MATH236 — Weeks 1&2

Paths

Chen notes, chapter 7

7.1 Paths—Introduction

In this chapter we discuss paths in $\mathbb{R}^n$; in particular we are interested in $\mathbb{R}^2$ and $\mathbb{R}^3$, since these can be easily drawn or visualised.

However, the ideas developed for these are easily extended to paths in $\mathbb{R}^n$, for $n > 3$.

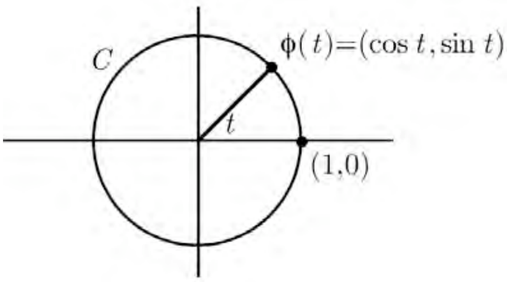
Before we give any formal definition, let us consider two examples.

Example 7.1.1 — tracing the unit circle in $\mathbb{R}^2$

Consider the unit circle $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ in $\mathbb{R}^2$.

At time $t = 0$, a starting at $(1, 0)$ starts to moves at constant speed along C in the anticlockwise direction, and first returns to the initial position at time $t = 2\pi$.

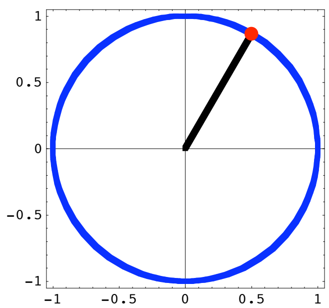
Knowing some trigonometry, it is easy to see that at any time $t \in [0, 2\pi]$ the position of the particle is given by $(x, y) = (\cos t, \sin t)$.



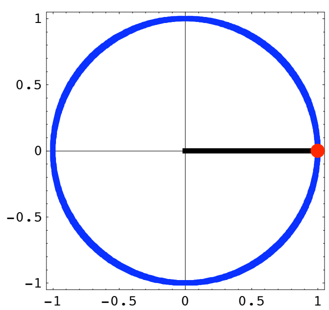
The motion of the particle can be completely described by a function $\boldsymbol{\phi} : [0, 2\pi] \to \mathbb{R}^2 : \boldsymbol{\phi}(t) = (\cos t, \sin t)$.

Note that $C = \boldsymbol{\phi}([0, 2\pi]) \stackrel{\text{def}}{=} \{\boldsymbol{\phi}(t) : t \in [0, 2\pi]\}$ is the ***range*** of the function.

graphic



movie



Example 7.1.2 — along a ray in $\mathbb{R}^3$

Consider a particle moving away from the origin $\mathbf{0} = (0, 0, 0)$ at time $t = 0$ in the direction of the unit vector $\boldsymbol{u} \in \mathbb{R}^3$ with constant acceleration a, and hence speed $t\,a$, at any given time $t \geq 0$.

In this case, the distance of the particle from the origin at time t is given by $\frac{1}{2}\,a\,t^2$, and so its position is given by $\boldsymbol{\phi}(t) = \frac{1}{2}\,a\,t^2\,\boldsymbol{u}$.

Suppose that we trace the movement of this particle from $t = 0$ to $t = T$. Then we are interested in a function

$$\boldsymbol{\phi} : [0, T] \to \mathbb{R}^3 : \phi(t) = \tfrac{1}{2}\,a\,t^2\,\boldsymbol{u}.$$

The ***range*** of this function is given by $\boldsymbol{\phi}([0, T]) = \left\{\frac{1}{2}\,a\,t^2\,\boldsymbol{u} : t \in [0, T]\right\}$. This is a line segment joining the origin $\mathbf{0}$ and the point $\frac{1}{2}\,a\,T^2\,\boldsymbol{u}$.

Note that the functions in Examples 7.1.1 and 7.1.2 do not just trace out curves. They also give the position of the particles at any time within the stated time interval.

