Restriction categories as enriched categories ☆

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A B S T R A C T

Restriction categories were introduced as a simple equational axiomatisation for categories of partial maps such as those which arise in the foundations of computability theory. A restriction structure on a category is given by an operation obeying four simple axioms, that assigns to each morphism of the category an endomorphism of its domain to be thought of as the partial identity representing the degree of definition.

In this paper, we show that restriction categories can be seen as a kind of enriched category; this allows their theory to be studied by way of the enrichment. Unlike most enrichments, ours is based not on a monoidal category, but rather a *weak double category* in the sense of Grandis–Paré, and provides—from a purely mathematical perspective—an example of such an enrichment arising in nature. Beyond exhibiting restriction categories as enriched categories, we show that varying the base of this enrichment also allows the important notions of *join* and *range* restriction category to be understood in the same manner.

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1. Introduction

The notion of *restriction category* provides an axiomatic setting for the study of notions of partiality. Whilst partiality occurs in many places in mathematics, it arises in the foundations of Computer Science quite brutally. The category whose objects are finite powers of the natural numbers, and whose morphisms are the partial recursive functions is a key example of a restriction category. That the basic results of computability theory could be proven axiomatically from the algebraic structure of this restriction category led Di Paola and Heller in [13] to advocate the study of *abstract computability* or, as they termed it, "recursion theory without elements". Different models of computability, such as ones based on (non-standard) partial combinatory algebras, or coming from domain theory, also form restriction categories, and one can measure the degree of divergence between these models by comparing properties of the associated restriction categories; see [4] for an overview.

Concretely, a restriction category is a category equipped with a restriction operator assigning to every morphism $f : A \to B$ an endomorphism $\bar{f} : A \to A$, to be thought of as the partial identity on A representing the domain of definition of f. The behaviour of a restriction operator is completely determined by just four equations, which makes this presentation of partiality particularly amenable to formal reasoning. Beyond their foundational value in computability theory, restriction categories also find applications in complexity theory [8], and to partial or local structures in mathematics, for example in inverse semigroup theory [10], and in algebraic and differential geometry [2,7].

The history of the theory of partiality—and, in particular, of the restriction axioms—is complicated, as many different groups independently discovered the basic axioms coming from very different perspectives. The restriction axioms

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themselves were discovered very early by Karl Menger [22] who was motivated by partiality in calculus [21] and were subsequently developed abstractly in the semigroup community where the axioms were rediscovered several times. This history is described in [18,10]. On the categorical side much of the development can be traced to the development of abstract computability and realizability [25,1,13] which stimulated the explicit development of a categorical theory of partiality [24,23]. Partiality, in this development, was often studied within the context of other structure. An early categorical formulation of "pure" partiality, however, was given by Marco Grandis [15], when, as part of a series of related notions motivated by the manifold construction, he introduced "e-cohesive categories". This notion was then rediscovered, with the axiomatisation given here, and explicitly linked to partiality by the first author and Steve Lack who developed the theory in a series of papers [5,6,9].

A restriction category is a particular kind of structured category, and so many aspects of ordinary category theory have "restriction" correlates: thus there are notions of restriction functors and natural transformation, of restriction limits and colimits, and so on. The generalisation from ordinary categories to restriction categories requires some thought; so, for example, whilst limits and colimits in ordinary categories behave in a completely dual manner, the same is not true of restriction limits and restriction colimits—essentially because the notion of restriction category is not self-dual. It is therefore reasonable to ask how one may justify the validity of these generalisations.

In this article, we answer this question by exhibiting restriction categories as a particular kind of *enriched category*. This means that aspects of the theory of restriction categories can be read off from the corresponding aspects of enriched category theory. Whilst we shall not do this here, we will in subsequent work exploit these observations to define *weighted* restriction limits and colimits, using them to exhibit categories of sheaves, of schemes, of manifolds, and so on, as "free cocompletions in the restriction world".

In fact, the kind of enrichment we consider is a little more subtle than is standard. Enriched category theory typically starts with a well-behaved monoidal category $\mathcal{V}$, and considers enriched categories whose homs are not sets but objects of $\mathcal{V}$; this is the approach of [19]. In the early 1980s, Walters [28,29] found applications in sheaf theory for a more general kind of enrichment wherein the monoidal category $\mathcal{V}$ is replaced by a bicategory $\mathcal{W}$. The enrichment necessary to describe restriction categories is more general still, involving an enrichment base which is a *weak double category* in the sense of [16]. Such enrichments have also featured recently in [14,26].

One advantage of the enriched approach is that we can deal with variants of the notion of restriction category by appropriately altering the base of the enrichment. In this paper, we show how to do this for the notions of *join* restriction category [2,17] and *range* restriction category [3]. In a join restriction category, compatible parallel families of partial maps can be patched together: structure which is familiar and important in domain theory and also in algebraic geometry. On the other hand, in a range restriction category, each morphism gives rise to a partial identity on its *codomain* as well as on its domain, which describes the morphism's *range* of definition; such structure is again familiar as it arises in Kleene's first model of computability.

We now give a brief overview of the contents of this paper. In Section 2 we recall the basic restriction notions, and introduce the (localic, hyperconnected) factorisation system[1] on restriction functors. The hyperconnected restriction functors are, intuitively, those which reflect as well as preserve the restriction structure; the localic morphisms are abstractly characterised as those orthogonal on the left to the hyperconnected ones. However, we are able to provide an explicit description of the localic morphisms, and also of the (localic, hyperconnected) factorisation of a restriction functor.

In Section 3, we recall the *fundamental functor* associated to every restriction category $\mathcal{C}$; this is a canonical restriction functor from $\mathcal{C}$ into a particular restriction category $\mathcal{S}$. We give a universal characterisation of the fundamental functor by showing that it is, to within isomorphism, the unique hyperconnected functor $\mathcal{C} \to \mathcal{S}$.

In Section 4, we recall the construction which assigns a 2-category $\Gamma\mathcal{C}$ to any restriction category $\mathcal{C}$; we see that the hyperconnected restriction functors $\mathcal{C} \to \mathcal{D}$ correspond to the 2-functors $\Gamma\mathcal{C} \to \Gamma\mathcal{D}$ which are local discrete fibrations (discrete fibrations on each hom-category). Combining this with the results of Section 2, we show that the definition of restriction category can be recast in purely 2-categorical terms: a restriction category corresponds to a local discrete fibration of 2-categories whose codomain is $\Gamma\mathcal{S}$.

In Section 5, we break off briefly to describe the appropriate notions of enrichment required for our main result; and then in Section 6, we give that result, exhibiting restriction categories, functors and natural transformations as enriched categories, functors and natural transformations over a suitable enrichment base. We make essential use of a result due to Richard Wood in collaboration with the first author,[2] characterising local discrete fibrations over a given 2-category as categories enriched in an associated bicategory.

Finally, in Sections 7 and 8, we show how the notions of range and join restriction category, as described above, can also be captured as categories enriched over a suitably modified base.

2. Restriction categories and the (localic, hyperconnected) factorisation

We begin by recalling from [5] some basic facts and results. A *restriction category* is a category $\mathcal{C}$ equipped with an operation assigning to each map $f : A \to B$ in $\mathcal{C}$ a map $\bar{f} : A \to A$, subject to the four axioms:

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[1] While this factorisation has clear categorical precedents, we were not able to find reference to it in the inverse or restriction semigroup literature.
[2] Unpublished, though advertised in [30].

(R1) $f\bar{f} = f$ for all $f : A \to B$;

(R2) $\bar{f}\bar{g} = \bar{g}\bar{f}$ for all $f : A \to B$ and $g : A \to C$;

(R3) $\overline{g\bar{f}} = \bar{g}\bar{f}$ for all $f : A \to B$ and $g : A \to C$;

(R4) $\bar{g}f = f\overline{gf}$ for all $f : A \to B$ and $g : B \to C$.

A functor $F : \mathcal{C} \to \mathcal{D}$ between restriction categories is a *restriction functor* if $F\bar{f} = \overline{Ff}$ for all $f : A \to B$ in $\mathcal{C}$. The restriction categories and restriction functors are the objects and 1-cells of a 2-category **rCat**, whose 2-cells are *total* natural transformations; a map $f : A \to B$ in a restriction category is called *total* if $\bar{f} = 1_A$, and a natural transformation is called total if all its components are.

The examples that follow are drawn from [5, §2.1.3].

Examples 2.1.

 (i) The category **Set**$_p$ of sets and partial functions is a restriction category, where to each partial function $f : A \rightharpoonup B$ we assign the partial function $\bar{f} : A \rightharpoonup A$ defined by taking $\bar{f}(a)$ to be a if $f(a)$ is defined, and to be undefined otherwise.

 (ii) The category **Rec** with objects, the natural numbers, and morphisms $n \to m$, partial recursive functions $\mathbb{N}^n \to \mathbb{N}^m$, is a restriction category; the restriction structure is as in (i), bearing in mind that $\bar{f}$ will be partial recursive whenever f is. **Rec** is the canonical example of a *Turing category*: for more details, see [4].

 (iii) The category **Top**$_p$, of topological spaces and continuous functions defined on some open subset of their domain, is a restriction category. The restriction structure is given as in (i).

 (iv) Generalising (i) and (iii), let $\mathcal{D}$ be a category equipped with a class $\mathcal{M}$ of monics which is closed under composition, stable under pullback, and contains the identities. There is a restriction category **Par**$(\mathcal{D}, \mathcal{M})$ with objects those of $\mathcal{D}$, and as morphisms $X \rightarrowtail Y$, isomorphism-classes of spans $f : X \leftarrow Z \to Y : g$ with left leg in $\mathcal{M}$. The restriction of such a span (f, g) is $(f, f) : X \rightarrowtail X$.

 (v) An *inverse monoid* is a monoid M such that, for each $x \in M$, there exists a unique element x^*, the *partial inverse* of x, satisfying $xx^*x = x$ and $x^*xx^* = x^*$; the basic example is the set of injective partial endofunctions of some set A. An inverse monoid can be seen as a one object restriction category where the restriction structure is given by $\bar{x} = xx^*$. More generally, an *inverse category* is a restriction category in which every map x is a *partial isomorphism*, meaning that there is a map x^* such that $\bar{x} = x^*x$ and $\overline{x^*} = xx^*$; such partial inverses can be shown to be unique if they exist. Inverse categories stand in the same relationship to restriction categories as groupoids do to ordinary categories; a one object inverse category is an inverse monoid.

 (vi) Consider the category $\mathcal{S}$ whose objects are meet-semilattices, and whose morphisms from A to B are stable maps $A \leftarrow B$ (note the reversal of direction); here, a *stable map* of meet-semilattices is one that preserves binary meets, but not necessarily the top element. $\mathcal{S}$ is a restriction category under the structure which to a stable map $f : A \leftarrow B$ assigns the morphism $\bar{f} : A \leftarrow A$ given by $\bar{f}(a) = a \wedge f(\top)$. This restriction category in fact has a privileged role in the theory, which will be made clear in Section 3 below.

The following result gives some basic consequences of the definition of restriction category that we will need later; the (essentially trivial) proofs can be found in Lemmas 2.1 and 2.2 of [5].

Lemma 2.2. *Let $\mathcal{C}$ be a restriction category. We have that*

 (i) *each $\bar{f}$ is idempotent, and $\bar{\bar{f}} = \bar{f}$;*

 (ii) $\overline{\bar{g}f} = \overline{gf}$;

 (iii) *total maps are closed under composition*;

 (iv) *if gf is total then f is total.*

Maps of the form $\bar{f}$ in a restriction category are called *restriction idempotents*; by (ii) above, they are equally well the endomorphisms e satisfying $\bar{e} = e$. We write $\mathcal{O}(A)$ for the set of restriction idempotents on an object A. When $\mathcal{C} = \textbf{Set}_p$, $\mathcal{O}(A)$ is isomorphic to the power-set of A; when $\mathcal{C} = \textbf{Top}_p$, $\mathcal{O}(A)$ is isomorphic to the open-set lattice of A; when $\mathcal{C} = \textbf{Par}(\mathcal{D}, \mathcal{M})$, $\mathcal{O}(A)$ is the set of $\mathcal{M}$-subobjects of A; when $\mathcal{C} = \textbf{Rec}$, $\mathcal{O}(n)$ is isomorphic to the set of recursively enumerable subsets of $\mathbb{N}^n$; and when $\mathcal{C} = \mathcal{S}$, $\mathcal{O}(A)$ is isomorphic to A itself.

Note also that, if $\mathcal{C}$ is a restriction category, we may partially order each of its homsets by taking $f \leqslant g$ just when $g\bar{f} = f$; the informal meaning being that f is obtained from g by restricting it to some smaller domain of definition. This partial ordering is preserved by composition, and also by the action on homs of any restriction functor.

