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The low-dimensional structures formed by tricategories

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Abstract

We form tricategories and the homomorphisms between them into a bicategory, whose 2-cells are certain degenerate tritransformations. We then enrich this bicategory into an example of a three-dimensional structure called a *locally cubical bicategory*, this being a bicategory enriched in the monoidal 2-category of pseudo double categories. Finally, we show that every sufficiently well-behaved locally cubical bicategory gives rise to a tricategory, and thereby deduce the existence of a tricategory of tricategories.

1. *Introduction*

A major impetus behind many developments in 2-dimensional category theory has been the observation that, just as the fundamental concepts of set theory are categorical in nature, so the fundamental concepts of category theory are 2-categorical in nature. In other words, if one wishes to study categories "in the small" – as mathematical entities in their own right rather than as universes of discourse – then a profitable way of doing this is by studying the 2-categorical properties of **Cat**, the 2-category of all categories.[1]

Once one moves from the study of categories to the study of (possibly weak) n-categories, it is very natural to generalise the above maxim, and to assert that *the fundamental concepts of n-category theory are $(n+1)$-categorical in nature.* This is a profitable thing to do: for example, consider the coherence theorem for bicategories [**18**], which in its simplest form states that

Every bicategory is biequivalent to a 2*-category.*

A priori, this is merely a statement about individual bicategories; but we may also read it as a statement about the tricategory of bicategories **Bicat**, since "biequivalent" may be read

[1] Here, and elsewhere, we will adopt a common-sense attitude to set-theoretic issues, assuming a sufficient supply of Grothendieck universes and leaving it to the reader to qualify entities with suitable constraints on their size.

as "internally biequivalent in the tricategory **Bicat**."[2] Thus another way of stating the above would be to say that the 2-categories are biequivalence-dense in **Bicat**.

This maxim permeates almost all research in higher-dimensional category theory, and so we draw attention to it here, not in order to point out where we might use it, but rather where we might *not* use it. For instance, consider once again the coherence theorem for bicategories. We may restate it slightly more tightly as:

Every bicategory is biequivalent to a 2*-category*
via an identity-on-objects biequivalence.

The restriction to identity-on-objects biequivalences affords us an interesting simplification, since, as pointed out in [**17**], we can express such a biequivalence as a mere *equivalence* in a suitable 2-category, which we denote by $\mathbf{Bicat}_2$. The 0-cells of $\mathbf{Bicat}_2$ are the bicategories; the 1-cells are the homomorphisms between them; and the 2-cells are the *icons* of [**16**]. These are degenerate oplax natural transformations whose every 1-cell component is an identity: we will meet them in more detail in Section 2 below.

With the help of the 2-category $\mathbf{Bicat}_2$, the coherence theorem for bicategories can be made into a 2-categorical, rather than a tricategorical, statement: namely, that the 2-categories are equivalence-dense in $\mathbf{Bicat}_2$ (cf. [**17**, theorem 5·4]). This is a somewhat tighter result; moreover, the 2-category $\mathbf{Bicat}_2$ is much simpler to work with than the tricategory **Bicat**. Thus we should revise our general maxim, and acknowledge that *some* of the fundamental concepts of n-category theory may be expressible using fewer than $(n + 1)$ dimensions. Consequently, when we study n-categories, it may be useful to form them not only into an $(n + 1)$-category, but also into suitable lower-dimensional structures. It is the purpose of this paper to do this in the case $n = 3$. We construct both a bicategory of tricategories $\mathbf{Tricat}_2$ and a tricategory of tricategories $\mathbf{Tricat}_3$: where in both cases, the 2-cells are suitably scaled-up analogues of the bicategorical icons mentioned above.

In [**16**], Lack gives a number of motivations for studying the 2-category $\mathbf{Bicat}_2$ of bicategories, lax functors, and icons. Many of these motivations have obvious analogues one dimension higher. For instance, the coherence theorem for tricategories can be restated as

Every tricategory is internally biequivalent to a **Gray***-category in the tricategory* $\mathbf{Tricat}_3$.

On the other hand, coherence for tricategories internal to the *bicategory* $\mathbf{Tricat}_2$ is an open question. The structures $\mathbf{Tricat}_2$ and $\mathbf{Tricat}_3$ also provide avenues for studying the simplicial nerves of tricategories, thus allowing comparisons with work of Street [**21**] to be pursued in dimension three. Moreover, it is shown in [**16**] that the 2-category of monoidal categories embeds nicely in $\mathbf{Bicat}_2$; and similarly, we show that the tricategory of monoidal bicategories – as constructed in [**4**] – embeds nicely in $\mathbf{Tricat}_3$.

An outline of the paper is as follows. In Section 2, we construct a bicategory of tricategories, $\mathbf{Tricat}_2$. The construction is straightforward and computational. The 2-cells of this bicategory we call *ico-icons*: they can be seen as doubly degenerate oplax tritransformations whose 0- and 1-cell components are identities. More explicitly, they exist only between

[2] In practice, one would tend to use the local definition of biequivalence, wherein $\mathcal{B}$ is biequivalent to $\mathcal{B}'$ if there exists a homomorphism $F\colon \mathcal{B} \to \mathcal{B}'$ which is biessentially surjective on objects and locally an equivalence of categories; but as long as we assume the axiom of choice, the difference between the two definitions is merely one of presentation.

trihomomorphisms which agree on 0- and 1-cells, and are given by a collection of (not necessarily invertible) 3-cell components together with coherence data and axioms.

In Section 3, we describe a tricategory of tricategories, $\mathbf{Tricat}_3$. The first candidate we consider for its 2-cells are the *oplax icons*, which are singly degenerate oplax tritransformations: they exist only between trihomomorphisms which agree on 0-cells, and are given by a collection of (not necessarily invertible) 2- and 3-cell components together with coherence data and axioms. These generalise the 2-cells of $\mathbf{Tricat}_2$, since every ico-icon is an oplax icon: indeed, the ico-icons are precisely the *i*dentity *c*omponents *o*plax icons. However, oplax icons turn out to be too lax to compose properly: the same phenomenon which occurs if one tries to replace the weak transformations in the tricategory **Bicat** with oplax transformations. Thus instead we take the 2-cells of $\mathbf{Tricat}_3$ to be the smaller class of *pseudo-icons*: these being oplax icons whose 3-dimensional data is invertible.

Although we describe the tricategory $\mathbf{Tricat}_3$ in Section 3, we do not complete its construction. One reason is that we want to avoid giving unenlightening tricategorical coherence computations as far as possible, to which end, we would like to reuse the work we did in Section 2; and though intuitively this is not a problem, technically it is rather troublesome. A second reason is that we wish to explain an unusual discrepancy, namely that the bicategory $\mathbf{Tricat}_2$ carries some information which the tricategory $\mathbf{Tricat}_3$ cannot, in that an ico-icon (2-cell of $\mathbf{Tricat}_2$) cannot be viewed as a pseudo-icon (2-cell of $\mathbf{Tricat}_3$) unless it is *invertible*.

In Sections 4–6 we describe a general mechanism which allows us to clear up both of the above issues. This begins in Section 4 with the introduction of a new kind of three-dimensional categorical structure which we call a *locally cubical bicategory*. Like a tricategory, it has 0-, 1-, 2- and 3-cells; but the 2-cells come in two different kinds, vertical and horizontal, whilst the 3-cells are cubical in nature. Moreover, the coherence axioms that are to be satisfied are of a bicategorical, rather than a tricategorical kind, and so the resultant structure is computationally more tractable than a tricategory. As a first application of this theory, we are able to show quite easily that the totality of bicategories (and more generally the totality of *pseudo double categories* in the sense of [**10**]) form a locally cubical bicategory.

Section 5 then describes a locally cubical bicategory of tricategories which we denote by $\mathfrak{Tricat}_3$. The construction is once again straightforward and computational, and reuses the work done in Section 2. The objects and 1-cells of $\mathfrak{Tricat}_3$ are just tricategories and trihomomorphisms; the vertical 2-cells are the ico-icons from $\mathbf{Tricat}_2$; the horizontal 2-cells are the pseudo-icons from $\mathbf{Tricat}_3$; whilst the 3-cells are "cubical icon modifications". In particular, $\mathfrak{Tricat}_3$ is a rich enough structure to encode all the information from both $\mathbf{Tricat}_2$ and $\mathbf{Tricat}_3$. This resolves the second of the issues mentioned above.

In order to resolve the first issue, we appeal to a general theory which allows us to construct tricategories out of sufficiently well-behaved locally cubical bicategories: more precisely, those with the property that every invertible vertical 2-cell gives rise to a horizontal 2-cell. This general theory is described in detail in Section 6; whilst in Section 7, we are able to apply it to the locally cubical bicategory $\mathfrak{Tricat}_3$, thereby deducing the existence of the tricategory of tricategories $\mathbf{Tricat}_3$. Additionally, we identify the tricategory of monoidal bicategories inside of $\mathbf{Tricat}_3$.

Notation. We follow [**1**] and [**14**] where it concerns 2- and bicategories: so in particular, our oplax natural transformations $\alpha\colon F \Rightarrow G$ have 2-cell components given by $\alpha_f\colon \alpha_B.Ff \Rightarrow Gf.\alpha_A$. We will tend to use either juxtaposition or the connective "." to denote composition, relying on context to sort out precisely which sort of composition is intended. When it comes to tricategories, our primary references are [**9**] and [**12**], but with a preference for the "algebraic" presentation of the latter: though we will not use this algebraicity in any essential way.

We will also make use of *pasting diagrams* of 2-cells inside bicategories. Such diagrams are only well-defined up to a choice of bracketing of their boundary, and so we assume such a choice to have been made wherever necessary. Occasionally we will need to use similar pasting diagrams of 2-cells in a tricategory, and the same caveat holds, only more so: here, the diagram is only well-defined up to a choice of order in which the pasting should be performed; and again, we assume such a choice to have been made. We adopt one further convention regarding pasting diagrams. Suppose we are given a 2-cell $\alpha\colon h(gf) \Rightarrow h'(g'f')$ in a bicategory $\mathcal{B}$, thus:

$$\begin{tikzcd}
& X \arrow[r, "g"] & Y \arrow[dr, "h"] & \\
W \arrow[ur, "f"] \arrow[dr, "f'"'] & & & Z, \\
& X' \arrow[r, "g'"'] & Y' \arrow[ur, "h'"'] &
\arrow[from=1-2, to=3-3, phantom, "\Downarrow\alpha"]
\end{tikzcd}$$

together with a homomorphism of bicategories $F\colon \mathcal{B} \to \mathcal{C}$. Applying F to α yields a 2-cell $F(h(gf)) \Rightarrow F(h'(g'f'))$ of $\mathcal{C}$, but frequently, we will be more interested in the 2-cell

$$\begin{tikzcd}
& FX \arrow[r, "Fg"] & FY \arrow[dr, "Fh"] & \\
FW \arrow[ur, "Ff"] \arrow[dr, "Ff'"'] & & & FZ; \\
& FX' \arrow[r, "Fg'"'] & FY' \arrow[ur, "Fh'"'] &
\arrow[from=1-2, to=3-3, phantom, "\Downarrow"]
\end{tikzcd}$$

obtained by pasting $F\alpha$ with suitable coherence constraints for the homomorphism F: and we will consistently denote the 2-cell obtained in this way by $\overline{F\alpha}$.

2. *A bicategory of tricategories*

We begin by describing the lowest-dimensional structure into which tricategories and their homomorphisms can form themselves. At first, one might think that this would be a category; but unfortunately, composition of trihomomorphisms fails to be associative on the nose, as it requires one to compose 1-cells in a hom-bicategory, which is itself not an associative operation. Consequently, the best we can hope for is a *bicategory* of tricategories, which we will denote by $\mathbf{Tricat}_2$.

The simplest such bicategory would have trihomomorphisms as its 1-cells and *blips* as its 2-cells. According to [**9**], blips are very degenerate tritransformations which can only exist between two trihomomorphisms $F, G\colon \mathcal{S} \to \mathcal{T}$ which agree on 0-, 1-, 2- and 3-cells. Though one might think that this forces F and G to be the same, they can in fact differ with respect to certain pieces of coherence data: and a "blip" is the means by which one measures these differences.

However, if we are going to form a bicategory of tricategories, it may as well be the most general possible one; and so we will consider more general sorts of both 1- and 2-cells. Let us begin by looking at the 1-cells.

Definition 1. Let $\mathcal{S}$ and $\mathcal{T}$ be tricategories. A **lax homomorphism** $F\colon \mathcal{S} \to \mathcal{T}$ is a lax morphism of tricategories in the sense of [**9**], all of whose coherence 3-cells are invertible. Hence F consists of:

(i) a function $F\colon \mathrm{ob}\,\mathcal{S} \to \mathrm{ob}\,\mathcal{T}$;
(ii) homomorphisms of bicategories $F_{A,B}\colon \mathcal{S}(A, B) \to \mathcal{T}(FA, FB)$;
(iii) 2-cells $\iota_A\colon I_{FA} \to FI_A$;
(iv) 2-cells $\chi_{f,g}\colon Fg.Ff \Rightarrow F(gf)$, pseudo-natural in f and g;
(v) invertible modifications ω, δ and γ witnessing the coherence of ι and χ;

all subject to the axioms for a morphism of tricategories as found in [**9**].

The notion of lax homomorphism is a sensible one from many angles. We can compose lax homomorphisms just as we would compose homomorphisms of tricategories. If we are given a pair of monoidal bicategories [**6**] which we view as one-object tricategories, then the lax homomorphisms between them are the natural bicategorical generalisation of a lax monoidal functor (*weak monoidal homomorphisms*, in the terminology of [**6**]). Lax homomorphisms from the terminal tricategory into $\mathcal{T}$ classify *pseudomonads* in $\mathcal{T}$ – that is, monads whose associativity and unit laws have been weakened to hold up to coherent isomorphism, and in a similar vein we may use lax homomorphisms to give a succinct definition of an *enriched bicategory* in the sense of [**3**, **15**] – that is, of a bicategory "enriched in a tricategory", which is a one-dimension-higher version of a category enriched in a bicategory [**1**, Section 5.5], which is in turn a generalisation of the familiar notion of a category enriched in a monoidal category. We shall see a little more of enriched bicategories in Section 4.

We now turn to the 2-cells of $\mathbf{Tricat}_2$. The most informative precedent here is the corresponding notion one dimension down: the *icons* of [**16**, **17**]. As mentioned in the Introduction, these are degenerate oplax transformations between homomorphisms of bicategories which agree on 0-cells. To be precise, given two such homomorphisms of bicategories $F, G\colon \mathcal{B} \to \mathcal{C}$, an **icon** $\alpha\colon F \Rightarrow G$ is given by specifying for each 1-cell $f\colon A \to B$ of $\mathcal{B}$, a 2-cell $\alpha_f\colon Ff \Rightarrow Gf$ of $\mathcal{C}$ such that:

(i) for each 2-cell $\sigma\colon f \Rightarrow g$ of $\mathcal{B}$, the following diagram commutes:

$$\begin{tikzcd} Ff \arrow[r, Rightarrow, "\alpha_f"] \arrow[d, Rightarrow, "F\sigma"'] & Gf \arrow[d, Rightarrow, "G\sigma"] \\ Fg \arrow[r, Rightarrow, "\alpha_g"'] & Gg; \end{tikzcd}$$

(ii) for each object $A \in \mathcal{B}$, the following diagram commutes:

$$\begin{tikzcd} \mathrm{id}_{FA} \arrow[r, Rightarrow, "\cong"] \arrow[d, equal] & F\mathrm{id}_A \arrow[d, Rightarrow, "\alpha_{\mathrm{id}_A}"] \\ \mathrm{id}_{GA} \arrow[r, Rightarrow, "\cong"'] & G\mathrm{id}_A; \end{tikzcd}$$

(iii) for each pair of composable 1-cells $f\colon A \to B$, $g\colon B \to C$ in $\mathcal{B}$, the following diagram commutes:

$$\begin{tikzcd} Fg.Ff \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "\alpha_g.\alpha_f"'] & F(gf) \arrow[d, Rightarrow, "\alpha_{gf}"] \\ Gg.Gf \arrow[r, Rightarrow, "\cong"'] & G(gf); \end{tikzcd}$$

where the arrows labelled with $\cong$ witness the pseudo-functoriality of F and G. There is a bijection between icons $F \Rightarrow G$ and those oplax natural transformations $F \Rightarrow G$ whose components are all identities (hence the name: *i*dentity *c*omponent *o*plax *n*atural transformation); however, icons differ crucially from the oplax natural transformations representing them in regard to the manner of their *composition*. Indeed, composition of oplax natural transformations is only associative up to invertible modification, whilst icons admit a strictly associative composition; and it is this which allows bicategories, homomorphisms and icons to form a 2-category $\mathbf{Bicat}_2$.

The 2-cells of $\mathbf{Tricat}_2$ we are about to describe – the *ico-icons* – can be seen as higher-dimensional analogues of these bicategorical icons. They are doubly degenerate oplax tritransformations between lax trihomomorphisms which agree on both 0- and 1-cells. Here again, composition of ico-icons will not simply be composition of tritransformations, but rather a modified form of that composition which is *strictly* associative. The choice of the name ico-icon will be explained by Proposition 4 below.