Definition: path

By a ***path*** in $\mathbb{R}^n$ we mean a function of the type $\boldsymbol{\phi} : [A, B] \to \mathbb{R}^n$, where $A, B \in \mathbb{R}$ with $A < B$.

The range $\boldsymbol{\phi}([A, B]) \overset{\text{def}}{=} \{\boldsymbol{\phi}(t) : t \in [A, B]\} \subseteq \mathbb{R}^n$ of the function $\boldsymbol{\phi}$ is called a ***curve***, with ***initial point*** $\boldsymbol{\phi}(A)$, and ***terminal point*** $\boldsymbol{\phi}(B)$.
Suppose that for every $t \in [A, B]$ we have that $\boldsymbol{\phi}(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t))$, where $\phi_1(t), \phi_2(t), \ldots, \phi_n(t) \in \mathbb{R}$.

Then the functions $\phi_i(t) : [A, B] \to \mathbb{R}$ are called the ***components*** of the path $\boldsymbol{\phi}$.

Remarks

- It is usual to write $\boldsymbol{\phi}(t) = (x(t), y(t))$ and $\boldsymbol{\phi}(t) = (x(t), y(t), z(t))$, in the cases $n = 2$ and $n = 3$, respectively.
- Note the distinction between a *path* and a *curve*.

  Quite often different paths can share the same curve; e.g.,

  $\boldsymbol{\phi} : [0, 2\pi] \to \mathbb{R}^2$ with $\boldsymbol{\phi}(t) = (\cos t, \sin t)$

  $\boldsymbol{\psi} : [0, 1] \to \mathbb{R}^2$ with $\boldsymbol{\psi}(t) = (\cos(2\pi t), \sin(2\pi t))$

  $\boldsymbol{\eta} : [0, 1] \to \mathbb{R}^2$ with $\boldsymbol{\eta}(t) = (\cos(2\pi t^2), \sin(2\pi t^2))$.

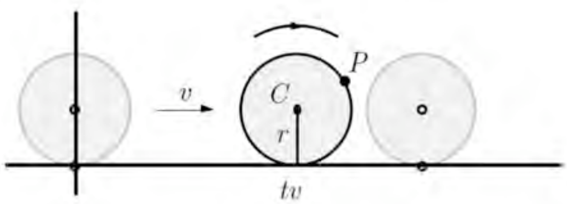
  satisfy $\boldsymbol{\phi}([0, 2\pi]) = \boldsymbol{\psi}([0, 1]) = \boldsymbol{\eta}([0, 1]) = C$, the unit circle in $\mathbb{R}^2$.
- Often we refer to the path $\boldsymbol{\phi}(t)$ without specifying the domain of definition of $\boldsymbol{\phi}$. This is really an abuse of notation which nevertheless can be convenient.

Example 7.1.3 — cycloid

Consider a circular disc of radius r standing on a level surface.

Let C denote the centre of the disc, and let P denote a fixed point on the rim of the disc.

Suppose that at time $t = 0$, the point P touches the surface, and is therefore directly below the point C. For convenience, let us assume that this point where the disc touches the surface at time $t = 0$ is the origin $(0, 0)$.



The disc now starts rolling to the right at constant speed v. We now wish to describe the path taken by the point P.

Clearly the point C is at position $(0, r)$ at time $t = 0$. Its position at time t is given by $(t\,v, r)$.

Note next that the circumference of the disc is $2\pi r$, and so the disc will have completed one revolution at time $t = 2\pi r/v$.

It follows that the angular speed of the disc is v/r.

Now let $\boldsymbol{\psi}(t)$ denote the relative position of P with respect to C. Clearly P rotates around C in a clockwise direction with angular speed v/r, so it follows that $\boldsymbol{\psi}(t) = (r\cos(-\frac{v\,t}{r} + \theta), r\sin(-\frac{v\,t}{r} + \theta))$, where $\theta \in \mathbb{R}$ is a constant.