We now introduce a class of restriction functors which will play an important role in what follows. Any restriction functor $F : \mathcal{C} \to \mathcal{D}$ sends restriction idempotents to restriction idempotents, and so induces, for each $A \in \mathcal{C}$ a mapping $\mathcal{O}(A) \to \mathcal{O}(FA)$. We define F to be *hyperconnected* if each such mapping is an isomorphism. Our terminology is drawn from topos theory, where a geometric morphism $f : \mathcal{E} \to \mathcal{F}$ is called hyperconnected if its inverse image functor $f^* : \mathcal{F} \to \mathcal{E}$ induces isomorphisms of subobject lattices $\textbf{Sub}_{\mathcal{F}}(A) \to \textbf{Sub}_{\mathcal{E}}(f^*A)$ for every $A \in \mathcal{E}$. We can make the analogy precise: to

each topos $\mathcal{E}$ we can associate the restriction category **Par**$(\mathcal{E}, \mathcal{M})$ as in Example 2.1(iii), with $\mathcal{M}$ the class of all monomorphisms in $\mathcal{E}$; and to each geometric morphism $f : \mathcal{E} \to \mathcal{F}$, we can, via its inverse image part, associate a restriction functor **Par**$(\mathcal{F}, \mathcal{M}) \to$ **Par**$(\mathcal{E}, \mathcal{M})$. This restriction functor will be hyperconnected just when the original f is a hyperconnected geometric morphism.

In the topos-theoretic context, we have a factorisation system (hyperconnected, localic); in the restriction setting, we have a corresponding factorisation (localic, hyperconnected). Note the reversal of the two classes: this is because restriction functors point in the "algebraic" direction whereas topos morphisms point in the opposite, "geometric" direction. In the restriction setting, we define a restriction functor $F : \mathcal{C} \to \mathcal{D}$ to be *localic* just when it is bijective on objects, and for every map $g : FA \to FB$ in $\mathcal{D}$, the poset of maps $f : A \to B$ in $\mathcal{C}$ with $g \leqslant Ff$ is downwards-directed (in particular, nonempty).

Proposition 2.3. *Localic and hyperconnected restriction functors are orthogonal.*

In the following proof, and throughout the rest of the paper, we will use the symbol . to denote composition of maps.

Proof. Given a commutative square in **rCat** as in

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{H} & \mathcal{E} \\
{\scriptstyle F}\downarrow & \nearrow{\scriptstyle J} & \downarrow{\scriptstyle G} \\
\mathcal{D} & \xrightarrow{K} & \mathcal{F}
\end{array}
$$

with F localic and G hyperconnected, we must show that there is a unique filler J as indicated making both triangles commute. We do so at the level of objects by taking $JX = H\tilde{X}$, where $\tilde{X}$ is the unique object of $\mathcal{C}$ with $F\tilde{X} = X$. On morphisms, given $f : X \to Y$ in $\mathcal{D}$, we note first that $KX = KF\tilde{X} = GH\tilde{X}$, so that $\overline{Kf} \in \mathcal{O}(GH\tilde{X})$; now since G is hyperconnected, there is a unique $e \in \mathcal{O}(H\tilde{X})$ with $Ge = \overline{Kf}$. Furthermore, since F is localic, there exists a morphism $h : \tilde{X} \to \tilde{Y}$ in $\mathcal{C}$ with $f \leqslant Fh$, and we now define $Jf = Hh.e$.

Note that $f \leqslant Fh$ implies $Kf \leqslant KFh$, whence $GJf = GHh.Ge = KFh.\overline{Kf} = Kf$, showing that the lower-right triangle commutes. Now $G(\overline{Jf}) = \overline{GJf} = \overline{Kf} = Ge$, whence by hyperconnectedness, $\overline{Jf} = e$. It follows that the definition of Jf is independent of the choice of h; for if h' is another map with $f \leqslant Fh'$, then by directedness, there exists $h'' \leqslant h, h'$ with $f \leqslant Fh''$, and by symmetry, it now suffices to show that $Hh.e = Hh''.e$. But both are $\leqslant Hh$, and both have the same restriction e, and so must coincide. It also follows that the upper triangle commutes; for when $f = Fg$ above, we have $e = \overline{Hg}$, and may take $h = g$, whence $JFg = Hg.\overline{Hg} = Hg$, as required.

We now show that J is functorial. When $f = 1_X$ above, we have $e = 1_{JX}$, and may take $h = 1_{\tilde{X}}$; whence $J(1_X) = 1_{JX}$ as required. When $f = f_1.f_2$, with $Jf_1 = Hh_1.e_1$ and $Jf_2 = Hh_2.e_2$, say, then we have $J(f_1.f_2) = Hh_1.Hh_2.e$, where $Ge = \overline{Kf_1.Kf_2}$. But by Lemma 2.2(ii), $\overline{Kf_1.Kf_2} = \overline{Kf_1}.Kf_2 = \overline{Ge_1}.Kf_2 = \overline{Ge_1}.GJf_2 = G(\overline{e_1.Jf_2})$, whence $e = \overline{e_1.Jf_2}$ by hyperconnectedness. So now we have $J(f_1.f_2) = Hh_1.Hh_2.e = Hh_1.Hh_2.\overline{e_1.Jf_2} = Hh_1.Hh_2.\overline{e_1}.Jf_2 = Hh_1.Hh_2.e_1.Hh_2.e_2 = Hh_1.e_1.Hh_2.e_2 = Jf_1.Jf_2$ as required.

Finally, we must verify that J is the *unique* diagonal filler for this square. Suppose that J' were another such. Clearly $JX = J'X$ on objects; on morphisms, given $f : X \to Y$ in $\mathcal{D}$ as above, we choose h with $f \leqslant Fh$, and now $Jf \leqslant JFh = Hh$ and $J'f \leqslant J'Fh = Hh$. But $G(\overline{J'f}) = \overline{Kf} = Ge$ implies $\overline{J'f} = e = \overline{Jf}$ and so $Jf = Hh.\overline{Jf} = Hh.\overline{J'f} = J'f$, as required. $\quad\square$

Proposition 2.4. rCat *admits* (*localic, hyperconnected*) *factorisations.*

Proof. We need to construct a factorisation

$$
\mathcal{C} \xrightarrow{H} \mathcal{E} \xrightarrow{K} \mathcal{D}
$$

for any restriction functor $F : \mathcal{C} \to \mathcal{D}$. We take the category $\mathcal{E}$ to have the same objects as $\mathcal{C}$, and morphisms $x \to y$ being equivalence classes of pairs (f, g) where $f : x \to y$ in $\mathcal{C}$ and $g : Fx \to Fy$ in $\mathcal{D}$ with $g \leqslant Ff$; the equivalence relation relates (f, g) and (f', g) just when there exists a pair (f'', g) with $f'' \leqslant f, f'$. The restriction of the equivalence class $[f, g]$ is $[\bar{f}, \bar{g}] = [1_x, \bar{g}]$; the functor H is the identity on objects and on morphisms sends f to $[f, Ff]$; whilst K acts as F does on objects, and on morphisms sends $[f, g]$ to g. The remaining details are straightforward. $\quad\square$

In fact, the localic and hyperconnected restriction functors enjoy a stronger orthogonality property than that described above, by virtue of the following result.

Proposition 2.5. *Each localic morphism is a codiscrete cofibration in* **rCat**; *which is to say that, whenever* $\alpha : H \Rightarrow GF$ *is a 2-cell in* **rCat** *with F localic, there exists a unique $J : \mathcal{D} \to \mathcal{E}$ and 2-cell $\beta : J \Rightarrow G$ with $JF = H$ and $\beta F = \alpha$, so that*

Proof. On objects, we define $JX = H\tilde{X}$ where, as before, $\tilde{X}$ is the unique object of $\mathcal{C}$ with $F\tilde{X} = X$. We take β to have components $\beta_X = \alpha_{\tilde{X}} : JX = H\tilde{X} \to GF\tilde{X} = GX$. To define J on morphisms, given $f : X \to Y$ in $\mathcal{D}$, we let $e = \overline{Gf.\beta_X}$, choose some $h : \tilde{X} \to \tilde{Y}$ with $f \leqslant Fh$, and now define $Jf = Hh.e$. Note that $\overline{Gf.\beta_X} \leqslant \overline{GFh.\beta_X} = \overline{\beta_Y.Hh} = \overline{\overline{\beta_Y}.Hh} = \overline{Hh}$ (using again Lemma 2.2(ii)), whence $\overline{Jf} = e$. It follows as in the proof of Proposition 2.3 that the definition of J is independent of the choice of h, that $JF = H$, and that J is a functor. Clearly $\beta F = \alpha$, and it only remains to show that β is in fact natural in f. Given f as above, we have $Jf = Hh.e$ as before. But now $\beta_Y.Jf = \alpha_{\tilde{Y}}.Hh.e = GFh.\alpha_{\tilde{X}}.e = GFh.\beta_X.\overline{Gf.\beta_X} = GFh.\overline{Gf}.\beta_X = Gf.\beta_X$ as required. $\square$

We thus obtain the following "enhanced" orthogonality property of localic and hyperconnected morphisms.

Corollary 2.6. *Given a 2-cell $\alpha : GH \Rightarrow KF$ in* **rCat** *with F localic and G hyperconnected, there is a unique $J : \mathcal{D} \to \mathcal{E}$ and $\beta : GJ \Rightarrow K$ with $JF = H$ and $\beta F = \alpha$, so that*

Proof. Apply Propositions 2.3 and 2.5. $\square$

3. The fundamental functor

In this section, we explain the privileged role that the restriction category $\mathcal{S}$ of Examples 2.1(vi) plays in the theory. A construction given in [5, §4.1] shows that every restriction category $\mathcal{C}$ admits a canonical restriction functor $\mathcal{O} : \mathcal{C} \to \mathcal{S}$, called the *fundamental functor* of $\mathcal{C}$. The following proposition summarises the main points of the construction.

Proposition 3.1. *Let $\mathcal{C}$ be a restriction category.*

(i) *For each $A \in \mathcal{C}$, the set $\mathcal{O}(A)$ of restriction idempotents on A is a meet-semilattice, with top element $1_A : A \to A$ and meet $e \wedge e' = ee'$.*
(ii) *For each map $f : A \to B$ in $\mathcal{C}$, the function $f^* : \mathcal{O}(A) \leftarrow \mathcal{O}(B)$ given by $e \mapsto \overline{ef}$ is a stable map of meet-semilattices.*
(iii) *The assignments $A \mapsto \mathcal{O}(A)$ and $f \mapsto f^*$ are the action on objects and morphisms of a functor $\mathcal{O} : \mathcal{C} \to \mathcal{S}$, the* fundamental functor *of $\mathcal{C}$.*
(iv) *The fundamental functor is a restriction functor.*

Proposition 3.2. *The fundamental functor $\mathcal{O} : \mathcal{C} \to \mathcal{S}$ of a restriction category $\mathcal{C}$ is a terminal object of* **rCat**$(\mathcal{C}, \mathcal{S})$.

Proof. We construct, for each restriction functor $F : \mathcal{C} \to \mathcal{S}$, a total natural transformation $\gamma : F \Rightarrow \mathcal{O}$. Its component $\gamma_A : F(A) \leftarrow \mathcal{O}(A)$ is given by $\gamma_A(e) = Fe(\top)$; this is top-preserving, since $\gamma_A(\top) = (F1_A)(\top) = \top$, and binary-meet-preserving, since $\gamma_A(e \wedge e') = \gamma_A(ee') = Fe'(Fe(\top)) = Fe'(\top) \wedge Fe(\top)$, the last equality holding because Fe' is a restriction idempotent. To show naturality, let $f : B \to A$ in $\mathcal{C}$; then $Ff(\gamma_A(e)) = Ff(Fe(\top)) = \overline{Ff}.Fe(\top) = F(\overline{ef})(\top) = \gamma_B(f^*e)$ as required. To show the uniqueness of γ, observe that for any $\delta : F \Rightarrow \mathcal{O}$ and any restriction idempotent $e : A \to A$ in $\mathcal{C}$, we have a naturality square as on the left in:

Evaluating at $\top \in \mathcal{O}(A)$, and observing that $\delta_A(\top) = \top$, as γ is total, we obtain the square on the right, which shows that necessarily $\delta_A(e) = Fe(\top)$. $\square$

By the preceding proposition, for any restriction functor $F : \mathcal{C} \to \mathcal{D}$, there is a unique total natural transformation φ fitting into a triangle

$$(3.1)$$

The components of φ are the mappings $\mathcal{O}(A) \to \mathcal{O}(FA)$ sending e to Fe, so that φ is invertible just when F is a hyperconnected restriction functor.