Definition 2. Given lax homomorphisms $F, G\colon \mathcal{S} \to \mathcal{T}$, an **ico-icon** $\alpha\colon F \Rightarrow G$ may exist only if F and G agree on objects and 1-cells of $\mathcal{S}$; and is then given by the following data:

(TD1) for each pair of objects $A, B \in \mathcal{S}$, an icon

$$\alpha_{A,B}\colon F_{A,B} \Longrightarrow G_{A,B}\colon \mathcal{S}(A,B) \longrightarrow \mathcal{T}(FA, FB)$$

(so in particular, for each 2-cell $\theta\colon f \Rightarrow g$ of $\mathcal{S}$, a 3-cell of $\mathcal{T}$:

$$\begin{tikzcd} Ff \arrow[r, Rightarrow, "F\theta"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\alpha_\theta"] & Fg \arrow[d, equal] \\ Gf \arrow[r, Rightarrow, "G\theta"'] & Gg \end{tikzcd} \quad);$$

(TD2) for each object A of $\mathcal{S}$, a 3-cell of $\mathcal{T}$:

$$\begin{tikzcd} I_{FA} \arrow[r, Rightarrow, "\iota^F_A"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow M^\alpha_A"] & FI_A \arrow[d, equal] \\ I_{GA} \arrow[r, Rightarrow, "\iota^G_A"'] & GI_A; \end{tikzcd}$$

(TD3) for each pair of composable 1-cells $f\colon A \to B$, $g\colon B \to C$ of $\mathcal{S}$, a 3-cell of $\mathcal{T}$:

$$\begin{tikzcd} Fg.Ff \arrow[r, Rightarrow, "\chi^F_{f,g}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \Pi^\alpha_{f,g}"] & F(gf) \arrow[d, equal] \\ Gg.Gf \arrow[r, Rightarrow, "\chi^G_{f,g}"'] & G(gf); \end{tikzcd}$$

subject to the following axioms:

(TA1) for each pair of 2-cells $\theta\colon f \Rightarrow g\colon B \to C$ and $\theta'\colon f' \Rightarrow g'\colon A \to B$ of $\mathcal{S}$, the following pasting equality holds:

$$
\begin{tikzcd}[column sep=small]
 & Ff.Ff' \arrow[r, Rightarrow, "\chi^F"] \arrow[dl, equal] \arrow[d, phantom, "\Swarrow\Pi^\alpha"] & F(ff') \arrow[dl, equal] \arrow[dr, Rightarrow, "F(\theta\theta')"] \arrow[dd, phantom, "\Swarrow\alpha_{\theta\theta'}"] & \\
Gf.Gf' \arrow[r, Rightarrow, "\chi^G"] \arrow[dr, Rightarrow, "G\theta.G\theta'"'] & G(ff') \arrow[dr, Rightarrow, "G(\theta\theta')"] \arrow[d, phantom, "\Searrow\chi^G"] & & F(gg') \arrow[dl, equal] \\
 & Gg.Gg' \arrow[r, Rightarrow, "\chi^G"'] & G(gg') &
\end{tikzcd}
$$

$$
=\quad
\begin{tikzcd}[column sep=small]
 & Ff.Ff' \arrow[r, Rightarrow, "\chi^F"] \arrow[dl, equal] \arrow[dr, Rightarrow, "F\theta.F\theta'"] & F(ff') \arrow[dr, Rightarrow, "F(\theta\theta')"] \arrow[d, phantom, "\Searrow\chi^F"] & \\
Gf.Gf' \arrow[dr, Rightarrow, "G\theta.G\theta'"'] \arrow[r, phantom, "\Swarrow\alpha_\theta.\alpha_{\theta'}"] & {} & Fg.Fg' \arrow[r, Rightarrow, "\chi^F"] \arrow[dl, equal] \arrow[d, phantom, "\Swarrow\Pi^\alpha"] & F(gg'); \arrow[dl, equal] \\
 & Gg.Gg' \arrow[r, Rightarrow, "\chi^G"'] & G(gg') &
\end{tikzcd}
$$

(TA2) for each 1-cell $f\colon A \to B$ of $\mathcal{S}$, the following pasting equality holds:

$$
\begin{tikzcd}[column sep=small]
 & FI_B.Ff \arrow[r, Rightarrow, "\chi^F"] \arrow[dr, phantom, "\Downarrow\gamma^F"] & F(I_B.f) \arrow[dr, Rightarrow, "F\mathfrak{l}"] & \\
I_{FB}.Ff \arrow[ur, Rightarrow, "\iota^F.1"] \arrow[d, equal] \arrow[dr, Rightarrow, "\mathfrak{l}"] & & & Ff \arrow[dl, equal] \arrow[d, equal] \\
I_{GB}.Gf \arrow[r, phantom, "="] \arrow[dr, Rightarrow, "\mathfrak{l}"'] & Ff \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "="] & Ff \arrow[r, phantom, "="] \arrow[d, equal] & Gf \arrow[dl, equal] \\
 & Gf \arrow[r, equal] & Gf &
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
 & FI_B.Ff \arrow[r, Rightarrow, "\chi^F"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \Pi^\alpha"] & F(I_B.f) \arrow[d, equal] \arrow[dr, Rightarrow, "F\mathfrak{l}"] & \\
I_{FB}.Ff \arrow[ur, Rightarrow, "\iota^F.1"] \arrow[d, equal] \arrow[r, phantom, "\Downarrow M^\alpha.1"] & GI_B.Gf \arrow[r, Rightarrow, "\chi^G"'] & G(I_B.f) \arrow[dr, Rightarrow, "G\mathfrak{l}"'] \arrow[r, phantom, "\Downarrow \alpha\mathfrak{l}"] & Ff \arrow[d, equal] \\
I_{GB}.Gf \arrow[ur, Rightarrow, "\iota^G.1"'] \arrow[dr, Rightarrow, "\mathfrak{l}"'] \arrow[rrr, phantom, "\Downarrow \gamma^G"] & & & Gf; \arrow[dl, equal] \\
 & Gf \arrow[r, equal] & Gf &
\end{tikzcd}
$$

(TA3) for each 1-cell $f \colon A \to B$ of $\mathcal{S}$, the following pasting equality holds:

$$
\begin{tikzcd}
 & Ff.FI_A \arrow[r, Rightarrow, "\chi^F"] & F(f.I_A) \arrow[dr, Rightarrow, "F\mathfrak{r}"] & \\
Ff.I_{FA} \arrow[ur, Rightarrow, "1.\iota^F"] \arrow[d, equal] \arrow[dr, Rightarrow, "\mathfrak{r}"] \arrow[rrr, phantom, "\Downarrow \delta^F"] & & & Ff \arrow[dl, equal] \arrow[d, equal] \\
Gf.I_{GA} \arrow[r, phantom, "="] \arrow[dr, Rightarrow, "\mathfrak{r}"'] & Ff \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "="] & Ff \arrow[d, equal] \arrow[r, phantom, "="] & Gf \arrow[dl, equal] \\
 & Gf \arrow[r, equal] & Gf &
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
 & Ff.FI_A \arrow[r, Rightarrow, "\chi^F"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow \Pi^\alpha"] & F(f.I_A) \arrow[d, equal] \arrow[dr, Rightarrow, "F\mathfrak{r}"] & \\
Ff.I_{FA} \arrow[ur, Rightarrow, "1.\iota^F"] \arrow[d, equal] \arrow[r, phantom, "\Downarrow 1.M^\alpha"] & Gf.GI_A \arrow[r, Rightarrow, "\chi^G"'] & G(f.I_A) \arrow[dr, Rightarrow, "G\mathfrak{r}"'] \arrow[r, phantom, "\Downarrow \alpha\mathfrak{r}"] & Ff \arrow[d, equal] \\
Gf.I_{GA} \arrow[ur, Rightarrow, "1.\iota^G"'] \arrow[dr, Rightarrow, "\mathfrak{r}"'] \arrow[rrr, phantom, "\Downarrow \delta^G"] & & & Gf; \arrow[dl, equal] \\
 & Gf \arrow[r, equal] & Gf &
\end{tikzcd}
$$

(TA4) for each triple f, g, h of composable 1-cells of $\mathcal{S}$, the following pasting equality holds:

$$\begin{tikzcd}
 & & F(hg).Ff \arrow[rr, Rightarrow, "\chi^F"] & & F((hg)f) \arrow[drr, Rightarrow, "F\mathfrak{a}"] & & \\
(Fh.Fg).Ff \arrow[urr, Rightarrow, "\chi^F.1"] \arrow[drr, Rightarrow, "\mathfrak{a}"'] \arrow[d, equal] & & & \Downarrow\omega^F & & & F(h(gf)) \arrow[d, equal] \\
(Gh.Gg).Gf \arrow[drr, Rightarrow, "\mathfrak{a}"'] & = & Fh.(Fg.Ff) \arrow[rr, Rightarrow, "1.\chi^F"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow 1.\Pi^\alpha"] & & Fh.F(gf) \arrow[urr, Rightarrow, "\chi^F"'] \arrow[d, equal] & \Downarrow\Pi^\alpha & G(h(gf)) \\
 & & Gh.(Gg.Gf) \arrow[rr, Rightarrow, "1.\chi^G"'] & & Gh.G(gf) \arrow[urr, Rightarrow, "\chi^G"'] & &
\end{tikzcd}$$

$$= \quad \begin{tikzcd}
 & & F(hg).Ff \arrow[rr, Rightarrow, "\chi^F"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\Pi^\alpha"] & & F((hg)f) \arrow[drr, Rightarrow, "F\mathfrak{a}"] \arrow[d, equal] & & \\
(Fh.Fg).Ff \arrow[urr, Rightarrow, "\chi^F.1"] \arrow[d, equal] & \Downarrow \Pi^\alpha.1 & G(hg).Gf \arrow[rr, Rightarrow, "\chi^G"'] & & G((hg)f) \arrow[drr, Rightarrow, "G\mathfrak{a}"'] & \Downarrow {}^{\alpha}\mathfrak{a} & F(h(gf)) \arrow[d, equal] \\
(Gh.Gg).Gf \arrow[urr, Rightarrow, "\chi^G.1"'] \arrow[drr, Rightarrow, "\mathfrak{a}"'] & & & \Downarrow\omega^G & & & G(h(gf)). \\
 & & Gh.(Gg.Gf) \arrow[rr, Rightarrow, "1.\chi^G"'] & & Gh.G(gf) \arrow[urr, Rightarrow, "\chi^G"'] & &
\end{tikzcd}$$

Observe that, because the raw data for an ico-icon is a collection of 3-cells in the target tricategory, there is no possibility of introducing a third dimension of structure given by "ico-icon modifications". To do this we have to look at *singly* degenerate, rather than *doubly* degenerate, oplax tritransformations. We do this in the next Section.

Now, in order to show that this collection of 0-, 1- and 2-cells forms a bicategory, we have to give additional *data* – vertical composition of 2-cells, horizontal composition of 1- and 2-cells and associativity and unitality constraints – subject to additional *axioms* – the category axioms for vertical composition, the middle-four interchange axiom and the pentagon and triangle axioms for the associativity and unit constraints.

We start with the vertical structure: the identity 2-cell $\mathrm{id}_F \colon F \Rightarrow F$ in $\mathbf{Tricat}_2$ we take to be given by the following data:

$$(\mathrm{id}_F)_{A,B} = \mathrm{id}_{F_{A,B}}, \quad M_A^{\mathrm{id}_F} = \mathrm{id}_{\iota_A^F} \quad \text{and} \quad \Pi_{f,g}^{\mathrm{id}_F} = \mathrm{id}_{\chi_{f,g}^F}.$$

Each of the axioms (TA1)–(TA4) now expresses that something is equal to itself pasted together with some identity 3-cells, which is clear enough. Next, given 2-cells $\alpha \colon F \Rightarrow G$

and $\beta\colon G \Rightarrow H$ in **Tricat**$_2$, we take $\beta\alpha\colon F \Rightarrow H$ to be given by the following data:

$$(\beta\alpha)_{A,B} = \beta_{A,B}.\alpha_{A,B}, \quad M_A^{\beta\alpha} = M_A^{\beta}.M_A^{\alpha} \quad \text{and} \quad \Pi_{f,g}^{\beta\alpha} = \Pi_{f,g}^{\beta}.\Pi_{f,g}^{\alpha}.$$

Each of the axioms (TA1)–(TA4) for this data follow from juxtaposing the corresponding axioms for α and β in a very straightforward manner. Moreover, because vertical composition of 3-cells in a tricategory is strictly associative and unital, so is the vertical composition of 2-cells in **Tricat**$_2$.

We turn now to the horizontal structure. Horizontal identities and composition for 1-cells are the identities and composition for lax homomorphisms as detailed in [**15**]; whilst given 2-cells $\alpha\colon F \Rightarrow F'\colon \mathcal{S} \to \mathcal{T}$ and $\beta\colon G \Rightarrow G'\colon \mathcal{T} \to \mathcal{U}$, their horizontal composite $\beta * \alpha\colon GF \Rightarrow G'F'\colon \mathcal{S} \to \mathcal{U}$ is given by:

(TD1) $(\beta * \alpha)_{A,B} := \beta_{A,B} * \alpha_{A,B}$, where $*$ on the right-hand side is the horizontal composite of the underlying icons in the 2-category **Bicat**$_2$ of the Introduction. In particular, given a 2-cell $\theta\colon f \Rightarrow g$ of $\mathcal{S}$, we have

$$(\beta * \alpha)_\theta = \begin{tikzcd} GFf \arrow[r, Rightarrow, "GF\theta"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta_{F\theta}"] & GFg \arrow[d, equal] \\ G'Ff \arrow[r, Rightarrow, "G'F\theta"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow G'\alpha_\theta"] & G'Fg \arrow[d, equal] \\ G'F'f \arrow[r, Rightarrow, "G'F'\theta"'] & G'F'g; \end{tikzcd}$$

(TD2)

$$M_A^{\beta*\alpha} := \begin{tikzcd} I_{GFA} \arrow[r, Rightarrow, "\iota^G"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow M^{\beta}_{FA}"] & GI_{FA} \arrow[r, Rightarrow, "G\iota^F"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta_{\iota^F}"] & GFI_A \arrow[d, equal] \\ I_{G'FA} \arrow[r, Rightarrow, "\iota^{G'}"] \arrow[d, equal] \arrow[dr, phantom, "="] & G'I_{FA} \arrow[r, Rightarrow, "G'\iota^F"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow G'M^{\alpha}_A"] & G'FI_A \arrow[d, equal] \\ I_{G'F'A} \arrow[r, Rightarrow, "\iota^{G'}"'] & G'I_{F'A} \arrow[r, Rightarrow, "G'\iota^{F'}"'] & G'F'I_A; \end{tikzcd}$$

(TD3)

$$\Pi_{f,g}^{\beta*\alpha} := \begin{tikzcd} GFg.GFf \arrow[r, Rightarrow, "\chi^G"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\Pi^{\beta}_{Ff,Fg}"] & G(Fg.Ff) \arrow[r, Rightarrow, "G\chi^F"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta_{\chi^F}"] & GF(gf) \arrow[d, equal] \\ G'Fg.G'Ff \arrow[r, Rightarrow, "\chi^{G'}"] \arrow[d, equal] \arrow[dr, phantom, "="] & G'(Fg.Ff) \arrow[r, Rightarrow, "G'\chi^F"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow G'\Pi^{\alpha}_{f,g}"] & G'F(gf) \arrow[d, equal] \\ G'F'g.G'F'f \arrow[r, Rightarrow, "\chi^{G'}"'] & G'(F'g.F'f) \arrow[r, Rightarrow, "G'\chi^{F'}"'] & G'F'(gf). \end{tikzcd}$$

We must check that these data satisfy (TA1)–(TA4). If we view the pasting equalities in these axioms as equating two ways round a cube or a hexagonal prism, then this verification is a matter of taking a suitable collection of such cubes and prisms for β and α and sticking them together in the right way. When realised in two dimensions, this amounts to displaying a succession of equalities of rather large pasting diagrams. We leave the task of reconstructing these to the reader.

Let us consider now the middle-four interchange axiom. Asking for this be satisfied amounts to checking that the other obvious way of defining $\beta * \alpha$ – via GF' rather than $G'F$ – gives the same answer; and this follows quickly from the middle-four interchange law in the hom-bicategories of $\mathcal{U}$, and the first icon axiom for β.

It remains to give the associativity and unit constraints a, l and r for **Tricat**$_2$. For the left unit constraint l, consider a lax homomorphism $F \colon \mathcal{S} \to \mathcal{T}$, and write F' for the composite $\mathrm{id}_{\mathcal{T}}.F \colon \mathcal{S} \to \mathcal{T}$. Now, F' agrees with F on 0-cells and on hom-bicategories, but differs in the remaining coherence data; indeed, we have

$$\iota_A^{F'} = \; I_{FA} \xRightarrow{\mathrm{id}_{I_{FA}}} I_{FA} \xRightarrow{\iota_A^F} FI_A$$

$$\text{and} \quad \chi_{f,g}^{F'} = \; Fg.Ff \xRightarrow{\mathrm{id}_{Fg.Ff}} Fg.Ff \xRightarrow{\chi_{f,g}^F} F(gf).$$

Thus we define a 2-cell $l_F \colon \mathrm{id}_{\mathcal{T}}.F \Rightarrow F$ in **Tricat**$_2$ as follows:

(TD1) $(l_F)_{A,B} = \mathrm{id}_{F_{A,B}} \colon F_{A,B} \Rightarrow F_{A,B}$;
(TD2) $M_A^{l_F}$ is the unit isomorphism $\iota_A^F.(\mathrm{id}_{I_{FA}}) \Rrightarrow \iota_A^F$ in the bicategory $\mathcal{T}(FA, FA)$;
(TD3) $\Pi_{f,g}^{l_F}$ is the unit isomorphism $\chi_{f,g}^F.(\mathrm{id}_{Fg.Ff}) \Rrightarrow \chi_{f,g}^F$ in the bicategory $\mathcal{T}(FA, FC)$.

Now each axiom (TA1)–(TA4) is a tautology which describes how we obtained $\chi^{F'}$, $\delta^{F'}$, $\gamma^{F'}$ and $\omega^{F'}$ from the corresponding data for F. The definition of r is dual to that of l, so we pass over it and onto the associativity constraint a. Consider three lax homomorphisms $F \colon \mathcal{R} \to \mathcal{S}$, $G \colon \mathcal{S} \to \mathcal{T}$ and $H \colon \mathcal{T} \to \mathcal{U}$ and the two composites $(HG)F$ and $H(GF) \colon \mathcal{R} \to \mathcal{U}$. As above, these agree on 0-cells and on hom-bicategories (and so we write their common value simply as HGF) but differ with respect to coherence data. This time we have:

$$\begin{aligned} \iota^{(HG)F} &= HG\iota^F.(H\iota^G.\iota^H), & \qquad & \iota^{H(GF)} = (HG\iota^F.H\iota^G).\iota^H, \\ \chi^{(HG)F} &= HG\chi^F.(H\chi^G.\chi^H) & \text{and} \qquad & \chi^{H(GF)} = (HG\chi^F.H\chi^G).\chi^H, \end{aligned}$$

where we omit the subscripts for clarity. Thus we take $a_{F,G,H} \colon (HG)F \Rightarrow H(GF)$ in **Tricat**$_2$ to be:

(TD1) $(a_{F,G,H})_{A,B} = \mathrm{id}_{(HGF)_{A,B}} \colon (HGF)_{A,B} \Rightarrow (HGF)_{A,B}$;
(TD2) $M_A^{a_{F,G,H}}$ is the associativity isomorphism

$$HG\iota_A^F.\big(H\iota_{FA}^G.\iota_{GFA}^H\big) \Rrightarrow \big(HG\iota_A^F.H\iota_{FA}^G\big).\iota_{GFA}^H$$

in the bicategory $\mathcal{U}(HGFA, HGFA)$;
(TD3) $\Pi_{f,g}^{a_{F,G,H}}$ is the associativity isomorphism

$$HG\chi_{f,g}^F.\big(H\chi_{Ff,Fg}^G.\chi_{GFf,GFg}^H\big) \Rrightarrow \big(HG\chi_{f,g}^F.H\chi_{Ff,Fg}^G\big).\chi_{GFf,GFg}^H$$

in the bicategory $\mathcal{U}(HGFA, HGFC)$.

We must now verify axioms (TA1)–(TA4) for these data. For this we observe that the 3-cell data χ, γ, δ and ω for $H(GF)$ and for $(HG)F$ are, in fact, obtained as different bracketings of the same pasting diagram. So by the pasting theorem for bicategories, we can obtain the 3-cell data χ, γ, δ and ω for $H(GF)$ from that for $(HG)F$ by pasting with suitable associativity isomorphisms in the appropriate hom-bicategory of $\mathcal{U}$; and this is precisely what axioms (TA1)–(TA4) say.