Clearly $\psi(0) = (0, -r)$, so that $\cos\theta = 0$ and $\sin\theta = -1$, whence $\theta = -\frac{1}{2}\pi$. Hence

$$\psi(t) = \left(r\cos\left(-\frac{v\,t}{r} - \frac{1}{2}\pi\right), r\sin\left(-\frac{v\,t}{r} - \frac{1}{2}\pi\right)\right) = \left(-r\sin\frac{v\,t}{r}, -r\cos\frac{v\,t}{r}\right).$$

It follows that the actual position of P at time t is given by:

$$\phi(t) = (t\,v, r) + \psi(t) = \left(t\,v - r\sin\frac{v\,t}{r}, r - r\cos\frac{v\,t}{r}\right).$$

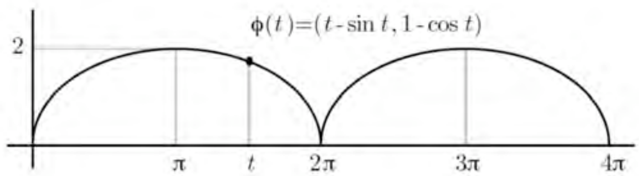
Suppose that $v = r = 1$. Then $\phi(t) = (t - \sin t, 1 - \cos t)$.

Clearly the point P touches the surface when $t = 2\,k\,\pi$, where k is any non-negative integer.

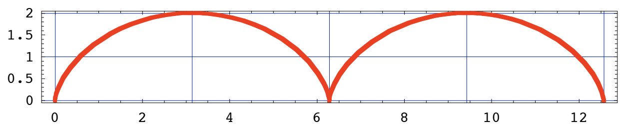
The image curve of the path $\phi : [A, B] \to \mathbb{R}^2 : \phi(t) = (t - \sin t, 1 - \cos t)$ is called a ***cycloid***.

Note that we have not specified the range for t in our discussion.

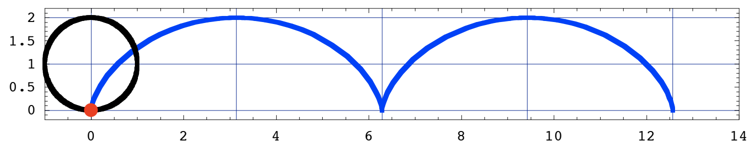
We can consider any interval $[A, B] \subseteq \mathbb{R}$, although to get a full picture, the interval should have length at least 2π.



graphic



movie



demos

- Animated Cycloid demo at the Wolfram demonstrations site
- local copy of the Cycloid demo.

7.2 Differentiable Paths

Definition: differentiable path

We say that a path $\phi : [A, B] \to \mathbb{R}^n$ is ***differentiable*** if the limit $\lim_{h\to 0} \frac{\phi(t+h) - \phi(t)}{h}$ exists for every $t \in [A, B]$, with the obvious restriction to one-sided limits at the endpoints of the interval $[A, B]$. In this case, the vector

$$\phi'(t) = \frac{d}{d\,t}\,\phi(t) = \lim_{h\to 0} \frac{\phi(t+h) - \phi(t)}{h}$$

is called the ***velocity vector*** of the path ϕ, and the quantity $\|\,\phi'(t)\,\|$ is called the ***speed*** of the path ϕ.

Remarks

- Note that we have borrowed some terminology from physics. This is entirely natural, as this area of mathematics is, to a large extent, motivated by the study of various problems in physics.

- Note that if the path is given by $\boldsymbol{\phi}(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t))$, then the velocity vector is given by $\boldsymbol{\phi}'(t) = (\phi_1'(t), \phi_2'(t), \ldots, \phi_n'(t))$ and the speed is given by: $\|\boldsymbol{\phi}'(t)\| = \left(|\phi_1'(t)|^2 + |\phi_2'(t)|^2 + \ldots + |\phi_n'(t)|^2\right)^{\frac{1}{2}}$.
- Note the special notation in the cases $n = 2$ and $n = 3$; e.g., $\|\boldsymbol{\phi}'(t)\| = \left(|x'(t)|^2 + |y'(t)|^2 + |z'(t)|^2\right)^{\frac{1}{2}}$ for a path in $\mathbb{R}^3$.
- The velocity vector $\boldsymbol{\phi}'(t)$ is a vector *tangent to the path* $\boldsymbol{\phi}(t)$ at time t. If C is the curve of the path $\boldsymbol{\phi}(t)$ and $\boldsymbol{\phi}'(t) \neq 0$, then $\boldsymbol{\phi}'(t)$ is a vector tangent to the curve C at the point $\boldsymbol{\phi}(t) \in C$.