Proposition 3.3. *Let $\mathcal{C}$ be a restriction category, and $F : \mathcal{C} \to \mathcal{S}$ a restriction functor. Then the following are equivalent*:

(i) *F is hyperconnected*;
(ii) *F is a terminal object in the category $\mathbf{rCat}(\mathcal{C}, \mathcal{S})$*;
(iii) *F has a (necessarily unique) isomorphism to the fundamental functor of $\mathcal{C}$.*

Proof. By the preceding result, the fundamental functor $\mathcal{O}_{\mathcal{C}}$ of $\mathcal{C}$ is a terminal object of $\mathbf{rCat}(\mathcal{C}, \mathcal{S})$ and so there is a unique total transformation $\gamma : F \Rightarrow \mathcal{O}_{\mathcal{C}}$; asking this to be invertible is equivalent both to (ii) and to (iii). To show that it is also equivalent to (i), we decompose γ as a composite 2-cell

where φ is as above, and δ is obtained using terminality of the fundamental functor of $\mathcal{S}$. For any $X \in \mathcal{S}$, the component $\delta_X : X \leftarrow \mathcal{O}_{\mathcal{S}}(X)$ is the map sending φ to $\varphi(\top)$, and this is invertible, with inverse $\mathcal{O}_{\mathcal{S}}(X) \leftarrow X$ sending e to $e \wedge (-)$. Thus δ is invertible, from which it follows that γ is invertible just when φ is—which is to say, just when F is hyperconnected. $\square$

The following result gives our first reformulation of the notion of restriction category; in and of itself it is of scant interest, but it prepares the ground for our second reformulation in the following section.

Proposition 3.4. *The 2-category $\mathbf{rCat}$ of restriction categories is 2-equivalent to the 2-category $\mathbf{rCat}'$ whose objects are hyperconnected restriction functors $F : \mathcal{C} \to \mathcal{S}$, whose morphisms are diagrams*

in $\mathbf{rCat}$, and whose 2-cells $(H, \gamma) \to (H', \gamma')$ are total natural transformations $\theta : H \to H'$ with $\gamma'.G\theta = \gamma$.

Proof. There is an obvious forgetful 2-functor $U : \mathbf{rCat}' \to \mathbf{rCat}$. It follows easily from the fact that a hyperconnected morphism into $\mathcal{S}$ is terminal in its hom-category that U is 2-fully faithful; it is moreover surjective on objects, since every restriction category admits a hyperconnected morphism to $\mathcal{S}$, namely, its fundamental functor. $\square$

Remark 3.5. Note that $\mathbf{rCat}'$ is the full sub-2-category of the lax slice 2-category $\mathbf{rCat} /\!\!/ \mathcal{S}$ whose 0-cells are the hyperconnected functors. Thus we may write $\mathbf{rCat}' = \mathbf{rCat} /\!\!/_h \mathcal{S}$. This occasions various remarks:

(1) $\mathbf{rCat}'$ is, in fact, a reflective sub-2-category of the lax slice 2-category. Given a restriction functor $\mathcal{C} \to \mathcal{S}$, its reflection into $\mathbf{rCat}'$ is the hyperconnected part of its (localic, hyperconnected) factorisation, whilst given a lax triangle over $\mathcal{S}$, its reflection into $\mathbf{rCat}'$ is obtained using the enhanced orthogonality property of Corollary 2.6.
(2) It also follows from Corollary 2.6 that the lax slice $\mathbf{rCat} /\!\!/ \mathcal{S}$ is 2-equivalent to the full sub-2-category of $\mathbf{rCat}^{\to}$ (the *strict* arrow category) whose objects are the localic morphisms.
(3) For any restriction category $\mathbf{Y}$ we can form $\mathbf{rCat} /\!\!/_h \mathbf{Y}$ as a reflective sub-2-category of the lax slice 2-category. The objects of this 2-category are then restriction categories with a "fundamental functor" to $\mathbf{Y}$: this means its lattices of restriction idempotents "live" in the restriction category $\mathbf{Y}$. Thus, for example, if $\mathbf{Y} = \mathbf{Top}_p$ then each idempotent semilattice $\mathcal{O}(X)$ would be identified with the locale of open sets of a topological space and, furthermore, each map would have to behave like a partial continuous map on these open sets.

4. Restriction categories as local discrete fibrations

As we have already seen, the homs of any restriction category can be partially ordered in a manner which is preserved by composition. This allows us to view each restriction category as a locally partially ordered 2-category; as each restriction functor preserves these partial orders, we therefore obtain a forgetful 2-functor $\Gamma : \textbf{rCat} \to \textbf{2-Cat}$ from the 2-category of restriction categories to the 2-category of 2-categories. In this section, we study the relationship between this forgetful functor and the notions introduced in the previous section.

We first describe a class of maps in $\textbf{2-Cat}$ which correspond to the hyperconnected morphisms in $\textbf{rCat}$. Recall that a functor $p : \mathcal{E} \to \mathcal{B}$ is called a *discrete fibration* if, for every $e \in \mathcal{E}$ and map $\gamma : b \to pe$ in $\mathcal{B}$, there exists a unique map γ' in $\mathcal{E}$ with $p(\gamma') = \gamma$. A 2-functor $F : \mathcal{K} \to \mathcal{L}$ between 2-categories is called a *local discrete fibration* if each functor $\mathcal{K}(X, Y) \to \mathcal{L}(FX, FY)$ is a discrete fibration.[3]

Proposition 4.1. *A restriction functor $F : \mathcal{C} \to \mathcal{D}$ is hyperconnected if and only if $\Gamma F : \Gamma \mathcal{C} \to \Gamma \mathcal{D}$ is a local discrete fibration.*

Proof. Suppose first that F is hyperconnected. We must show for each $A, B \in \mathcal{C}$ that the functor $(\Gamma F)_{A,B} : (\Gamma \mathcal{C})(A, B) \to (\Gamma \mathcal{D})(FA, FB)$ is a discrete fibration. Thus given $g \in \mathcal{C}(A, B)$ and $k \leqslant Fg$ in $\mathcal{D}(FA, FB)$, we must show that there is a unique $f \leqslant g$ in $\mathcal{C}(A, B)$ with $Ff = k$. Because F is hyperconnected, there exists a unique $e \in \mathcal{O}(A)$ with $Fe = \bar{k}$. Now take $f = ge$; clearly we have $f \leqslant g$, and moreover $Ff = Fg.Fe = Fg.\bar{k} = k$. Finally, if $f' \leqslant g$ with $Ff' = k$, then from $F(\overline{f'}) = \overline{Ff'} = \bar{k} = Fe$ we deduce that $\overline{f'} = e$, so that $f' = g\overline{f'} = ge = f$, as required.

Suppose conversely that ΓF is a local discrete fibration. We must show that for each $A \in \mathcal{C}$, the induced mapping $\mathcal{O}(A) \to \mathcal{O}(FA)$ is an isomorphism. So let $e : FA \to FA$ be a restriction idempotent. We have $e \leqslant F(1_A)$ and so a unique $e' : A \to A$ with $e' \leqslant 1_A$ and $Fe' = e$. We have $e' = 1_A.\overline{e'} = \overline{e'}$, so that e' is a restriction idempotent over e; moreover, if e'' is another restriction idempotent over e then $e'' \leqslant 1_A$ and $Fe'' = e$ imply that $e'' = e$ by uniqueness of liftings. $\square$

Remark 4.2. The discrete fibrations are the right class of the *comprehensive* factorisation system [27] on $\textbf{Cat}$, whose corresponding left class comprises the *final* functors. As both discrete fibrations and final functors are stable under finite products, the comprehensive factorisation induces on $\textbf{2-Cat}$ a factorisation system (bijective on objects and locally final, local discrete fibration). In light of the preceding result, applying this factorisation to maps in the image of Γ yields the (localic, hyperconnected) factorisations described in Proposition 2.4 above.

The importance of local discrete fibrations is that they allow us to lift restriction structure from the codomain to the domain:

Proposition 4.3. *Let $\mathcal{D}$ be a restriction category, $\mathcal{C}$ a 2-category, and $F : \mathcal{C} \to \Gamma \mathcal{D}$ a local discrete fibration. There is a unique restriction category $\hat{\mathcal{C}}$ and a unique restriction functor $\hat{F} : \hat{\mathcal{C}} \to \mathcal{D}$ such that $\Gamma \hat{\mathcal{C}} = \mathcal{C}$ and $\Gamma \hat{F} = F$.*

Proof. First observe that, because $\Gamma \mathcal{D}$ is locally partially ordered and F is a local discrete fibration, $\mathcal{C}$ is also locally partially ordered. We take the underlying category of $\hat{\mathcal{C}}$ to be the underlying 1-category of $\mathcal{C}$, and equip it with the following restriction structure. For each map $f : A \to B$ in $\mathcal{C}$, we have $\overline{Ff} \leqslant F(1_A) \in \mathcal{D}(FA, FA)$ and so, because F is a local discrete fibration, have a unique $\bar{f} \leqslant 1_X$ in $\mathcal{C}(A, A)$ with $F(\bar{f}) = \overline{Ff}$. We now check the axioms (R1)–(R4):

(R1) $\bar{f} \leqslant 1_A$ implies $f\bar{f} \leqslant f$; and since $F(f\bar{f}) = Ff.F\bar{f} = Ff.\overline{Ff} = Ff$, we conclude by the uniqueness of liftings that $f\bar{f} = f$.

(R2) $\bar{f} \leqslant 1_A$ and $\bar{g} \leqslant 1_A$ implies $\bar{f}\bar{g} \leqslant 1_A$ and $\bar{g}\bar{f} \leqslant 1_A$. But $F(\bar{g}\bar{f}) = \overline{Fg}\,\overline{Ff} = \overline{Fg}\,\overline{Ff} = F(\bar{f}\bar{g})$ and so by uniqueness of liftings, $\bar{g}\bar{f} = \bar{f}\bar{g}$.

(R3) Now we have $\overline{g\bar{f}} \leqslant 1_A$ and $\bar{g}\bar{f} \leqslant 1_A$, but $F(\overline{g\bar{f}}) = \overline{Fg.\overline{Ff}} = \overline{Fg}.\overline{Ff} = F(\bar{g}\bar{f})$, whence by uniqueness of liftings, $\overline{g\bar{f}} = \bar{g}\bar{f}$.

(R4) Finally, we have $\bar{g}f \leqslant f$ and $f\overline{gf} \leqslant f$, but $F(\bar{g}f) = \overline{Ff}.Fg = Ff.\overline{Fg.Ff} = F(f\overline{gf})$, whence again by uniqueness of liftings, $\bar{g}f = f\overline{gf}$.

Thus $\hat{\mathcal{C}}$ is a restriction category as required. To show that $\Gamma \hat{\mathcal{C}} = \mathcal{C}$, we must check that $f \leqslant g$ in $\mathcal{C}(A, B)$ just when $g\bar{f} = f$. Now because $\bar{f} \leqslant 1_A$, we also have $g\bar{f} \leqslant g$, and so $g\bar{f} = f$ implies $f \leqslant g$. Conversely, if $f \leqslant g$, then also $Ff \leqslant Fg$, which is to say that $Fg.\overline{Ff} = Ff$. But now we have $f \leqslant g$ and also $g\bar{f} \leqslant g$, but $F(g\bar{f}) = Fg.\overline{Ff} = Ff$, whence, by uniqueness of liftings, $g\bar{f} = f$. Finally, it is clear that F lifts to a restriction functor $\hat{F} : \hat{\mathcal{C}} \to \mathcal{D}$.

It remains to show that $\hat{\mathcal{C}}$ and $\hat{F}$ are unique over $\mathcal{C}$ and F. So suppose there is given some other restriction structure $f \mapsto \tilde{f}$ inducing the 2-category structure of $\mathcal{C}$. Then for each $f : A \to B$, we have $\tilde{f} \leqslant 1_A$ and $\bar{f} \leqslant 1_A$; but now $F(\tilde{f}) = \overline{Ff} = F(\bar{f})$ and so by uniqueness of liftings, we have $\tilde{f} = \bar{f}$. Thus $\hat{\mathcal{C}}$ is unique; the uniqueness of $\hat{F}$ is now immediate. $\square$

---

[3] In general, if φ is some property of functors, then a 2-functor $F : \mathcal{K} \to \mathcal{L}$ may be called *locally φ* if each $\mathcal{K}(X, Y) \to \mathcal{L}(FX, FY)$ is φ.

The preceding two results allow us to reformulate the notion of hyperconnected restriction functor in purely 2-categorical terms. The following result allows us to do similarly for the notion of total natural transformation; in its statement, recall that a map f in a 2-category $\mathcal{K}$ is called a *discrete fibration* if for each $X \in \mathcal{K}$, the functor $\mathcal{K}(X, f)$ is a discrete fibration of categories.

Proposition 4.4. *Let $\mathcal{C}$ be a restriction category. A map $f : A \to B$ in $\mathcal{C}$ is total if and only if it is a discrete fibration in $\Gamma\mathcal{C}$.*

Proof. If $f : A \to B$ is a discrete fibration in $\Gamma\mathcal{C}$, then in particular, composition with it reflects identity 2-cells; whence from $\bar{f} \leqslant 1_A$ and $f.\bar{f} = f = f.1_A$ we deduce that $\bar{f} = 1_A$, so that f is total. Suppose conversely that f is total. We must show that for every $b : X \to B$ and $a : X \to A$ with $b \leqslant fa$, there exists a unique $c \leqslant a$ with $fc = b$. Because $b \leqslant fa$, we have $b = fa\bar{b}$, so that taking $c = a\bar{b}$, the above two conditions are clearly satisfied. To show uniqueness of c, suppose that $d \leqslant a$ with $fd = b$. Then we have $\bar{b} = \overline{fd} = \overline{\bar{f}d} = \bar{d}$ by totality of f and Lemma 2.2(ii) so that $d = a\bar{d} = a\bar{b} = c$ as desired.　□

Combining the above results, we have:

Theorem 4.5. *The 2-category* **rCat** *of restriction categories is 2-equivalent to the 2-category* **rCat**″ *whose objects are local discrete fibrations $F : \mathcal{C} \to \Gamma\mathcal{S}$; whose 1-cells are diagrams*

with H a 2-functor and γ a 2-natural transformation whose components are discrete fibrations; and whose 2-cells $(H, \gamma) \to (H', \gamma')$ are 2-natural transformations $\theta : H \to H'$ with $\gamma'.G\theta = \gamma$.