It remains to check the naturality of l, r and a, and the pentagon and triangle identities. For the naturality of l, we must show that for any 2-cell $\alpha\colon F \Rightarrow G$ of $\mathbf{Tricat}_2$, the following diagram commutes:

$$\begin{tikzcd} \mathrm{id}_{\mathcal{T}}.F \arrow[r, Rightarrow, "l_F"] \arrow[d, Rightarrow, "\mathrm{id}_{\mathcal{T}}.\alpha"'] & F \arrow[d, Rightarrow, "\alpha"] \\ \mathrm{id}_{\mathcal{T}}.G \arrow[r, Rightarrow, "l_G"'] & G. \end{tikzcd}$$

We easily verify that the left-hand 2-cell $\alpha' = \mathrm{id}_{\mathcal{T}}.\alpha$ has components $\alpha'_\theta = \alpha_\theta$, $M^{\alpha'}_A = M^{\alpha}_A.(\mathrm{id}_{I_{FA}})$ and $\Pi^{\alpha'}_{f,g} = \Pi^{\alpha}_{f,g}.(\mathrm{id}_{Ff.Fg})$; therefore the naturality of l is a consequence of the naturality of the left unit constraints in the hom-bicategories of $\mathcal{T}$; and dually for r. For the naturality of a, we must show that the following diagram commutes in $\mathbf{Tricat}_2$ for all suitable 2-cells α, β and ϵ:

$$\begin{tikzcd} (HG)F \arrow[r, Rightarrow, "a_{F,G,H}"] \arrow[d, Rightarrow, "(\alpha\beta)\epsilon"'] & H(GF) \arrow[d, Rightarrow, "\alpha(\beta\epsilon)"] \\ (H'G')F' \arrow[r, Rightarrow, "a_{F',G',H'}"'] & H'(G'F'), \end{tikzcd}$$

for which we must show that (TD1)–(TD3) agree for the two ways around this square. For (TD1) this is trivial; so consider (TD2). For both $(\alpha\beta)\epsilon$ and $\alpha(\beta\epsilon)$, we obtain this datum by pasting together the same 3×3 diagram of 3-cells; the only difference being the manner in which we bracket together the boundary of this diagram. Thus the commutativity of the above square with respect to (TD2) is a further instance of the pasting theorem for bicategories. (TD3) is obtained in a similar manner.

Finally, it is not hard to verify that the pentagon and triangle identities for a, l and r follow from instances of the pentagon and triangle identities in the hom-bicategories of the target tricategory. This completes the definition of the bicategory $\mathbf{Tricat}_2$.

3. *Towards a tricategory of tricategories*

We now wish to describe a tricategory of tricategories $\mathbf{Tricat}_3$. This will have the same 0-cells and 1-cells as $\mathbf{Tricat}_2$, but will have 2-cells with one fewer level of degeneracy, which consequently admit a notion of 3-cell between them. Although we introduce the 2- and 3-cells of $\mathbf{Tricat}_3$ in this Section, we will not actually prove that we obtain a tricategory from them until we reach Section 7. As explained in the Introduction, we do this for two reasons. Firstly, so that we can set up some machinery which will allow us to avoid checking all the tricategorical coherence axioms by hand; and secondly, in order to investigate the curious fact that $\mathbf{Tricat}_3$ does not really extend $\mathbf{Tricat}_2$, in that not every 2-cell of the latter gives rise to a 2-cell of the former.

We now begin our description of $\mathbf{Tricat}_3$. Its objects and 1-cells are, as stated above, tricategories and lax trihomomorphisms. The 2-cells are to be "singly degenerate oplax tritransformations". The most obvious way of interpreting this notion would be as follows:

Definition 3. Let there be given lax homomorphisms of tricategories $F, G\colon \mathcal{S} \to \mathcal{T}$; then an **oplax icon** $\alpha\colon F \Rrightarrow G$ may exist only if F and G agree on objects whereupon it consists of the following data:

(ID1) for each A and B in $\mathcal{S}$, an oplax natural transformation

$$\alpha_{A,B} \colon F_{A,B} \Longrightarrow G_{A,B} \colon \mathcal{S}(A, B) \longrightarrow \mathcal{T}(FA, FB)$$

(so in particular, for each 1-cell $f \colon A \to B$ of $\mathcal{S}$, we have a 2-cell $\alpha_f \colon Ff \Rightarrow Gf$ of $\mathcal{T}$, and for each 2-cell $\theta \colon f \Rightarrow g$ of $\mathcal{S}$, a 3-cell

$$\begin{tikzcd} Ff \arrow[r, Rightarrow, "F\theta"] \arrow[d, Rightarrow, "\alpha_f"'] \arrow[dr, phantom, "\Downarrow\alpha_\theta"] & Fg \arrow[d, Rightarrow, "\alpha_g"] \\ Gf \arrow[r, Rightarrow, "G\theta"'] & Gg \end{tikzcd} \quad);$$

(ID2) for each object A of $\mathcal{S}$, a 3-cell of $\mathcal{T}$:

$$\begin{tikzcd} I_{FA} \arrow[r, Rightarrow, "\iota^F_A"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow M^\alpha_A"] & FI_A \arrow[d, Rightarrow, "\alpha_{I_A}"] \\ I_{GA} \arrow[r, Rightarrow, "\iota^G_A"'] & GI_A; \end{tikzcd}$$

(ID3) for each A, B and C in $\mathcal{S}$, a modification

$$\begin{tikzcd} F(–) \otimes F(?) \arrow[r, Rightarrow, "\chi^F"] \arrow[d, Rightarrow, "\alpha_{(–)}\otimes\alpha_{(?)}"'] \arrow[dr, phantom, "\Downarrow\Pi^\alpha_{A,B,C}"] & F\big((–) \otimes (?)\big) \arrow[d, Rightarrow, "\alpha_{((–)\otimes(?))}"] \\ G(–) \otimes G(?) \arrow[r, Rightarrow, "\chi^G"'] & G\big((–) \otimes (?)\big), \end{tikzcd}$$

where, for instance, $F(–) \otimes F(?)$ represents the homomorphism

$$\mathcal{S}(B, C) \times \mathcal{S}(A, B) \xrightarrow{F\times F} \mathcal{T}(FB, FC) \times \mathcal{T}(FA, FB) \xrightarrow{\otimes} \mathcal{T}(FA, FC)$$

(so in particular, for each pair of composable 1-cells $f \colon A \to B$, $g \colon B \to C$ of $\mathcal{S}$, we have a 3-cell of $\mathcal{T}$:

$$\begin{tikzcd} Fg.Ff \arrow[r, Rightarrow, "\chi^F_{f,g}"] \arrow[d, Rightarrow, "\alpha_g.\alpha_f"'] \arrow[dr, phantom, "\Downarrow\Pi^\alpha_{f,g}"] & F(gf) \arrow[d, Rightarrow, "\alpha_{gf}"] \\ Gg.Gf \arrow[r, Rightarrow, "\chi^G_{f,g}"'] & G(gf) \end{tikzcd} \quad).$$

These data are subject to the following axioms:

(IA1) for each 1-cell $f\colon A \to B$ of $\mathcal{S}$, the following pasting equality holds:

$$\begin{tikzcd}
& FI_B.Ff \arrow[r, Rightarrow, "\chi^F"] & F(I_B.f) \arrow[dr, Rightarrow, "F\mathfrak{l}"] & \\
I_{FB}.Ff \arrow[ur, Rightarrow, "\iota^F.1"] \arrow[d, Rightarrow, "1.\alpha_f"'] \arrow[dr, Rightarrow, "\mathfrak{l}"] \arrow[rrr, phantom, "\Downarrow\gamma^F"] & & & Ff \arrow[d, Rightarrow, "\alpha_f"] \\
I_{GB}.Gf \arrow[dr, Rightarrow, "\mathfrak{l}"'] \arrow[r, phantom, "\cong"] & Ff \arrow[r, equal] \arrow[d, Rightarrow, "\alpha_f"] \arrow[dr, phantom, "="] & Ff \arrow[ur, equal] \arrow[d, Rightarrow, "\alpha_f"] \arrow[r, phantom, "="] & Gf \\
& Gf \arrow[r, equal] & Gf \arrow[ur, equal] &
\end{tikzcd}$$

$$= \begin{tikzcd}
& FI_B.Ff \arrow[r, Rightarrow, "\chi^F"] \arrow[d, Rightarrow, "\alpha_{I_B}.\alpha_f"'] \arrow[r, phantom, shift right=6ex, "\Downarrow\Pi^\alpha"] & F(I_B.f) \arrow[dr, Rightarrow, "F\mathfrak{l}"] \arrow[d, Rightarrow, "\alpha_{I_B.f}"'] & \\
I_{FB}.Ff \arrow[ur, Rightarrow, "\iota^F.1"] \arrow[d, Rightarrow, "1.\alpha_f"'] \arrow[r, phantom, "\Downarrow\overline{M^\alpha.1}"] & GI_B.Gf \arrow[r, Rightarrow, "\chi^G"'] & G(I_B.f) \arrow[dr, Rightarrow, "G\mathfrak{l}"'] \arrow[r, phantom, "\Downarrow{}^{\alpha}\mathfrak{l}"] & Ff \arrow[d, Rightarrow, "\alpha_f"] \\
I_{GB}.Gf \arrow[ur, Rightarrow, "\iota^G.1"'] \arrow[dr, Rightarrow, "\mathfrak{l}"'] \arrow[rrr, phantom, "\Downarrow\gamma^G"] & & & Gf; \\
& Gf \arrow[r, equal] & Gf \arrow[ur, equal] &
\end{tikzcd}$$

(IA2) for each 1-cell $f\colon A \to B$ of $\mathcal{S}$, the following pasting equality holds:

$$\begin{tikzcd}
& Ff.FI_A \arrow[r, Rightarrow, "\chi^F"] & F(f.I_A) \arrow[dr, Rightarrow, "F\mathfrak{r}"] & \\
Ff.I_{FA} \arrow[ur, Rightarrow, "1.\iota^F"] \arrow[d, Rightarrow, "\alpha_f.1"'] \arrow[dr, Rightarrow, "\mathfrak{r}"] \arrow[rrr, phantom, "\Downarrow\delta^F"] & & & Ff \arrow[d, Rightarrow, "\alpha_f"] \\
Gf.I_{GA} \arrow[dr, Rightarrow, "\mathfrak{r}"'] \arrow[r, phantom, "\cong"] & Ff \arrow[r, equal] \arrow[d, Rightarrow, "\alpha_f"] \arrow[dr, phantom, "="] & Ff \arrow[ur, equal] \arrow[d, Rightarrow, "\alpha_f"] \arrow[r, phantom, "="] & Gf \\
& Gf \arrow[r, equal] & Gf \arrow[ur, equal] &
\end{tikzcd}$$

$$
= \quad
\begin{tikzcd}
& Ff.FI_A \arrow[r, Rightarrow, "\chi^F"] \arrow[d, Rightarrow, "\alpha_f.\alpha_{I_A}"'] \arrow[r, phantom, shift right=6, "\Downarrow\Pi^\alpha"] & F(f.I_A) \arrow[d, Rightarrow, "\alpha_{f.I_A}"'] \arrow[dr, Rightarrow, "F\mathfrak{r}"] & \\
Ff.I_{FA} \arrow[ur, Rightarrow, "1.\iota^F"] \arrow[d, Rightarrow, "\alpha_f.1"'] \arrow[r, phantom, "\Downarrow\overline{1.M^\alpha}"] & Gf.GI_A \arrow[r, Rightarrow, "\chi^G"'] & G(f.I_A) \arrow[dr, Rightarrow, "G\mathfrak{r}"'] \arrow[r, phantom, "\Downarrow\alpha\mathfrak{r}"] & Ff \arrow[d, Rightarrow, "\alpha_f"] \\
Gf.I_{GA} \arrow[ur, Rightarrow, "1.\iota^G"'] \arrow[dr, Rightarrow, "\mathfrak{r}"'] \arrow[rrr, phantom, "\Downarrow\delta^G"] & & & Gf; \\
& Gf \arrow[r, equal] & Gf \arrow[ur, equal] &
\end{tikzcd}
$$

(IA3) for each triple f, g, h of composable 1-cells of $\mathcal{S}$, the following pasting equality holds:

$$
\begin{tikzcd}
& F(hg).Ff \arrow[r, Rightarrow, "\chi^F"] & F((hg)f) \arrow[dr, Rightarrow, "F\mathfrak{a}"] & \\
(Fh.Fg).Ff \arrow[ur, Rightarrow, "\chi^F.1"] \arrow[d, Rightarrow, "(\alpha_h.\alpha_g).\alpha_f"'] \arrow[dr, Rightarrow, "\mathfrak{a}"] \arrow[rrr, phantom, "\Downarrow\omega^F"] & & & F(h(gf)) \arrow[d, Rightarrow, "\alpha_{h(gf)}"] \\
(Gh.Gg).Gf \arrow[dr, Rightarrow, "\mathfrak{a}"'] \arrow[r, phantom, "\cong"] & Fh.(Fg.Ff) \arrow[r, Rightarrow, "1.\chi^F"] \arrow[d, Rightarrow, "\alpha_h.(\alpha_g.\alpha_f)"] \arrow[r, phantom, shift right=8, "\Downarrow\overline{1.\Pi^\alpha}"] & Fh.F(gf) \arrow[ur, Rightarrow, "\chi^F"] \arrow[d, Rightarrow, "\alpha_h.\alpha_{gf}"] \arrow[r, phantom, "\Downarrow\Pi^\alpha"] & G(h(gf)) \\
& Gh.(Gg.Gf) \arrow[r, Rightarrow, "1.\chi^G"'] & Gh.G(gf) \arrow[ur, Rightarrow, "\chi^G"'] &
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
& F(hg).Ff \arrow[r, Rightarrow, "\chi^F"] \arrow[d, Rightarrow, "\alpha_{hg}.\alpha_f"] \arrow[r, phantom, shift right=6, "\Downarrow\Pi^\alpha"] & F((hg)f) \arrow[d, Rightarrow, "\alpha_{(hg)f}"] \arrow[dr, Rightarrow, "F\mathfrak{a}"] & \\
(Fh.Fg).Ff \arrow[ur, Rightarrow, "\chi^F.1"] \arrow[d, Rightarrow, "(\alpha_h.\alpha_g).\alpha_f"'] \arrow[r, phantom, "\Downarrow\overline{\Pi^\alpha.1}"] & G(hg).Gf \arrow[r, Rightarrow, "\chi^G"'] & G((hg)f) \arrow[dr, Rightarrow, "G\mathfrak{a}"'] \arrow[r, phantom, "\Downarrow\alpha\mathfrak{a}"] & F(h(gf)) \arrow[d, Rightarrow, "\alpha_{h(gf)}"] \\
(Gh.Gg).Gf \arrow[ur, Rightarrow, "\chi^G.1"'] \arrow[dr, Rightarrow, "\mathfrak{a}"'] \arrow[rrr, phantom, "\Downarrow\omega^G"] & & & G(h(gf)). \\
& Gh.(Gg.Gf) \arrow[r, Rightarrow, "1.\chi^G"'] & Gh.G(gf) \arrow[ur, Rightarrow, "\chi^G"'] &
\end{tikzcd}
$$

The definition of oplax icon generalises that of ico-icon, in that:

PROPOSITION 4. *Let $F, G\colon \mathcal{S} \to \mathcal{T}$ be lax homomorphisms. Then the ico-icons $F \Rightarrow G$ are in bijection with the class of oplax icons $\alpha\colon F \Rrightarrow G$ for which each component $\alpha_f\colon Ff \Rightarrow Gf$ is an identity 2-cell: they are the i̲dentity c̲omponents o̲plax icons.*

Unfortunately, oplax icons do not provide a suitable notion of 2-cell for our tricategory **Tricat**$_3$. The reason is that although oplax icons may be "whiskered" with lax homomorphisms on each side, these whiskerings do not give rise to a well-defined composition of oplax icons along a 0-cell boundary. Indeed, if we are given a diagram

$$\begin{tikzcd} \mathcal{S} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{T} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{U} \end{tikzcd}$$

of lax homomorphisms and oplax icons, then there are two canonical ways of composing it up which need not agree, even up to isomorphism. The same phenomenon occurs if one tries to form a tricategory of bicategories whose 2-cells are oplax natural transformations. In order to obtain a tricategory, we therefore restrict attention to a suitable subclass of the oplax icons:

Definition 5. Let $F, G\colon \mathcal{S} \to \mathcal{T}$ be lax homomorphisms. By a **pseudo-icon** $\alpha\colon F \Rrightarrow G$ we mean an oplax icon α for which each 3-cell α_θ, M^α_A, and $\Pi^\alpha_{f,g}$ is invertible.

These pseudo-icons are to be the 2-cells of **Tricat**$_3$. Note that, although every ico-icon gives rise to an oplax icon, it is only the *invertible* ico-icons which give rise to pseudo-icons. We now turn to the 3-cells of **Tricat**$_3$.

Definition 6. Given pseudo-icons $\alpha, \beta\colon F \Rrightarrow G$, a **pseudo-icon modification** $\Gamma\colon \alpha \Rrightarrow \beta$ consists in the following data:

(MD1) For each A, B in $\mathcal{S}$, a modification $\Gamma_{A,B}\colon \alpha_{A,B} \Rrightarrow \beta_{A,B}$ (and so in particular, for each 1-cell $f\colon A \to B$ of $\mathcal{S}$, a 3-cell $\Gamma_f\colon \alpha_f \Rrightarrow \beta_f$ of $\mathcal{T}$);

subject to the following axioms:

(MA1) for each object A of $\mathcal{S}$, the following pasting equality holds:

$$\begin{tikzcd}[column sep=small]
 & FI_A \arrow[r, Rightarrow, "\alpha_{I_A}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\Gamma_{I_A}"] & GI_A \arrow[d, equal] & & & FI_A \arrow[r, Rightarrow, "\alpha_{I_A}"] & GI_A \arrow[d, equal] \\
I_{FA} \arrow[ur, Rightarrow, "\iota^F_A"] \arrow[d, equal] & FI_A \arrow[r, Rightarrow, "\beta_{I_A}"] \arrow[d, phantom, "\Swarrow M^\beta_A"] & GI_A & = & I_{FA} \arrow[ur, Rightarrow, "\iota^F_A"] \arrow[r, equal] \arrow[d, equal] & I_{GA} \arrow[ur, Rightarrow, "\iota^G_A"'] \arrow[u, phantom, "\Swarrow M^\alpha_A"] \arrow[d, equal] & GI_A; \\
I_{FA} \arrow[r, equal] \arrow[ur, Rightarrow, "\iota^F_A"] & I_{GA} \arrow[ur, Rightarrow, "\iota^G_A"'] & & & I_{FA} \arrow[r, equal] & I_{GA} \arrow[ur, Rightarrow, "\iota^G_A"'] &
\end{tikzcd}$$

(MA2) for each pair of composable 1-cells $f\colon A \to B$, $g\colon B \to C$ of $\mathcal{S}$, the following pasting equality holds:

$$
\begin{tikzcd}
& F(gf) \arrow[r, Rightarrow, "\alpha_{gf}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\Gamma_{gf}"] & G(gf) \arrow[d, equal] \\
Fg.Ff \arrow[ur, Rightarrow, "\chi^F_{f,g}"] \arrow[d, equal] & F(gf) \arrow[r, Rightarrow, "\beta_{gf}"] \arrow[d, phantom, "\Swarrow\Pi^{\beta}_{f,g}"] & G(gf) \\
Fg.Ff \arrow[ur, Rightarrow, "\chi^F_{f,g}"'] \arrow[r, Rightarrow, "\beta_g.\beta_f"'] & Gg.Gf \arrow[ur, Rightarrow, "\chi^G_{f,g}"'] &
\end{tikzcd}
\;=\;
\begin{tikzcd}
& F(gf) \arrow[r, Rightarrow, "\alpha_{gf}"] \arrow[d, phantom, "\Swarrow\Pi^{\alpha}_{f,g}"] & G(gf) \arrow[d, equal] \\
Fg.Ff \arrow[ur, Rightarrow, "\chi^F_{f,g}"] \arrow[r, Rightarrow, "\alpha_g.\alpha_f"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\Gamma_g.\Gamma_f"] & Gg.Gf \arrow[ur, Rightarrow, "\chi^G_{f,g}"'] \arrow[d, equal] & G(gf). \\
Fg.Ff \arrow[r, Rightarrow, "\beta_g.\beta_f"'] & Gg.Gf \arrow[ur, Rightarrow, "\chi^G_{f,g}"'] &
\end{tikzcd}
$$

THEOREM 7. *There is a tricategory* **Tricat**$_3$ *with objects being tricategories;* 1*-cells, lax homomorphisms;* 2*-cells, pseudo-icons; and* 3*-cells, pseudo-icon modifications.*

It would certainly be possible to prove this result at this point in the paper: we would simply follow the same path as in Section 2, first defining the various kinds of composition we need, then the various pieces of coherence data, and finally checking the coherence axioms these must satisfy. However, rather than doing this directly, we would like to reuse some of the results we proved about **Tricat**$_2$.