Example 7.2.1 — cycloid

For the cycloid $\boldsymbol{\phi}(t) = (t - \sin t, 1 - \cos t)$ described in Example 7.1.3, the velocity vector is given by $\boldsymbol{\phi}'(t) = (1 - \cos t, \sin t)$.

Note that $1 - \cos t = 0$ implies that $\sin t = 0$, so the velocity is never vertical.

The speed of the path is $\|\boldsymbol{\phi}'(t)\| = ((1 - \cos t)^2 + \sin^2 t)^{\frac{1}{2}} = (2 - 2\cos t)^{\frac{1}{2}}$.

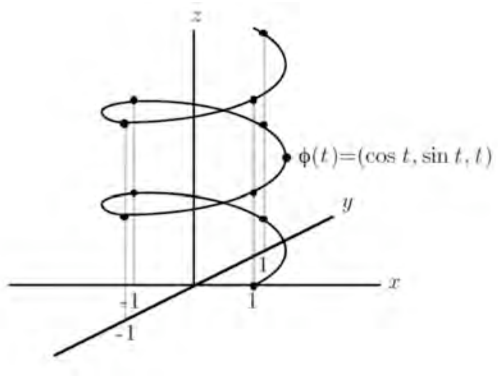
This is minimum and zero when $\cos t = 1$, at which the point P touches the surface. The speed is maximum when $\cos t = -1$, at which the point P is at the maximum height.

Example 7.2.2 — helix

To study the path $\boldsymbol{\phi}(t) = (\cos t, \sin t, t)$ in $\mathbb{R}^3$, we first of all consider just the first two components by studying the path $\boldsymbol{\psi}(t) = (\cos t, \sin t)$ in $\mathbb{R}^2$. This path describes a circle on the plane, followed in the anticlockwise direction.

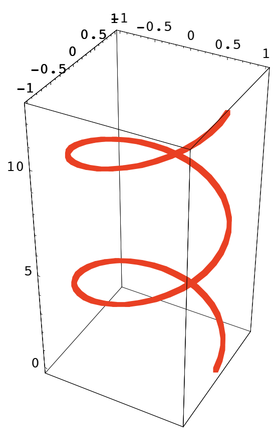
The third component t describes an increase in height with time if we think of the third component as being the vertical component.

It follows that if we consider the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$, then the path $\boldsymbol{\phi}(t)$ wraps around this cylinder in an anticlockwise direction, with the third component increasing. The curve of the path $\boldsymbol{\phi}(t)$ is called a ***helix***.



The path has velocity vector $\boldsymbol{\phi}'(t) = (-\sin t, \cos t, 1)$ and speed $\|\boldsymbol{\phi}'(t)\| = (\sin^2 t + \cos^2 t + 1^2)^{\frac{1}{2}} = \sqrt{2}$, so the path has constant speed.

graphic



Suppose that $\boldsymbol{\phi}(t)$ is a differentiable path. We have already indicated that if $\boldsymbol{\phi}'(t) \neq 0$, then it is a vector tangent to the path at the point $\boldsymbol{\phi}(t)$. We have ...

Theorem 7A

Suppose that $\phi(t)$ is a differentiable path in $\mathbb{R}^n$. Then the tangent line to the path at the point $\phi(t_0)$ is given by $L(\lambda) = \phi(t_0) + \lambda\, \phi'(t_0)$, provided that $\phi'(t) \neq 0$.