Proof. It suffices to show that **rCat**″ is 2-equivalent to the 2-category **rCat**′ of Proposition 3.4. By Proposition 4.1, each hyperconnected morphism $F : \mathcal{C} \to \mathcal{S}$ induces a local discrete fibration $\Gamma F : \Gamma\mathcal{C} \to \Gamma\mathcal{S}$, and by Proposition 4.4, this assignation provides the action on objects of a 2-functor **rCat**′ $\to$ **rCat**″. By Propositions 4.1 and 4.3, this 2-functor is surjective on objects; we claim it is also 2-fully faithful. To show fully faithfulness on 1-cells, consider a diagram

in **2**-**Cat**, with all components of γ discrete fibrations. By Proposition 4.4, each γ_A is a total map. We must show that $H = \Gamma\hat{H}$ for a unique restriction functor $\hat{H} : \mathcal{C} \to \mathcal{D}$; it will then follow that γ is the image under Γ of the unique total natural transformation $G\hat{H} \to F$. Clearly $\hat{H}$ must be defined as H is on objects and morphisms, and so all that is needed is to verify that $\overline{Hf} = H\bar{f}$ for each $f : A \to B$ in $\mathcal{C}$. Since $\overline{Hf} \leqslant 1_{HA}$ and $H\bar{f} \leqslant 1_{HA}$, both are restriction idempotents on HA in $\mathcal{D}$; since G is hyperconnected, it suffices to show that $G(\overline{Hf}) = GH\bar{f}$ in $\mathcal{S}$. Observe first that, for any map $k : X \to Y$ in $\mathcal{C}$, we have $\overline{GHk} = \overline{\gamma_Y.GHk} = \gamma_Y.GHk = Fk.\gamma_X$, using naturality and totality of γ together with Lemma 2.2(ii). Using this and the fact that F and G are restriction functors, we calculate that

$$G(\overline{Hf}) = \overline{GHf} = \overline{\overline{Ff}.\gamma_A} = \overline{\overline{Ff}.\gamma_A} = \overline{F\bar{f}.\gamma_A} = \overline{GH\bar{f}} = GH\bar{f}$$

in $\mathcal{S}$, as required. This shows that $U : $ **rCat**′ $\to$ **rCat**″ is fully faithful on 1-cells. As for 2-cells, suppose we are given $\theta : (\Gamma H, \Gamma\gamma) \Rightarrow (\Gamma H', \Gamma\gamma') : (\Gamma\mathcal{C}, \Gamma F) \to (\Gamma\mathcal{D}, \Gamma G)$ in **rCat**″. We must show that $\theta = \Gamma\hat{\theta}$ for a unique $\hat{\theta}$, which will clearly be the case so long as θ has total components in $\mathcal{D}$. Since G is hyperconnected, it reflects totality, and so it is enough to show that each $G\theta_X$ is a total map of $\mathcal{S}$. But $\gamma'_X.G\theta_X = \gamma_X$ in $\mathcal{S}$, and γ_X and γ'_X are both total, whence also $G\theta_X$ by Lemma 2.2(iv).　□

5. Notions of enriched category

We are now in a position to explain how restriction categories may be viewed as enriched categories, but before doing so, must break off to discuss briefly the notion of enrichment we shall need, which, as explained in the introduction, is a little more subtle than usual. If $\mathcal{V}$ is a monoidal category, there is a well-known notion of *category enriched in $\mathcal{V}$* or *$\mathcal{V}$-category* [19], whose homs are given by objects from $\mathcal{V}$ rather than by sets. It was first in the work of Walters [29] that it was realised that a more general kind of enrichment was fruitful, in which the monoidal category $\mathcal{V}$ is replaced by a bicategory $\mathcal{W}$. In this case, a *$\mathcal{W}$-category $\mathcal{C}$* is given by:

- a set of objects $\mathrm{ob}\,\mathcal{C}$;
- a function $|-| : \mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{W}$;
- for each $x, y \in \mathrm{ob}\,\mathcal{C}$, a 1-cell $\mathcal{C}(x, y) : |x| \to |y|$ in $\mathcal{W}$;
- for each $x \in \mathrm{ob}\,\mathcal{C}$, a 2-cell $j_x : 1_{|x|} \Rightarrow \mathcal{C}(x, x)$ in $\mathcal{W}$; and
- for each $x, y, z \in \mathrm{ob}\,\mathcal{C}$, a 2-cell $m_{xyz} : \mathcal{C}(y, z) \otimes \mathcal{C}(x, y) \Rightarrow \mathcal{C}(x, z)$ in $\mathcal{W}$

with j and m satisfying the usual associativity and unitality axioms; there are corresponding notions of $\mathcal{W}$-functor and $\mathcal{W}$-transformation. The use to which Walter put these concepts was in representing sheaves on a given locale (more generally, site) as categories enriched in an associated bicategory $\mathcal{W}$.

For our purposes, it turns out that whilst we may identify restriction categories with $\mathcal{W}$-categories for a suitable $\mathcal{W}$, this correspondence does not extend to the functors and natural transformations between them; there are more restriction functors between two restriction categories than there are $\mathcal{W}$-functors between the corresponding $\mathcal{W}$-categories. To rectify this, we will consider enrichment over a yet more general kind of base, namely that of a *weak double category*. Enrichments of this and even more general sorts were considered in [20].

Recall from [16] that a *weak double category* $\mathbb{W}$ is a pseudo-category object in **Cat**: it is given by collections of *objects* $A, B, \dots$, *vertical morphisms* $f, g, \dots : A \to B$, *horizontal morphisms* $U, V, \dots : A \nrightarrow B$ and *cells*

$$
\begin{array}{ccc}
A & \xrightarrow{\;\;U\;\;} & C \\
f\downarrow & \;\Downarrow\alpha & \downarrow g \\
B & \xrightarrow[\;\;V\;\;]{} & D
\end{array}
\tag{5.1}
$$

together with composition and identity operations for vertical and horizontal arrows, and for cells along vertical and horizontal boundaries. Composition of vertical arrows is strictly associative and unital, whilst that of horizontal arrows is only associative and unital up to *globular cells*: ones whose vertical boundaries are identities. Every bicategory can be seen as a weak double category with only identity vertical arrows; conversely, the objects, horizontal arrows, and globular cells in any weak double category form a bicategory, the *underlying bicategory* of the weak double category.

We now describe the notions of category, functor and natural transformation enriched in a weak double category $\mathbb{W}$. Firstly, a $\mathbb{W}$-category is simply a category enriched over the underlying bicategory of $\mathbb{W}$. For $\mathbb{W}$-categories $\mathcal{C}$ and $\mathcal{D}$, a $\mathbb{W}$-*functor* $\mathcal{C} \to \mathcal{D}$ is given by:

- a function $H : \mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{D}$;
- vertical morphisms $H_x : |x| \to |Hx|$ for each $x \in \mathrm{ob}\,\mathcal{C}$;
- cells

$$
\begin{array}{ccc}
|x| & \xrightarrow{\;\mathcal{C}(x,y)\;} & |y| \\
H_x\downarrow & \;\Downarrow H_{x,y} & \downarrow H_y \\
|Hx| & \xrightarrow[\;\mathcal{D}(Hx,Hy)\;]{} & |Hy|
\end{array}
$$

for each $x, y \in \mathrm{ob}\,\mathcal{C}$, satisfying the two functoriality axioms

$$
\begin{array}{ccc}
|x| & \xrightarrow{\;1_{|x|}\;} & |x| \\
\| & \;\Downarrow j_x & \| \\
|x| & \xrightarrow[\;\mathcal{C}(x,x)\;]{} & |x| \\
H_x\downarrow & \;\Downarrow H_{x,x} & \downarrow H_x \\
|Hx| & \xrightarrow[\;\mathcal{D}(Hx,Hx)\;]{} & |Hx|
\end{array}
\quad=\quad
\begin{array}{ccc}
|x| & \xrightarrow{\;1_{|x|}\;} & |x| \\
H_x\downarrow & \;\Downarrow 1_{H_x} & \downarrow H_x \\
|Hx| & \xrightarrow[\;1_{|Hx|}\;]{} & |Hx| \\
\| & \;\Downarrow j_{Hx} & \| \\
|Hx| & \xrightarrow[\;\mathcal{D}(Hx,Hx)\;]{} & |Hx|
\end{array}
$$

and

$$
\begin{array}{ccccc}
|x| & \xrightarrow{\;\mathcal{C}(x,y)\;} & |y| & \xrightarrow{\;\mathcal{C}(y,z)\;} & |z| \\
H_x\downarrow & \;\Downarrow H_{x,y} & \downarrow H_y \;\;\Downarrow H_{y,z} & & \downarrow H_z \\
|Hx| & \xrightarrow[\;\mathcal{D}(Hx,Hy)\;]{} & |Hy| & \xrightarrow[\;\mathcal{D}(Hy,Hz)\;]{} & |Hz| \\
\| & & \;\Downarrow m_{Hx,Hy,Hz} & & \| \\
|Hx| & & \xrightarrow[\;\mathcal{D}(Hx,Hz)\;]{} & & |Hz|
\end{array}
\quad=\quad
\begin{array}{ccccc}
|x| & \xrightarrow{\;\mathcal{C}(x,y)\;} & |y| & \xrightarrow{\;\mathcal{C}(y,z)\;} & |z| \\
\| & & \;\Downarrow m_{x,y,z} & & \| \\
|x| & & \xrightarrow[\;\mathcal{C}(x,z)\;]{} & & |Hz| \\
H_x\downarrow & & \;\Downarrow H_{x,z} & & \downarrow H_z \\
|Hx| & & \xrightarrow[\;\mathcal{D}(Hx,Hz)\;]{} & & |Hz|
\end{array}
$$

We can compose $\mathbb{W}$-functors in an evident way, and so obtain a category $\mathbb{W}$-**Cat** of $\mathbb{W}$-categories and $\mathbb{W}$-functors. Now a $\mathbb{W}$-*natural transformation* $\alpha : H \Rightarrow K : \mathcal{C} \to \mathcal{D}$ is given by a collection of cells

$$
\begin{array}{ccc}
|x| & \xrightarrow{\;1_{|x|}\;} & |x| \\
H_x \downarrow & \Downarrow \alpha_x & \downarrow K_x \\
|Hx| & \xrightarrow[\mathcal{D}(Hx, Kx)]{} & |Kx|
\end{array}
$$

satisfying the following naturality axiom for all $x, y \in \mathrm{ob}\,\mathcal{C}$.

$\mathbb{W}$-transformations $H \Rightarrow K$ and $K \Rightarrow L$ compose by placing their 2-cell data horizontally adjacent and using the unit constraints of $\mathbb{W}$; the precomposition of a $\mathbb{W}$-transformation by a $\mathbb{W}$-functor is defined by substitution; whilst postcomposition of a $\mathbb{W}$-transformation by a $\mathbb{W}$-functor is given by placing their 2-cell data vertically adjacent. In this way, the $\mathbb{W}$-categories, $\mathbb{W}$-functors and $\mathbb{W}$-natural transformations form a 2-category $\mathbb{W}$-**Cat**. In particular, when $\mathbb{W}$ is the weak double category associated to a bicategory $\mathcal{W}$, we obtain a 2-category $\mathcal{W}$-**Cat** of $\mathcal{W}$-categories, $\mathcal{W}$-functors and $\mathcal{W}$-transformations, as in [29].

6. Restriction categories as enriched categories

We now return to the task of exhibiting restriction categories as enriched categories. The one remaining ingredient we require is a construction due to Brian Day [11]. Given a locally small 2-category $\mathcal{K}$, we consider the bicategory $\mathcal{PK}$ whose objects are those of $\mathcal{K}$, and whose hom-categories are given by $\mathcal{PK}(A, B) := [\mathcal{K}(A, B)^{\mathrm{op}}, \mathbf{Set}]$. Writing $\mathcal{Y}$ for the Yoneda embedding into a presheaf category, the identity map in $\mathcal{PK}(B, B)$ is the representable $\mathcal{Y}(1_B)$, and composition $\mathcal{PK}(B, C) \times \mathcal{PK}(A, B) \to \mathcal{PK}(A, C)$ is Day convolution: it is determined by the requirement that it be cocontinuous in each variable and defined on representables by $\mathcal{Y}g.\mathcal{Y}f = \mathcal{Y}(g.f)$. There is a homomorphism of bicategories $\mathcal{Y} : \mathcal{K} \to \mathcal{PK}$ which is the identity on objects, and on each hom-category is the Yoneda embedding. Using the equivalence between presheaves on a small category $\mathcal{C}$ and discrete fibrations over $\mathcal{C}$ in **Cat**, it is now not difficult to derive the following result, due to Richard Wood in collaboration with the first author; we shall prove a generalisation of it as Proposition 6.2 below.

