [A_1, B_1]$, is called an ***elementary region of type 1***.



A region of the type:

(2) $\quad R = \{(x, y) \subseteq \mathbb{R}^2 \ : \ y \in [A_2, B_2] \text{ and } \psi_1(y) \le x \le \psi_2(y)\},$

where the functions $\psi_1 : [A_2, B_2] \to \mathbb{R}$ and $\psi_2 : [A_2, B_2] \to \mathbb{R}$ are continuous in the interval $[A_2, B_2]$ and where $\psi_1(y) \le \psi_2(y)$ for every $y \in [A_2, B_2]$, is called an ***elementary region of type 2***.



Furthermore, an ***elementary region of type 3*** is one which is of *both* type 1 and type 2; in other words, one that can be described by both (1) and (2).

Green's theorem relates a line integral along a simple closed curve C in $\mathbb{R}^2$ to a double integral over the region R enclosed by the curve. We say that C has ***positive orientation*** if the region R is on the left when we follow the curve C, and has ***negative orientation*** otherwise. For example, a circle followed in the anticlockwise direction has positive orientation with respect to the region it encloses.

Theorem 12A: Green's theorem

Suppose that $R \subseteq \mathbb{R}^2$ is an elementary region of type 3, with boundary curve C followed with positive orientation. Suppose further that the functions $P : R \to \mathbb{R}$ and $Q : R \to \mathbb{R}$ are both continuously differentiable. Then

(3) $\quad \int_C (P\,dx + Q\,dy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$

Remarks

- Consider the vector field $\boldsymbol{F} = (P(x, y), Q(x, y))$ in $\mathbb{R}^2$. Then (3) can be written as $\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \displaystyle\int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$. Note that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is the scalar curl of the vector field $\boldsymbol{F} = (P, Q)$.

- Consider a vector field $\boldsymbol{F}(x, y, z) = (P(x, y), Q(x, y), 0)$ in $\mathbb{R}^3$, and imagine the region R to be a surface S on the xy-plane, with boundary curve C. Then we have

  (4) $\quad \int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \int_C (P(x, y), Q(x, y), 0) \cdot (dx, dy, dz) = \displaystyle\int\int_R \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \cdot (0, 0, 1)\,dx\,dy = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$

  On the other hand, we can parametrize the surface S by the function $\Phi : R \to \mathbb{R}^3 : (x, y) \mapsto (x, y, 0)$. Then $\boldsymbol{t}_x = (1, 0, 0)$ and $\boldsymbol{t}_y = (0, 1, 0)$, so that $\boldsymbol{t}_x \times \boldsymbol{t}_y = (0, 0, 1)$.

Hence

(5) $\int_{\Phi} (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} = \int\int_R \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \cdot (0, 0, 1)\, dx\, dy = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy.$

If we take the oriented surface S to have normal vector in the positive z-direction, then

(6) $\int_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} = \int_{\Phi} (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S}.$

Combining (4),(5),(6), we conclude that $\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \int_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S}$. This is known as ***Stokes' theorem***. We shall study this in Section 12.2.

- Replacing Q by P and replacing P by $-Q$ in (3), we obtain

(7) $\int\int_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\, dy = \int_C (P\, dy - Q\, dx).$

Consider now a vector field $\boldsymbol{F} = (P(x, y), Q(x, y))$ in $\mathbb{R}^2$. Then

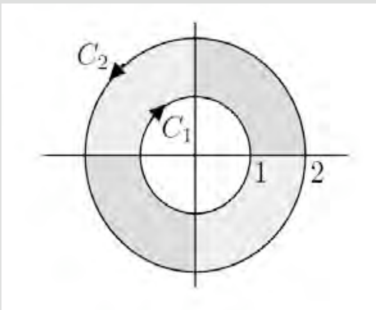
(8) $\int\int_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\, dy = \int\int_R (\operatorname{div} \boldsymbol{F})\, dx\, dy.$

Next, suppose that $\boldsymbol{\phi}$ is an orientation preserving parametrization of C. Then a tangent vector at a point $(x(t), y(t))$ is given by $(x'(t), y'(t))$. Rotating this vector in the clockwise direction by an angle $\frac{1}{2}\pi$ gives an outward normal vector to C at the point $(x(t), y(t))$. This outward normal vector is $(y'(t), -x'(t))$, with unit vector $\boldsymbol{n} = \frac{(y'(t), -x'(t))}{\|(y'(t), -x'(t))\|}$. It follows that

(9) $\int_C (P\, dy - Q\, dx) = \int_C \boldsymbol{F} \cdot \boldsymbol{n}\, ds.$

Combining (7),(8),(9), we obtain $\int\int_R (\operatorname{div} \boldsymbol{F})\, dx\, dy = \int_C \boldsymbol{F} \cdot \boldsymbol{n}\, ds$. This is the 2-dimensional version of Gauss' ***Divergence Theorem*** which we shall study in Section 12.3.

- Green's theorem can be extended to regions R which are finite unions of essentially disjoint elementary regions of type 3. For example, consider the annulus $R = \{(x, y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 4\}$. We can cut R into four subregions of type 3 by the lines $x = 0$ and $y = 0$.



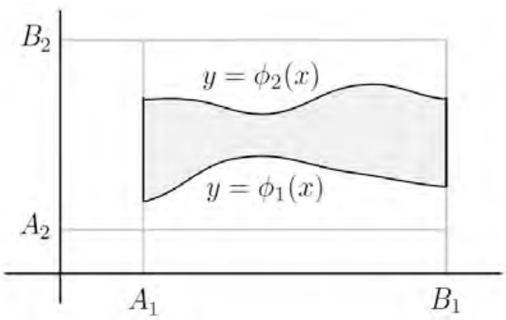
The boundary curve is now the union of the two circles $C_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and $C_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}$, with C_1 followed in the clockwise direction and C_2 followed in the anticlockwise direction.

Proof of Theorem 12A

Consider first of all the integral:

(10) $\int_C P\, dx.$

Since R is an elementary region of type 3, it is also an elementary region of type 1, and so can be described in the form (1):



The boundary curve C of this region can be split into four parts. There are two straight line segments: from $(A_1, \phi_2(A_1))$ to $(A_1, \phi_1(A_1))$ and from $(B_1, \phi_1(B_1))$ to $(B_1, \phi_2(B_1))$. There are also two curves $C_1 = \{(x, \phi_1(x)) : x \in [A_1, B_1]\}$ and $C_2 = \{(x, \phi_2(x)) : x \in [A_1, B_1]\}$, followed from $(A_1, \phi_1(A_1))$ to $(B_1, \phi_1(B_1))$ and from $(B_1, \phi_2(B_1))$ to $(A_1, \phi_2(A_1))$ respectively.

The contribution from the two *straight line* segments to the integral (10) is zero, since $dx = 0$ on these two line segments. It follows that

$$\int_C P\, dx = \int_{C_1} P\, dx + \int_{C_2} P\, dx = \int_{A_1}^{A_2} P(x, \phi_1(x))\, dx + \int_{A_2}^{A_1} P(x, \phi_2(x))\, dx = -\int_{A_1}^{A_2} (P(x, \phi_2(x)) - P(x, \phi_1(x)))\, dx.$$

On the other hand, it follows from Fubini's theorem that $\int\int_R \frac{\partial P}{\partial y}\, dx\, dy = \int_{A_1}^{B_1}\left(\int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial P}{\partial y}\, dy\right) dx = \int_{A_1}^{B_1} (P(x, \phi_2(x)) - P(x, \phi_1(x)))\, dx$, by the Fundamental theorem of calculus. Hence

(11) $$\int_C P\, dx = -\int\int_R \frac{\partial P}{\partial y}\, dx\, dy.$$

Similarly, it can be proved that

(12) $$\int_C Q\, dy = \int\int_R \frac{\partial Q}{\partial x}\, dx\, dy.$$

The formula (3) now follows on combining (11) and (12).

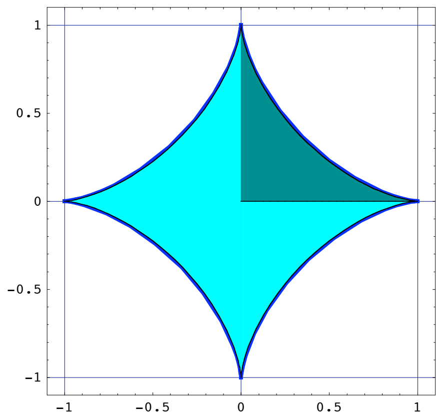
Example 12.1.1 — area of a region

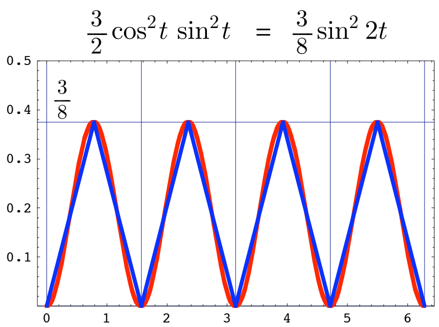
Consider the special case when $P(x, y) = -\frac{1}{2} y$ and $Q(x, y) = \frac{1}{2} x$. Then (3) becomes $\frac{1}{2}\int_C (x\, dy - y\, dx) = \int\int_R 1\, dx\, dy$. This is equal to the area of R.