Indeed, we have already shown that that the composition of lax homomorphisms is associative up to an invertible ico-icon. Each invertible ico-icon witnessing this associativity gives rise to a corresponding pseudo-icon in **Tricat**$_3$; and so by taking these pseudo-icons as our witnesses for associativity in **Tricat**$_3$, we might hope to be able to reuse the coherence work we did in Section 2.

However, matters are not quite this simple. If we take the unique sensible definition of vertical composition of pseudo-icons, then we find that the composition of two invertible ico-icons *qua* ico-icon does not agree with their composite *qua* pseudo-icon. In particular, the invertible ico-icons witnessing associativity in **Tricat**$_2$ become mere equivalence pseudo-icons in **Tricat**$_3$, whilst each commutative diagram of coherence 2-cells in **Tricat**$_2$ gives rise to a diagram in **Tricat**$_3$ which may commute only up to an invertible 3-cell.

Intuitively, it is clear that this should not be a problem, and that we should still be able to "read off" the coherence for **Tricat**$_3$ from that for **Tricat**$_2$, but to make this precise we must turn our intuition into a mathematical principle. In order to motivate how we will do this, let us examine more closely why the naive approach does not work.

The problem is essentially that the putative tricategory **Tricat**$_3$ does not include *all* of the data carried by the mere bicategory **Tricat**$_2$. This occurs at the level of basic cell data – since not every ico-icon is a pseudo-icon – but more importantly, at the level of compositional data: the data for the strictly associative composition of ico-icons from **Tricat**$_2$ is no longer present in **Tricat**$_3$.

The solution we give to this problem is to describe a categorical structure into which tricategories, lax homomorphisms, pseudo-icons and modifications may be formed which is richer than **Tricat**$_3$, and in particular retains *all* the data from **Tricat**$_2$. This categorical structure is not a tricategory, but rather what we call a *locally cubical bicategory*. This is a genuinely weak three-dimensional structure whose coherence laws are particularly simple:

they have a bicategorical rather than tricategorical flavour. In particular, the locally cubical bicategory of tricategories that we construct will be able to take its coherence data directly from $\mathbf{Tricat}_2$.

The existence of the desired tricategory of tricategories $\mathbf{Tricat}_3$ now follows from a general result (given in Section 6) which says that any well-behaved locally cubical bicategory gives rise to a tricategory in a canonical way. This result can be seen as a crystallisation of the intuition we had above that we should be able to "read off" the coherence of $\mathbf{Tricat}_3$ from $\mathbf{Tricat}_2$.

4. *Locally cubical bicategories*

The purpose of this section is to define the locally cubical bicategories alluded to at the end of the previous section. Like tricategories, these are weak categorical structures comprised of 0-, 1-, 2- and 3-cells; however, the 2-cells come in two varieties, *horizontal* and *vertical*, whilst the 3-cells are cubical in nature. Composition of vertical 2-cells is strictly associative; that of horizontal 2-cells is only so up to an invertible 3-cell; whilst the associativity constraints for 1-cells are given by *vertical* 2-cells, and are of a bicategorical, up-to-isomorphism, rather than a tricategorical, up-to-equivalence, kind. A locally cubical bicategory may be described succinctly as a "bicategory weakly enriched in pseudo double categories"; and our task in this section will be to expand upon this description.

The concept of *strict* double category is due to Ehresmann. It is an example of the notion of *double model* for an essentially-algebraic theory, this being a model of the theory in its own category of (**Set**-based) models. Thus a double category – which is a double model of the theory of categories – is a category object in **Cat**.

The theory of categories is somewhat special, since its category of (**Set**-based) models may be enriched to a 2-category, so that, as well as *strict* category objects in **Cat**, we may also consider *pseudo* category objects: and these are the pseudo double categories which we will be interested in.

Definition 8. A **pseudo double category** [**10**] $\mathfrak{C}$ is given by specifying a collection of *objects* $x, y, z, \ldots$, a collection of *vertical* 1*-cells* between objects, which we write as $a\colon x \to y$, a collection of *horizontal 1-cells* between objects, which we write as $f\colon x \nrightarrow y$, and a collection of *2-cells*, each of which is bounded by a square of horizontal and vertical arrows, and which we write as:

$$\begin{tikzcd} x \arrow[r, "f", "\shortmid"{marking}] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\alpha"] & w \arrow[d, "b"] \\ y \arrow[r, "g"', "\shortmid"{marking}] & z, \end{tikzcd}$$

or sometimes simply as $\alpha\colon f \Rightarrow g$. Moreover, we must give:

(i) identities and composition for vertical 1-cells, $\mathrm{id}_x\colon x \to x$ and $(a, b) \mapsto ab$, making the objects and vertical arrows into a category $\mathcal{C}_0$;

(ii) vertical identities and composition for 2-cells, $\mathrm{id}_f\colon f \Rightarrow f$ and $(\beta, \alpha) \mapsto \beta\alpha$:

$$\begin{tikzcd} x \arrow[r, "f", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_x"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_f"] & y \arrow[d, "\mathrm{id}_y"] \\ x \arrow[r, "f"', "\shortmid"{marking}] & y \end{tikzcd} \quad ; \quad \begin{tikzcd} u \arrow[r, "f", "\shortmid"{marking}] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\alpha"] & x \arrow[d, "b"] \\ v \arrow[r, "g"', "\shortmid"{marking}] \arrow[d, "c"'] \arrow[dr, phantom, "\Downarrow\beta"] & y \arrow[d, "d"] \\ w \arrow[r, "h"', "\shortmid"{marking}] & z \end{tikzcd} \longmapsto \begin{tikzcd} u \arrow[r, "f", "\shortmid"{marking}] \arrow[d, "ca"'] \arrow[dr, phantom, "\Downarrow\beta\alpha"] & x \arrow[d, "db"] \\ w \arrow[r, "h"', "\shortmid"{marking}] & z \end{tikzcd}$$

making the horizontal arrows and 2-cells into a category $\mathcal{C}_1$ for which "vertical source" and "vertical target" become functors $s, t \colon \mathcal{C}_1 \to \mathcal{C}_0$;

(iii) identities and composition for horizontal 1-cells, $I_x \colon x \nrightarrow x$ and $(g, f) \mapsto gf$;

(iv) horizontal identities and composition for 2-cells, $I_a \colon I_x \Rightarrow I_y$ and $(\beta, \alpha) \mapsto \beta * \alpha$:

$$
\begin{tikzcd}
x \arrow[r, "I_x", "\shortmid"{marking}] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow I_a"] & x \arrow[d, "a"] \\
y \arrow[r, "I_y"', "\shortmid"{marking}] & y
\end{tikzcd}
\quad ; \quad
\begin{tikzcd}
u \arrow[r, "f", "\shortmid"{marking}] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\alpha"] & v \arrow[r, "g", "\shortmid"{marking}] \arrow[d, "b"] \arrow[dr, phantom, "\Downarrow\beta"] & w \arrow[d, "c"] \\
x \arrow[r, "h"', "\shortmid"{marking}] & y \arrow[r, "k"', "\shortmid"{marking}] & z
\end{tikzcd}
\longmapsto
\begin{tikzcd}
u \arrow[r, "gf", "\shortmid"{marking}] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\beta*\alpha"] & w \arrow[d, "c"] \\
x \arrow[r, "kh"', "\shortmid"{marking}] & z,
\end{tikzcd}
$$

satisfying functoriality constraints: firstly, $I_{(-)}$ is a functor $\mathcal{C}_0 \to \mathcal{C}_1$, which says that we have $I_{\mathrm{id}_x} = \mathrm{id}_{I_x}$ and $I_{ab} = I_a.I_b$ and secondly, horizontal composition is a functor $* \colon \mathcal{C}_1 {}_s\!\times_t \mathcal{C}_1 \to \mathcal{C}_1$ which says that $\mathrm{id}_g * \mathrm{id}_f = \mathrm{id}_{gf}$ and $(\delta * \gamma).(\beta * \alpha) = (\delta\beta) * (\gamma\alpha)$.

(v) horizontal unitality and associativity constraints given by 2-cells

$$
\begin{tikzcd}
x \arrow[r, "I_y.f", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_x"'] \arrow[dr, phantom, "\Downarrow\mathfrak{l}_f"] & y \arrow[d, "\mathrm{id}_y"] \\
x \arrow[r, "f"', "\shortmid"{marking}] & y
\end{tikzcd}
,\quad
\begin{tikzcd}
x \arrow[r, "f.I_x", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_x"'] \arrow[dr, phantom, "\Downarrow\mathfrak{r}_f"] & y \arrow[d, "\mathrm{id}_y"] \\
x \arrow[r, "f"', "\shortmid"{marking}] & y
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
x \arrow[r, "h(gf)", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_x"'] \arrow[dr, phantom, "\Downarrow\mathfrak{a}_{f,g,h}"] & z \arrow[d, "\mathrm{id}_z"] \\
x \arrow[r, "(hg)f"', "\shortmid"{marking}] & z,
\end{tikzcd}
$$

natural in f, g and h, and invertible as arrows of $\mathcal{C}_1$. These 2-cells must obey two laws: the pentagon law, which equates the two routes from $k(h(gf))$ to $((kh)g)f$, and the triangle law, which equates the two routes from $g.(I_y.f)$ to gf.

Pseudo double categories are sometimes also known as *weak* double categories; they are a special case of Verity's more general notion of *double bicategory* [**22**]. A more comprehensive reference on pseudo double categories is [**10**]: though be aware that we interchange its usage of the terms "horizontal" and "vertical" to give a better fit with the usual 2-categorical terminology. Since the only sorts of double categories we will be concerned with in this paper are the pseudo ones, we may sometimes choose to write simply "double category", leaving the qualifier "pseudo" understood.

Some simple examples of pseudo double categories are $\mathfrak{Cat}$, the pseudo double category of "categories, functors, profunctors and transformations", $\mathfrak{Rng}$, the pseudo double category of "rings, ring homomorphisms, bimodules and skew-linear maps", and the pseudo double category $\mathfrak{Span}(\mathcal{C})$ of "objects, morphisms, spans and span morphisms" in a category with pullbacks $\mathcal{C}$. These are typical examples of pseudo double categories, in that they have notions of *homomorphism* and *bimodule* as their respective vertical and horizontal 1-cells. Any bicategory $\mathcal{B}$ gives us a pseudo double category $\mathbb{U}(\mathcal{B})$ with only identity vertical 1-cells, whilst any pseudo double category $\mathfrak{C}$ gives a bicategory $\mathbb{H}(\mathfrak{C})$ upon throwing away the non-identity vertical 1-cells, and all the 2-cells except for those whose vertical source and target are identity arrows. We will refer to such 2-cells as **globular 2-cells**; they are also sometimes known as *special* 2-cells.

Just as in the theory of bicategories, the appropriate notion of morphism between pseudo double categories only preserves horizontal composition up to comparison 2-cells, the most important case being the *homomorphisms*, for which these 2-cells are invertible. We can define a homomorphism between small pseudo double categories in terms of a pseudomorphism of pseudocategory objects, but just as easy is to give the elementary description:

Definition 9. A **homomorphism of pseudo double categories** $F\colon \mathfrak{C} \to \mathfrak{D}$ is given by assignations on objects, 1-cells and 2-cells which preserve source and target and are functorial with respect to vertical composition of 1- and 2-cells, together with comparison 2-cells

$$\begin{tikzcd} Fx \arrow[r, "I_{Fx}", "|"'] \arrow[d, "\mathrm{id}_{Fx}"'] \arrow[dr, phantom, "\Downarrow \mathrm{m}_x"] & Fx \arrow[d, "\mathrm{id}_{Fx}"] \\ Fx \arrow[r, "FI_x"', "|"] & Fx \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Fx \arrow[r, "Fg.Ff", "|"'] \arrow[d, "\mathrm{id}_{Fx}"'] \arrow[dr, phantom, "\Downarrow \mathrm{m}_{f,g}"] & Fz \arrow[d, "\mathrm{id}_{Fz}"] \\ Fx \arrow[r, "F(gf)"', "|"] & Fz \end{tikzcd}$$

which are invertible as arrows of $\mathcal{D}_1$, and natural in x, respectively g and f. Moreover, we require the commutativity of three familiar diagrams, which equate, respectively, the two possible ways of going from $Ff.I_{Fx}$ to Ff, from $I_{Fy}.Ff$ to Ff and from $Fh.(Fg.Ff)$ to $F((hg)f)$.

With the obvious notion of composition and identities, we obtain a category **DblCat** of (possibly large) pseudo double categories and homomorphisms between them. If we write **Bicat** for the category of bicategories and homomorphisms, then the assignations $\mathcal{B} \mapsto \mathbb{U}(\mathcal{B})$ and $\mathfrak{C} \mapsto \mathbb{H}(\mathfrak{C})$ described above extend to a pair of adjoint functors $\mathbb{U} \dashv \mathbb{H}\colon$ **DblCat** $\to$ **Bicat**, for which the composite $\mathbb{HU}$ is the identity; we can thus view **Bicat** as a coreflective subcategory of **DblCat**.

Now, **DblCat** is in fact the underlying ordinary category of a 2-category whose 2-cells are the so-called *vertical transformations*. We can understand these 2-cells by observing that there is a 2-monad on the 2-category **CatGph** $:= [\bullet \rightrightarrows \bullet,$ **Cat**$]$ whose strict algebras are small pseudo double categories, and whose algebra pseudomorphisms are the homomorphisms between them. The corresponding *algebra 2-cells* are precisely the vertical transformations. Spelling this out, we have:

Definition 10. A **vertical transformation** $\alpha\colon F \Rightarrow G$ between homomorphisms of pseudo double categories $F, G\colon \mathfrak{C} \to \mathfrak{D}$ is given by specifying, for each object $x \in \mathfrak{C}$, a vertical 1-cell $\alpha_x\colon Fx \to Gx$ of $\mathfrak{D}$ and for each horizontal 1-cell $f\colon x \to\!\!\!\!\!+\; y$ in $\mathfrak{C}$ a 2-cell

$$\begin{tikzcd} Fx \arrow[r, "Ff", "|"'] \arrow[d, "\alpha_x"'] \arrow[dr, phantom, "\Downarrow \alpha_f"] & Fy \arrow[d, "\alpha_y"] \\ Gx \arrow[r, "Gf"', "|"] & Gy \end{tikzcd}$$

of $\mathfrak{D}$, such that the α_x's are natural in morphisms of $\mathcal{D}_0$, the α_f's are natural in morphisms of $\mathcal{D}_1$, and the following diagrams commute:

$$\begin{tikzcd} I_{Fx} \arrow[r, Rightarrow, "\mathrm{m}_x^F"] \arrow[d, Rightarrow, "I_{\alpha_x}"'] & FI_x \arrow[d, Rightarrow, "\alpha_{I_x}"] \\ I_{Gx} \arrow[r, Rightarrow, "\mathrm{m}_x^G"'] & GI_x \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Fg.Ff \arrow[r, Rightarrow, "\mathrm{m}_{g,f}^F"] \arrow[d, Rightarrow, "\alpha_g * \alpha_f"'] & F(gf) \arrow[d, Rightarrow, "\alpha_{gf}"] \\ Gg.Gf \arrow[r, Rightarrow, "\mathrm{m}_{g,f}^G"'] & G(gf). \end{tikzcd}$$

In the case that $\mathfrak{C}$ and $\mathfrak{D}$ are bicategories, the vertical transformation between homomorphisms $\mathfrak{C} \to \mathfrak{D}$ are precisely the bicategorical icons of Section 1; however, the reader should carefully note that the coreflection of **DblCat** into **Bicat** does *not* enrich to a two-dimensional coreflection, since there is no way of coreflecting a general vertical transformation between homomorphisms of pseudo double categories into an icon between the corresponding homomorphisms of bicategories.

It follows from the algebraic description of **DblCat** that it admits a wide class of 2-dimensional limits, of which we will only be concerned with finite products. That **DblCat** admits these, makes it, of course, into a symmetric monoidal category, but the 2-dimensional aspect of these products means that we may view it instead as a *symmetric monoidal 2-category*: that is, a symmetric monoidal category whose tensor product is a 2-functor and whose coherence natural transformations are 2-natural transformations. What we now wish to describe is how we can use this monoidal 2-category **DblCat** as a suitable base for *enrichment*.

For any monoidal category $\mathcal{V}$, we have the well-known notion of a *category enriched in* $\mathcal{V}$ or $\mathcal{V}$*-category*, which instead of having hom-sets between 0-cells, has hom-objects drawn from $\mathcal{V}$, with the corresponding composition being expressed by morphisms of $\mathcal{V}$ subject to associativity and unitality laws. Now, if instead of a monoidal category $\mathcal{V}$ we begin with a *monoidal bicategory* $\mathcal{W}$ in the sense of [**6**], then we may generalise this definition to obtain the notion of *bicategory enriched in* $\mathcal{W}$ or $\mathcal{W}$*-bicategory* [**3**, **15**]. A $\mathcal{W}$-bicategory is like a bicategory, but instead of hom-categories between 0-cells, it has hom-objects drawn from $\mathcal{W}$: and instead of composition functors, it has composition morphisms drawn from $\mathcal{W}$, which are now required to be associative and unital only up to coherent 2-cells of $\mathcal{W}$.[3] Thus we can think of a $\mathcal{W}$-bicategory as being a "category weakly enriched in $\mathcal{W}$".

The simplest sort of enriched bicategory is a **Cat**-bicategory, which is just a (locally small) ordinary bicategory. Other examples are obtained by taking $\mathcal{W} = \mathcal{V}$-**Cat** for some monoidal category $\mathcal{V}$, for which a $\mathcal{W}$-bicategory has sets of 0- and 1-cells as usual, but now a $\mathcal{V}$-object of 2-cells between any parallel pair of 1-cells; by taking $\mathcal{W} =$ **Mod**, the bicategory of categories and profunctors, for which a $\mathcal{W}$-bicategory is a *probicategory* in the sense of Day [**5**]; and by taking $\mathcal{W}$ to be an ordinary monoidal category, viewed as a locally discrete monoidal bicategory, whereupon $\mathcal{W}$-bicategories reduce to categories enriched in $\mathcal{W}$. An account of the general theory of enriched bicategories can be found in [**15**], but we will need sufficiently little of it that we can easily arrange for our account to be self-contained:

Definition 11. A **locally cubical bicategory** is a bicategory enriched in the monoidal 2-category **DblCat**. Explicitly, it is given by the following data:

(LDD1) a collection ob $\mathfrak{B}$ of objects;
(LDD2) for every pair $A, B \in$ ob $\mathfrak{B}$, a pseudo double category $\mathfrak{B}(A, B)$;
(LDD3) for every $A \in$ ob $\mathfrak{B}$, a unit homomorphism

$$\llcorner I_x \lrcorner : 1 \to \mathfrak{B}(A, A);$$

(LDD4) for every triple $A, B, C \in$ ob $\mathfrak{B}$, a composition homomorphism

$$\otimes : \mathfrak{B}(B, C) \times \mathfrak{B}(A, B) \to \mathfrak{B}(A, C);$$

[3] Note that this differs from the notion of "category enriched in a bicategory" studied in [**20**]; these are the *polyads* of [**1**], and are essentially categories enriched in a monoidal category where that monoidal category happens to be spread out over many objects.