Proposition 6.1. *(See [30].) If $\mathcal{K}$ is a locally small 2-category, then the 2-category $\mathcal{PK}$-**Cat** is 2-equivalent to the 2-category* **2-Cat** $/_{\mathrm{ldf}}\,\mathcal{K}$ *whose objects are local discrete fibrations $F : \mathcal{C} \to \mathcal{K}$ with $\mathcal{C}$ locally small, and whose 1- and 2-cells are 2-functors and 2-natural transformations commuting with the projections to $\mathcal{K}$.*

Comparing this result with Theorem 4.5, we see that restriction categories may be identified with $\mathcal{P}(\Gamma\mathcal{S})$-enriched categories; but that, as anticipated above, restriction functors between restriction categories are rather more general than $\mathcal{P}(\Gamma\mathcal{S})$-functors between the corresponding $\mathcal{P}(\Gamma\mathcal{S})$-categories. To rectify this, we shall consider enrichment in a weak double category obtained by a double-categorical analogue[4] of Day's construction.

Suppose that $\mathcal{K}$ is a locally small 2-category; we construct a weak double category $\mathbb{PK}$ as follows. Its objects are those of $\mathcal{K}$, a vertical arrow $A \to B$ is a discrete fibration $B \to A$ in $\mathcal{K}$, a horizontal arrow $A \nrightarrow B$ is a presheaf $U \in [\mathcal{K}(A, B)^{\mathrm{op}}, \mathbf{Set}]$, whilst a cell of the form (5.1) is a 2-cell

$$
\begin{array}{ccc}
A & \xrightarrow{\;U\;} & C \\
\mathcal{Y}f \uparrow & \Downarrow \alpha & \uparrow \mathcal{Y}g \\
B & \xrightarrow[\;V\;]{} & D
\end{array}
\tag{6.1}
$$

---

[4] The construction is very natural, but to the authors' knowledge has not appeared in print before.

in $\mathcal{PK}$. Composition of vertical morphisms is as in $\mathcal{K}$, that of horizontal morphisms is as in $\mathcal{PK}$ and cell composition is given by pasting in $\mathcal{PK}$. It is easy to see that the underlying bicategory of $\mathbb{P}\mathcal{K}$ is isomorphic to $\mathcal{PK}$.

Proposition 6.2. *If $\mathcal{K}$ is a locally small 2-category, then $\mathbb{P}\mathcal{K}$-**Cat** is 2-equivalent to the 2-category **2**-**Cat** $/\!\!/_{\mathrm{ldf}} \mathcal{K}$ whose objects are local discrete fibrations $F : \mathcal{C} \to \mathcal{K}$ with $\mathcal{C}$ locally small; whose morphisms are diagrams*

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\quad H \quad} & \mathcal{D} \\
& {\scriptstyle F} \searrow {\stackrel{\Longleftarrow}{\gamma}} \swarrow {\scriptstyle G} & \\
& \mathcal{K} &
\end{array}$$
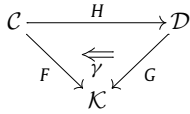
with H a 2-functor and γ a 2-natural transformation whose components are discrete fibrations; and whose 2-cells $(H, \gamma) \to (H', \gamma')$ are 2-natural transformations $\theta : H \to H'$ with $\gamma'.G\theta = \gamma$.

To prove this, we will need an alternate description of the cells in $\mathcal{PK}$. In the statement of the following result, we introduce the notation $\pi_X : \mathrm{el}\, X \to \mathcal{C}$ to denote the discrete fibration corresponding to a presheaf $X \in [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$: thus objects of $\mathrm{el}\, X$ are pairs $(c \in \mathcal{C}, x \in Xc)$, and morphisms $f : (c, x) \to (d, y)$ are maps $f : c \to d$ in $\mathcal{C}$ for which $(Xf)(d) = c$.

Lemma 6.3. *To give a cell* (6.1) *is equally to give a functor α fitting into a commutative diagram*
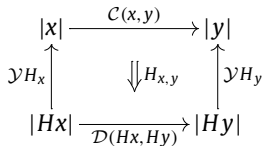
$$\begin{array}{ccc}
\mathrm{el}\, U & \xrightarrow{\quad\quad\quad \pi_U \quad\quad\quad} & \mathcal{K}(A, C) \\
{\scriptstyle \alpha} \downarrow & & \downarrow {\scriptstyle \mathcal{K}(f, C)} \\
\mathrm{el}\, V & \xrightarrow[\pi_V]{} \mathcal{K}(B, D) \xrightarrow[\mathcal{K}(B, g)]{} & \mathcal{K}(B, C)
\end{array} \qquad (6.2)$$

Proof. Because composition in $\mathcal{PK}$ is Day convolution, to give a cell (6.1) is equally to give a map $\mathrm{Lan}_{\mathcal{K}(f,C)}(U) \to \mathrm{Lan}_{\mathcal{K}(B,g)}(V)$ in $[\mathcal{K}(B, C)^{\mathrm{op}}, \mathbf{Set}]$. Now $\mathcal{K}(B, g)$ is a discrete fibration, since g is, and thus the bottom edge of (6.2) is the discrete fibration corresponding to $\mathrm{Lan}_{\mathcal{K}(B,g)}(V)$. To give a map α as indicated is equally to give a map from π_U to the pullback of the bottom face along $\mathcal{K}(f, C)$; which is to give a map of presheaves $U \to \mathcal{K}(f, C)^*(\mathrm{Lan}_{\mathcal{K}(B,g)}(V))$, or equally, a map $\mathrm{Lan}_{\mathcal{K}(f,C)}(U) \to \mathrm{Lan}_{\mathcal{K}(B,g)}(V)$, as required. $\square$

We are now ready to give:

Proof of Proposition 6.2. We define a 2-functor $\Lambda : \mathbb{P}\mathcal{K}$-**Cat** $\to$ **2**-**Cat** $/\!\!/_{\mathrm{ldf}} \mathcal{K}$ as follows. For $\mathcal{C}$ a $\mathbb{P}\mathcal{K}$-enriched category, we let $\Lambda\mathcal{C}$ be the 2-category whose objects are those of $\mathcal{C}$, and whose hom-category $(\Lambda\mathcal{C})(x, y)$ is the category of elements of $\mathcal{C}(x, y) : \mathcal{K}(|x|, |y|)^{\mathrm{op}} \to \mathbf{Set}$. The composition and identities of $\Lambda\mathcal{C}$ are induced by the composition and identity cells of $\mathcal{C}$, and there is a local discrete fibration $\pi_{\mathcal{C}} : \Lambda\mathcal{C} \to \mathcal{K}$ which on objects, sends x to $|x|$ and on hom-categories is the discrete fibration $(\Lambda\mathcal{C})(x, y) \to \mathcal{K}(|x|, |y|)$. This defines the action of Λ on objects.

Now let $H : \mathcal{C} \to \mathcal{D}$ be a $\mathbb{P}\mathcal{K}$-functor. We thus have a function $H : \mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{D}$, discrete fibrations $H_x : |Hx| \to |x|$ in $\mathcal{K}$ for each $x \in \mathcal{C}$, and 2-cells

$$\begin{array}{ccc}
|x| & \xrightarrow{\quad \mathcal{C}(x,y) \quad} & |y| \\
{\scriptstyle \mathcal{Y}H_x} \uparrow & \Downarrow {\scriptstyle H_{x,y}} & \uparrow {\scriptstyle \mathcal{Y}H_y} \\
|Hx| & \xrightarrow[\mathcal{D}(Hx, Hy)]{} & |Hy|
\end{array}$$

in $\mathcal{PK}$ for each $x, y \in \mathcal{C}$. By Lemma 6.3, to give the 2-cell $H_{x,y}$ is equally well to give a functor $(\Lambda H)_{x,y} : (\Lambda\mathcal{C})(x, y) \to (\Lambda\mathcal{D})(Hx, Hy)$ fitting into a commutative diagram

$$\begin{array}{ccc}
(\Lambda\mathcal{C})(x, y) & \xrightarrow{\quad\quad\quad (\pi_{\mathcal{C}})_{x,y} \quad\quad\quad} & \mathcal{K}(|x|, |y|) \\
{\scriptstyle (\Lambda H)_{x,y}} \downarrow & & \downarrow {\scriptstyle \mathcal{K}(H_x, 1)} \\
(\Lambda\mathcal{D})(Hx, Hy) & \xrightarrow[(\pi_{\mathcal{D}})_{Hx, Hy}]{} \mathcal{K}(|Hx|, |Hy|) \xrightarrow[\mathcal{K}(1, H_y)]{} & \mathcal{K}(|Hx|, |y|)
\end{array} \qquad (6.3)$$

We therefore obtain a 2-functor $\Lambda H : \Lambda\mathcal{C} \to \Lambda\mathcal{D}$ which sends x to Hx, and on hom-categories, is given by these functors $(\Lambda H)_{x,y}$; the 2-functor axioms for ΛH are implied by the $\mathbb{P}\mathcal{K}$-functor axioms for H. Moreover, the maps $H_x : |Hx| \to |x|$ are the components of a 2-natural transformation $\gamma_H : \pi_{\mathcal{D}}.\Lambda H \Rightarrow \pi_{\mathcal{C}}$; the naturality of these components is expressed precisely by the commutativity of the diagrams (6.3). This defines the action of Λ on morphisms.

Finally, let $\alpha : H \Rightarrow K : \mathcal{C} \to \mathcal{D}$ be a $\mathbb{P}\mathcal{K}$-natural transformation. Thus we have a family of 2-cells

$$
\begin{array}{ccc}
|x| & \xrightarrow{\;1_{|x|}\;} & |x| \\
{\scriptstyle \mathcal{Y}H_x}\big\uparrow & \Downarrow\alpha_x & \big\uparrow{\scriptstyle \mathcal{Y}K_x} \\
|Hx| & \xrightarrow[\;\mathcal{D}(Hx,Kx)\;]{} & |Kx|
\end{array}
$$

in $\mathbb{P}\mathcal{K}$, satisfying one naturality axiom. To give α_x is equally to give a morphism $\mathcal{Y}H_x \to \mathrm{Lan}_{\mathcal{K}(1,K_x)}(\mathcal{D}(Hx, Kx))$ in $[\mathcal{K}(|Hx|, |x|)^{\mathrm{op}}, \mathbf{Set}]$ and so, applying the Yoneda lemma and passing to categories of elements, a morphism $(\Lambda\alpha)_x$ fitting into a diagram

$$
\begin{array}{ccc}
1 & =\!=\!=\!=\!=\!=\!=\!= & 1 \\
{\scriptstyle (\Lambda\alpha)_x}\big\downarrow & & \big\downarrow{\scriptstyle H_x} \\
(\Lambda\mathcal{D})(Hx, Kx) & \xrightarrow[(\pi_{\mathcal{D}})_{Hx,Kx}]{} \mathcal{K}(|Hx|, |Kx|) & \xrightarrow[\mathcal{K}(1,K_x)]{} \mathcal{K}(|Hx|, |x|)
\end{array}
$$

The $\mathbb{P}\mathcal{K}$-naturality of α implies that the maps $(\Lambda\alpha)_x$ are components of a 2-natural transformation $\Lambda H \Rightarrow \Lambda K : \Lambda\mathcal{C} \to \Lambda\mathcal{D}$; whilst the commutativity of the displayed rectangles implies that $\gamma_K . (\pi_{\mathcal{D}} . \Lambda\alpha) = \gamma_H$. This defines the action of Λ on 2-cells. It is now not hard to show that the Λ so defined is essentially surjective on objects, and 2-fully faithful, hence a 2-equivalence. $\square$

If we now define $\mathbb{R} := \mathbb{P}(\Gamma\mathcal{S})$, we immediately conclude from Proposition 6.2 and Theorem 4.5 that:

Theorem 6.4. *The* 2-*category of restriction categories is* 2-*equivalent to the* 2-*category of* $\mathbb{R}$-*enriched categories.*

7. Join restriction categories

We have shown that restriction categories may be seen as categories enriched over a certain base; by appropriately changing that base, we will now see that certain variants of the notion of restriction category are also expressible as enriched categories. In this section we consider join restriction categories, which are restriction categories in which compatible families of parallel morphisms can be patched together.