Suppose now that R is the region bounded by the hypocycloid C of four cusps, given by the equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ and parametrized by $\boldsymbol{\phi} : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\cos^3 t, \sin^3 t)$. Then $dx = -3\cos^2 t\, \sin t\, dt$ and $dy = 3\sin^2 t\, \cos t\, dt$. Hence the area of the region bounded by the hypocycloid is given by

$$\frac{1}{2}\int_C (x\, dy - y\, dx) = \frac{1}{2}\int_0^{2\pi} (3\cos^4 t\, \sin^2 t + 3\cos^2 t\, \sin^4 t)\, dt = \frac{3}{2}\int_0^{2\pi} \cos^2 t\, \sin^2 t\, dt$$
$$= \frac{3}{8}\int_0^{2\pi} \sin^2 2t\, dt = \frac{3}{16}\int_0^{4\pi} \frac{1}{2}(1 - \cos 2\theta)\, d\theta = \frac{3}{4}\int_0^{\pi} \frac{1}{2}(1 - \cos 2\theta)\, d\theta = \frac{3}{8}[\theta]_0^{\pi} = \frac{3}{8}\pi.$$

graphic





Example 12.1.2

Let $P(x, y) = x^2 y \cos x + 2 x y \sin x - y^2 e^x$ and $Q(x, y) = x^2 \sin x - 2 y e^x$. Then $\frac{\partial Q}{\partial x} = 2x\sin x + x^2 \cos x - 2 y e^x = \frac{\partial P}{\partial y}$.

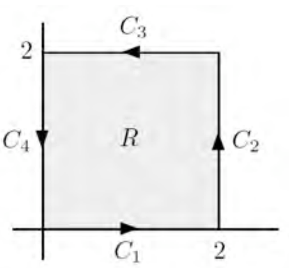
It follows from Green's theorem that

(13) $$\int_C (P\, dx + Q\, dy) = 0$$

for the boundary curve C of *any* elementary region of type 3. Note that (13) holds if C is the boundary curve of any elementary region of type 3 in which the equality $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ holds. In particular this holds when $\boldsymbol{F} = (P(x, y), Q(x, y))$ is a vector field in $\mathbb{R}^2$ such that $\boldsymbol{F} = \nabla f$, for some continuously differentiable function $f(x, y)$.

Example 12.1.3

Let $P(x, y) = x^2 - x y^3$ and $Q(x, y) = y^2 - 2 x y$, and let R denote the square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$ and $(0, 2)$.



The boundary curve is then $C = C_1 \cup C_2 \cup C_3 \cup C_4$, where C_1, C_2, C_3, C_4 are the four sides of R followed in the anticlockwise direction with initial point $(0, 0)$, and can be parametrized respectively by

$$\boldsymbol{\phi}_1 : [0,2] \to \mathbb{R}^2 : t \mapsto (t, 0), \quad \boldsymbol{\phi}_2 : [0,2] \to \mathbb{R}^2 : t \mapsto (2, t), \quad \boldsymbol{\phi}_3 : [0,2] \to \mathbb{R}^2 : t \mapsto (2-t, 2), \quad \boldsymbol{\phi}_4 : [0,2] \to \mathbb{R}^2 : t \mapsto (0, 2-t).$$

We have

$$\int_{C_1} (x\,dy - y\,dx) = \int_{C_1} (P, Q)\cdot(1, 0)\,dt = \int_0^2 P(t, 0)\,dt = \int_0^2 t^2\,dt = \tfrac{1}{3}[t^3]_0^{\,2} = \tfrac{8}{3},$$
$$\int_{C_2} (x\,dy - y\,dx) = \int_{C_2} (P, Q)\cdot(0, 1)\,dt = \int_0^2 Q(2, t)\,dt = \int_0^2 (t^2 - 4t)\,dt = [\tfrac{1}{3}t^3 - 2t^2]_0^{\,2} = -5\tfrac{1}{3},$$
$$\int_{C_3} (x\,dy - y\,dx) = \int_{C_3} (P, Q)\cdot(-1, 0)\,dt = -\int_0^2 P(2-t, 2)\,dt = -\int_0^2 (2-t)(-6-t)\,dt = [12t - 2t^2 - \tfrac{1}{3}t^3]_0^{\,2} = 13\tfrac{1}{3},$$
$$\int_{C_4} (x\,dy - y\,dx) = \int_{C_4} (P, Q)\cdot(0, -1)\,dt = -\int_0^2 Q(0, 2-t)\,dt = -\int_0^2 (2-t)^2\,dt = -[4t - 2t^2 + \tfrac{1}{3}t^3]_0^{\,2} = -\tfrac{8}{3}.$$

Hence $\int_C (P\,dx + Q\,dy) = \int_{C_1}(P\,dx + Q\,dy) + \int_{C_2}(P\,dx + Q\,dy) + \int_{C_3}(P\,dx + Q\,dy) + \int_{C_4}(P\,dx + Q\,dy) = \frac{8}{3} - 5\frac{1}{3} + 13\frac{1}{3} - \frac{8}{3} = 8$.

This calculation can be somewhat simplified by noting that $dx = 0$ on C_2 and C_4, while $dy = 0$ on C_1 and C_3, and that the parametrizations are linear on each of the sides of the square. Hence we can write down directly:

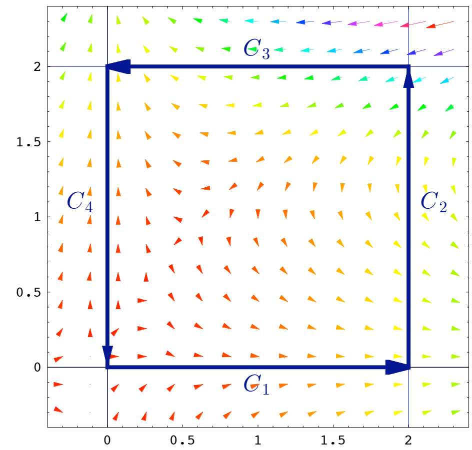
$$\begin{aligned}\int_C (x\,dy - y\,dx) &= \int_0^2 P(x, 0)\,dx + \int_0^2 Q(0, y)\,dy - \int_0^2 P(2-x, 0)\,dx - \int_0^2 Q(0, y)\,dy \\ &= \int_0^2 x^2\,dx + \int_0^2 (y^2 - 4y)\,dy - \int_0^2 (x^2 - 8x)\,dx - \int_0^2 y^2\,dy = \tfrac{8}{3} + (\tfrac{8}{3} - 8) - (\tfrac{8}{3} - 16) - \tfrac{8}{3} = 8.\end{aligned}$$

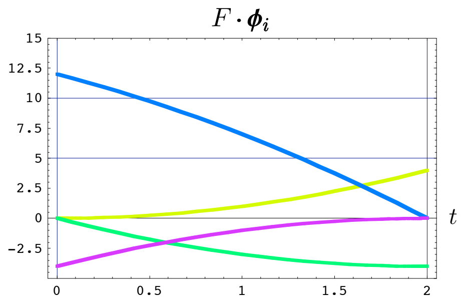
On the other hand, we have

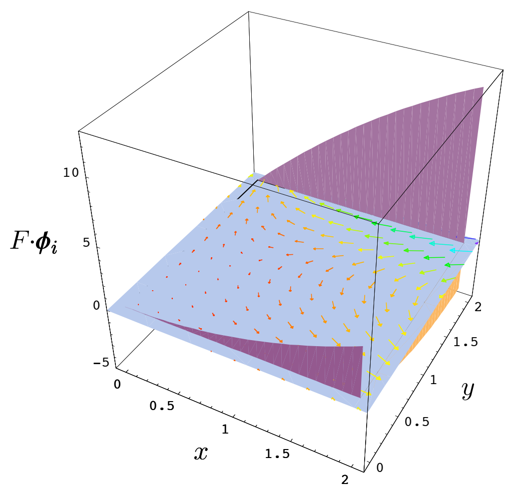
$$\begin{aligned}\int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy &= \int\int_R (3xy^2 - 2y)\,dx\,dy = \int_0^2 \left(\int_0^2 (3xy^2 - 2y)\,dx\right) dy = \int_0^2 \left(\left([\tfrac{3}{2}x^2]_0^{\,2}\right) y^2 - ([2x]_0^{\,2})\,y\right) dy \\ &= \int_0^2 (6y^2 - 4y)\,dy = [2y^3 - 2y^2]_0^{\,2} = 8.\end{aligned}$$

This verifies Green's theorem.

visualisation







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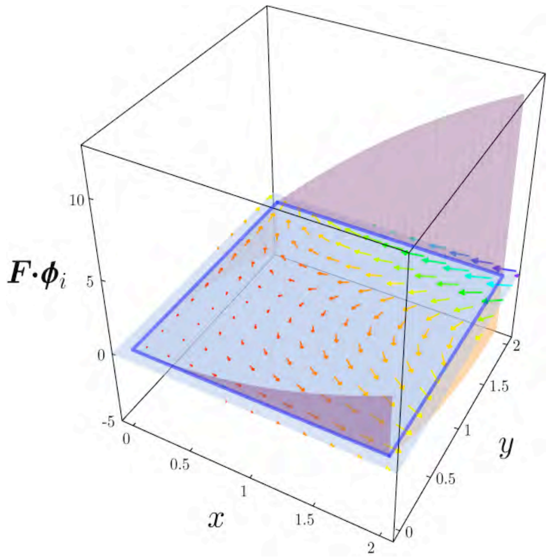
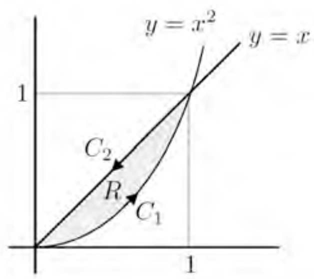


Example 12.1.4

Let $P(x, y) = x\,y + y^2$ and $Q(x, y) = x^2$, and let R denote the region bounded by the line $y = x$ and the parabola $y = x^2$.