Example 7.2.3 — tangent to the helix

The equation of the tangent line to the helix $\phi(t) = (\cos t, \sin t, t)$ at the point $\phi(t_0)$ is given by

$$L(\lambda) = \phi(t_0) + \lambda\, \phi'(t_0) = (\cos t_0, \sin t_0, t_0) + \lambda(-\sin t_0, \cos t_0, 1).$$

Suppose that $t_0 = 2\pi$. Then $\phi(2\pi) = (1, 0, 2\pi)$, and the tangent line becomes

$$L(\lambda) = \phi(2\pi) + \lambda\, \phi'(2\pi) = (1, 0, 2\pi) + \lambda(0, 1, 1).$$

Writing $L(\lambda) = (x, y, z)$, we have $x = 1$, $y = \lambda$ and $z = 2\pi + \lambda$.

It follows that the tangent line to the helix at the point $(1, 0, 2\pi)$ is given by: $x = 1$ and $z = y + 2\pi$.

Try to visualize this from the pictures in Example 7.2.2.

Example 7.2.4 — tangent to the cycloid

The equation of the tangent line to the cycloid $\phi(t) = (t - \sin t, 1 - \cos t)$ at the point $\phi(t_0)$ is given by
$L(\lambda) = \phi(t_0) + \lambda\, \phi'(t_0) = (t_0 - \sin t_0, 1 - \cos t_0) + \lambda(1 - \cos t_0, \sin t_0)$.

Suppose that $t_0 = 2\pi$. Then $\phi(2\pi) = (2\pi, 0)$ and $L(\lambda) = (2\pi, 0) + \lambda(0, 0) = (2\pi, 0)$, clearly not the equation of a line.

Observe that since $\phi'(t) = (0, 0)$, then Theorem 7A does not apply in this case.

Example 7.2.5 — escape from the helix

Let us return to the helix discussed in Examples 7.2.2 and 7.2.3.

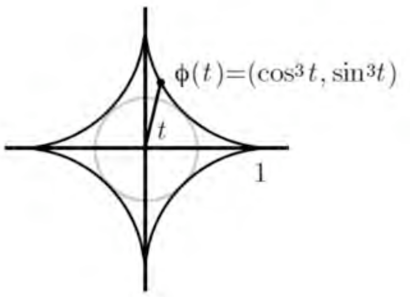
Suppose that a particle follows the helix from $t = 0$ to $t = 2\pi$ and then flies off at constant velocity on a tangent at $t = 2\pi$.

We wish to determine the position of the particle at $t = 4\pi$. Note that the particle is at position $\phi(2\pi) = (1, 0, 2\pi)$ when $t = 2\pi$, with tangential velocity $\phi'(2\pi) = (0, 1, 1)$. It follows that its position at $t = 4\pi$ must be given by

$$\phi(2\pi) + (4\pi - 2\pi)\phi'(2\pi) = (1, 0, 2\pi) + 2\pi(0, 1, 1) = (1, 2\pi, 4\pi).$$

Example 7.2.6 — hypocycloid of 4 cusps

Consider the ***hypocycloid of four cusps*** $\phi : [0, 2\pi] \to \mathbb{R}^2$ with $\phi(t) = (\cos^3 t, \sin^3 t)$.

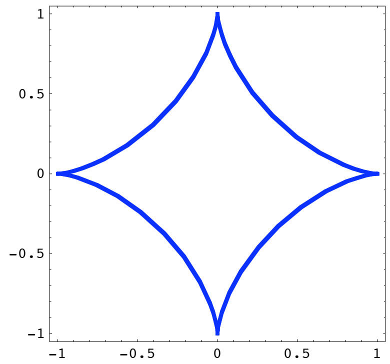


This path has velocity vector $\phi'(t) = (-3\cos^2 t \sin t, 3\sin^2 t \cos t)$ and speed

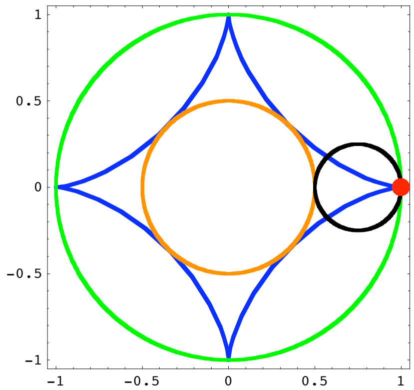
$$\|\phi'(t)\| = (9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t)^{\frac{1}{2}} = 3\,|\cos t \sin t\,|.$$