(LDD5) for every pair $A, B \in \operatorname{ob} \mathfrak{B}$, invertible vertical transformations

$$\begin{tikzcd}
\mathfrak{B}(A,B) \times \mathfrak{B}(A,A) \arrow[dr, "\otimes"'] & \mathfrak{B}(A,B) \arrow[l, "1\times\llcorner I_A \lrcorner"'] \arrow[r, "\llcorner I_B \lrcorner\times 1"] \arrow[d, "1"] \arrow[dl, phantom, "\overset{r}{\Rightarrow}"] \arrow[dr, phantom, "\overset{l}{\Leftarrow}"] & \mathfrak{B}(B,B) \times \mathfrak{B}(A,B); \arrow[dl, "\otimes"] \\
& \mathfrak{B}(A,B) &
\end{tikzcd}$$

(LDD6) for every quadruple $A, B, C, D \in \operatorname{ob} \mathfrak{B}$, an invertible vertical transformation

$$\begin{tikzcd}
\mathfrak{B}(C,D) \times \mathfrak{B}(B,C) \times \mathfrak{B}(A,B) \arrow[r, "1\times\otimes"] \arrow[d, "\otimes\times 1"'] \arrow[dr, phantom, "\Downarrow a"] & \mathfrak{B}(C,D) \times \mathfrak{B}(A,C) \arrow[d, "\otimes"] \\
\mathfrak{B}(B,D) \times \mathfrak{B}(A,B) \arrow[r, "\otimes"'] & \mathfrak{B}(A,D).
\end{tikzcd}$$

Subject to the following two axioms:

(LDA1) For each triple of objects A, B, C of $\mathfrak{B}$, the following pasting equality holds:

$$\begin{tikzcd}
\mathfrak{B}^2 \arrow[r, "1\times\llcorner I_B \lrcorner\times 1"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow 1\times l"] & \mathfrak{B}^3 \arrow[r, "1\times\otimes"] \arrow[d, "1\times\otimes"] & \mathfrak{B}^2 \arrow[d, "\otimes"] \\
\mathfrak{B}^2 \arrow[r, "1"'] & \mathfrak{B}^2 \arrow[r, "\otimes"'] & \mathfrak{B}
\end{tikzcd}
=
\begin{tikzcd}
\mathfrak{B}^2 \arrow[r, "1\times\llcorner I_B \lrcorner\times 1"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow r\times 1"] & \mathfrak{B}^3 \arrow[r, "1\times\otimes"] \arrow[d, "\otimes\times 1"] \arrow[dr, phantom, "\Downarrow a"] & \mathfrak{B}^2 \arrow[d, "\otimes"] \\
\mathfrak{B}^2 \arrow[r, "1"'] & \mathfrak{B}^2 \arrow[r, "\otimes"'] & \mathfrak{B},
\end{tikzcd}$$

where $\mathfrak{B}^2$ and $\mathfrak{B}^3$ abbreviate the appropriate products of hom-double categories;

(LDA2) For each quintuple of objects A, B, C, D, E of $\mathfrak{B}$, the following pasting equality holds:

$$\begin{tikzcd}
\mathfrak{B}^4 \arrow[r, "1\times 1\times\otimes"] \arrow[d, "\otimes\times 1\times 1"'] \arrow[dr, phantom, "\cong"] & \mathfrak{B}^3 \arrow[d, "\otimes\times 1"] \arrow[dr, "1\times\otimes"] & \\
\mathfrak{B}^3 \arrow[r, "1\times\otimes"'] \arrow[dr, "\otimes\times 1"'] & \mathfrak{B}^2 \arrow[dr, "\otimes"] \arrow[d, phantom, "\Downarrow a"] \arrow[r, phantom, "\Downarrow a"] & \mathfrak{B}^2 \arrow[d, "\otimes"] \\
& \mathfrak{B}^2 \arrow[r, "\otimes"'] & \mathfrak{B}
\end{tikzcd}
=
\begin{tikzcd}
\mathfrak{B}^4 \arrow[r, "1\times 1\times\otimes"] \arrow[d, "\otimes\times 1\times 1"'] \arrow[dr, "1\times\otimes\times 1"] & \mathfrak{B}^3 \arrow[dr, "1\times\otimes"] \arrow[d, phantom, "\Downarrow 1\times a"] & \\
\mathfrak{B}^3 \arrow[dr, "\otimes\times 1"'] \arrow[r, phantom, "\Downarrow a\times 1"] & \mathfrak{B}^3 \arrow[r, "1\times\otimes"'] \arrow[d, "\otimes\times 1"] \arrow[dr, phantom, "\Downarrow a"] & \mathfrak{B}^2 \arrow[d, "\otimes"] \\
& \mathfrak{B}^2 \arrow[r, "\otimes"'] & \mathfrak{B},
\end{tikzcd}$$

where we observe the same convention regarding $\mathfrak{B}^4$, $\mathfrak{B}^3$ and $\mathfrak{B}^2$.

It may be helpful to extract a description of the various sorts of composition that a **DblCat**-bicategory possesses. The 0-cells, 1-cells and vertical 2-cells form an ordinary bicategory. Next come the the horizontal 2-cells, which can be composed with each other along either a 1-cell boundary or a 0-cell boundary, with both compositions being associative up to an invertible globular 3-cell; moreover, the corresponding "middle four interchange" law only holds up to an invertible globular 3-cell. Finally, the 3-cells themselves can be composed with each other along the two different types of 2-cell boundary and along 0-cell boundaries; and these operations are strictly associative modulo the associativity of the boundaries.

A *one-object* locally cubical bicategory amounts to a **monoidal double category** [**8**, **11**, **19**] – that is, a pseudo double category with an up-to-isomorphism tensor product on it. In particular, any double category with *finite products* in the appropriate double categorical sense[4] becomes a monoidal double category under the cartesian tensor product. The double categories $\mathfrak{Cat}$, $\mathfrak{Span}(\mathcal{C})$ (where $\mathcal{C}$ is a category with finite limits) and $\mathfrak{Rng}$ are all monoidal in this way: though in the case of $\mathfrak{Rng}$, there is another natural monoidal structure which is derived from the tensor product on the category of rings.

For a non-degenerate example of a locally cubical bicategory, we turn to **DblCat** itself. As demonstrated in [**8**], we may define an internal hom 2-functor

$$\mathfrak{Hom}(-,?)\colon \mathbf{DblCat}^{\mathrm{op}} \times \mathbf{DblCat} \longrightarrow \mathbf{DblCat}$$

for which $\mathfrak{Hom}(\mathfrak{C},\mathfrak{D})$ is the following double category. Its objects are homomorphisms $\mathfrak{C} \to \mathfrak{D}$, and its vertical 1-cells $\alpha\colon F \Rightarrow G$ are the vertical transformations between them. Its horizontal 1-cells $\alpha\colon F \Rrightarrow G$ are the *horizontal pseudo-natural transformations*, whose components at an object $x \in \mathfrak{C}$ are given by horizontal 1-cells $\alpha_x\colon Fx \nrightarrow Gx$ of $\mathfrak{D}$, together with pseudo-naturality data like that for a pseudo-natural transformation of bicategories; and indeed, in the case that $\mathfrak{C}$ and $\mathfrak{D}$ are bicategories the two notions coincide. Finally, the 2-cells of $\mathfrak{Hom}(\mathfrak{C},\mathfrak{D})$ are the *cubical modifications*, which are bounded by two horizontal and two vertical transformations and whose basic data consists of giving, for each object of the source, a 2-cell of the target bounded by the components of these transformations: Definition 13 below makes this explicit in the special case where $\mathfrak{C}$ and $\mathfrak{D}$ are bicategories.

When we say that $\mathfrak{Hom}(-,?)$ acts as an internal hom, we are affirming a universal property: namely, that for each $\mathfrak{C}$ the 2-functor $(-) \times \mathfrak{C}\colon \mathbf{DblCat} \to \mathbf{DblCat}$ is left biadjoint to $\mathfrak{Hom}(\mathfrak{C},-)$, so that what we have is a *biclosed* monoidal bicategory in the sense of [**6**]. Now, in [**15**], it is demonstrated that, just as any closed monoidal category can be viewed as a category enriched over itself, so any biclosed monoidal bicategory can be viewed as a bicategory enriched over itself, with the hom-objects being given by the biclosed structure. Applying this result to the monoidal 2-category **DblCat**, we obtain a locally cubical bicategory $\mathfrak{DblCat}$, with 0-cells being the pseudo double categories; 1-cells, the homomorphisms; vertical 2-cells, the vertical transformations; horizontal 2-cells, the horizontal pseudo-natural transformations; and 3-cells the cubical modifications.

In particular, if we restrict our attention to those pseudo double categories lying in the image of the embedding $\mathbb{U}\colon \mathbf{Bicat} \to \mathbf{DblCat}$ then we obtain:

COROLLARY 12. *There is a locally cubical bicategory* $\mathfrak{Bicat}$ *which has as* 0*-cells, bicategories; as* 1*-cells, homomorphisms; as vertical* 2*-cells, bicategorical icons; as horizontal* 2*-cells, pseudo-natural transformations; and as* 3*-cells, cubical modifications.*

Whilst the 0-, 1- and 2-cells of $\mathfrak{Bicat}$ are familiar, the same is not true of the 3-cells; and since we will need them in Definition 16 below, it is worth giving an explicit description.

Definition 13. Suppose that $F, G, H, K\colon \mathcal{B} \to \mathcal{C}$ are homomorphisms of bicategories; that $\alpha\colon F \Rrightarrow G$ and $\beta\colon H \Rrightarrow K$ are pseudo-natural transformations; and that $\gamma\colon F \Rightarrow H$

[4] By which we mean a *pseudo-functorial choice of double products* in the sense of [**10**]. Such pseudo double categories are slightly stricter versions of the *cartesian bicategories* of [**2**], which, although they are presented in a globular way, are essentially cubical structures.

and $\delta \colon G \Rightarrow K$ are bicategorical icons. Then a **cubical modification**

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "\alpha", "\shortmid"{marking}] \arrow[d, Rightarrow, "\gamma"'] \arrow[dr, phantom, "\Downarrow\Gamma"] & G \arrow[d, Rightarrow, "\delta"] \\
H \arrow[r, Rightarrow, "\beta"', "\shortmid"{marking}] & K
\end{tikzcd}$$

is given by specifying, for every object $A \in \mathcal{B}$ a 2-cell $\Gamma_A \colon \alpha_A \Rightarrow \beta_A$, in such a way that for every 1-cell $f \colon A \to B$ of $\mathcal{B}$, the following pasting equality holds:

$$\begin{tikzcd}
 & FB \arrow[rr, "\alpha_B"] \arrow[dr, phantom, "\overset{\alpha_f}{\Rightarrow}"] & & GB \arrow[d, equal] \\
FA \arrow[ur, "Ff"] \arrow[rr, "\alpha_A"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\Gamma_A"] & & GA \arrow[ur, "Gf"'] \arrow[d, equal] \arrow[r, phantom, "\Downarrow\delta_f"] & KB \\
HA \arrow[rr, "\beta_A"'] & & KA; \arrow[ur, "Kf"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & FB \arrow[rr, "\alpha_B"] \arrow[d, equal] \arrow[drr, phantom, "\Downarrow\Gamma_B"] & & GB \arrow[d, equal] \\
FA \arrow[ur, "Ff"] \arrow[d, equal] \arrow[r, phantom, "\Downarrow\gamma_f"] & HB \arrow[rr, "\beta_B"'] \arrow[dr, phantom, "\overset{\beta_f}{\Rightarrow}"] & & KB. \\
HA \arrow[ur, "Hf"] \arrow[rr, "\beta_A"'] & & KA \arrow[ur, "Kf"'] &
\end{tikzcd}$$

In particular, to give a *globular* 3-cell of $\mathfrak{Bicat}$ is precisely to give a modification between pseudo-natural transformations in the standard sense; and so $\mathfrak{Bicat}$ is rich enough to encode faithfully all the cells and all of the forms of composition which feature in the tricategory of bicategories, but is able to do so using coherence whose complexity does not rise above the bicategorical level.

Pleasing as this is, we should note that not every tricategory can be reduced to a locally cubical bicategory in this way; for example, given a bicategory $\mathcal{B}$ with bipullbacks, we may form the tricategory **Span**($\mathcal{B}$) of spans in $\mathcal{B}$. In this tricategory, 1-cell composition is given by bipullback, and so is only determined up-to-equivalence, rather than up-to-isomorphism; so evidently, it will be inexpressible as a locally cubical bicategory.

Remark 14. There are two canonical ways of forming a tricategory of bicategories, corresponding to the two canonical ways of composing a pair of strong transformations along a 0-cell boundary: however, Proposition 12 exhibited a single canonical locally cubical bicategory of bicategories. The discrepancy is resolved if we observe that to obtain this **DblCat**-bicategory we must fix a choice of biclosed structure on **DblCat**, and that there are two canonical ways of doing this, depending on how we choose the counit maps $\mathfrak{Hom}(\mathfrak{B}, \mathfrak{C}) \times \mathfrak{B} \to \mathfrak{C}$ for the biadjunctions in question.

5. *A locally cubical bicategory of tricategories*

We now return to our study of tricategories with the goal of forming them into a locally cubical bicategory. The result we will prove in this section is:

THEOREM 15. *There is a locally cubical bicategory $\mathfrak{Tricat}_3$ with* 0*-cells being tricategories;* 1*-cells, lax homomorphisms; vertical* 2*-cells, ico-icons; horizontal* 2*-cells, pseudo-icons; and* 3*-cells, cubical pseudo-icon modifications.*

We have already met the lax homomorphisms (Definition 1), the ico-icons (Definition 2) and the pseudo-icons (Definition 5); however, we have not yet introduced the *cubical* *pseudo-icon modifications*. These generalise the (globular) pseudo-icon modifications of Definition 6 as follows:

Definition 16. Let F, G, F', $G' \colon \mathcal{S} \to \mathcal{T}$ be lax homomorphisms of tricategories, let $\alpha \colon F \nRrightarrow G$ and $\beta \colon F' \nRrightarrow G'$ be pseudo-icons, and let $\gamma \colon F \Rightarrow F'$ and $\delta \colon G \Rightarrow G'$ be ico-icons. Then a **cubical pseudo-icon modification**

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\gamma"'] \arrow[dr, phantom, "\Downarrow\Gamma"] & G \arrow[d, Rightarrow, "\delta"] \\ F' \arrow[r, Rightarrow, "\beta"'] & G' \end{tikzcd}$$

consists in the following data:

(MD1) for each A, B in $\mathcal{S}$, a cubical modification (cf. Definition 13)

$$\begin{tikzcd} F_{A,B} \arrow[r, Rightarrow, "\alpha_{A,B}"] \arrow[d, Rightarrow, "\gamma_{A,B}"'] \arrow[dr, phantom, "\Downarrow\Gamma_{A,B}"] & G_{A,B} \arrow[d, Rightarrow, "\delta_{A,B}"] \\ F'_{A,B} \arrow[r, Rightarrow, "\beta_{A,B}"'] & G'_{A,B} \end{tikzcd}$$

(and so in particular, for each 1-cell $f \colon A \to B$ of $\mathcal{S}$, a 3-cell $\Gamma_f \colon \alpha_f \Rrightarrow \beta_f$ of $\mathcal{T}$);

subject to the following axioms:

(MA1) for each object A of $\mathcal{S}$, the following pasting equality holds:

$$\begin{tikzcd} & FI_A \arrow[r, Rightarrow, "\alpha_{I_A}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\Gamma_{I_A}"] & GI_A \arrow[d, equal] \\ I_{FA} \arrow[ur, Rightarrow, "\iota^F_A"] \arrow[d, equal] \arrow[r, phantom, "\Downarrow M^\gamma_A"] & F'I_A \arrow[r, Rightarrow, "\beta_{I_A}"] \arrow[dr, phantom, "\Swarrow M^\beta_A"] & G'I_A \\ I_{F'A} \arrow[ur, Rightarrow, "\iota^{F'}_A"'] \arrow[r, equal] & I_{G'A} \arrow[ur, Rightarrow, "\iota^{G'}_A"'] & \end{tikzcd} = \begin{tikzcd} & FI_A \arrow[r, Rightarrow, "\alpha_{I_A}"] \arrow[d, phantom, "\Swarrow M^\alpha_A"] & GI_A \arrow[d, equal] \\ I_{FA} \arrow[ur, Rightarrow, "\iota^F_A"] \arrow[d, equal] \arrow[r, equal] & I_{GA} \arrow[ur, Rightarrow, "\iota^G_A"'] \arrow[d, equal] \arrow[r, phantom, "\Downarrow M^\delta_A"] & G'I_A; \\ I_{F'A} \arrow[r, equal] & I_{G'A} \arrow[ur, Rightarrow, "\iota^{G'}_A"'] & \end{tikzcd}$$

(MA2) for each pair of composable 1-cells $f \colon A \to B$, $g \colon B \to C$ of $\mathcal{S}$, the following pasting equality holds:

$$\begin{tikzcd} & F(gf) \arrow[r, Rightarrow, "\alpha_{gf}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\Gamma_{gf}"] & G(gf) \arrow[d, equal] \\ Fg.Ff \arrow[ur, Rightarrow, "\chi^F_{f,g}"] \arrow[d, equal] \arrow[r, phantom, "\Downarrow\Pi^\gamma_{f,g}"] & F'(gf) \arrow[r, Rightarrow, "\beta_{gf}"] \arrow[dr, phantom, "\Swarrow\Pi^\beta_{f,g}"] & G'(gf) \\ F'g.F'f \arrow[ur, Rightarrow, "\chi^{F'}_{f,g}"'] \arrow[r, Rightarrow, "\beta_g.\beta_f"'] & G'g.G'f \arrow[ur, Rightarrow, "\chi^{G'}_{f,g}"'] & \end{tikzcd} = \begin{tikzcd} & F(gf) \arrow[r, Rightarrow, "\alpha_{gf}"] \arrow[d, phantom, "\Swarrow\Pi^\alpha_{f,g}"] & G(gf) \arrow[d, equal] \\ Fg.Ff \arrow[ur, Rightarrow, "\chi^F_{f,g}"] \arrow[d, equal] \arrow[r, Rightarrow, "\alpha_g.\alpha_f"] \arrow[dr, phantom, "\Downarrow\Gamma_g.\Gamma_f"] & Gg.Gf \arrow[ur, Rightarrow, "\chi^G_{f,g}"'] \arrow[d, equal] \arrow[r, phantom, "\Downarrow\Pi^\delta_{f,g}"] & G'(gf). \\ F'g.F'f \arrow[r, Rightarrow, "\beta_g.\beta_f"'] & G'g.G'f \arrow[ur, Rightarrow, "\chi^{G'}_{f,g}"'] & \end{tikzcd}$$

The first step in the proof of Theorem 15 will be to give the local structure:

PROPOSITION 17. *Let $\mathcal{S}$ and $\mathcal{T}$ be tricategories. Then the lax homomorphisms, ico-icons, pseudo-icons and cubical pseudo-icon modifications from $\mathcal{S}$ to $\mathcal{T}$ form a pseudo double category $\mathfrak{Tricat}_3(\mathcal{S},\mathcal{T})$.*

Proof. Underlying each lax homomorphism, ico-icon, pseudo-icon or cubical pseudo-icon modification is an indexed family of homomorphisms of bicategories, bicategorical icons, pseudo-natural transformations, or cubical modifications, respectively: thus our approach will be to lift the compositional structure from the pseudo double categories $\mathfrak{Hom}(\mathfrak{C},\mathfrak{D})$ as defined preceding Corollary 12.