The boundary curve is then $C = C_1 \cup C_2$, where C_1 is part of a parabola from $(0, 0)$ to $(1, 1)$ and C_2 is the part of the line from $(1, 1)$ to $(0, 0)$. The curves C_1 and C_2 can be parametrized respectively by $\boldsymbol{\phi}_1 : [0, 1] \to \mathbb{R}^2 : t \mapsto (t, t^2)$, $\boldsymbol{\phi}_2 : [0, 1] \to \mathbb{R}^2 : t \mapsto (1 - t, 1 - t)$.

We have $\int_{C_1} (x\,dy - y\,dx) = \int_{C_1} (P, Q)\cdot(1, 2t)\,dt = \int_0^1 ((x(t)\,y(t) + y(t)^2) + 2t(x(t)^2))\,dt = \int_0^1 (3t^3 + t^4)\,dt = \left[\frac{3}{4}t^4 + \frac{1}{5}t^5\right]_0^1 = \frac{19}{20}$,

$\int_{C_2} (x\,dy - y\,dx) = \int_{C_2} (P, Q)\cdot(-1, -1)\,dt = -\int_0^1 (x(t)\,y(t) + y(t)^2 + x(t)^2)\,dt = -\int_0^1 3t^2\,dt = -[t^3]_0^1 = -1$.

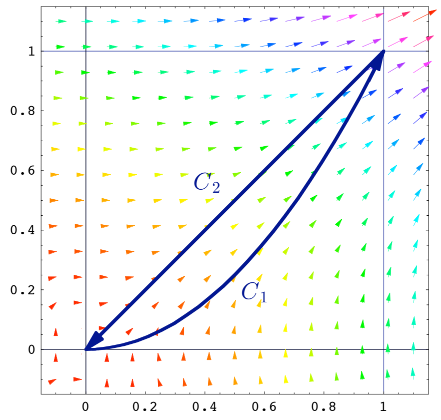
Hence $\int_C (P\,dx + Q\,dy) = \int_{C_1} (P\,dx + Q\,dy) + \int_{C_2} (P\,dx + Q\,dy) = \frac{19}{20} - 1 = -\frac{1}{20}$.

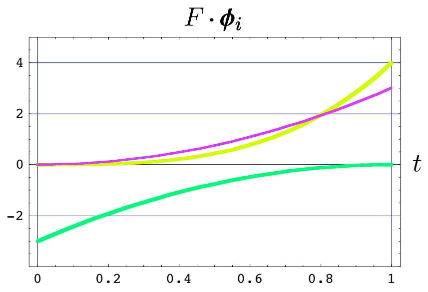
On the other hand, we have $\int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int\int_R (2x - (x + 2y))\,dx\,dy = \int_0^1 \left(\int_{x^2}^x (x - 2y)\,dy\right) dx = \int_0^1 (([y]_{x^2}^x)\,x - ([y^2]_{x^2}^x))\,dx$

$= \int_0^1 (x^2 - x^3 - x^2 + x^4)\,dx = \left[\frac{1}{5}x^5 - \frac{1}{4}x^4\right]_0^1 = -\frac{1}{20}$.

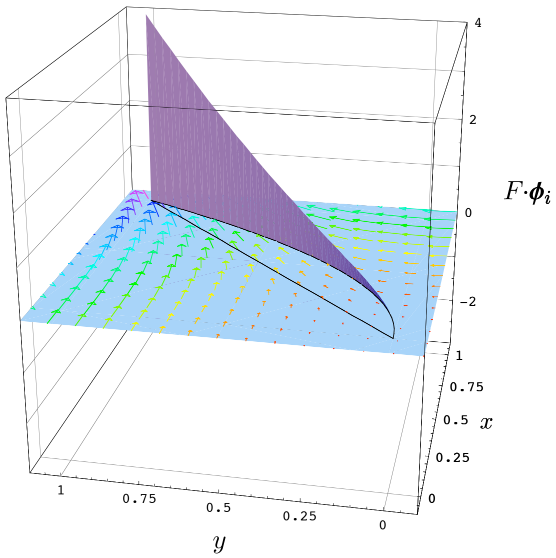
This verifies Green's theorem.

visualisation





The magenta curve is from C_2^-, the reverse of the green plot from C_2, for easier comparison with the yellow plot from C_1.



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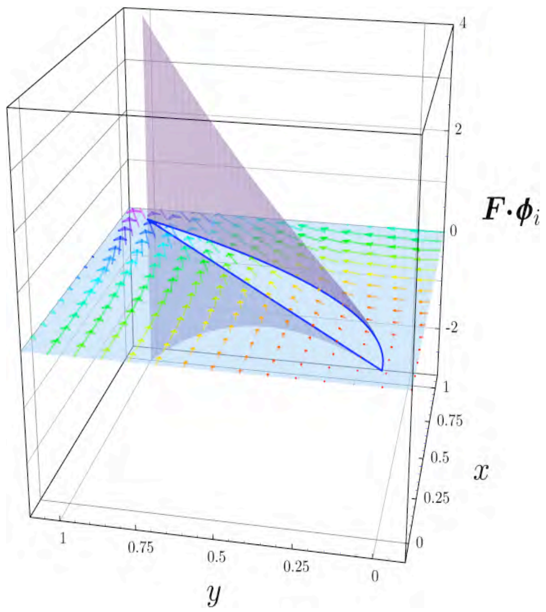


12.2 Stokes' Theorem

Stokes's theorem relates a line integral along a simple closed curve C in $\mathbb{R}^3$ to a surface integral over a surface S with boundary curve C. A special case of it is Green's theorem discussed in the last section.

Clearly any relationship between the line integral and the surface integral requires a convention concerning the orientation of the curve C with respect to the orientation of the surface S. We use the ***right-hand-thumb rule***: extend the thumb on our right hand and close the fingers; if the thumb points in the direction of the chosen normal of S, then the curve C is said to have ***positive orientation*** if it follows the direction of the fingers. In other words, if we follow the curve C in positive orientation, then the surface S is on the *left*.

Theorem 12B: (Stokes' Theorem)

Suppose that $S \subset \mathbb{R}^3$ is an oriented surface, defined by an orientation preserving parametrization $\Phi : R \to \mathbb{R}^3$ for some elementary region $R \subseteq \mathbb{R}^2$ of type 3, and with boundary curve C followed with positive orientation. Suppose further that the vector field $\boldsymbol{F}$ is continuously differentiable in S. Then $\int_C \boldsymbol{F}\cdot d\boldsymbol{s} = \int_S (\operatorname{curl}\boldsymbol{F})\cdot d\boldsymbol{S}$.

We shall not give a rigorous proof here. Instead, we only very roughly give an outline of the main ideas, and show that the result may be deduced from Green's theorem. In the sketch below, we often make extra assumptions which are not normally necessary.

Heuristics of Theorem 12B

Write $\boldsymbol{F} = (F_1, F_2, F_3)$. Then

(14) $\int_C \boldsymbol{F}\cdot d\boldsymbol{s} = \int_C (F_1, F_2, F_3)\cdot d\boldsymbol{s} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$, and

(15) $\int_S (\nabla\times\boldsymbol{F})\cdot d\boldsymbol{S} = \int_S (\nabla\times(F_1,0,0))\cdot d\boldsymbol{S} + \int_S (\nabla\times(0,F_2,0))\cdot d\boldsymbol{S} + \int_S (\nabla\times(0,0,F_3))\cdot d\boldsymbol{S}$.

Suppose that a parametrization of S is given by: $\Phi : R \to S \subset \mathbb{R}^3$ whereby $(u,v) \mapsto (x(u,v), y(u,v), z(u,v))$.

Let C' denote the boundary of R, and consider the integral $\int_C F_1\,dx$. Since $dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$ it follows from Green's theorem that

$$\int_C F_1\,dx = \int_{C'}\left(F_1 \frac{\partial x}{\partial u}du + F_1\frac{\partial x}{\partial v}dv\right) = \int\int_R \left(\frac{\partial}{\partial u}\left(F_1\frac{\partial x}{\partial v}\right) - \frac{\partial}{\partial v}\left(F_1\frac{\partial x}{\partial u}\right)\right)du\,dv$$
$$= \int\int_R\left(\frac{\partial F_1}{\partial u}\frac{\partial x}{\partial v} + F_1\frac{\partial^2 x}{\partial u\,\partial v} - \frac{\partial F_1}{\partial v}\frac{\partial x}{\partial u} - F_1\frac{\partial^2 x}{\partial v\,\partial u}\right)du\,dv = \int\int_R\left(\frac{\partial F_1}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial F_1}{\partial v}\frac{\partial x}{\partial u}\right)du\,dv.$$

Next note that $\frac{\partial F_1}{\partial u} = \frac{\partial F_1}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial u}$ and $\frac{\partial F_1}{\partial v} = \frac{\partial F_1}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial v}$ so that

$$\frac{\partial F_1}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial F_1}{\partial v}\frac{\partial x}{\partial u} = \left(\frac{\partial F_1}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial u}\right)\frac{\partial x}{\partial v} - \left(\frac{\partial F_1}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial v}\right)\frac{\partial x}{\partial u}$$
$$= \left(\frac{\partial F_1}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial u}\right)\frac{\partial x}{\partial v} - \left(\frac{\partial F_1}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F_1}{\partial z}\frac{\partial z}{\partial v}\right)\frac{\partial x}{\partial u} = \frac{\partial F_1}{\partial y}\left(\frac{\partial y}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial x}{\partial u}\right) + \frac{\partial F_1}{\partial z}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right)$$
$$= \frac{\partial F_1}{\partial z}\frac{\partial(z,x)}{\partial(u,v)} - \frac{\partial F_1}{\partial y}\frac{\partial(x,y)}{\partial(u,v)}.$$

Hence

(16) $\int_C F_1\,dx = \int\int_R\left(\frac{\partial F_1}{\partial z}\frac{\partial(z,x)}{\partial(u,v)} - \frac{\partial F_1}{\partial y}\frac{\partial(x,y)}{\partial(u,v)}\right)du\,dv$.