A parallel pair of maps $f, g : A \rightrightarrows B$ in a restriction category $\mathcal{C}$ are said to be *compatible* if $f\bar{g} = g\bar{f}$; this says that f and g agree when restricted to their domain of mutual definition. A family of maps $(f_i : A \to B \mid i \in I)$ is called *compatible* if pairwise so. We call $\mathcal{C}$ a *join restriction category* [2,17] if every compatible family of maps in $\mathcal{C}(A, B)$ admits a join $\bigvee_i f_i$ with respect to the restriction partial order, and these joins satisfy the axioms:

(J1) $\overline{\bigvee_i f_i} = \bigvee_i \overline{f_i}$;
(J2) $(\bigvee_i f_i)g = \bigvee_i (f_i g)$ for all $g : A' \to A$ in $\mathcal{C}$;
(J3) $h(\bigvee_i f_i) = \bigvee_i (h f_i)$ for all $h : B \to B'$ in $\mathcal{C}$.

In fact, it turns out that the third of these axioms is a consequence of the other two; see [17, Lemma 3.1.8].

Examples 7.1.

(i) The category $\mathbf{Set}_p$ is a join restriction category. A family of partial functions $(f_i : A \rightharpoonup B \mid i \in I)$ is compatible if $f_i(a) = f_j(a)$ whenever f_i and f_j are both defined at a, and the union of such a family is defined by

$$
\left(\bigvee_i f_i\right)(a) = \begin{cases} f_i(a) & \text{if there exists } i \in I \text{ with } f_i(a) \text{ defined;} \\ \text{undefined} & \text{otherwise.} \end{cases}
$$

(ii) The category $\mathbf{Top}_p$ is a join restriction category; the joins in $\mathbf{Top}_p(A, B)$ are precisely what one needs to verify that the presheaf of B-valued continuous functions on A is in fact a sheaf.

(iii) The category $\mathbf{Rec}$ of partial recursive functions is not a join restriction category, but it is in the obvious sense a *finite* join restriction category. One may patch together finitely many compatible partial recursive functions by suitably interleaving their calculations; but the same is not true of an infinite compatible family unless that family is itself recursively indexed.

(iv) Let $j\mathcal{S}$ denote the subcategory of $\mathcal{S}$ whose objects are frames—complete posets verifying the infinite distributive law $a \wedge \bigvee_i b_i = \bigvee_i a \wedge b_i$—and whose morphisms from A to B are stable maps $A \leftarrow B$ that preserve all joins. If A is a frame and $a \in A$, then it is easy to see that the restriction idempotent $a \wedge (-) : A \leftarrow A$ in $\mathcal{S}$ lies in $j\mathcal{S}$; whence $j\mathcal{S}$ is a hyperconnected restriction subcategory of $\mathcal{S}$. It is moreover a join restriction category. A family of maps $(f_i : A \leftarrow B)$ is compatible just when $f_i(b) \wedge f_j(\top) = f_j(b) \wedge f_i(\top)$ for all $i, j \in I$ and $b \in B$; the union of such a family is defined pointwise by $(\bigvee_i f_i)(b) = \bigvee_i f_i(b)$.

By a *join restriction functor*, we mean a restriction functor $F : \mathcal{C} \to \mathcal{D}$ between join restriction categories that satisfies $F(\bigvee_i f_i) = \bigvee_i (F f_i)$ for each compatible family of maps. It is easy to verify that join restriction functors are closed under composition; thus join restriction categories, join restriction functors and total natural transformations form a sub-2-category **jrCat** of **rCat**.

We will now exhibit **jrCat** as 2-equivalent to a 2-category of enriched categories, following the same progression of transformations as in the preceding sections. We begin with a result which is a straightforward consequence of the join restriction axioms.

Proposition 7.2. *Let $\mathcal{C}$ be a join restriction category. Then*:

  (i) *For each $A \in \mathcal{C}$, $\mathcal{O}(A)$ is a frame.*
 (ii) *For each $f : A \to B$ in $\mathcal{C}$, $f^* : \mathcal{O}(B) \to \mathcal{O}(A)$ preserves all joins.*
(iii) *The induced factorisation $\mathcal{O} : \mathcal{C} \to j\mathcal{S}$ of the fundamental functor of $\mathcal{C}$ is a hyperconnected join restriction functor.*

The next step is an analogue of Proposition 3.2.

Proposition 7.3. *If $\mathcal{C}$ is a join restriction category, then the fundamental functor $\mathcal{O} : \mathcal{C} \to j\mathcal{S}$ is a terminal object of $\mathbf{jrCat}(\mathcal{C}, j\mathcal{S})$.*

Proof. Let $F : \mathcal{C} \to j\mathcal{S}$ be a join restriction functor; as in the proof of Proposition 3.2, the total natural transformation $\gamma : F \Rightarrow \mathcal{O}$ must have components $\gamma_A : FA \leftarrow \mathcal{O}(A)$ given by $\gamma_A(e) = Fe(\top)$. The only extra point to check is that each component γ_A preserves joins, for which we calculate that:

$$\gamma_A\left(\bigvee_i f_i\right) = F\left(\bigvee_i f_i\right)(\top) = \left(\bigvee_i F f_i\right)(\top) = \bigvee_i \left(F f_i(\top)\right) = \bigvee_i \left(\gamma_A(f_i)\right). \qquad \square$$

It is now an identical argument to that of Section 2 to conclude that:

Theorem 7.4. *The 2-category $\mathbf{jrCat}$ of join restriction categories is 2-equivalent to the 2-category $\mathbf{jrCat}' = \mathbf{jrCat}/\!/_h \, j\mathcal{S}$ whose objects are hyperconnected join restriction functors $F : \mathcal{C} \to j\mathcal{S}$, whose morphisms are diagrams*
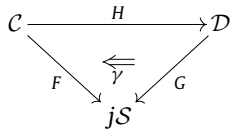
in $\mathbf{jrCat}$, and whose 2-cells $(H, \gamma) \to (H', \gamma')$ are total natural transformations $\theta : H \to H'$ with $\gamma'.G\theta = \gamma$.

To give our next reformulation of join restriction categories, we shall identify a notion to which they bear the same relation as do restriction categories to 2-categories. By an *fc-site*, we mean a category with finite connected limits equipped with a Grothendieck topology. By a *2-fc-site*, we mean a 2-category $\mathcal{K}$ whose hom-categories are fc-sites and for which pre- and postcomposition by 1-cells preserves both covers and finite connected limits.

Proposition 7.5. *If $\mathcal{C}$ is a join restriction category, then $\Gamma\mathcal{C}$ is a 2-fc-site, where a family of maps $(f_i \leqslant f \mid i \in I)$ in $\Gamma\mathcal{C}(A, B)$ is covering just when $\bigvee_i f_i = f$.*

Proof. $\Gamma\mathcal{C}(A, B)$ is a poset and so trivially has equalisers. It also has pullbacks: if $f, g \leqslant h$ then $f \times_h g$ is $f\bar{g} = g\bar{f}$. Now if $(f_i \leqslant f \mid i \in I)$ is a covering family in $\Gamma\mathcal{C}(A, B)$, then for any $g \leqslant f$, the pullback family $(f_i \bar{g} \leqslant g \mid i \in I)$ is again covering since $\bigvee_i (f_i \bar{g}) = (\bigvee_i f_i)\bar{g} = f\bar{g} = g$ as required. Thus each $\Gamma\mathcal{C}(A, B)$ is an fc-site. It remains to show that whiskering by 1-cells preserves covers and finite connected limits. Preservation of covers follows from (J2) and (J3); equalisers are trivially preserved; and as for pullbacks, these are clearly preserved by postcomposition, whilst precomposition preserves them by (R4). $\square$

If $\mathcal{B}$ is an fc-site, then we call a functor $p : \mathcal{E} \to \mathcal{B}$ *étale* if it is a discrete fibration and the corresponding presheaf $\mathcal{B}^{\mathrm{op}} \to \mathbf{Set}$ is a j-sheaf. If $\mathcal{K}$ is a 2-fc-site, then we call a 2-functor $\mathcal{L} \to \mathcal{K}$ *locally étale* if each induced functor on homs is étale.

Proposition 7.6. *Let $F : \mathcal{C} \to \mathcal{D}$ be a hyperconnected restriction functor between join restriction categories. Then F preserves joins if and only if $\Gamma F : \Gamma\mathcal{C} \to \Gamma\mathcal{D}$ is locally étale.*

Proof. For each $A, B \in \mathcal{C}$, we know by Proposition 4.1 that $(\Gamma F)_{A,B} : \Gamma \mathcal{C}(A, B) \to \Gamma \mathcal{D}(FA, FB)$ is a discrete fibration. It will be étale precisely when for every cover $(g_i \leqslant g \mid i \in I)$ in $\Gamma D(FA, FB)$, each matching family of elements over it in $\mathcal{C}(A, B)$ admits a unique patching. To give a matching family over the g_i's is to give maps $(f_i \in \mathcal{C}(A, B) \mid i \in I)$ with $Ff_i = g_i$ for each i, and such that $f_i \mid_j = f_j \mid_i$ for all $i, j \in I$; here $f_i \mid_j$ is obtained, using the discrete fibration property of ΓF, as the unique element of $\mathcal{C}(A, B)$ with $f_i \mid_j \leqslant f_i$ and $F(f_i \mid_j) = g_i \times_g g_j$. We know from above that $g_i \times_g g_j = g_i \overline{g_j}$ and thus $f_i \mid_j = f_i \overline{f_j}$; so to say that the f_i's are a matching family is to say that they are compatible in the sense defined above.

Now let F preserve joins. For any matching family $(f_i \mid i \in I)$ as above, the element $\bigvee_i f_i$ is a patching: it satisfies $f_i \leqslant \bigvee_i f_i$ for each i and $F(\bigvee_i f_i) = \bigvee_i Ff_i = \bigvee_i g_i = g$. Moreover, if h is another patching, so satisfying $f_i \leqslant h$ for each i and $Fh = g$, then $\bigvee_i f_i \leqslant h$ and $Fh = g = F(\bigvee_i f_i)$ implies that $h = \bigvee_i f_i$, since discrete fibrations reflect identities. Thus ΓF is locally étale.

Suppose conversely that ΓF is locally étale, and let $(f_i : A \to B \mid i \in I)$ be a compatible family. We always have $\bigvee_i (Ff_i) \leqslant F(\bigvee_i f_i)$ and so it suffices to show the converse inequality. The f_i's are a matching family for the cover $(Ff_i \leqslant \bigvee_i Ff_i \mid i \in I)$, and so admit a unique patching k satisfying $Fk = \bigvee_i (Ff_i)$ and $f_i \leqslant k$ for each i. Thus $\bigvee_i f_i \leqslant k$ and so $F(\bigvee_i f_i) \leqslant \bigvee_i (Ff_i)$ as required. $\square$

The following is now the analogue in this context of Proposition 4.3.

Proposition 7.7. *Let $\mathcal{D}$ be a join restriction category, and let $F : \mathcal{C} \to \Gamma \mathcal{D}$ be locally étale. Then the unique restriction functor $\hat{F} : \hat{\mathcal{C}} \to \mathcal{D}$ lifting F is a join restriction functor between join restriction categories.*

Proof. If $(f_i : A \to B \mid i \in I)$ are a compatible family in $\hat{\mathcal{C}}$, then they form a matching family for the cover $(Ff_i \leqslant \bigvee_i Ff_i \mid i \in I)$ in $\mathcal{D}$, and so admit a unique patching f satisfying $Ff = \bigvee_i (Ff_i)$ and $f_i \leqslant f$ for each i. We claim that $f = \bigvee_i f_i$. Indeed, if $k : A \to B$ with $f_i \leqslant k$ for each i, then $\bigvee_i (Ff_i) \leqslant Fk$; now because ΓF is a local discrete fibration, we obtain $h \leqslant k$ with $Fh = \bigvee_i (Ff_i)$. But $Ff_i \leqslant \bigvee_i (Ff_i)$ for each i implies $f_i \leqslant h$ for each i, whence h is a patching for the f_i's. So $f = h \leqslant k$ as required. The join restriction category axioms for $\hat{\mathcal{C}}$ now follow by an argument identical in form to that of Proposition 4.3. It is immediate that $\hat{F}$ is a join restriction functor. $\square$

In our next result, we call a map f in a 2-fc-site $\mathcal{K}$ *étale* if $\mathcal{K}(X, f)$ is an étale functor for each $X \in \mathcal{K}$.