We begin with the vertical structure of $\mathfrak{Tricat}_3(\mathcal{S},\mathcal{T})$. We have already seen in Section 2 that the lax homomorphisms and ico-icons from $\mathcal{S}$ to $\mathcal{T}$ form a category; we must show the same is true of the pseudo-icons and the cubical pseudo-icon modifications. So for each pseudo-icon $\alpha\colon F \Rrightarrow G$ we must give a cubical pseudo-icon modification

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "\alpha" description] \arrow[d, Rightarrow, "\mathrm{id}_F"'] \arrow[dr, phantom, "\Downarrow \mathrm{id}_\alpha"] & G \arrow[d, Rightarrow, "\mathrm{id}_G"] \\
F \arrow[r, Rightarrow, "\alpha"' description] & G;
\end{tikzcd}$$

which we take to be given by the identity family of cubical modifications $(\mathrm{id}_\alpha)_{A,B} = \mathrm{id}_{\alpha_{A,B}}\colon \alpha_{A,B} \Rrightarrow \alpha_{A,B}$. The axioms (MA1) and (MA2) are clear, since every occurence of Γ reduces to an identity 3-cell. Next, given cubical pseudo-icon modifications

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "\alpha" description] \arrow[d, Rightarrow, "\sigma"'] \arrow[dr, phantom, "\Downarrow \Gamma"] & F' \arrow[d, Rightarrow, "\sigma'"] \\
G \arrow[r, Rightarrow, "\beta" description] \arrow[d, Rightarrow, "\tau"'] \arrow[dr, phantom, "\Downarrow \Delta"] & G' \arrow[d, Rightarrow, "\tau'"] \\
H \arrow[r, Rightarrow, "\gamma"' description] & H',
\end{tikzcd}$$

we must provide a vertical composite $\Delta\Gamma\colon \alpha \Rrightarrow \gamma$, which we do by composing their underlying families of cubical modifications:

$$(\Delta\Gamma)_{A,B} = \Delta_{A,B}.\Gamma_{A,B}\colon \alpha_{A,B} \Rrightarrow \gamma_{A,B}.$$

Now the axioms (MA1) and (MA2) follow from an application of the corresponding axiom for Δ followed by the corresponding axiom for Γ. Associativity and unitality of this composition follow from that for composition of cubical modifications.

We next describe the horizontal identities of $\mathfrak{Tricat}_3(\mathcal{S},\mathcal{T})$. Firstly, for each lax homomorphism $F\colon \mathcal{S} \to \mathcal{T}$, we must give an identity pseudo-icon $I_F\colon F \Rrightarrow F$. This has (ID1) given by the family $(I_F)_{A,B} = \mathrm{id}_{F_{A,B}}\colon F_{A,B} \Rightarrow F_{A,B}$ whilst $M_A^{I_F}$ and $\Pi^{I_F}_{A,B,C}$ are given by unnamed coherence isomorphisms in the hom-bicategories of $\mathcal{T}$. Secondly, for each ico-icon $\alpha\colon F \Rightarrow G$, we must give a cubical pseudo-icon modification

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "I_F" description] \arrow[d, Rightarrow, "\alpha"'] \arrow[dr, phantom, "\Downarrow I_\alpha"] & F \arrow[d, Rightarrow, "\alpha"] \\
G \arrow[r, Rightarrow, "I_G"' description] & G;
\end{tikzcd}$$

which we do by taking (MD1) to be given by the identity family of cubical modifications $\mathrm{id}_{\mathrm{id}_{Ff}} : \mathrm{id}_{Ff} \Rrightarrow \mathrm{id}_{Gf}$. Each of the axioms (IA1)–(IA3) for I_F and (MA1)–(MA2) for I_α now asserts that some 3-cell is equal to itself when pasted with such unnamed coherence cells, and this follows from coherence for bicategories. Finally, we must check functoriality of $I_{(-)}$, which is immediate.

We now come to the horizontal composition of $\mathfrak{Tricat}_3(\mathcal{S}, \mathcal{T})$. First, for each pair of pseudo-icons $\alpha\colon F \Rrightarrow G$ and $\beta\colon G \Rrightarrow H$, we must give a composite pseudo-icon $\beta\alpha\colon F \Rrightarrow H$. We do this as follows:

(ID1) $(\beta\alpha)_{A,B} = \beta_{A,B}.\alpha_{A,B}\colon F_{A,B} \Rightarrow H_{A,B}$;

(ID2) $M^{\beta\alpha}_A$ is the pasting:

$$\begin{tikzcd}
I_{FA} \arrow[r, Rightarrow, "\iota^F_A"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow M^\alpha_A"] & FI_A \arrow[d, Rightarrow, "\alpha_{I_A}"] \\
I_{GA} \arrow[r, Rightarrow, "\iota^G_A"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow M^\beta_A"] & GI_A \arrow[d, Rightarrow, "\beta_{I_A}"] \\
I_{HA} \arrow[r, Rightarrow, "\iota^H_A"'] & HI_A;
\end{tikzcd}$$

(ID3) $\Pi^\alpha_{A,B,C}$ is the pasting:

$$\begin{tikzcd}
F(-) \otimes F(?) \arrow[r, Rightarrow, "\chi^F"] \arrow[d, Rightarrow, "\alpha_{(-)}\otimes\alpha_{(?)}"'] \arrow[dd, Rightarrow, bend right=60, "(\beta\alpha)_{(-)}\otimes(\beta\alpha)_{(?)}"'] \arrow[dr, phantom, "\Downarrow\Pi^\alpha_{A,B,C}"] & F\big((-) \otimes (?)\big) \arrow[d, Rightarrow, "\alpha_{((-)\otimes(?))}"] \\
\cong\ G(-) \otimes G(?) \arrow[r, Rightarrow, "\chi^G"] \arrow[d, Rightarrow, "\beta_{(-)}\otimes\beta_{(?)}"'] \arrow[dr, phantom, "\Downarrow\Pi^\beta_{A,B,C}"] & G\big((-) \otimes (?)\big) \arrow[d, Rightarrow, "\beta_{((-)\otimes(?))}"] \\
H(-) \otimes H(?) \arrow[r, Rightarrow, "\chi^H"'] & H\big((-) \otimes (?)\big).
\end{tikzcd}$$

Showing that these data satisfy axioms (IA1)–(IA3) is almost as simple as placing the corresponding diagrams for β and α alongside each other; though not quite, since there are a number of auxiliary coherence results we need to prove first. For instance, in order to prove (IA1) we must show that:

$$\begin{tikzcd}
I_{FB}.Ff \arrow[r, Rightarrow, "\iota^F.1"] \arrow[d, Rightarrow, "1.\alpha_f"'] \arrow[dd, Rightarrow, bend right=60, "1.(\beta\alpha)_f"'] \arrow[dr, phantom, "\Downarrow\overline{M^\alpha.1}"] & FI_B.Ff \arrow[d, Rightarrow, "\alpha_{I_B}.\alpha_f"] \arrow[dd, Rightarrow, bend left=60, "(\beta\alpha)_{I_B}.(\beta\alpha)_f"] \\
\cong\ I_{GB}.Gf \arrow[r, Rightarrow, "\iota^G.1"] \arrow[d, Rightarrow, "1.\beta_f"'] \arrow[dr, phantom, "\Downarrow\overline{M^\beta.1}"] & GI_B.Gf\ \cong \arrow[d, Rightarrow, "\beta_{I_B}.\beta_f"] \\
I_{HB}.Hf \arrow[r, Rightarrow, "\iota^H.1"'] & HI_B.Hf
\end{tikzcd}$$

$$= \quad \begin{tikzcd}
I_{FB}.Ff \arrow[r, Rightarrow, "\iota^F.1"] \arrow[d, Rightarrow, "1.(\beta\alpha)_f"'] \arrow[dr, phantom, "\Downarrow\overline{M^{\beta\alpha}.1}"] & FI_B.Ff \arrow[d, Rightarrow, "(\beta\alpha)_{I_B}.(\beta\alpha)_f"] \\
I_{HB}.Hf \arrow[r, Rightarrow, "\iota^H.1"'] & HI_B.Hf
\end{tikzcd}$$

holds; and similarly for (IA2) and (IA3). These derivations are straightforward bicategorical manipulations and left to the reader.

Secondly, for each diagram of cubical pseudo-icon modifications

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\alpha", "|" marking] \arrow[d, Rightarrow, "\sigma"'] \arrow[dr, phantom, "\Downarrow\Gamma"] & G \arrow[r, Rightarrow, "\beta", "|" marking] \arrow[d, Rightarrow, "\tau"] \arrow[dr, phantom, "\Downarrow\Delta"] & H \arrow[d, Rightarrow, "\upsilon"] \\ F' \arrow[r, Rightarrow, "\alpha'"', "|" marking] & G' \arrow[r, Rightarrow, "\beta'"', "|" marking] & H' \end{tikzcd}$$

we must give a cubical pseudo-icon modification $\Delta * \Gamma \colon \beta\alpha \Rrightarrow \beta'\alpha' \colon F \Rightarrow H$. We do this by taking

$$(\Delta * \Gamma)_{A,B} = \Delta_{A,B} * \Gamma_{A,B} \colon \beta_{A,B}\alpha_{A,B} \Rrightarrow \beta'_{A,B}.\alpha'_{A,B},$$

where $*$ on the right-hand side is horizontal composition of cubical modifications in the pseudo double category $\mathfrak{Hom}(\mathcal{S}(A, B), \mathcal{T}(FA, FB))$. Explicitly, for any 1-cell f of the tricategory $\mathcal{S}$, the 3-cell $(\Delta * \Gamma)_f$ of the tricategory $\mathcal{T}$ is given by the pasting

$$\begin{tikzcd} Ff \arrow[r, Rightarrow, "\alpha_f"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\Gamma_f"] & Gf \arrow[r, Rightarrow, "\beta_f"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\Delta_f"] & Hf \arrow[d, equal] \\ Ff \arrow[r, Rightarrow, "\alpha'_f"'] & Gf \arrow[r, Rightarrow, "\beta'_f"'] & Hf, \end{tikzcd}$$

and thus (MA1) and (MA2) for $\Delta * \Gamma$ follow by placing the corresponding axioms for Γ and Δ beside each other, together with some very simple manipulation with unnamed coherence cells. Finally, we must check functoriality of the horizontal composition functor, which is just the middle-four interchange law. This will hold in $\mathfrak{Tricat}_3(\mathcal{S}, \mathcal{T})$ because it does in each double category $\mathfrak{Hom}(\mathcal{S}(A, B), \mathcal{T}(FA, FB))$.

It remains only to give the unitality and associativity constraints for the pseudo double category $\mathfrak{Tricat}_3(\mathcal{S}, \mathcal{T})$. So let there be given pseudo-icons $\alpha \colon F \Rightarrow G$, $\beta \colon G \Rightarrow H$ and $\gamma \colon H \Rightarrow K$. Then:

(i) the associativity constraint $\mathfrak{a}_{\alpha,\beta,\gamma} \colon (\gamma\beta)\alpha \Rrightarrow \gamma(\beta\alpha)$ has component modification $(\mathfrak{a}_{\alpha,\beta,\gamma})_{A,B}$ given by the associativity constraint $\mathfrak{a}_{\alpha_{A,B},\beta_{A,B},\gamma_{A,B}}$ in the double category $\mathfrak{Hom}(\mathcal{S}(A, B), \mathcal{T}(FA, FB))$;

(ii) the left unitality constraint $\mathfrak{l}_\alpha \colon \mathrm{id}_G.\alpha \Rrightarrow \alpha$ has component modification $(\mathfrak{l}_\alpha)_{A,B}$ given by the left unitality constraint $\mathfrak{l}_{\alpha_{A,B}}$ in $\mathfrak{Hom}(\mathcal{S}(A, B), \mathcal{T}(FA, FB))$;

(iii) the right unitality constraint $\mathfrak{r}_\alpha \colon \alpha.\mathrm{id}_G \Rrightarrow \alpha$ has component modification $(\mathfrak{r}_\alpha)_{A,B}$ given by the right unitality constraint $\mathfrak{r}_{\alpha_{A,B}}$ in $\mathfrak{Hom}(\mathcal{S}(A, B), \mathcal{T}(FA, FB))$.

The naturality of these constraints in α, β and γ is inherited from the hom-double categories $\mathfrak{Hom}(\mathcal{S}(A, B), \mathcal{T}(FA, FB))$; and that these data satisfy the axioms (MA1) and (MA2) is also straightforward. In the case of $\mathfrak{a}_{\alpha,\beta,\gamma}$, for example, we see that $M_A^{\gamma(\beta\alpha)}$ and $\Pi_{f,g}^{\gamma(\beta\alpha)}$ can be obtained from $M_A^{(\gamma\beta)\alpha}$ and $\Pi_{f,g}^{(\gamma\beta)\alpha}$ by pasting with unnamed coherence isomorphisms; but the components of $\mathfrak{a}_{\alpha,\beta,\gamma}$ are built from the selfsame coherence isomorphisms, and so the result follows from the coherence theorem for bicategories. □

In order for the pseudo double categories $\mathfrak{Tricat}_3(\mathcal{S}, \mathcal{T})$ we have just defined to provide homs for the locally cubical bicategory $\mathfrak{Tricat}_3$, we must define double homomorphisms which provide top-level identities and composition. The double homomorphism

$$\llcorner I_{\mathcal{T}} \lrcorner \colon 1 \longrightarrow \mathfrak{Tricat}_3(\mathcal{T}, \mathcal{T})$$

is straightforward; it sends the unique object of the terminal pseudo double category to the identity lax homomorphism $\mathrm{id}_{\mathcal{T}}\colon \mathcal{T} \to \mathcal{T}$, the unique vertical 1-cell to the identity ico-icon on $\mathrm{id}_{\mathcal{T}}$; the unique horizontal 1-cell to the identity pseudo-icon on $\mathrm{id}_{\mathcal{T}}$; and the unique 2-cell to the identity cubical pseudo-icon modification on this. The coherence data for this homomorphism is obtained from unitality constraints in $\mathfrak{Tricat}_3(\mathcal{T}, \mathcal{T})$, and so the homomorphism axioms follow from coherence for bicategories. We must now give the composition double homomorphism

$$\otimes\colon \mathfrak{Tricat}_3(\mathcal{T}, \mathcal{U}) \times \mathfrak{Tricat}_3(\mathcal{S}, \mathcal{T}) \longrightarrow \mathfrak{Tricat}_3(\mathcal{S}, \mathcal{U}).$$

The general approach to defining this will be similar to that adopted in the proof of Proposition 17. There, we defined the compositional structure on $\mathfrak{Tricat}_3(\mathcal{S}, \mathcal{T})$ by lifting it from the pseudo double categories $\mathfrak{Hom}(\mathcal{S}(A, B), \mathcal{T}(FA, FB))$: here, we will define $\otimes$ by lifting the double homomorphisms

$$\begin{aligned}\mathfrak{Hom}(\mathcal{T}(FA, FB), \mathcal{U}(GFA, GFB)) \times \mathfrak{Hom}(\mathcal{S}(A, B), \mathcal{T}(FA, FB))\\ \longrightarrow \mathfrak{Hom}(\mathcal{S}(A, B), \mathcal{U}(GFA, GFB))\end{aligned}$$

which provide composition in the locally cubical bicategory $\mathfrak{Bicat}$ of Proposition 12.

In detail, $\otimes$ is given as follows. On objects and vertical 1-cells, it is given by the composition law for **Tricat**$_2$. On horizontal 1-cells, we consider pseudo-icons $\alpha\colon F \Rrightarrow F'\colon \mathcal{S} \to \mathcal{T}$ and $\beta\colon G \Rrightarrow G'\colon \mathcal{T} \to \mathcal{U}$, for which the composite pseudo-icon $\beta \otimes \alpha\colon GF \Rrightarrow G'F'$ is given as follows:

(ID1) $(\beta \otimes \alpha)_{A,B} = \beta_{FA,FB} \otimes \alpha_{A,B}$, where $\otimes$ on the right-hand side is one of the two canonical choices for horizontal composition of pseudo-natural transformations; for concreteness let us take

$$(\beta \otimes \alpha)_f = GFf \xRightarrow{\beta_{Ff}} G'Ff \xRightarrow{G'\alpha_f} G'F'f$$

and

$$(\beta \otimes \alpha)_\theta = \begin{tikzcd} GFf \arrow[r, Rightarrow, "GF\theta"] \arrow[d, Rightarrow, "\beta_{Ff}"'] \arrow[dr, phantom, "\Downarrow\beta_{F\theta}"] & GFg \arrow[d, Rightarrow, "\beta_{Fg}"] \\ G'Ff \arrow[r, Rightarrow, "G'F\theta"] \arrow[d, Rightarrow, "G'\alpha_f"'] \arrow[dr, phantom, "\Downarrow G'\alpha_\theta"] & G'Fg \arrow[d, Rightarrow, "G'\alpha_g"] \\ G'F'f \arrow[r, Rightarrow, "G'F'\theta"'] & G'F'g \end{tikzcd};$$

(ID2) $M_A^{\beta\otimes\alpha}$ is the following 3-cell:

$$\begin{tikzcd} I_{GFA} \arrow[r, Rightarrow, "\iota^G"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow M^\beta_{FA}"] & GI_{FA} \arrow[r, Rightarrow, "G\iota^F"] \arrow[d, Rightarrow, "\beta_{I_{FA}}"] \arrow[dr, phantom, "\Downarrow\beta_{\iota^F}"] & GFI_A \arrow[d, Rightarrow, "\beta_{FI_A}"] \\ I_{G'FA} \arrow[r, Rightarrow, "\iota^{G'}"] \arrow[d, equal] \arrow[dr, phantom, "="] & G'I_{FA} \arrow[r, Rightarrow, "G'\iota^F"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\overline{G'M^\alpha_A}"] & G'FI_A \arrow[d, Rightarrow, "G'\alpha_{I_A}"] \\ I_{G'F'A} \arrow[r, Rightarrow, "\iota^{G'}"'] & G'I_{F'A} \arrow[r, Rightarrow, "G'\iota^{F'}"'] & G'F'I_A \end{tikzcd};$$

(ID3) $\Pi^{\beta\otimes\alpha}_{A,B,C}$ is the pseudo-natural transformation with the following components:

$$\begin{tikzcd}[column sep=large]
GFg.GFf \arrow[r, Rightarrow, "\chi^G"] \arrow[d, Rightarrow, "\beta_{Fg}.\beta_{Ff}"'] \arrow[dd, Rightarrow, bend right=60, "(\beta\otimes\alpha)_g.(\beta\otimes\alpha)_f"'] \arrow[dr, phantom, "\Downarrow\Pi^{\beta}_{Ff,Fg}"] & G(Fg.Ff) \arrow[r, Rightarrow, "G\chi^F"] \arrow[d, Rightarrow, "\beta_{Fg.Ff}"] \arrow[dr, phantom, "\Downarrow\beta_{\chi^F}"] & GF(gf) \arrow[d, Rightarrow, "\beta_{F(gf)}"] \\
\cong \quad G'Fg.G'Ff \arrow[r, Rightarrow, "\chi^{G'}"] \arrow[d, Rightarrow, "G'\alpha_g.G'\alpha_f"'] \arrow[dr, phantom, "\cong"] & G'(Fg.Ff) \arrow[r, Rightarrow, "G'\chi^F"] \arrow[d, Rightarrow, "G'(\alpha_g.\alpha_f)"] \arrow[dr, phantom, "\Downarrow\overline{G'\Pi^{\alpha}_{f,g}}"] & G'F(gf) \arrow[d, Rightarrow, "G'\alpha_{gf}"] \\
G'F'g.G'F'f \arrow[r, Rightarrow, "\chi^{G'}"'] & G'(F'g.F'f) \arrow[r, Rightarrow, "G'\chi^{F'}"'] & G'F'(gf).
\end{tikzcd}$$

The proof that these data satisfy axioms (IA1)–(IA3) consists once again in building large cubes or hexagonal prisms from smaller ones, together with some simple manipulation with unnamed coherence cells: and once again, we leave this task to the reader.