Now

(17) $\int_S(\nabla\times(F_1,0,0))\cdot d\boldsymbol{S} = \int_S\left(0, \frac{\partial F_1}{\partial z}, -\frac{\partial F_1}{\partial y}\right)\cdot d\boldsymbol{S} = \int_R\left(0, \frac{\partial F_1}{\partial z}, -\frac{\partial F_1}{\partial y}\right)\cdot\left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)}\right)du\,dv = \int_R\left(\frac{\partial F_1}{\partial z}\frac{\partial(z,x)}{\partial(u,v)} - \frac{\partial F_1}{\partial y}\frac{\partial(x,y)}{\partial(u,v)}\right)du\,dv.$

Combining (16) and (17) gives

(18) $\int_C F_1\,dx = \int_S(\nabla\times(F_1,0,0))\cdot d\boldsymbol{S}$.

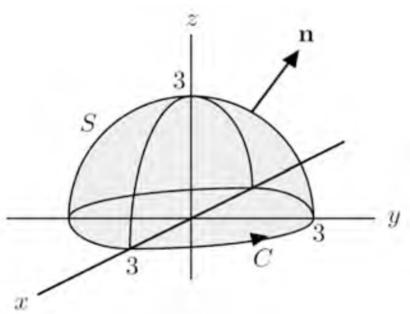
Similarly we get that

(19) $\int_C F_2\,dy = \int_S(\nabla\times(0,F_2,0))\cdot d\boldsymbol{S}$ and $\int_C F_3\,dz = \int_S(\nabla\times(0,0,F_3))\cdot d\boldsymbol{S}$.

Thus Stokes' theorem follows on combining (14), (15), (18) and (19).

Example 12.2.1 — spherical cap

Let S denote the upper hemispherical surface of the sphere $x^2+y^2+z^2=9$, with outward-pointing normal.



Then the boundary curve C is given by $x^2 + y^2 = 9$, followed in the anticlockwise direction.

Consider the vector field $\boldsymbol{F}(x, y, z) = (2y, 3x, -z^2)$. Let us first of all evaluate the integral: $\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$.

By using the orientation-preserving parametrization $\boldsymbol{\phi} : [0, 2\pi] \to \mathbb{R}^3$ whereby $t \mapsto (3\cos t, 3\sin t, 0)$, we have

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \int_0^{2\pi} \boldsymbol{F}(\boldsymbol{\phi}(t)) \cdot \boldsymbol{\phi}'(t)\, dt = \int_0^{2\pi} (6\sin t, 9\cos t, 0) \cdot (-3\sin t, 3\cos t, 0)\, dt = 9\int_0^{2\pi} (3\cos^2 t - 2\sin^2 t)\, dt$$
$$= 9\int_0^{2\pi} \tfrac{1}{2}(1 - 3\cos 2t)\, dt = 9\pi.$$

Next, let us evaluate the integral: $\int_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S}$.

Consider the parametrization $\Phi : R \to \mathbb{R}^3$ whereby $(u, v) \mapsto (3\sin u \cos v, 3\sin u \sin v, 3\cos u)$, where $R = [0, \frac{1}{2}\pi] \times [0, 2\pi]$.

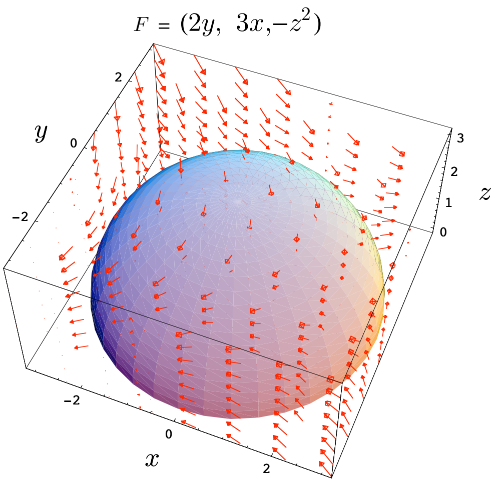
We have $\boldsymbol{t}_u \times \boldsymbol{t}_v = (9\sin^2 u \cos v, 9\sin^2 u \sin v, 9\cos u \sin u) = 3\sin u\, \Phi(u, v)$, so that Φ is an orientation-preserving parametrization of S.

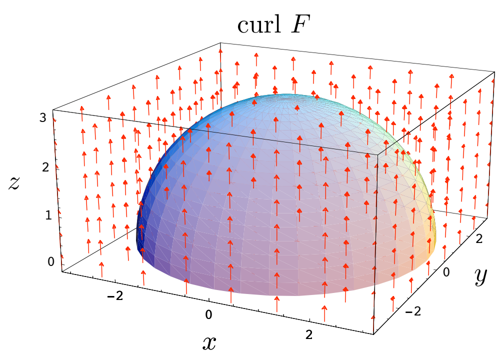
It is easy to see that $\nabla \times \boldsymbol{F} = (0, 0, 3-2) = (0, 0, 1)$, so

$$\int_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} = \int\int_R (0, 0, 1) \cdot 3\sin u\, \Phi(u, v)\, du\, dv = \int\int_R 9\sin u \cos u\, du\, dv = 9\left(\int_0^{\frac{1}{2}\pi} \sin u \cos u\, du\right)\left(\int_0^{2\pi} 1\, dv\right)$$
$$= 9\pi \int_0^{\frac{1}{2}\pi} \sin 2u\, du = \tfrac{9}{2}\pi \int_0^{\pi} \sin 2u\, d(2u) = \tfrac{9}{2}\pi\left(\int_0^{\pi} \sin\theta\, d\theta\right) = \tfrac{9}{2}\pi\,[-\cos\theta]_0^{\pi} = 9\pi.$$

This verifies Stokes' theorem.

visualisation





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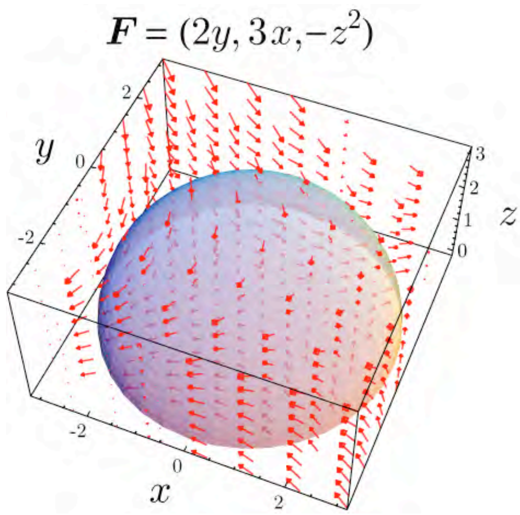
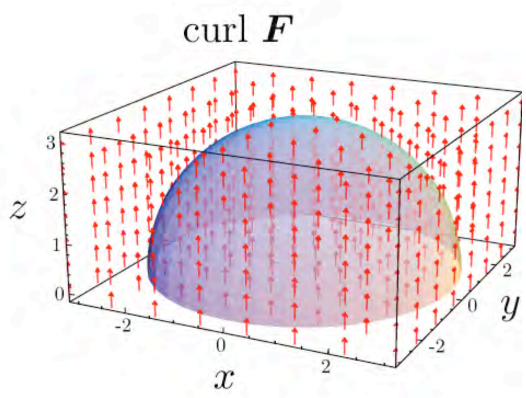
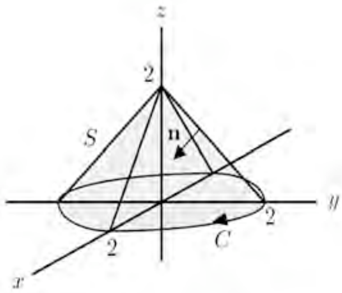




Example 12.2.2 — conical cap

Let S denote the surface of the cone $z = 2 - \sqrt{x^2 + y^2}$ above the $x\,y$-plane, with inward-pointing normal.



Then the boundary curve C is give by $x^2 + y^2 = 4$, followed in the clockwise direction.

Consider the vector field $\boldsymbol{F}(x, y, z) = (x - z, x^3 + y z, -3 x y^2)$. Let us first of all evaluate the integral: $\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$.

By using the *orientation-reversing* parametrization $\boldsymbol{\phi} : [0, 2\pi] \to \mathbb{R}^3$ whereby $t \mapsto (2\cos t, 2\sin t, 0)$, we have

$$\begin{aligned}\int_C \boldsymbol{F} \cdot d\boldsymbol{s} &= -\int_0^{2\pi} \boldsymbol{F}(\boldsymbol{\phi}(t)) \cdot \boldsymbol{\phi}'(t)\, dt = -\int_0^{2\pi} (2\cos t, 8\cos^3 t, *) \cdot (-2\sin t, 2\cos t, 0)\, dt = -4\int_0^{2\pi} (4\cos^4 t - \cos t \sin t)\, dt \\ &= -16\int_0^{2\pi} \cos^4 t\, dt = -16\int_0^{2\pi} \tfrac{1}{4}(\cos 2t + 1)^2\, dt = -8\pi - 4\int_0^{2\pi} \cos^2 2t\, dt = -8\pi - 4\int_0^{2\pi} \tfrac{1}{2}(\cos 4t + 1)\, dt \\ &= -12\pi.\end{aligned}$$

Next, let us evaluate the integral: $\int_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S}$.