Proposition 7.8. *Let $\mathcal{C}$ be a restriction category. A map $f : A \to B$ in $\mathcal{C}$ is total if and only if it is an étale map in $\Gamma \mathcal{C}$.*

Proof. If $h : A \to B$ is étale in $\Gamma \mathcal{C}$, then it is in particular a discrete fibration, and so total by Proposition 4.4. Conversely, if h is total, then it is a discrete fibration in $\Gamma \mathcal{C}$; to show it is étale, let $X \in \mathcal{C}$ and consider $(g_i \leqslant g \mid i \in I)$ a cover in $\Gamma \mathcal{C}(X, B)$. Arguing as in Proposition 7.6, to give a matching family over the g_i's is to give a compatible family $(f_i \in \mathcal{C}(X, A) \mid i \in I)$ with $hf_i = g_i$ for each i. We must show that there is a unique f with $f_i \leqslant f$ for each i and with $hf = g$. For existence, we may take $f = \bigvee_i f_i$; for uniqueness, observe that any f with these properties satisfies $\bar{f} = \overline{hf} = \bar{g} = \bigvee_i \bar{g}_i = \bigvee_i \overline{hf_i} = \bigvee_i \overline{f_i}$ and also $\bigvee_i f_i \leqslant f$, whence $f = f\bar{f} = f \bigvee_i \overline{f_i} = \bigvee_i f_i$, as required. $\square$

We can now give an analogue of Theorem 4.5:

Theorem 7.9. *The 2-category **jrCat** of join restriction categories is 2-equivalent to the 2-category **jrCat''** whose objects are locally étale maps $F : \mathcal{C} \to \Gamma(j\mathcal{S})$; whose 1-cells are diagrams*
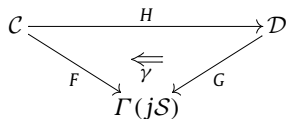
with H a 2-functor and γ a 2-natural transformation with étale components, and whose 2-cells $(H, \gamma) \to (H', \gamma')$ are 2-natural transformations $\theta : H \to H'$ with $\gamma'.G\theta = \gamma$.

Proof. Arguing as in Theorem 4.5, it suffices to construct a 2-equivalence between **jrCat''** and the **jrCat'** of Theorem 7.4. By Proposition 7.6, there is a 2-functor **jrCat'** $\to$ **jrCat''**, which by Propositions 7.6 and 7.7 is surjective on objects. To show 2-fully faithfulness, consider a diagram
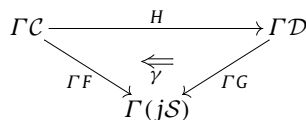
with γ a 2-natural transformation whose components are discrete fibrations. Postcomposing with the local discrete fibration $\Gamma(j\mathcal{S}) \to \Gamma\mathcal{S}$ and applying Theorem 4.5, we conclude that H lifts to a restriction functor $\mathcal{C} \to \mathcal{D}$; we must furthermore show that it preserves joins. Given a compatible family in $\mathcal{C}(A, B)$, we always have $\bigvee_i Hf_i \leqslant H(\bigvee_i f_i)$, so it suffices to show that $\overline{\bigvee_i Hf_i} = \overline{H(\bigvee_i f_i)}$, or equally that $\bigvee_i H\bar{f}_i = H(\bigvee_i \bar{f}_i)$. Since G is hyperconnected, it suffices to prove this last equality on postcomposition with G; which follows from the calculation

$$G\left(\bigvee_i H\bar{f}_i\right) = \bigvee_i GH\bar{f}_i = \bigvee_i(F\bar{f}_i.\gamma_A) = \left(\bigvee_i F\bar{f}_i\right).\gamma_A = F\left(\bigvee_i \bar{f}_i\right).\gamma_A = GH\left(\bigvee_i \bar{f}_i\right)$$

in $j\mathcal{S}$, whose second and fifth equalities arise as in the proof of Theorem 4.5. This proves fully faithfulness on 1-cells; that on 2-cells is as before. $\square$

Finally, we are ready to show how join restriction categories may be viewed as enriched categories. In preparation for this, we recall a result of Brian Day allowing us to "locally localise" a bicategory.

Proposition 7.10. *(See [12].) Let $\mathcal{L}$ be a bicategory and Σ a class of 2-cells of $\mathcal{L}$ closed under whiskering with 1-cells on each side. Suppose that for each $A, B \in \mathcal{L}$, the objects in $\mathcal{L}(A, B)$ which are orthogonal to $\Sigma \cap \mathcal{L}(A, B)$ constitute a full reflective subcategory $\mathcal{L}_\Sigma(A, B)$ with reflector $\ell_{A,B} : \mathcal{L}(A, B) \to \mathcal{L}_\Sigma(A, B)$, say. Then there is a bicategory $\mathcal{L}_\Sigma$ with homs given as above and composition law*

$$\mathcal{L}_\Sigma(B, C) \times \mathcal{L}_\Sigma(A, B) \hookrightarrow \mathcal{L}(B, C) \times \mathcal{L}(A, B) \xrightarrow{\circ} \mathcal{L}(A, C) \xrightarrow{\ell} \mathcal{L}_\Sigma(A, C).$$

Moreover, the maps $\ell_{A,B}$ assemble into a homomorphism of bicategories $\ell : \mathcal{L} \to \mathcal{L}_\Sigma$ which exhibit $\mathcal{L}_\Sigma$ as the localisation of $\mathcal{L}$ with respect to the class of 2-cells Σ.

The particular instance of this result which will be relevant for us is the following.

Proposition 7.11. *If $\mathcal{K}$ is a 2-fc-site, then there is a bicategory $\mathbf{Sh}(\mathcal{K})$ with the same objects as $\mathcal{K}$, and with hom-categories $\mathbf{Sh}(\mathcal{K})(A, B) = \mathbf{Sh}(\mathcal{K}(A, B))$.*

Proof. We apply the preceding result, taking $\mathcal{L} = \mathcal{P}\mathcal{K}$ and taking Σ to comprise all 2-cells inverted by sheafification in each hom. The result is immediate so long as we can show that Σ is stable under whiskering by 1-cells. It is enough to do so with respect to whiskering by representable 1-cells, since every 1-cell in $\mathcal{P}\mathcal{K}$ can be written as a colimit of representables; composition in $\mathcal{P}\mathcal{K}$ is cocontinuous in each variable; and Σ is stable under colimits.

Given $f : A \to B$ in $\mathcal{K}$, the precomposition functor $(-) \circ \mathcal{Y}f : \mathcal{P}\mathcal{K}(B, C) \to \mathcal{P}\mathcal{K}(A, C)$ is given by left Kan extension along $\mathcal{K}(f, C)$. We must show that this maps Σ into itself. Equivalently, we may show that the composite

$$\mathcal{P}\mathcal{K}(B, C) \xrightarrow{\mathrm{Lan}_{\mathcal{K}(f,C)}} \mathcal{P}\mathcal{K}(A, C) \xrightarrow{\ell} \mathbf{Sh}(\mathcal{K}(A, C))$$

extends through $\mathbf{Sh}(\mathcal{K}(B, C))$. This latter category is equally well the localisation of $\mathcal{P}\mathcal{K}(B, C)$ at the covering sieves for its topology, so it suffices to show that the displayed composite inverts each covering sieve; or equally that $\mathrm{Lan}_{\mathcal{K}(f,C)}$ sends covering sieves to maps in Σ. By assumption, $\mathcal{K}(f, C)$ preserves finite connected limits, and thus so does $\mathrm{Lan}_{\mathcal{K}(f,C)}$. In particular it preserves monomorphisms; of course, it also preserves colimits, and so it preserves epi-mono factorisations.

Now if $\phi \rightarrowtail \mathcal{Y}g$ is the sieve generated by a family $(\alpha_i : g_i \to g \mid i \in I)$ in $\mathcal{K}(B, C)$, then it is the second half of the epi-mono factorisation of $\sum_i \mathcal{Y}g_i \to \mathcal{Y}g$; thus its image under $\mathrm{Lan}_{\mathcal{K}(f,C)}$ is the second half of the epi-mono factorisation of $\sum_i \mathcal{Y}(g_i f) \to \mathcal{Y}(gf)$, and hence a covering sieve, since $(-) \circ f$ preserves covers. This proves that Σ is stable under precomposition with 1-cells; replacing $\mathcal{K}$ by $\mathcal{K}^{\mathrm{op}}$ proves the same for postcomposition. $\square$

We may now prove an analogue of Proposition 6.1, identifying $\mathbf{Sh}(\mathcal{K})$-enriched categories with locally étale 2-functors $\mathcal{C} \to \mathcal{K}$, and using this, we may identify join restriction categories with $\mathbf{Sh}(\Gamma(j\mathcal{S}))$-enriched categories. However, as before, the enriched functors between $\mathbf{Sh}(\mathcal{K})$-enriched categories are too limited to capture the general join restriction functors. As before, we rectify this by passing to a suitable weak double category, and enriching over that.

Suppose, then, that $\mathcal{K}$ is a 2-fc-site; we define a weak double category $\mathbb{S}\mathbf{h}(\mathcal{K})$ as follows. Its objects are those of $\mathcal{K}$, a vertical arrow $A \to B$ is an étale map $B \to A$ in $\mathcal{K}$, a horizontal arrow $A \nrightarrow B$ is a sheaf $U \in \mathbf{Sh}(\mathcal{K}(A, B))$, whilst a cell of the form (5.1) is a 2-cell

$$\begin{array}{ccc} A & \xrightarrow{U} & C \\ \ell(\mathcal{Y}f) \Big\uparrow & \Downarrow \alpha & \Big\uparrow \ell(\mathcal{Y}g) \\ B & \xrightarrow{V} & D \end{array}$$

in $\mathbf{Sh}(\mathcal{K})$. Composition of vertical morphisms is as in $\mathcal{K}$, that of horizontal morphisms is as in $\mathbf{Sh}(\mathcal{K})$ and cell composition is given by pasting in $\mathbf{Sh}(\mathcal{K})$.

An argument identical in form to that of Proposition 6.2 now proves that:

Proposition 7.12. *If $\mathcal{K}$ is a 2-fc-site, then $\mathbb{S}\mathbf{h}(\mathcal{K})$-**Cat** is 2-equivalent to the 2-category $\mathbf{2}$-**Cat** $/\!\!/_{\text{lét}}\mathcal{K}$ whose objects are locally étale 2-functors $F : \mathcal{C} \to \mathcal{K}$ with $\mathcal{C}$ locally small; whose morphisms are diagrams*

with H a 2-functor and γ a 2-natural transformation with étale components; and whose 2-cells $(H, \gamma) \to (H', \gamma')$ are 2-natural transformations $\theta : H \to H'$ with $\gamma'.G\theta = \gamma$.

Thus, defining $\mathbf{j}\mathbb{R} := \mathbb{S}\mathbf{h}(\Gamma(j\mathcal{S}))$, we immediately conclude from Proposition 7.12 and Theorem 7.9 that:

Theorem 7.13. *The 2-category of join restriction categories is 2-equivalent to the 2-category of $\mathbf{j}\mathbb{R}$-enriched categories.*

8. Range restriction categories

In this final section, we turn our attention from join restriction categories to *range restriction categories* [3]. A range restriction category is a restriction category $\mathcal{C}$ equipped with a *range* operator assigning to each map $f : A \to B$ a map $\hat{f} : B \to B$, subject to the following four axioms:

(RR1) $\overline{\hat{f}} = \hat{f}$ for all $f : A \to B$;
(RR2) $\hat{f}f = f$ for all $f : A \to B$;
(RR3) $\widehat{\hat{g}f} = \bar{g}\hat{f}$ for all $f : A \to B$ and $g : B \to C$;
(RR4) $g\hat{f} = \widehat{gf}$ for all $f : A \to B$ and $g : B \to C$.

A *range restriction functor* $F : \mathcal{C} \to \mathcal{D}$ is a restriction functor which also satisfies $F(\hat{f}) = \widehat{Ff}$ for each arrow f of $\mathcal{C}$. We write **rrCat** for the 2-category of range restriction categories, range restriction functors and total natural transformations.

By (RR1), each map $\hat{f}$ in a range restriction category is a restriction idempotent, whose intended interpretation is as the image of the map f. For instance, the category **Set**$_p$ of sets and partial functions is a range restriction category, when equipped with the range operator which to a partial function $f : A \rightharpoonup B$ assigns the partial function $\hat{f} : B \rightharpoonup B$ with

$$\hat{f}(b) = \begin{cases} b & \text{if } f(a) = b \text{ for some } a \in A; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Although we have presented it as extra structure, having a range operator is in fact a property of a restriction category:

Lemma 8.1. *(Cf. [3, §2.11].) A restriction category bears at most one range operator.*

Proof. Suppose that $(\hat{\ })$ and $(\tilde{\ })$ are two such operators. Then

$$\hat{f} = \widehat{\tilde{f}f} = \tilde{f}\hat{f} = \hat{f}\tilde{f} = \widetilde{\hat{f}f} = \tilde{f} . \qquad \square$$

The crucial fact for our purposes is that the property of having a range operator can be expressed purely in terms of the fundamental functor. Given a poset B and $b \in B$, we write $B/b \subseteq B$ for the downset $\{x \in B \mid x \leqslant b\}$. Any stable map of meet-semilattices $g : A \to B$ factors through $B/g(\top)$ as $g' : A \to B/g(\top)$, say, and by a *local left adjoint* for g, we mean a monotone map f which is left adjoint to g'. We now define a stable map of meet-semilattices $g : A \to B$ to be *open* [3, Definition 2.5] if it has a local left adjoint $f : B/g(\top) \to A$ satisfying *Frobenius reciprocity*:

$$f\big(g(a) \wedge b\big) = a \wedge f(b) \quad \text{for all } a \in A \text{ and } b \leqslant g(\top) \in B.$$

It is now easy to check that identity maps are open, composites of open maps are open, and that if $f \leqslant g$ and g is open, then so is f; from which it follows that the subcategory $o\mathcal{S}$ of $\mathcal{S}$ with all objects but only open maps is a hyperconnected sub-restriction category.