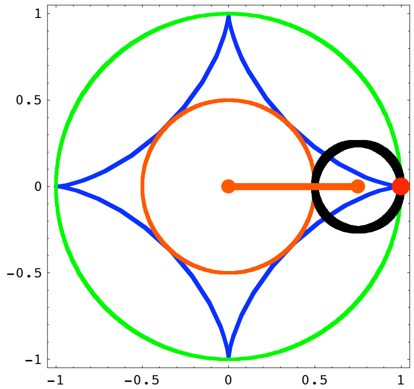
Note that while the hypocycloid is a differentiable path, its curve has cusps. However, the velocity and speed are zero at these cusps.

graphic



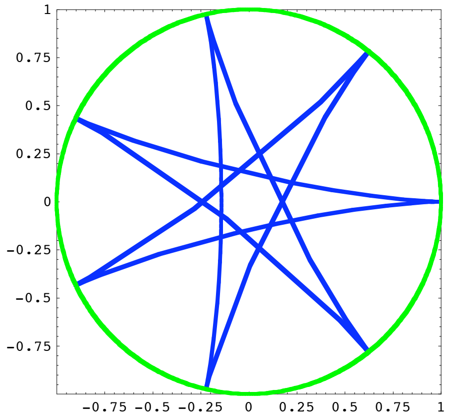
movie — hypocycloid

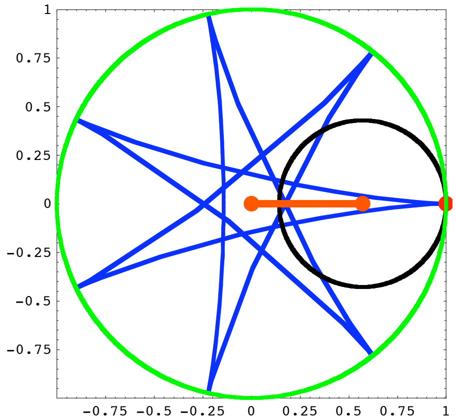




more general hypocycloids — spirograph

In[683]:=
```
cusps = 7 / 3; cycles = 3;
```





We state without proof the following two theorems. The proofs are not diffcult, and follow by applying the usual differentiation rules to the components.

Theorem 7B — derivatives of paths, chain rule, etc.

Suppose that $\phi(t)$ and $\psi(t)$ are differentiablepaths in $\mathbb{R}^n$. Suppose furtherthat $a(t)$ and $b(t)$ are differentiable real-valuedfunctions.

Then (a) $(\phi(t)+\psi(t))' = \phi'(t)+\psi'(t)$;

(b) $(a(t)\,\phi(t))' = a'(t)\,\phi(t) + a(t)\,\phi'(t)$;

(c) $(\phi(t)\cdot\psi(t))' = \phi'(t)\cdot\psi(t)+\phi(t)\cdot\psi'(t)$; and

(d) $\phi(a(t))' = a'(t)\,\phi'(a(t))$.

The above represent the sum rule, scalar multiplication rule, dot product rule and chain rule respectively.

Note also the vector product rule below which is valid only in $\mathbb{R}^3$:

Theorem 7C — cross-product rule

Suppose that $\phi(t)$ and $\psi(t)$ are differentiablepaths in $\mathbb{R}^3$. Then $(\phi(t)\times\psi(t))' = \phi'(t)\times\psi(t)+\phi(t)\times\psi'(t)$.

7.3 Arc Length

In this section, we are interested in calculating the length of the curve followed by a path. To motivate this, note that the speed $\|\boldsymbol{\phi}'(t)\|$ of a path $\boldsymbol{\phi}(t)$ is the rate of change of distance with respect to time.