Consider the parametrization $\Phi : R \to \mathbb{R}^3$ whereby $(u, v) \mapsto (u \cos v, u \sin v, 2 - u)$, where $R = [0, 2] \times [0, 2\pi]$. We have

$$\boldsymbol{t}_u \times \boldsymbol{t}_v = (\cos v, \sin v, -1) \times (-u \sin v, u \cos v, 0) = (u \cos v, u \sin v, u) = (x(u, v), y(u, v), 2 - z(u, v))$$

so that Φ is an *orientation-reversing* parametrization of S.

Since $\nabla \times \boldsymbol{F} = (-6 x y - y, -1 + 3 y^2, 3 x^2)$, it follows that

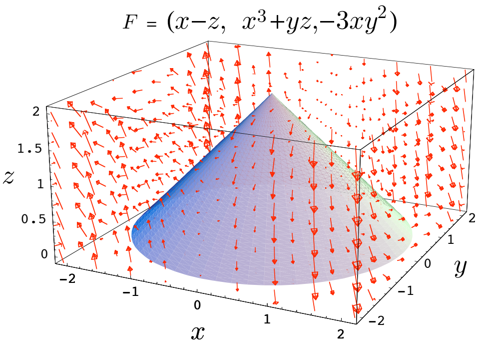
$$\begin{aligned}(\nabla \times \boldsymbol{F}) \cdot (\boldsymbol{t}_u \times \boldsymbol{t}_v) &= (-6 x y - y, -1 + 3 y^2, 3 x^2) \cdot (x, y, u) = 3 x^2 u - y + 3 y^3 - x y - 6 x^2 y \\ &= 3 u^3 \sin^3 v + 3 u^3 \cos^2 v (1 - 2\sin v) - u^2 \cos v \sin v - u \sin v.\end{aligned}$$

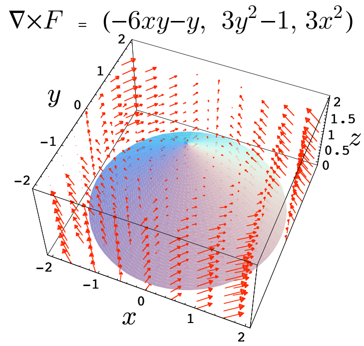
Each term is separable in u and v, mostly giving 0 when integrated over $[0, 2\pi]$ in the angle v, so the surface integral evaluates easily as:

$$\begin{aligned}\int_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} &= -\int_\Phi (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} = -\int\int_R (\nabla \times \boldsymbol{F}) \cdot (\boldsymbol{t}_u \times \boldsymbol{t}_v)\, du\, dv = -\int_0^2 \left(\int_0^{2\pi} (\nabla \times \boldsymbol{F}) \cdot (\boldsymbol{t}_u \times \boldsymbol{t}_v)\, dv\right) du \\ &= -\int_0^2 \left(\int_0^{2\pi} 3 u^3 \cos^2 v\, dv\right) du = -3\left(\int_0^2 u^3\, du\right)\left(\int_0^{2\pi} \cos^2 v\, dv\right) = -3 \times \left(\tfrac{1}{4}[u^4]_0^2\right) \times \pi \\ &= -12\pi.\end{aligned}$$

This verifies Stokes's theorem.

visualisation





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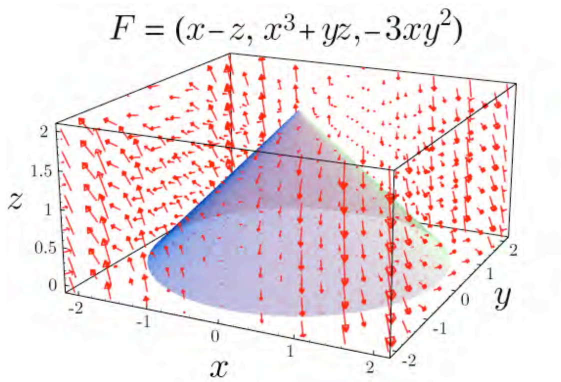
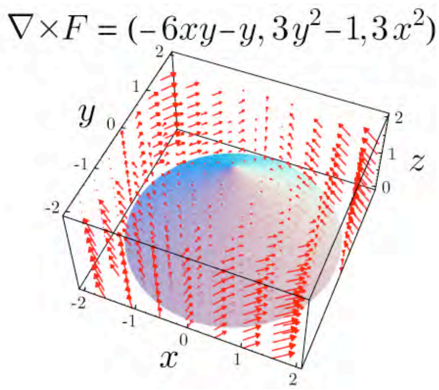




Gradient fields

Suppose that $\boldsymbol{F} = \nabla f$ is a gradient vector field in $\mathbb{R}^3$. Then it follows from Theorem 9A that for any continuously differentiable path $\boldsymbol{\phi} : [A, B] \to \mathbb{R}^3$ such that the composition function $\boldsymbol{F} \circ \boldsymbol{\phi} : [A, B] \to \mathbb{R}^3$ is continuous, we have $\int_{\boldsymbol{\phi}} \boldsymbol{F} \cdot d\boldsymbol{s} = f(\boldsymbol{\phi}(B)) - f(\boldsymbol{\phi}(A))$. In other words, the value of the integral depends only on the endpoints of the path $\boldsymbol{\phi}$. With the help of Stokes' theorem, we can characterize gradient vector fields.

Theorem 12C — characterization of gradient fields

Suppose that $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a continuously differentiable vector field. Then the following statements are equivalent:

(a) For any oriented simple *closed* curve C, we have $\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = 0$.

(b) For *any* two oriented simple curves C_1 and C_2 with the *same initial point* and the *same terminal point*, we have $\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{s}$.

(c) There exists a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\boldsymbol{F} = \nabla f$ *everywhere* in $\mathbb{R}^3$.

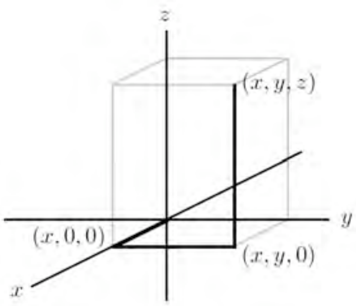
(d) We have $\nabla \times \boldsymbol{F} = 0$ *everywhere* in $\mathbb{R}^3$.

Sketch of proof

We shall show that (a)⇒(b)⇒(c)⇒(d)⇒(a).

To show that (a)⇒(b), let C be the curve C_1 followed by C_2^-; then C is closed. If C is simple, then $\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{s} - \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_C \boldsymbol{F} \cdot d\boldsymbol{s} = 0$. If C is not simple, then an elaboration of this argument will give the same result.

To show that (b)⇒(c), let C be any oriented simple curve with initial point $(0, 0, 0)$ and terminal point (x, y, z), and write $f(x, y, z) = \int_C \boldsymbol{F} \cdot d\boldsymbol{s}$. Since (b) holds, $f(x, y, z)$ is independent of the choice of C. In particular, we can take C to be the line segment from $(0, 0, 0)$ to $(x, 0, 0)$, followed by the line segment from $(x, 0, 0)$ to $(x, y, 0)$, followed by the line segment from $(x, y, 0)$ to (x, y, z).



Assume first of all that x, y, z are all positive. Then the three line segments can be parametrized respectively by

$$\boldsymbol{\phi}_1 : [0, x] \to \mathbb{R}^3 \text{ whereby } t \mapsto (t, 0, 0) \qquad \boldsymbol{\phi}_2 : [0, y] \to \mathbb{R}^3 \text{ whereby } t \mapsto (x, t, 0) \qquad \boldsymbol{\phi}_3 : [0, z] \to \mathbb{R}^3 \text{ whereby } t \mapsto (x, y, t),$$

so that writing $\boldsymbol{F} = (F_1, F_2, F_3)$, we have $f(x, y, z) = \int_0^x F_1(t, 0, 0)\, dt + \int_0^y F_2(x, t, 0)\, dt + \int_0^z F_3(x, y, t)\, dt$. With a little modification in the argument, this last formula can be shown to hold even if x, y, z are not all positive.

By the Fundamental theorem of calculus, we clearly have $\frac{\partial f}{\partial z} = F_3$. By using different paths, it can be shown that $\frac{\partial f}{\partial x} = F_1$ and $\frac{\partial f}{\partial y} = F_2$, so that $\nabla f = \boldsymbol{F}$.

That (c)⇒(d) is proved in Theorem 8G.

Finally, to prove that (d)⇒(a), we simply apply Stokes' theorem with any surface S whose boundary is C.

Remarks

- In the statement of Theorem 12C, it is possible to assume that the vector field $\boldsymbol{F}$ is continuously differentiable in $\mathbb{R}^3$, except possibly at a finite number of points. The proof only needs minor modification.

- There is a 2-dimensional version of Theorem 12C. Recall that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is the scalar curl of a vector field $\boldsymbol{F} = (P, Q)$ in $\mathbb{R}^2$. Thus there exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\boldsymbol{F} = \nabla f$ *everywhere* in $\mathbb{R}^2$ if and only if $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ everywhere in $\mathbb{R}^2$. Here Green's theorem plays the role of Stokes' theorem in establishing the result. However, we *cannot allow exceptions* to the condition that $\boldsymbol{F}$ be continuously differentiable in $\mathbb{R}^2$.

- Theorem 12C is in some sense the converse of Theorem 8G. Recall now Theorem 8F, that for any twice continuously differentiable vector field $\boldsymbol{F}$ in $\mathbb{R}^3$, we have $\nabla \cdot (\nabla \times \boldsymbol{F}) = 0$. One can prove (see Theorem12F below) that if $\boldsymbol{G}$ is a vector field continuously differentiable everywhere in $\mathbb{R}^3$ with $\nabla \cdot \boldsymbol{G} = 0$, then there exists a vector field $\boldsymbol{F}$ in $\mathbb{R}^3$ such that $\boldsymbol{G} = \nabla \times \boldsymbol{F}$.