Finally we must define the action of $\otimes$ on pseudo-icon modifications. Given two such:

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "\alpha", "\shortmid" marking] \arrow[d, Rightarrow, "\sigma"'] \arrow[dr, phantom, "\Downarrow\Gamma"] & F' \arrow[d, Rightarrow, "\sigma'"] \\
H \arrow[r, Rightarrow, "\gamma"', "\shortmid" marking] & H'
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
G \arrow[r, Rightarrow, "\beta", "\shortmid" marking] \arrow[d, Rightarrow, "\tau"'] \arrow[dr, phantom, "\Downarrow\Delta"] & G' \arrow[d, Rightarrow, "\tau'"] \\
K \arrow[r, Rightarrow, "\delta"', "\shortmid" marking] & K'
\end{tikzcd}$$

in the hom-double categories $\mathfrak{Tricat}_3(\mathcal{S},\mathcal{T})$ and $\mathfrak{Tricat}_3(\mathcal{T},\mathcal{U})$ respectively, we define their composite $\Delta\otimes\Gamma\colon \beta\otimes\alpha \Rrightarrow \delta\otimes\gamma$ to be given by horizontally composing their underlying families of cubical modifications in the locally cubical bicategory $\mathfrak{Bicat}$:

$$(\Delta\otimes\Gamma)_{A,B} = \Gamma_{FA,FB}\otimes\Delta_{A,B}\colon \beta_{FA,FB}\otimes\alpha_{A,B} \Rrightarrow \delta_{FA,FB}\otimes\gamma_{A,B}.$$

So in particular, for any 1-cell $f\colon A\to B$ of $\mathcal{S}$, we have $(\Delta\otimes\Gamma)_f$ given by the following pasting

$$\begin{tikzcd}
GFf \arrow[r, Rightarrow, "\beta_{Ff}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\Delta_{Ff}"] & G'Ff \arrow[r, Rightarrow, "G'\alpha_f"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\tau'_{\alpha_f}"] & G'F'f \arrow[d, equal] \\
KFf \arrow[r, Rightarrow, "\delta_{Ff}"] \arrow[d, equal] \arrow[dr, phantom, "="] & K'Ff \arrow[r, Rightarrow, "K'\alpha_f"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow K'\Gamma_f"] & K'F'f \arrow[d, equal] \\
KHf \arrow[r, Rightarrow, "\delta_{Hf}"'] & K'Hf \arrow[r, Rightarrow, "K'\gamma_f"'] & K'H'f.
\end{tikzcd}$$

Proving axioms (MA1) and (MA2) for this data amounts to constructing a further succession of pasting equalities which traverse the interior of a $2\times2\times2$ cube, using:

(i) the corresponding axioms (MA1) or (MA2) for Δ and Γ,
(ii) the cubical modification axioms for the components of Δ,
(iii) the icon axioms for the components of τ',
(iv) the pseudo-natural transformation axioms for the components of δ,
(v) and some further calculus with unnamed coherence isomorphisms.

Functoriality of this composition with respect to vertical composition is inherited from that of horizontal composition of cubical modifications in $\mathfrak{Bicat}$.

It remains to exhibit the pseudo-functoriality constraints for $\otimes$; so let there be given lax homomorphisms and pseudo-icons

$$\begin{tikzcd}[column sep=large]
\mathcal{S} \arrow[r, bend left=50, "F"] \arrow[r, "F'" description] \arrow[r, bend right=50, "F''"'] \arrow[r, phantom, shift left=3.5ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3.5ex, "\Downarrow\gamma"] & \mathcal{T} \arrow[r, bend left=50, "G"] \arrow[r, "G'" description] \arrow[r, bend right=50, "G''"'] \arrow[r, phantom, shift left=3.5ex, "\Downarrow\beta"] \arrow[r, phantom, shift right=3.5ex, "\Downarrow\delta"] & \mathcal{U}.
\end{tikzcd}$$

We must exhibit invertible globular icon modifications

$$i_{(G,F)} \colon \mathrm{id}_{GF} \Rrightarrow \mathrm{id}_G \otimes \mathrm{id}_F \colon GF \Longrightarrow GF \quad \text{and} \quad m_{(\beta,\alpha),(\delta,\gamma)} \colon (\delta \otimes \gamma)(\beta \otimes \alpha) \Rrightarrow (\delta\beta) \otimes (\gamma\alpha);$$

and to do this, we take their respective (A, B)-components to be the invertible modifications witnessing pseudo-functoriality of horizontal composition in the following diagram of homomorphisms and pseudo-natural transformations:

$$\begin{tikzcd}[column sep=huge]
\mathcal{S}(A, B) \arrow[r, bend left=40, "F_{A,B}"] \arrow[r, "F'_{A,B}" description] \arrow[r, bend right=40, "F''_{A,B}"'] \arrow[r, phantom, shift left=3.5ex, "\Downarrow\alpha_{A,B}"] \arrow[r, phantom, shift right=3.5ex, "\Downarrow\gamma_{A,B}"] & \mathcal{T}(FA, FB) \arrow[r, bend left=40, "G_{FA,FB}"] \arrow[r, "G'_{FA,FB}" description] \arrow[r, bend right=40, "G''_{FA,FB}"'] \arrow[r, phantom, shift left=3.5ex, "\Downarrow\beta_{FA,FB}"] \arrow[r, phantom, shift right=3.5ex, "\Downarrow\delta_{FA,FB}"] & \mathcal{U}(GFA, GFB).
\end{tikzcd}$$

We must check that these data satisfy axioms (MA1) and (MA2). The proof is straightforward manipulation using the pseudo-naturality axioms for δ and the modification axioms for Π^δ. Finally, the naturality of the maps $m_{(\beta,\alpha),(\delta,\gamma)}$ in all variables follows componentwise; as do the coherence axioms which m and i must satisfy.

In order to complete the definition of $\mathfrak{Tricat}_3$, all that remains is to give the associativity and unitality constraints for top-level composition, and to check the triangle and pentagon axioms. At the level of 1-cells and vertical 2-cells, these are the corresponding constraints from **Tricat**$_2$; whilst at the level of horizontal 2-cells and 3-cells, we suppose given trihomomorphisms and pseudo-icons

$$\begin{tikzcd}
\mathcal{R} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{S} \arrow[r, bend left=50, "G"] \arrow[r, bend right=50, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{T} \arrow[r, bend left=50, "H"] \arrow[r, bend right=50, "H'"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathcal{U},
\end{tikzcd}$$

and must exhibit an invertible pseudo-icon modification

$$\begin{tikzcd}[column sep=large, row sep=large]
(HG)F \arrow[r, Rightarrow, "(\gamma\otimes\beta)\otimes\alpha"] \arrow[d, Rightarrow, "a_{F,G,H}"'] \arrow[dr, phantom, "\Downarrow a_{\alpha,\beta,\gamma}"] & (H'G')F' \arrow[d, Rightarrow, "a_{F',G',H'}"] \\
H(GF) \arrow[r, Rightarrow, "\gamma\otimes(\beta\otimes\alpha)"'] & H'(G'F')
\end{tikzcd}$$

where $a_{F,G,H}$ and $a_{F',G',H'}$ are the corresponding constraints from **Tricat**$_2$. So we take the (A, B)th component of this pseudo-icon modification to be the cubical modification providing the associativity constraint for the composition

$$\begin{tikzcd}[column sep=large]
\mathcal{R}(A, B) \arrow[r, bend left=20, "F_{A,B}"] \arrow[r, bend right=20, "F'_{A,B}"'] \arrow[r, phantom, "\Downarrow\alpha_{A,B}"] & \mathcal{S}(FA, FB) \arrow[r, bend left=20, "G_{FA,FB}"] \arrow[r, bend right=20, "G'_{FA,FB}"'] \arrow[r, phantom, "\Downarrow\beta_{FA,FB}"] & \mathcal{T}(GFA, GFB) \arrow[r, bend left=20, "H_{GFA,GFB}"] \arrow[r, bend right=20, "H'_{GFA,GFB}"'] \arrow[r, phantom, "\Downarrow\gamma_{GFA,GFB}"] & \mathcal{U}(HGFA, HGFB)
\end{tikzcd}$$

in the locally cubical bicategory $\mathfrak{Bicat}$. We must check that these data satisfy the axioms for an icon modification; let us do only (MA2), since (MA1) is similar. We first observe that the 3-cells $\Pi_A^{(\gamma\otimes\beta)\otimes\alpha}$ and $\Pi_A^{\gamma\otimes(\beta\otimes\alpha)}$ are obtained by pasting together what is essentially the same 3×3 diagram of 3-cells, and some trivial calculus with unnamed coherence cells shows that they are precisely the same diagram, modulo rewriting of the boundary, so that the latter 3-cell may be obtained from the former by pasting with unnamed coherence cells. But this is precisely the content of axiom (MA2).

Finally, we must check that these icon modifications $a_{\alpha,\beta,\gamma}$ are natural in in α, β and γ, and satisfy the pentagon and triangle equalities. Each of these follows componentwise from the corresponding facts in $\mathfrak{Bicat}$. This completes the definition of $\mathfrak{Tricat}_3$.

6. *From locally cubical bicategories to tricategories*

In the previous Section, we constructed a locally cubical bicategory of tricategories which we called $\mathfrak{Tricat}_3$. Recall from Section 3 that one reason for doing this was so that we could deduce the existence of the tricategory $\mathbf{Tricat}_3$. The purpose of this section is to describe the general machinery which will allow us to do this.

The construction takes a well-behaved locally cubical bicategory $\mathfrak{B}$ and builds a tricategory out of it. This tricategory will have the same 0- and 1-cells as $\mathfrak{B}$; as 2-cells, the horizontal 2-cells of $\mathfrak{B}$; and as 3-cells, the globular 3-cells of $\mathfrak{B}$. The main point of interest is the construction of the tricategorical associativity constraints, which are to be given by *horizontal* 2-cells of $\mathfrak{B}$. Since the associativity constraints in $\mathfrak{B}$ are given by *vertical* 2-cells, we will need some kind of linkage between the two types of 2-cell in order to proceed.

Definition 18. A pseudo double category $\mathfrak{C}$ is **fibrant** if the functor $(s, t)\colon \mathcal{C}_1 \to \mathcal{C}_0 \times \mathcal{C}_0$ is an isofibration.

Recall here that a functor $F\colon \mathcal{A} \to \mathcal{B}$ between categories is an *isofibration* if whenever we have a object $a \in \mathcal{A}$ and isomorphism $\phi\colon Fa \to b$ in $\mathcal{B}$, there exists an object $c \in \mathcal{A}$ and isomorphism $\theta\colon a \to c$ such that $Fc = b$ and $F\theta = \phi$. Thus a pseudo double category $\mathfrak{C}$ is fibrant just when every diagram like (a) below with f and g isomorphisms has a filler like (b) for which the 2-cell θ is invertible as an arrow of $\mathcal{C}_1$:

$$(a)\quad \begin{tikzcd} x \arrow[d, "f"'] & x' \arrow[d, "g"] \\ y \arrow[r, "k"', "|"] & y' \end{tikzcd} \qquad \rightsquigarrow \qquad (b)\quad \begin{tikzcd} x \arrow[r, "h", "|"'] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\theta"] & x' \arrow[d, "g"] \\ y \arrow[r, "k"', "|"] & y'. \end{tikzcd}$$

Thus fibrancy is precisely the property which [**10**] refers to as *horizontal invariance*. We may reformulate this property in various useful ways, and since detailed accounts of this process may be found in [**7**] or [**11**], we restrict ourselves here to recording those equivalent formulations which will be useful to us.

For the first, we consider the case of the above filling condition where g and k are both identities: given a vertical map $f\colon x \to y$ of $\mathfrak{C}$, it asserts the existence of a horizontal 1-cell $\overline{f}$ and a 2-cell ϵ_f fitting into the diagram:

$$\begin{tikzcd} x \arrow[r, "\overline{f}", "|"'] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\epsilon_f"] & y \arrow[d, "\mathrm{id}_y"] \\ y \arrow[r, "I_y"', "|"] & y. \end{tikzcd}$$

From this, we may define a further 2-cell η_f as the composite

$$
\begin{tikzcd}
x \arrow[r, "I_x"] \arrow[d, "\mathrm{id}_x"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & x \arrow[d, "f"] \\
x \arrow[r, "\overline{f}"'] & y
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
x \arrow[r, "I_x"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow I_f"] & x \arrow[d, "f"] \\
y \arrow[r, "\overline{f}"] \arrow[d, "f^{-1}"'] \arrow[dr, phantom, "\Downarrow\epsilon_f^{-1}"] & y \arrow[d, "\mathrm{id}_y"] \\
x \arrow[r, "\overline{f}"'] & y.
\end{tikzcd}
$$

Now the pair (η_f, ϵ_f) satisfy the triangle identities:

$$\epsilon_f.\eta_f = I_f\colon I_x \Longrightarrow I_y \quad \text{and} \quad \epsilon_f * \eta_f = (l^{-1}r)_{\overline{f}}\colon \overline{f}.I_x \Longrightarrow I_y.\overline{f}$$

and so, in the terminology of [**11**], f and $\overline{f}$ are *orthogonal companions*; which gives us the "only if" direction of:

PROPOSITION 19. *A pseudo double category $\mathfrak{C}$ is fibrant iff every vertical isomorphism has an orthogonal companion.*

For the "if" direction, suppose that we are given a diagram like (a); then we can complete it to a diagram like (b) by taking h to be $\overline{g^{-1}}.(k.\overline{f})$ and θ to be the 2-cell:

$$\overline{g^{-1}}.(k.\overline{f}) \xRightarrow{(I_g.\epsilon_{g^{-1}})*(\mathrm{id}_k*\epsilon_f)} I_{y'}.(k.I_y) \xRightarrow{I_{y'}*r_k} I_{y'}.k \overset{l_k}{\Rrightarrow} k.$$

Thus each of $\mathfrak{Cat}$, $\mathfrak{Rng}$ and $\mathfrak{Span}(\mathcal{C})$ is a fibrant double category: for $\mathfrak{Cat}$, the horizontal companion of a functor $F\colon \mathcal{C} \to \mathcal{D}$ is the profunctor $\overline{F}(-, ?) = \mathcal{D}(-, F?)$; for $\mathfrak{Rng}$, the companion of a homomorphism $f\colon R \to S$ is S itself, viewed as a left S-, right R-module; and in $\mathfrak{Span}(\mathcal{C})$, the companion of a morphism $f\colon C \to D$ is the span $C \xleftarrow{id} C \xrightarrow{f} D$. Observe that in all of these examples, it is arbitrary vertical morphisms, and not just the isomorphisms, which have companions: such pseudo double categories are essentially the *pro-arrow equipments* of [**23**, **24**]. A more detailed analysis of this correspondence may be found in Appendix C of [**19**].

PROPOSITION 20. *Let $\mathfrak{C}$ be a fibrant double category equipped with a choice of orthogonal companion for every vertical isomorphism. Then the assignation $f \mapsto \overline{f}$ underlies an identity-on-objects homomorphism of bicategories*

$$\overline{(\ )}\colon V^{iso}(\mathfrak{C}) \longrightarrow H(\mathfrak{C}),$$

where $V^{iso}(\mathfrak{C})$ is the category of objects and vertical isomorphisms in $\mathfrak{C}$. Moreover, if we are given vertical isomorphisms $f\colon w \to y$ and $g\colon x \to z$ in $\mathfrak{C}$, then pasting with η_f and ϵ_g induces a bijection between the set of 2-cells of the form (c) and the set of 2-cells of the form (d) :

$$
(c) \quad
\begin{tikzcd}
x \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & x' \arrow[d, "g"] \\
y \arrow[r, "k"'] & y'
\end{tikzcd}
\qquad \text{and} \qquad (d) \quad
\begin{tikzcd}
x \arrow[r, "\overline{g}.h"] \arrow[d, "\mathrm{id}_x"'] \arrow[dr, phantom, "\Downarrow\overline{\alpha}"] & y' \arrow[d, "\mathrm{id}_{y'}"] \\
x \arrow[r, "k.\overline{f}"'] & y';
\end{tikzcd}
$$

and $\overline{\alpha}$ is invertible as an arrow of $\mathcal{C}_1$ just when α is. Furthermore, these bijections satisfy four evident axioms expressing their functoriality with respect to vertical and horizontal composition of 2-cells.

If we remove the restriction to vertical isomorphisms, then the structure described in this Proposition is that of a *pseudo folding structure* in the sense of [**7**]. The proof of the Proposition is straightforward manipulation, and it is not hard to prove a converse – namely, that from a homomorphism of bicategories $\overline{(\ )}\colon V^{\mathrm{iso}}(\mathfrak{C}) \to H(\mathfrak{C})$ and a bijective assignation $\alpha \mapsto \overline{\alpha}$ on 2-cells satisfying the four functoriality axioms, one may define a choice of orthogonal companion for every vertical isomorphism. A proof of this correspondence may be extracted from the pages leading up to [**7**, theorem 3.28].

Definition 21. The 2-category $\mathbf{DblCat}_f$ has objects being fibrant pseudo double categories equipped with a choice of orthogonal companions; as 1-cells, the homomorphisms between the underlying double categories; and as 2-cells, the vertical transformations between them.