Definition: velocity & arc-length differentials

Suppose that $\boldsymbol{\phi} : [A, B] \to \mathbb{R}^n$ is a differentiable path. The ***velocity differential*** is given by

$$d\mathbf{s} = \boldsymbol{\phi}'(t)\, dt = (\phi_1'(t), \phi_2'(t), \ldots, \phi_n'(t))\, dt.$$

The corresponding ***arc-length differential*** is given by

$$ds = \|\boldsymbol{\phi}'(t)\|\, dt = (|\phi_1'(t)|^2 + |\phi_2'(t)|^2 + \ldots + |\phi_n'(t)|^2)^{\frac{1}{2}}\, dt.$$

Remarks

- The velocity differential describes an infinitesimal displacement of a particle following the path $\boldsymbol{\phi}$. The arc-length differential describes the magnitude of this infinitesimal displacement.
- In $\mathbb{R}^2$ and $\mathbb{R}^3$, we have velocity differentials $d\mathbf{s} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) dt$ and $d\mathbf{s} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) dt$ and arc-length differentials $ds = \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)^{\frac{1}{2}} dt$ and $ds = \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right)^{\frac{1}{2}} dt$ respectively.

Definition: arc-length

Suppose that $\boldsymbol{\phi} : [A, B] \to \mathbb{R}^n$ is a continuously differentiable path. Then the quantity $\int_{t=A}^{t=B} 1\, ds = \int_A^B \|\boldsymbol{\phi}'(t)\|\, dt$ is called the ***arc length*** of the path $\boldsymbol{\phi}$.

Remark

- Note that if $\boldsymbol{\phi}(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t))$, then $\int_A^B \|\boldsymbol{\phi}'(t)\|\, dt = \int_A^B (|\phi_1'(t)|^2 + \ldots + |\phi_n'(t)|^2)^{\frac{1}{2}}\, dt.$

Example 7.3.1 — arc length of the cycloid

The cycloid $\boldsymbol{\phi} : [0, 2\pi] \to \mathbb{R}^2 : \boldsymbol{\phi}(t) = (t - \sin t, 1 - \cos t)$ has arc length:

$$\int_0^{2\pi} \|\boldsymbol{\phi}'(t)\|\, dt = \int_0^{2\pi} (2 - 2\cos t)^{\frac{1}{2}}\, dt = 2\int_0^{\pi} (2 - 2(1 - 2\sin^2(\tfrac{1}{2}t)))^{\frac{1}{2}}\, d(\tfrac{1}{2}t) = 2\int_0^{\pi} 2\sin(\tfrac{1}{2}t)\, d(\tfrac{1}{2}t)$$
$$= 4[-\cos\theta]_0^{\pi} = 8.$$

Example 7.3.2 — arc length of the helix

The helix $\boldsymbol{\phi} : [0, 2\pi] \to \mathbb{R}^3 : \boldsymbol{\phi}(t) = (\cos t, \sin t, t)$ has arc length

$$\int_0^{2\pi} \|\boldsymbol{\phi}'(t)\|\, dt = \int_0^{2\pi} (1 + (-\sin t)^2 + \cos^2 t)^{\frac{1}{2}}\, dt = \int_0^{2\pi} \sqrt{2}\, dt$$
$$= 2\sqrt{2}\,\pi \approx 8.8857658763 \ldots$$

Example 7.3.3 — arc length of the hypocycloid

The hypocycloid of four cusps $\boldsymbol{\phi} : [0, 2\pi] \to \mathbb{R}^2 : \boldsymbol{\phi}(t) = (\cos^3 t, \sin^3 t)$ has arc length:

$$\int_0^{2\pi} \|\boldsymbol{\phi}'(t)\|\, dt = \int_0^{2\pi} 3(\cos^4 t \sin^2 t + \cos^2 t \sin^4 t)^{\frac{1}{2}}\, dt = 3\int_0^{2\pi} (\cos^2 t \sin^2 t)^{\frac{1}{2}}\, dt$$
$$= 12\int_0^{\frac{1}{2}\pi} \cos t \sin t\, dt = 3\int_0^{\pi} \sin(2t)\, d(2t)$$
$$= 3[-\cos\theta]_0^{\pi} = 6.$$