12.3 Gauß' (Divergence) Theorem

symmetric elementry regions

Gauss' theorem relates a surface integral over a closed surface S in $\mathbb{R}^3$ to a volume integral over a region V with boundary surface S.We shall be concerned with regions in $\mathbb{R}^3$ of the type

(20) $\quad V = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in R \text{ and } \phi_1(x, y) \le z \le \phi_2(x, y)\},$

where R is an elementary region in $\mathbb{R}^2$, and where the functions $\phi_1 : R \to \mathbb{R}^2$ and $\phi_2 : R \to \mathbb{R}^2$ are continuous, with $\phi_1(x, y) \le \phi_2(x, y)$ for every $(x, y) \in R$.

There are two other types, one with y bounded between continuous functions of (x, z) in an elementary region, the other with x bounded between continuous functions of (y, z) in an elementary region.

A region in $\mathbb{R}^3$ which can be simultaneously described in all these three ways is called a ***symmetric elementary region*** in $\mathbb{R}^3$.

Clearly we can evaluate triple integrals of continuous functions over such regions; see Section 5.7 (studied in MATH235).

Theorem 12D: Gauss' Theorem

Suppose that $V \subseteq \mathbb{R}^3$ is a symmetric elementary region, with boundary surface S oriented with outward normal. Suppose further that a vector field $\boldsymbol{F}$ is continuously differentiable on V. Then $\int_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_V (\operatorname{div} \boldsymbol{F})\, dx\, dy\, dz$.

Remarks

- Sometimes, we write: $\int_S \boldsymbol{F} \cdot d\boldsymbol{S} = \int_V (\operatorname{div} \boldsymbol{F})\, dV$.
- Gauss' theorem is in fact valid for any region V which can be expressed as a union of finitely-many essentially disjoint symmetric elementary regions.
- We shall see that the proof of Gauss' theorem is very similar to that of Green's theorem.

Sketch of proof of Gauss' Theorem

Write $\boldsymbol{F} = (F_1, F_2, F_3)$. Then

(21) $$\int_S \boldsymbol{F} \cdot d\boldsymbol{S} = \int_S (F_1, F_2, F_3) \cdot d\boldsymbol{S} = \int_S ((F_1, 0, 0) + (0, F_2, 0) + (0, 0, F_3)) \cdot d\boldsymbol{S}$$
$$= \int_S (F_1, 0, 0) \cdot d\boldsymbol{S} + \int_S (0, F_2, 0) \cdot d\boldsymbol{S} + \int_S (0, 0, F_3) \cdot d\boldsymbol{S}$$

and

(22) $$\iiint_V (\operatorname{div} \boldsymbol{F})\, dx\, dy\, dz = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz$$
$$= \iiint_V \frac{\partial F_1}{\partial x}\, dx\, dy\, dz + \iiint_V \frac{\partial F_2}{\partial y}\, dx\, dy\, dz + \iiint_V \frac{\partial F_3}{\partial z}\, dx\, dy\, dz.$$

We shall show first of all that

(23) $$\int_S (0, 0, F_3) \cdot d\boldsymbol{S} = \iiint_V \frac{\partial F_3}{\partial z}\, dx\, dy\, dz.$$

Since V is a symmetric elementary region, it can be described in the form (20), so that

(24) $$\iiint_V \frac{\partial F_3}{\partial z}\, dx\, dy\, dz = \iint_R \left(\int_{\phi_1(x,y)}^{\phi_2(x,y)} \frac{\partial F_3}{\partial z}\, dz\right) dx\, dy = \iint_R (F_3(x, y, \phi_2(x, y)) - F_3(x, y, \phi_1(x, y)))\, dx\, dy.$$

On the other hand, the boundary surface S can be partitioned into six surfaces, with:

bottom surface: $S_1 = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in R \text{ and } z = \phi_1(x, y)\},$

top surface: $S_2 = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in R \text{ and } z = \phi_2(x, y)\},$

and four side surfaces: S_3, S_4, S_5, S_6 corresponding to the four edges of the elementary region R.

The normal vectors to the surfaces S_3, S_4, S_5, S_6 are all horizontal, with no component in the z-direction. Hence $\int_{S_3} (0, 0, F_3) \cdot d\boldsymbol{S} = \int_{S_4} (0, 0, F_3) \cdot d\boldsymbol{S} = \int_{S_5} (0, 0, F_3) \cdot d\boldsymbol{S} = \int_{S_6} (0, 0, F_3) \cdot d\boldsymbol{S} = 0$, and so

(25) $$\int_S (0, 0, F_3) \cdot d\boldsymbol{S} = \int_{S_1} (0, 0, F_3) \cdot d\boldsymbol{S} + \int_{S_2} (0, 0, F_3) \cdot d\boldsymbol{S}.$$

The surface S_1 can be parametrized by $\Phi : R \to S_1 \subset \mathbb{R}^3$ whereby $(x, y) \mapsto (x, y, \phi_1(x, y))$, with normal vector

$$\boldsymbol{t}_x \times \boldsymbol{t}_y = \left(1, 0, \frac{\partial \phi_1}{\partial x}\right) \times \left(0, 1, \frac{\partial \phi_1}{\partial y}\right) = \left(-\frac{\partial \phi_1}{\partial x}, -\frac{\partial \phi_1}{\partial y}, 1\right).$$

Hence Φ is an orientation-reversing parametrization of S_1, and so

(26) $$\int_{S_1} (0, 0, F_3) \cdot d\boldsymbol{S} = -\iint_R (0, 0, F_3) \cdot \left(-\frac{\partial \phi_1}{\partial x}, -\frac{\partial \phi_1}{\partial y}, 1\right) dx\, dy = -\iint_R F_3(x, y, \phi_1(x, y))\, dx\, dy.$$

The surface S_2 can be parametrized by $\Psi : R \to S_2 \subset \mathbb{R}^3$ whereby $(x, y) \mapsto (x, y, \phi_2(x, y))$, with normal vector

$$\boldsymbol{t}_x \times \boldsymbol{t}_y = \left(1, 0, \frac{\partial \phi_2}{\partial x}\right) \times \left(0, 1, \frac{\partial \phi_2}{\partial y}\right) = \left(-\frac{\partial \phi_2}{\partial x}, -\frac{\partial \phi_2}{\partial y}, 1\right).$$

Hence Ψ is an orientation-preserving parametrization of S_2, and so

(27) $$\int_{S_2} (0, 0, F_3) \cdot d\boldsymbol{S} = \int\int_R (0, 0, F_3) \cdot \left(-\frac{\partial \phi_2}{\partial x}, -\frac{\partial \phi_2}{\partial y}, 1\right) dx\, dy = \int\int_R F_3(x, y, \phi_2(x, y))\, dx\, dy.$$

The formula (23) now follows on combining (24), (25), (26) and (27).

Similarly, we have

(28) $$\int_S (F_1, 0, 0) \cdot d\boldsymbol{S} = \int\int\int_V \frac{\partial F_1}{\partial x}\, dx\, dy\, dz \quad \text{and} \quad \int_S (0, F_2, 0) \cdot d\boldsymbol{S} = \int\int\int_V \frac{\partial F_2}{\partial y}\, dx\, dy\, dz.$$

Gauss' theorem now follows on combining (21), (22), (23) and (28).

Example 12.3.1 — spherical volume

Let V denote the unit ball $x^2 + y^2 + z^2 \le 1$ in $\mathbb{R}^3$. Then the boundary surface S is given by $x^2 + y^2 + z^2 = 1$. Consider the vector field $\boldsymbol{F}(x, y, z) = (2x, y^2, z^2)$.

Let us first of all calculate the integral: $\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$.

The surface S can be parametrized by $\Phi : R \to \mathbb{R}^3$ whereby $(u, v) \mapsto (\sin u \cos v, \sin u \sin v, \cos u)$, where $R = [0, \pi] \times [0, 2\pi]$, and where $\boldsymbol{t}_u \times \boldsymbol{t}_v = (\sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u) = (\sin u)\, \Phi(u, v)$. This is an orientation-preserving parametrization, hence

$$\begin{aligned}
\int_S \boldsymbol{F} \cdot d\boldsymbol{S} &= \int\int_R \boldsymbol{F}(\sin u \cos v, \sin u \sin v, \cos u) \cdot (\sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u)\, du\, dv \\
&= \int\int_R (2 \sin u \cos v, \sin^2 u \sin^2 v, \cos^2 u) \cdot (\sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u)\, du\, dv \\
&= \int\int_R (2 \sin^3 u \cos^2 v + \sin^4 u \sin^3 v + \sin u \cos^3 u)\, du\, dv \\
&= 2\left(\int_0^\pi \sin^3 u\, du\right)\left(\int_0^{2\pi} \cos^2 v\, dv\right) + \left(\int_0^\pi \sin^4 u\, du\right)\left(\int_0^{2\pi} \sin^3 v\, dv\right) + \left(\int_0^\pi \sin u \cos^3 u\, du\right)\left(\int_0^{2\pi} 1\, dv\right) \\
&= 2\pi \int_0^\pi (1 - \cos^2 u) \sin u\, du + 0 + 2\pi \int_{-1}^1 h^3\, dh = 2\pi \left[1 - \tfrac{1}{3} h^3\right]_{-1}^{+1} + 0 + 0 \\
&= 2\pi \times 2 \times \tfrac{2}{3} = \tfrac{8}{3}\pi.
\end{aligned}$$

Next, note that
$$\begin{aligned}
\int\int\int_V (\operatorname{div} \boldsymbol{F})\, dx\, dy\, dz &= \int\int\int_V (2 + 2y + 2z)\, dx\, dy\, dz = 2\int\int\int_V (1 + y + z)\, dx\, dy\, dz = 2\int\int\int_V 1\, dx\, dy\, dz + 0 + 0 \\
&= 2 \times \tfrac{4}{3}\pi = \tfrac{8}{3}\pi,
\end{aligned}$$

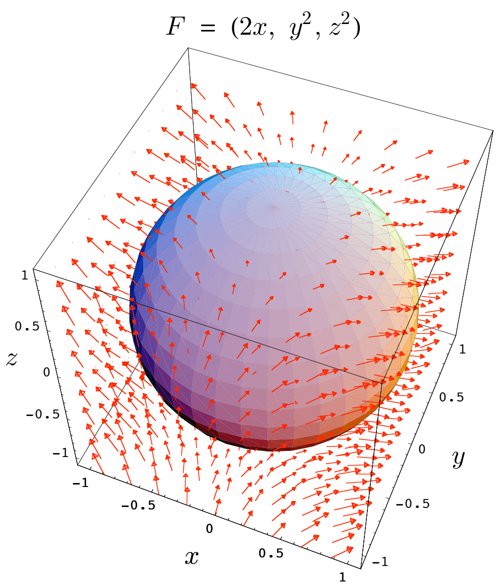
since the volume of the unit sphere is equal to $\frac{4}{3}\pi$. This verifies Gauss' theorem.