One may reasonably ask why we do not require the 1-cells $F\colon \mathfrak{C} \to \mathfrak{D}$ of $\mathbf{DblCat}_f$ to respect the choices of orthogonal companions in $\mathfrak{C}$ and $\mathfrak{D}$. The reason is that, in fact, *any* homomorphism between objects of $\mathbf{DblCat}_f$ will automatically respect these choices in a unique way. To make this explicit, let us say that a homomorphism $F\colon \mathfrak{C} \to \mathfrak{D}$ between objects of $\mathbf{DblCat}_f$ is a **fibrant homomorphism** if, for every invertible vertical 1-cell $f\colon x \to y$ of $\mathfrak{C}$, there is given an invertible globular 2-cell

$$\mu_f\colon F(\overline{f}) \Longrightarrow \overline{Ff}\colon Fx \nrightarrow Fy$$

of $\mathfrak{D}$, subject to three axioms. The first two equate, respectively, the two possible 2-cells in $\mathfrak{D}$ from I_{Fx} to $\overline{F(\mathrm{id}_x)}$; and from $F(\overline{g}) \,.\, F(\overline{f})$ to $\overline{F(gf)}$. The third axiom concerns a 2-cell α of the type (c) above, and equates the two globular 2-cells

$$\begin{tikzcd}
Fx \arrow[r, "Fh", "\shortmid"{marking}] \arrow[d, "F(\overline{f})"', "\shortmid"{marking}] \arrow[d, bend right=80, "\overline{Ff}"', "\shortmid"{marking}] \arrow[d, phantom, bend right=40, "\overset{\Leftarrow}{\mu_f}"] \arrow[dr, phantom, "\Downarrow F(\overline{\alpha})"] & x' \arrow[d, "F(\overline{g})", "\shortmid"{marking}] \\
Fy \arrow[r, "Fk"', "\shortmid"{marking}] & Fy'
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
Fx \arrow[r, "Fh", "\shortmid"{marking}] \arrow[d, "\overline{Ff}"', "\shortmid"{marking}] \arrow[dr, phantom, "\Downarrow \overline{F\alpha}"] & x' \arrow[d, "\overline{Fg}", "\shortmid"{marking}] \arrow[d, bend left=80, "F(\overline{g})", "\shortmid"{marking}] \arrow[d, phantom, bend left=40, "\overset{\Leftarrow}{\mu_f}"] \\
Fy \arrow[r, "Fk"', "\shortmid"{marking}] & Fy'.
\end{tikzcd}$$

Now, given any homomorphism $F\colon \mathfrak{C} \to \mathfrak{D}$ between objects of $\mathbf{DblCat}_f$, we may make it into a fibrant homomorphism as follows. Given an invertible vertical arrow $f\colon x \to y$ of $\mathfrak{C}$, we can consider the globular 2-cell

$$\overline{F\epsilon_f}\colon \overline{\mathrm{id}_{Fy}}.F(\overline{f}) \Longrightarrow FI_y.\overline{Ff}$$

of $\mathfrak{D}$; and since both $\overline{\mathrm{id}_{Fy}}$ and FI_y are isomorphic to I_{Fy}, we obtain from this a globular 2-cell $\mu_f\colon F(\overline{f}) \Rightarrow \overline{Ff}$, which is easily checked to satisfy the three axioms. And in fact, this is the only possible structure of fibrant homomorphism on F: for given an arbitrary such structure, applying the third axiom to the 2-cells ϵ_f in $\mathfrak{C}$ shows that the maps μ_f must coincide with those defined above. A similar argument applies to the 2-cells of $\mathbf{DblCat}_f$.

A conceptual explanation of why this should be the case is that $\mathbf{DblCat}_f$ is, in some sense, the 2-category of algebras for a particularly simple kind of 2-dimensional monad on $\mathbf{DblCat}$, the kind which [**13**] calls *pseudo-idempotent*: and such monads have the property that the forgetful functor from the 2-category of algebras and algebra pseudomorphisms to the underlying base 2-category is 2-fully faithful. The qualifier "in some sense" covers a slight wrinkle in this explanation: namely, that the 2-monad which gives rise to $\mathbf{DblCat}_f$ lives not on $\mathbf{DblCat}$ but on $\mathbf{DblCat}_{\mathrm{str}}$, the 2-category of pseudo double categories and *strict* homomorphisms between them, so that making this argument rigourous would require a little more work.

Definition 22. We will say that a locally cubical bicategory is **locally fibrant** just when each of its hom-double categories is fibrant.

In particular, a monoidal double category is locally fibrant just when its underlying pseudo double category is fibrant, so that all of our examples of monoidal double categories are locally fibrant. The locally cubical bicategory $\mathfrak{DblCat}$ is easily seen *not* to be locally fibrant; on the other hand, we may show that, for pseudo double categories $\mathfrak{C}$ and $\mathfrak{D}$, if $\mathfrak{D}$ is fibrant then so is $\mathfrak{Hom}(\mathfrak{C},\mathfrak{D})$. It follows that the locally cubical bicategory $\mathfrak{DblCat}_f$, of fibrant double categories and all cells between them, is itself locally fibrant; and since any bicategory is trivially fibrant, that the locally cubical bicategory $\mathfrak{Bicat}$ is too.

We will now show that every locally fibrant locally cubical bicategory gives rise to a tricategory. We begin with a technical result:

PROPOSITION 23. *Let* $\mathbf{DblCat}_g$ *be the maximal sub-2-category of* $\mathbf{DblCat}_f$ *with only invertible 2-cells. Then the functor of mere categories* $\mathbf{DblCat}_g \hookrightarrow \mathbf{DblCat} \xrightarrow{\mathbb{H}} \mathbf{Bicat}$ *can be extended to a trihomomorphism*

$$\mathbb{H}\colon \mathbf{DblCat}_g \longrightarrow \mathbf{Bicat}.$$

Proof. First we define $\mathbb{H}$ on cells. This is already done for 0- and 1-cells, and since $\mathbf{DblCat}_g$ has no non-trivial 3-cells, it remains only to define it on 2-cells. So let there be given an invertible vertical transformation $\alpha\colon F \Rightarrow G\colon \mathfrak{C} \to \mathfrak{D}$. We define a pseudo-natural transformation $\mathbb{H}\alpha\colon \mathbb{H}F \Rightarrow \mathbb{H}G$ by taking

$$(\mathbb{H}\alpha)_x = \overline{\alpha_x}\colon Fx \longrightarrow Gx \qquad \text{and} \qquad (\mathbb{H}\alpha)_f = \begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, "\overline{\alpha_x}"'] \arrow[dr, phantom, "\Downarrow\overline{\alpha_f}"] & Fy \arrow[d, "\overline{\alpha_y}"] \\ Gx \arrow[r, "Gf"'] & Gy. \end{tikzcd}$$

The transformation axioms for $\mathbb{H}\alpha$ follow straightforwardly from the vertical transformation axioms for α and the functoriality of $\overline{(\ )}$ with respect to 2-cell composition. Next we ensure that $\mathbb{H}$ is locally a homomorphism of bicategories, which entails giving modifications $i_F\colon \mathrm{id}_{\mathbb{H}F} \Rrightarrow \mathbb{H}(\mathrm{id}_F)$ and $m_{\alpha,\beta}\colon \mathbb{H}\beta.\mathbb{H}\alpha \Rrightarrow \mathbb{H}(\beta.\alpha)$. These will have 2-cell components

$$(i_F)_x\colon \mathrm{id}_{Fx} \Rightarrow \overline{\mathrm{id}_{Fx}} \qquad \text{and} \qquad (m_{\alpha,\beta})_x\colon \overline{\beta_x}\,.\,\overline{\alpha_x} \Rightarrow \overline{\beta_x\alpha_x}$$

in $\mathbb{H}\mathfrak{D}$ given by the pseudo-functoriality constraints for $\overline{(\ )}$. The coherence axioms for these data therefore follow pointwise. Next, we must give adjoint pseudo-natural equivalences

$$\chi_{\mathfrak{C},\mathfrak{D},\mathfrak{E}}\colon \mathbb{H}(\text{–}) \otimes \mathbb{H}(?) \Longrightarrow \mathbb{H}(\text{–}\otimes ?)\colon \mathbf{DblCat}_g(\mathfrak{C},\mathfrak{D}) \times \mathbf{DblCat}_g(\mathfrak{B},\mathfrak{C}) \longrightarrow \mathbf{Bicat}(\mathbb{H}\mathfrak{B},\mathbb{H}\mathfrak{D}).$$

Observe that the homomorphisms $\mathbb{H}(\text{–}) \otimes \mathbb{H}(?)$ and $\mathbb{H}(\text{–} \otimes ?)$ agree on objects, and thus we may consider icons between them: in particular, any *invertible* icon between them will give rise to an adjoint pseudo-natural equivalence, and so to give χ it suffices to give invertible icons $\chi\colon \mathbb{H}(\text{–}) \otimes \mathbb{H}(?) \Rightarrow \mathbb{H}(\text{–} \otimes ?)$. So consider a pair of horizontally composable 2-cells

$$\begin{tikzcd} \mathfrak{B} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathfrak{C} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathfrak{D} \end{tikzcd}$$

in **DblCat**$_g$: we must give a modification $\chi_{\alpha,\beta}\colon \mathbb{H}\beta * \mathbb{H}\alpha \Rrightarrow \mathbb{H}(\beta * \alpha)$. Now, these two pseudo-natural transformations have respective x-components given by

$$(\mathbb{H}\beta * \mathbb{H}\alpha)_x = GFx \xrightarrow{\overline{\beta_{Fx}}} G'Fx \xrightarrow{G'\overline{\alpha_x}} G'F'x$$
$$\text{and } \mathbb{H}(\beta * \alpha)_x = GFx \xrightarrow{\overline{G'\alpha_x.\beta_{Fx}}} G'F'x,$$

and so we take $(\chi_{\alpha,\beta})_x$ to be the 2-cell

$$G'\overline{\alpha_x} \,.\, \overline{\beta_{Fx}} \overset{\mu_{\alpha_x}.1}{\Longrightarrow} \overline{G'\alpha_x} \,.\, \overline{\beta_{Fx}} \overset{\cong}{\Longrightarrow} \overline{G'\alpha_x.\beta_{Fx}}$$

of $\mathbb{H}\mathfrak{D}$. The modification axioms for $\chi_{\alpha,\beta}$ follow from the third fibrant homomorphism axiom and the functoriality axioms for $\overline{(\ )}$ with respect to 2-cell composition. We must verify that these components $\chi_{\alpha,\beta}$ satisfy the three axioms making χ into an icon. The first is vacuous, whilst the second and third follow by a diagram chase using the axioms for a fibrant homomorphism. We argue entirely analogously in order to give the adjoint equivalences $\iota\colon I_{\mathbb{H}x} \Rightarrow \mathbb{H}(I_x)$.

Next we must give invertible modifications ω, δ and γ. In the case of ω, for instance, this involves giving invertible modifications

$$\begin{tikzcd}
& (\mathbb{H}(-) \otimes \mathbb{H}(?)) \otimes \mathbb{H}(*) \arrow[dl, Rightarrow, "\chi\otimes 1"'] \arrow[dr, Rightarrow, "\mathfrak{a}"] & \\
\mathbb{H}(- \otimes ?) \otimes \mathbb{H}(*) \arrow[d, Rightarrow, "\chi"'] \arrow[rr, phantom, "\overset{\omega}{\Rrightarrow}"] & & \mathbb{H}(-) \otimes (\mathbb{H}(?) \otimes \mathbb{H}(*)) \arrow[d, Rightarrow, "1\otimes\chi"] \\
\mathbb{H}((- \otimes ?) \otimes *) \arrow[dr, Rightarrow, "\mathbb{H}\mathfrak{a}"'] & & \mathbb{H}(-) \otimes \mathbb{H}(? \otimes *) \arrow[dl, Rightarrow, "\chi"] \\
& \mathbb{H}(- \otimes (? \otimes *)). &
\end{tikzcd}$$

To do this, observe first that every pseudo-natural transformation bounding this diagram may also be viewed as an icon. We already know this for χ and hence also for $1 \otimes \chi$ and $\chi \otimes 1$; and it is so for $\mathfrak{a}$ and $\mathbb{H}\mathfrak{a}$ since composition of 1-cells in both **DblCat**$_g$ and **Bicat** is *strictly* associative. If we now compose all the 2-arrows in this diagram *qua* icons, we obtain two further icons $\sigma, \tau\colon (\mathbb{H}(-) \otimes \mathbb{H}(?)) \otimes \mathbb{H}(*) \Rightarrow \mathbb{H}(- \otimes (? \otimes *))$: and a long but straightforward diagram chase with the fibrant homomorphism axioms shows that these two icons are, in fact, equal.

On the other hand, if we compose the two sides *qua* pseudo-natural transformations, then the pseudo-naturals that we get will not necessarily be icons, but they will, at least, be *isomorphic* to icons, namely the icons σ and τ respectively. Thus we take ω to be the composite of the invertible modification from the left-hand side of this diagram to $\sigma = \tau$ and the invertible modification from τ to the right-hand side. We proceed similarly for the invertible modifications δ and γ.

The final thing to check are the two trihomomorphism axioms, equating certain pastings of 3-cells in **Bicat**. But all the 3-cells in question are either coherence 3-cells of **Bicat**; or component 3-cells of ω, δ and γ. But these latter 3-cells are in turn built from coherence

3-cells of **Bicat** and coherence 3-cells for the local homomorphisms $\overline{(\ )}$. The result thus follows by coherence for tricategories and bicategorical coherence for functors.

THEOREM 24. *Let $\mathfrak{C}$ be a locally fibrant locally cubical bicategory with chosen companions in each hom. Then there is a tricategory $\mathcal{T}$ with the same objects as $\mathfrak{C}$, and*

$$\mathcal{T}(A, B) = \mathbb{H}\big(\mathfrak{C}(A, B)\big).$$

Proof. We begin by observing that both $\mathbf{DblCat}_g$ and **Bicat** come equipped with finite product structure; and that the trihomomorphism $\mathbb{H}$ preserves the cartesian product of j-cells for $j = 0, 1, 2, 3$. Now, the top-level composition and identity functors for $\mathcal{T}$ are given by applying $\mathbb{H}$ to the corresponding data (LDD3) and (LDD4) for $\mathfrak{C}$:

$$1 = \mathbb{H}1 \xrightarrow{\mathbb{H}\llcorner I_A \lrcorner} \mathbb{H}\big(\mathfrak{C}(A, A)) = \mathcal{T}(A, A)$$

and

$$\mathcal{T}(B, C) \times \mathcal{T}(A, B) = \mathbb{H}\big(\mathfrak{C}(B, C) \times \mathfrak{C}(A, B)\big) \xrightarrow{\mathbb{H}\otimes} \mathbb{H}\big(\mathfrak{C}(A, C)\big) = \mathcal{T}(A, C).$$

To obtain the pseudo-natural adjoint equivalences $\mathfrak{a}$, $\mathfrak{l}$ and $\mathfrak{r}$ witnessing the associativity and unitality of this composition, we apply $\mathbb{H}$ to the corresponding data (LDD5) and (LDD6) for $\mathfrak{C}$. Since each of a, l and r is an adjoint equivalence (in fact, an isomorphism) in the relevant hom of $\mathbf{DblCat}_g$, the same will obtain for their images in **Bicat**; and because $\mathbb{H}$ strictly preserves both cartesian products and composition of 1-cells, these adjoint equivalences will have the correct sources and targets.

Next we must give the invertible modifications π, μ, λ and ρ. To obtain π, for example, we begin by applying $\mathbb{H}$ to the axiom (LDA2) for $\mathfrak{C}$. This yields an equality of 2-cells in **Bicat**; however, these 2-cells are not of the right form to be the source and target of π. In order to make them so, we may adjust by coherence 3-cells in **Bicat** whose existence is guaranteed by the coherence theorem for trihomomorphisms. Consequently, we may take π to be given by the composite of these coherence 3-cells; and similarly for μ, λ and ρ.

Finally, we must check the three tricategory axioms. These are normally stated in a "local" form, asserting the equality of certain pastings of 3-cells in the relevant hom-bicategories; but in this situation, it will be more appropriate to consider them in their "global" form. Each such axiom amounts to giving a diagram of 2- and 3-cells in **Bicat**, whose vertices are pasting diagrams built from copies of the 2-cells a, l and r, and whose arrows are 3-cells between those 2-cells, built from copies of π, μ, λ and ρ; and asserting that the two ways around this diagram coincide.

To show this, we consider the corresponding diagram for $\mathfrak{C}$. This is a diagram of 2- and 3-cells in $\mathbf{DblCat_g}$, which since $\mathbf{DblCat}_g$ has only identity 3-cells, must commute. Hence by applying $\mathbb{H}$ we obtain a commutative diagram in **Bicat**, which, unfortunately, has both the wrong vertices and the wrong arrows. Nonetheless, by the coherence theorem for functors, each "wrong" vertex admits an isomorphism 3-cell to the "right" vertex; and in such a way that composing these isomorphism 3-cells with the "wrong" arrows yields the "right" arrows.

Special cases of this theorem give us new proofs of some existing results. Restricting to the one-object case, we have the result that *any fibrant monoidal double category gives rise to a monoidal bicategory*; this statement and a sketch proof appear as [**19**, theorem B.4]. In particular, we obtain elegant proofs that the bicategories of rings and bimodules, of categories and profunctors, and of spans internal to a cartesian category $\mathcal{C}$ are all monoidal bicategories.[5] Finally, applying this theorem to the fibrant locally cubical bicategory $\mathfrak{Bicat}$, we deduce the existence of a tricategory of bicategories **Bicat**. Again, the result is not new, but the proof is, showing how the tricategory structure on **Bicat** may be induced from a piece of canonical, universally determined structure, namely the biclosed structure on **DblCat**.

7. *A tricategory of tricategories*

We are now finally in a position to prove Theorem 7, which asserts the existence of the tricategory of tricategories $\mathbf{Tricat}_3$. We will do this by applying the machinery of the previous section to the locally cubical bicategory $\mathfrak{Tricat}_3$. In order to do this, we must first prove that $\mathfrak{Tricat}_3$ is locally fibrant.

PROPOSITION 25. *Each pseudo double category $\mathfrak{Tricat}_3(\mathcal{S}, \mathcal{T})$ is fibrant.*

Proof. Suppose we are given an invertible ico-icon $\alpha\colon F \Rightarrow G\colon \mathcal{S} \to \mathcal{T}$. We must provide a pseudo-icon $\overline{\alpha}\colon F \Rrightarrow G$ and an invertible icon modification

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\overline{\alpha}"] \arrow[d, Rightarrow, "\alpha"'] \arrow[dr, phantom, "\Downarrow\epsilon_\alpha"] & G \arrow[d, Rightarrow, "\mathrm{id}_G"] \\ G \arrow[r, Rightarrow, "I_G"'] & G. \end{tikzcd}$$

Now by Proposition 4, there is a bijection between the ico-icons $F \Rightarrow G$ and the oplax icons $F \Rrightarrow G$ with identity 2-cell components: for which the invertible ico-icons on the one side correspond to the pseudo-icons on the other. Thus we take $\overline{\alpha}$ to be the pseudo-icon corresponding to α under this bijection.

To give the icon modification ϵ_α, we must give 3-cells $(\epsilon_\alpha)_f\colon \overline{\alpha}_f \Rrightarrow (I_G)_f$ of $\mathcal{T}$, forming the components of an ob $\mathcal{S} \times$ ob $\mathcal{S}$-indexed family of cubical modifications, and satisfying axioms (MA1) and (MA2). Since we have $\overline{\alpha}_f = (I_G)_f = \mathrm{id}_{Ff}\colon Ff \Rightarrow Ff$, we take $(\epsilon_\alpha)_f = \mathrm{id}_{\mathrm{id}_f}$. The cubical modification axioms and axioms (MA1) and (MA2) now follow by coherence for bicategories.

And so finally we obtain:

COROLLARY 26. *There is a tricategory* $\mathbf{Tricat}_3$ *with objects being tricategories; 1-cells, lax homomorphisms; 2-cells, pseudo-icons; and 3-cells, pseudo-icon modifications.*

COROLLARY 27. *The tricategory* **MonBicat** *of monoidal bicategories, weak monoidal functors, weak monoidal transformations, and monoidal modifications is triequivalent to the full sub-tricategory of* $\mathbf{Tricat}_3$ *consisting of those tricategories with a single object.*

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[5] For the last two of these examples, the machinery of [2] provides another elegant proof of this fact, and in fact goes further, showing that the monoidal bicategories in question are *symmetric* monoidal bicategories.

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