Proposition 8.2. *(Cf. [3, Proposition 2.13].) A restriction category $\mathcal{C}$ admits a range operator just when its fundamental functor $\mathcal{O} : \mathcal{C} \to \mathcal{S}$ factors through $o\mathcal{S}$.*

Proof. First suppose that $\mathcal{C}$ is a range restriction category. Then for each $f : A \to B$ in $\mathcal{C}$, we define a local left adjoint $f_!$ to f^* by $f_!(a) = \widehat{fa}$. To check adjointness, let $a \leqslant \bar{f} \in \mathcal{O}(A)$ and $b \in \mathcal{O}(B)$; we must show that $a \leqslant f^*(b)$ iff $f_!(a) \leqslant b$, i.e., that $\overline{bf}e = e$ iff $b\widehat{fa} = \widehat{fa}$. But if $b\widehat{fa} = a$, then

$$b\widehat{fa} = \widehat{bfa} = \widehat{f\overline{bf}a} = \widehat{fa};$$

and conversely, if $b\widehat{fa} = \widehat{fa}$, then since $a \leqslant \bar{f}$, we have

$$a = \bar{f}a = \overline{fa} = \overline{\widehat{fa}fa} = \overline{b\widehat{fa}fa} = \overline{bfa} = \overline{bf}a.$$

To check Frobenius reciprocity, we calculate that, for a and b as above, we have

$$f_!\big(f^*(b) \wedge a\big) = \widehat{f\overline{bf}a} = \widehat{bfa} = b\widehat{fa} = b \wedge f_!(a),$$

as required. This proves that if $\mathcal{C}$ admits a range operator, then its fundamental functor factors through $o\mathcal{S}$.

Suppose conversely that the fundamental functor of $\mathcal{C}$ factors through $o\mathcal{S}$. We define a range operator by $\hat{f} = f_!(\bar{f})$. Clearly this satisfies (RR1). For (RR2), note first that $\overline{f_!(\bar{f})}f = f^*(f_!(f^*(\top))) = f^*(\top) = \bar{f}$; whence $\hat{f}f = f_!(\bar{f})f = f f_!(\bar{f})f = f\bar{f} = f$ as required. For (RR3), let $f : A \to B$ and $g : B \to C$. It is easy to show that $\bar{g}_! : \mathcal{O}(B)/\bar{g} \to \mathcal{O}(B)$ is given by inclusion, and it follows that $(\bar{g}f)_!$ is just the restriction of $f_!$ to $\mathcal{O}(A)/(\bar{g} \wedge \bar{f})$. So now

$$\widehat{\bar{g}f} = (\bar{g}f)_!(\overline{\bar{g}f}) = f_!(\overline{\bar{g}f}) = f_!\big(f^*(\bar{g})\big) = f_!\big(f^*(\bar{g}) \wedge \bar{f}\big) = \bar{g} \wedge f_!(\bar{f}) = \bar{g}\hat{f}$$

as required, where the penultimate equality uses Frobenius reciprocity. Finally, for (RR4), let $f : A \to B$ and $g : B \to C$; it is again easy to show that $(g\hat{f})_!$ is the restriction of $g_!$ to $\mathcal{O}(B)/(\hat{f} \wedge \bar{g})$, and thus

$$\widehat{g\hat{f}} = (g\hat{f})_!(\overline{g\hat{f}}) = g_!(\bar{g}\hat{f}) = g_!\big(\bar{g} \wedge f_!(\bar{f})\big) = g_!\big(f_!(f^*(\bar{g}) \wedge \bar{f})\big) = g_!\big(f_!(\overline{gf})\big) = \widehat{gf}$$

as required, where the fourth equality again uses Frobenius reciprocity. $\square$

We now relate range restriction functors to the fundamental functor. A commutative square of stable maps as on the left in

$$
\begin{array}{ccc}
A & \xrightarrow{\;h\;} & B \\
f\downarrow & & \downarrow g \\
C & \xrightarrow{\;k\;} & D
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xleftarrow{\;h_!\;} & B/h(\top) \\
f\downarrow & & \downarrow g/h(\top) \\
C & \xleftarrow{\;k_!\;} & D/k(\top)
\end{array}
$$

is called *Beck–Chevalley* if f and g are total, h and k are open—with local left adjoints $h_!$ and $k_!$, say—and the square of stable maps on the right above is also commutative. If $F, G : \mathcal{C} \to \mathcal{S}$ are restriction functors taking values in open maps, then we call a total natural transformation $\gamma : F \Rightarrow G$ *Beck–Chevalley* if all of its naturality squares are Beck–Chevalley.

Proposition 8.3. *A restriction functor $F : \mathcal{C} \to \mathcal{D}$ between range restriction categories preserves the range operator just when the associated total natural transformation φ of* (3.1) *is Beck–Chevalley.*

Proof. The Beck–Chevalley condition says that, for all $f : A \to B$ in $\mathcal{C}$, we have commutativity on the left in:

$$
\begin{array}{ccc}
\mathcal{O}(A)/\bar{f} & \xrightarrow{\;\varphi_A/\bar{f}\;} & \mathcal{O}(FA)/\overline{Ff} \\
f_!\downarrow & & \downarrow (Ff)_! \\
\mathcal{O}(B) & \xrightarrow{\;\varphi_B\;} & \mathcal{O}(FB)
\end{array}
\qquad\qquad
\begin{array}{ccc}
a & \longmapsto & Fa \\
\downarrow & & \downarrow \\
\widehat{fa} & \longmapsto & F(\widehat{fa}) = \widehat{Ff}.Fa
\end{array}
$$

Evaluating at $a \leqslant \bar{f} \in \mathcal{O}(A)$ as on the right, this says that $\widehat{F(fa)} = F(\widehat{fa})$ for every $f : A \to B$ and $a \leqslant \bar{f} \in \mathcal{O}(A)$ in $\mathcal{C}$; which is easily equivalent to F's preserving the range operator. $\square$

From the preceding two results and Proposition 3.4, we thereby conclude that:

Theorem 8.4. *The 2-category **rrCat** of range restriction categories is 2-equivalent to the 2-category **rrCat**$'$ whose objects are hyper-connected restriction functors $F : \mathcal{C} \to \mathcal{S}$ which factor through $o\mathcal{S}$, whose morphisms are diagrams*
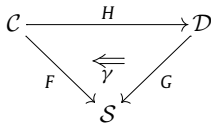
in **rCat** with γ Beck–Chevalley, and whose 2-cells $(H, \gamma) \to (H', \gamma')$ are total natural transformations $\theta : H \to H'$ with $\gamma'.G\theta = \gamma$.

Since range restriction categories are special kinds of restriction categories, they are equally well special kinds of $\mathbb{R}$-enriched categories, where we recall that $\mathbb{R} = \mathbb{P}(\Gamma S)$. What we will now show is that range restriction categories are in fact **r**$\mathbb{R}$-enriched categories for a suitable sub-double category **r**$\mathbb{R} \subset \mathbb{R}$, with this equivalence extending to the functors and natural transformations between them.

The objects and arrows of **r**$\mathbb{R}$ will be the same as $\mathbb{R}$. A horizontal arrow $A \to\!\!\!\to B$ of **r**$\mathbb{R}$ will be such an arrow in $\mathbb{R}$—thus a presheaf on $\Gamma(S)(A, B)$—whose associated discrete fibration el $U \to \Gamma(S)(A, B)$ factors through $\Gamma(oS)(A, B)$. A cell of **r**$\mathbb{R}$ may be described as follows. By Lemma 6.3, a cell of $\mathbb{R}$ of the form (5.1) can be identified with a functor α fitting into a commutative diagram
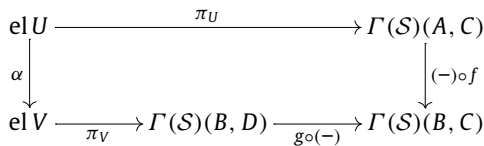
Such a cell will lie in **r**$\mathbb{R}$ if, firstly, U and V lie in **r**$\mathbb{R}$, so that π_U and π_V both take their image in open maps; and secondly, whenever given $(k, u) \in$ el U with $\alpha(k, u) = (k', u')$, say, the square
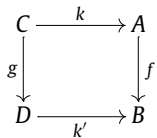
of stable maps is Beck–Chevalley.

Proposition 8.5. **r**$\mathbb{R}$ is a sub-double category of $\mathbb{R}$.

Proof. Because restriction idempotents in S are open maps, it follows that horizontal identities of $\mathbb{R}$ are in **r**$\mathbb{R}$. We next show that, given $U : A \to\!\!\!\to C$ and $W : C \to\!\!\!\to E$ in **r**$\mathbb{R}$, the composite WU is again in **r**$\mathbb{R}$. By the formula for Day convolution, an object of el WU lying over $m \in \Gamma(S)(A, E)$ is an element of the set

$$\int^{k,\ell} Uk \times W\ell \times \Gamma(S)(A, E)(m, \ell k)$$

so that an object of el (WU) can be represented by giving

$$(k : A \leftarrow C, \ u \in Uk) \in \text{el } U, \qquad (\ell : C \leftarrow E, \ w \in W\ell) \in \text{el } W \qquad \text{and} \qquad m \leqslant \ell k \text{ in } S. \tag{8.1}$$

Because U and W are in **r**$\mathbb{R}$, both k and ℓ are open, whence also $k\ell$. But now $m \leqslant \ell k$ implies that m is open. Thus el $WU \to \Gamma(S)(A, E)$ factors through $\Gamma(oS)(A, E)$ so that WU lies in **r**$\mathbb{R}$ as required.

It remains to show that cells of **r**$\mathbb{R}$ are closed under identities and composition. This is not hard to do for vertical and horizontal identities, and for composition along a horizontal boundary; we consider the case of composition along a vertical boundary in more detail. Given cells
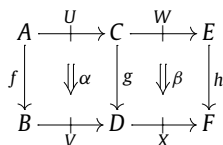
in **r**$\mathbb{R}$, we must show that their composite in $\mathbb{R}$ lies again in **r**$\mathbb{R}$. Now, given an object of el WU of the form (8.1), its image under $\beta\alpha :$ el $WU \to$ el XV is obtained as follows. If the images of (k, u) and (ℓ, w) under α and β are (k', u') and (ℓ', w') respectively, then we have a commutative diagram

$$\begin{array}{ccccc}
B' & \xleftarrow{\;k\;} & D & \xleftarrow{\;\ell'\;} & F \\
{\scriptstyle f}\big\uparrow & & {\scriptstyle g}\big\uparrow & & \big\uparrow{\scriptstyle h} \\
A & \xleftarrow{\;k\;} & C & \xleftarrow{\;\ell\;} & E
\end{array}$$

of stable maps, and a 2-cell $m \leqslant \ell k$ in $\mathcal{S}$. Because h is a discrete fibration in $\mathcal{S}$, we obtain from these data a stable map $m' : B \leftarrow F$ with $m' \leqslant \ell' k'$ and $hm' = mf$ in $\mathcal{S}$; explicitly, $m' = \ell' k'.\overline{mf}$. Now the image under $\beta\alpha$ of (8.1) is the object of $\mathrm{el}(XV)$ represented by the data

$$(k', u') \in \mathrm{el}\, V, \qquad (\ell', w') \in \mathrm{el}\, X \qquad \text{and} \qquad m' \leqslant \ell' k'.$$

For $\beta\alpha$ to lie in $\mathbf{r}\mathbb{R}$, we must show that the left-hand square of stable maps in

$$\begin{array}{ccc}
E \xrightarrow{\;m\;} A & & E \xrightarrow{\;\ell\;} C \xrightarrow{\;k\;} A \xrightarrow{\;\overline{m}\;} A \\
{\scriptstyle h}\big\downarrow \quad \big\downarrow{\scriptstyle f} \quad = & & {\scriptstyle h}\big\downarrow \quad \big\downarrow{\scriptstyle g} \quad \big\downarrow{\scriptstyle f} \quad \big\downarrow{\scriptstyle f} \\
F \xrightarrow[\;m'\;]{} B & & F \xrightarrow[\;\ell'\;]{} D \xrightarrow[\;k'\;]{} B \xrightarrow[\;\overline{mf}\;]{} B
\end{array}$$

is Beck–Chevalley. This square decomposes as the composite of the squares on the right. All three of those squares are Beck–Chevalley, the left two by assumption and the right-hand one since $(\overline{mf})_!$ and $(\overline{m})_!$ are given simply by inclusion. The result now follows from the easy observation that Beck–Chevalley squares compose. $\quad\square$

Given the manner in which we have defined $\mathbf{r}\mathbb{R}$, it is now an immediate consequence of Theorems 4.5 and 8.4 that:

Theorem 8.6. *The 2-category of range restriction categories is 2-equivalent to the 2-category of $\mathbf{r}\mathbb{R}$-enriched categories.*

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