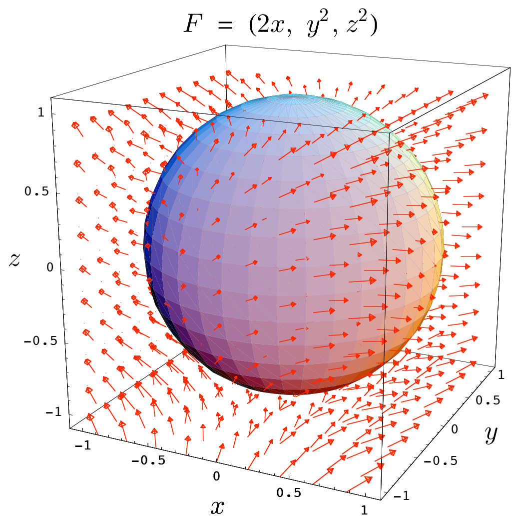
Here we have used that $\int\int\int_V y\, dx\, dy\, dz = \int\int\int_V z\, dx\, dy\, dz = \int\int\int_V x\, dx\, dy\, dz = 0$, which can be seen in various ways. (e.g., by symmetry —there is as much contributing negatively for $y < 0$, as positively for $y > 0$. That is, we are integrating an odd function over a symmetric domain.)

Alternatively, write $V = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 \le 1 \text{ and } -\sqrt{1 - x^2 - z^2} \le y \le \sqrt{1 - x^2 - z^2}\right\}$, so that

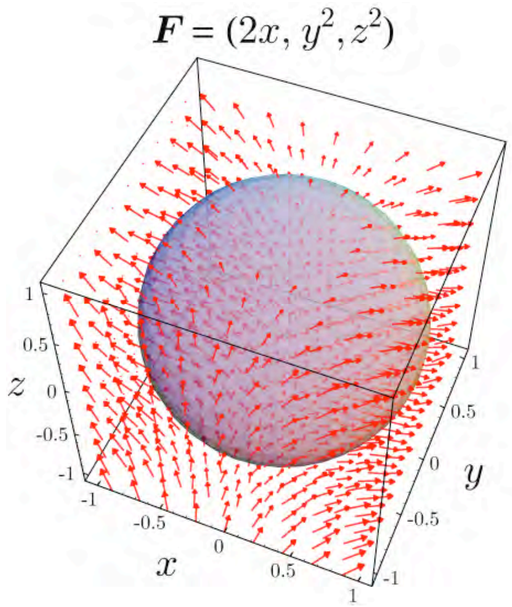
$$\int\int\int_V y\, dx\, dy\, dz = \int\int_{x^2+z^2 \le 1} \left(\int_{-\sqrt{1-x^2-z^2}}^{+\sqrt{1-x^2-z^2}} y\, dy\right) dx\, dz = \int\int_{x^2+z^2 \le 1} \left(\tfrac{1}{2}\left[y^2\right]_{-\sqrt{1-x^2-z^2}}^{+\sqrt{1-x^2-z^2}}\right) dx\, dz = 0.$$

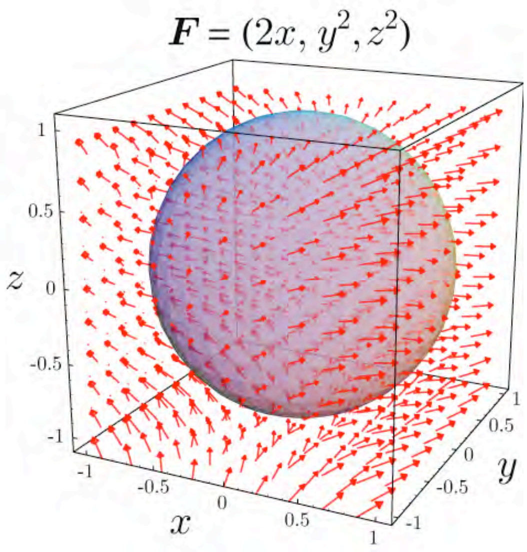
visualisation





with transparency





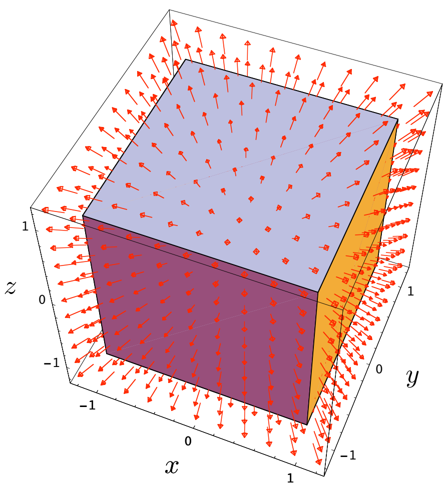
Example 12.3.2 — cubical volume

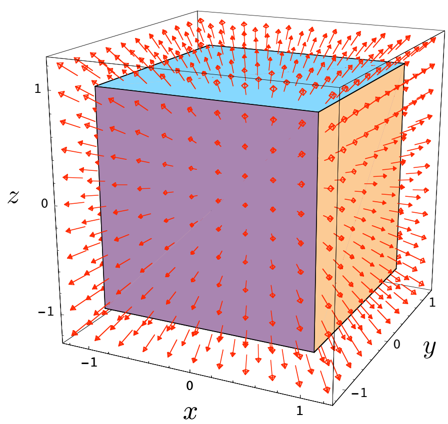
Let V be the cube with vertices $(\pm 1, \pm 1, \pm 1)$, with boundary surface S. Consider the vector field $\boldsymbol{F}(x, y, z) = (x, y, z)$. We have shown in Example 11.4.6 that $\int_S \boldsymbol{F} \cdot d\boldsymbol{S} = 24$. Now $\iiint_V (\operatorname{div} \boldsymbol{F})\, dx\, dy\, dz = 3 \iiint_V dx\, dy\, dz = 8 \times 3 = 24$. This verifies Gauss' theorem.

In fact, we can generalize this observation. Suppose that S is the boundary surface of any region V in $\mathbb{R}^3$ for which Gauss' theorem holds. Then

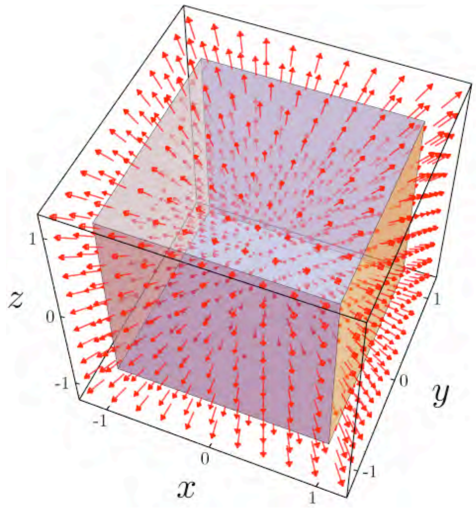
$$\int_S \boldsymbol{r} \cdot d\boldsymbol{S} = 3 \iiint_V dx\, dy\, dz = 3\, \mathrm{vol}(V), \quad \text{where } \boldsymbol{r} = (x, y, z) \text{ denotes the vector to points on } S.$$

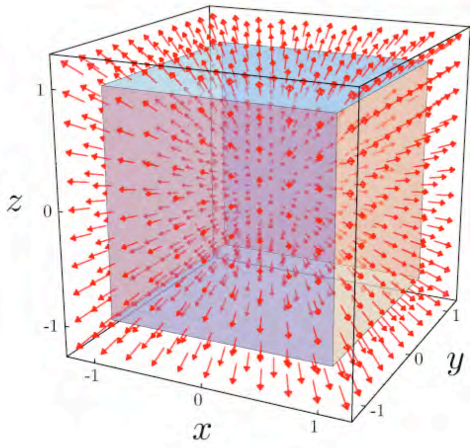
visualisation





with transparency





We conclude this chapter by proving the following famous result.

Theorem 12E: (Gauß' Law)

Suppose that $V \subseteq \mathbb{R}^3$ is a symmetric elementary region, with boundary surface S oriented with outward normal. Suppose further that $(0\,,0\,,0) \notin S$. Then $\int_S \frac{\boldsymbol{r}}{r^3} \cdot d\boldsymbol{S} = \begin{cases} 4\pi & \text{if } (0\,,0\,,0) \in V \\ 0 & \text{if } (0\,,0\,,0) \notin V, \end{cases}$ where $\boldsymbol{r} = (x\,,y\,,z)$ denotes the vector to points on S, and $r = \|\boldsymbol{r}\| = (x^2+y^2+z^2)^{\frac{1}{2}}$.

Sketch of proof

Suppose first of all that $(0\,,0\,,0) \notin V$. Then the vector field $\frac{\boldsymbol{r}}{r^3}$ is continuously differentiable on V, and so it follows from Gauss' theorem that $\int_S \frac{\boldsymbol{r}}{r^3} \cdot d\boldsymbol{S} = \iiint_V \operatorname{div}(\frac{\boldsymbol{r}}{r^3})\, dx\, dy\, dz$. It is easy to check that $\operatorname{div}(\frac{\boldsymbol{r}}{r^3}) = 0$ whenever $\boldsymbol{r} \neq 0$. The desired conclusion therefore holds in this case.

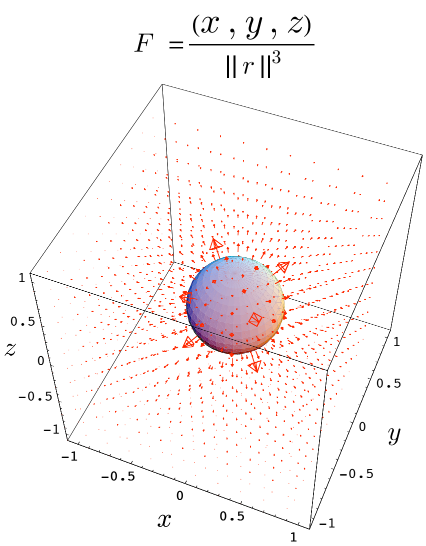
Suppose now that $(0\,,0\,,0) \in V$. Since $(0\,,0\,,0) \notin S$, it follows that there exists $\epsilon > 0$ such that the open ball $B(\epsilon)$, with centre $(0\,,0\,,0)$ and radius $\epsilon > 0$, satisfies $B(\epsilon) \subset V$. Now let $\Omega = V \setminus B(\epsilon)$, the region V with the open ball $B(\epsilon)$ removed. Clearly this region has boundary surface $S \cup T$, where T is the boundary surface of $B(\epsilon)$ with normal pointing towards $(0\,,0\,,0)$. Applying Gauss's theorem to this region Ω (note that Ω is *not* an elementary region), we have $\int_S \frac{\boldsymbol{r}}{r^3} \cdot d\boldsymbol{S} + \int_T \frac{\boldsymbol{r}}{r^3} \cdot d\boldsymbol{S} = \iiint_V \operatorname{div}(\frac{\boldsymbol{r}}{r^3})\, dx\, dy\, dz = 0$, so that $\int_S \frac{\boldsymbol{r}}{r^3} \cdot d\boldsymbol{S} = -\int_T \frac{\boldsymbol{r}}{r^3} \cdot d\boldsymbol{S}$.

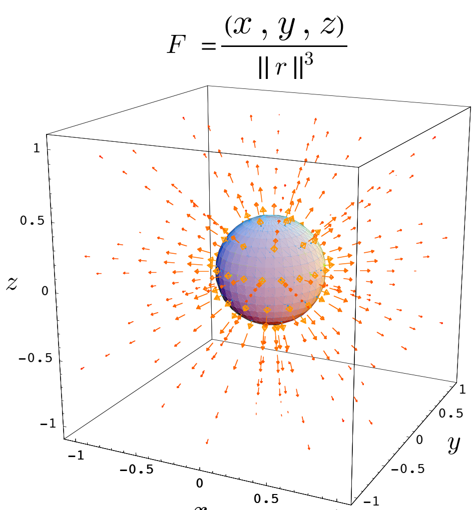
The boundary surface T can be parametrized by $\Phi : R \to \mathbb{R}^3$ whereby $(u\,,v) \mapsto \epsilon(\sin u \cos v\,, \sin u \sin v\,, \cos u)$, where $R = [0\,,\pi] \times [0\,,2\pi]$, and where $\boldsymbol{t}_u \times \boldsymbol{t}_v = \epsilon^2(\sin^2 u \cos v\,, \sin^2 u \sin v\,, \sin u \cos u) = \epsilon(\sin u)\,\Phi(u\,,v)$. This is an orientation-reversing parametrization, hence

$$-\int_T \frac{\boldsymbol{r}}{r^3} \cdot d\boldsymbol{S} = \iint_R \frac{\Phi(u\,,v)}{\epsilon^3} \cdot \epsilon(\sin u)\,\Phi(u\,,v)\, du\, dv = \frac{1}{\epsilon^2} \iint_R (\sin u)\,\Phi(u\,,v) \cdot \Phi(u\,,v)\, du\, dv = \iint_R \sin u\; du\, dv = 2\pi \int_0^\pi \sin u\, du = 4\pi\,.$$

This gives the desired conclusion.

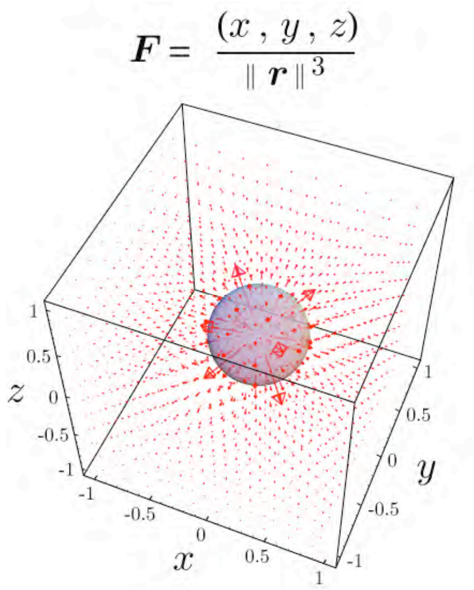
visualisation

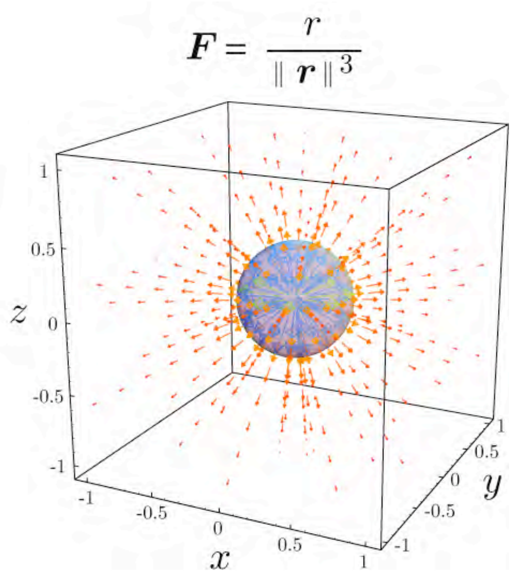




The image at right uses color-coding and a non-linear (monotonic) scaling of vector lengths, with the vectors distributed radially via an angular distribution, whereas that on the left shows vectors anchored to a cubical grid.

with transparency





Another Theorem

Theorem 12F:

Suppose that $\boldsymbol{F}$ is a vector field, defined and continuously differentiable everywhere in $\mathbb{R}^3$, satisfying $\nabla \cdot \boldsymbol{F} = 0$. Then there exists a continuously differentiable vector field $\boldsymbol{G}$ such that $\nabla \times \boldsymbol{G} = \boldsymbol{F}$.

Sketch of proof

Write $\boldsymbol{F} = (F_1, F_2, F_3)$ and define $\boldsymbol{G} = (G_1, G_2, G_3)$ by: $G_1(x, y, z) = \int_0^z F_2(x, y, t)\,dt - \int_0^y F_3(x, t, 0)\,dt$, $G_2(x, y, z) = -\int_0^z F_1(x, y, t)\,dt$, $G_3(x, y, z) = 0$. Then

$$\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} = 0 + \frac{\partial}{\partial z}\left(\int_0^z F_1(x, y, t)\,dt\right) = F_1(x, y, z)\,;$$

$$\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x} = \frac{\partial}{\partial z}\left(\int_0^z F_2(x, y, t)\,dt\right) - 0 = F_2(x, y, z)\,;$$

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = -\int_0^z\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right)dt + \frac{\partial}{\partial y}\left(\int_0^y F_3(x, t, 0)\,dt\right) = \int_0^z \frac{\partial F_3}{\partial z}(x, y, t)\,dt + F_3(x, y, 0) = F_3(x, y, z)\,.$$

Hence we have that $\nabla \times \boldsymbol{G}\,|_{(x,y,z)} = (F_1, F_2, F_3)\,|_{(x,y,z)} = \boldsymbol{F}(x, y, z)$, as required.

Remarks

- Whereas in Theorem 12C it is possible to assume that the vector field $\boldsymbol{F}$ is continuously differentiable in $\mathbb{R}^3$ except possibly at a finite number of points, this extension is not applicable here. The vector field $\boldsymbol{F}$ must be continuously differentiable *everywhere* in $\mathbb{R}^3$.

- Theorem 12F is in some sense the converse of Theorem 8F, which says that for any twice continuously differentiable vector field $\boldsymbol{G}$ in $\mathbb{R}^3$, we have $\nabla \cdot (\nabla \times \boldsymbol{G}) = 0$. Here we have proved that if $\boldsymbol{F}$ is a continuously differentiable vector field in $\mathbb{R}^3$ with $\nabla \cdot \boldsymbol{F} = 0$, then there exists a vector field $\boldsymbol{G}$ in $\mathbb{R}^3$ such that $\boldsymbol{F} = \nabla \times \boldsymbol{G}$.