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Lawvere theories, finitary monads and Cauchy-completion ☆

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ABSTRACT

We consider the equivalence of Lawvere theories and finitary monads on **Set** from the perspective of $\mathcal{F}$-enriched category theory, where $\mathcal{F}$ is the monoidal category of finitary endofunctors of **Set** under composition. We identify finitary monads with one-object $\mathcal{F}$-categories, and ordinary categories admitting finite powers (i.e., n-fold products of each object with itself) with $\mathcal{F}$-categories admitting a certain class Φ of absolute colimits; we then show that, from this perspective, the passage from a finitary monad to the associated Lawvere theory is given by completion under Φ-colimits. We also account for other phenomena from the enriched viewpoint: the equivalence of the algebras for a finitary monad with the models of the corresponding Lawvere theory; the functorial semantics in arbitrary categories with finite powers; and the existence of left adjoints to algebraic functors.

1. Introduction

At the heart of universal algebra is the notion of *equational theory*—a first-order theory without relation symbols whose axioms are exclusively of the form $(\forall x_1)\dots(\forall x_n)(\sigma = \tau)$. There is an elegant category-theoretic treatment of equational theories due to Lawvere [17]: for each equational theory $\mathcal{T}$, one may define a category $\mathbf{T}$ with finite powers (i.e., n-fold products $A^n = A \times \cdots \times A$ of each object with itself) such that models of $\mathcal{T}$ correspond to finite-power-preserving functors $\mathbf{T} \to \mathbf{Set}$. The objects of $\mathbf{T}$ are the distinct finite powers X^n of a fixed object X, whilst morphisms $X^n \to X^m$ are m-tuples of derived n-ary operations of $\mathcal{T}$; a category of this form is called a *Lawvere theory*.

A second way of treating equational theories categorically is using monads: with each equational theory $\mathcal{T}$, we associate a monad T on the category of sets whose algebras are exactly the $\mathcal{T}$-models. The value of T at a set X is given by the set of derived terms of the theory with free variables from X. Since each derived term involves only finitely many variables, the action of T is entirely determined by its behaviour on finite sets; formally, this says that the monad T is *finitary*, in the sense that its underlying endofunctor preserves filtered colimits.

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The formulations in terms of finitary monads and Lawvere theories are equivalent; more precisely, we have an equivalence of categories

$$\mathbf{Mnd}_f(\mathbf{Set}) \simeq \mathbf{Law} \tag{1.1}$$

which commutes (to within pseudonatural equivalence) with the functors sending a finitary monad to its category of algebras, and a Lawvere theory to its category of models (a *model* of a Lawvere theory $\mathbf{T}$ being, as above, a finite-power-preserving functor $\mathbf{T} \to \mathbf{Set}$). This equivalence was essentially established by Linton in [20]; for modern treatments the reader could consult, for instance, [1] or [24]; for a historical overview, see [9].

The equivalence (1.1) has been extended in many directions [21,24,23,15,2] to deal with other notions of algebraic structure: for example, ones with different kinds of arities for the operations, with different rules for handling variable contexts, or with different objects than sets bearing the structure. Yet as natural as these extensions are, they do not offer a compelling explanation as to why the monad–theory correspondence should exist in the first place.

This article will attempt such an explanation, making use of a seemingly unrelated insight of Lawvere: his description in [18] of metric spaces as enriched categories in the sense of [13]. His treatment emphasises particularly the process of completing an enriched category under *absolute* colimits, those colimit-types which are preserved by any functor. Applied to a metric space, seen as an enriched category, this completion yields the classical Cauchy completion, and the name *Cauchy-completion* has subsequently come to refer to the completion of any kind of enriched category under absolute colimits. A notable application of these ideas is [28], which identifies sheaves on a given site with certain Cauchy-complete categories enriched over an associated bicategory. Our application will use Cauchy-completion over a suitable enrichment base to explain the monad–theory correspondence.

In more detail, we will consider categories enriched in $\mathcal{F}$, the category of finitary endofunctors of $\mathbf{Set}$ with its compositional monoidal structure. Amongst the totality of such enriched categories, we find:

(1) Every finitary monad T on the category of sets; and
(2) Every ordinary category with finite powers, so in particular:
   (a) Every Lawvere theory $\mathbf{T}$;
   (b) The category $\mathbf{Set}$ of sets.

On the one hand, finitary monads as in (1) are precisely monoids in $\mathcal{F}$, thus, one-object $\mathcal{F}$-categories; on the other, we will identify ordinary categories as in (2) with the $\mathcal{F}$-categories admitting a certain class Φ of absolute colimits. The $\mathcal{F}$-categories of the form (2)—which we call *representable*—are reflective amongst all $\mathcal{F}$-categories, with reflector given by Cauchy-completion with respect to the class Φ; and the key to our reconstruction of the equivalence (1.1) is that this Cauchy-completion applied to a finitary monad T yields precisely the associated Lawvere theory $\mathbf{T}$.

This perspective also explains the interaction of the equivalence (1.1) with models. On viewing a finitary monad T or a Lawvere theory $\mathbf{T}$ as an $\mathcal{F}$-category, we find that the categories $\mathbf{Alg}(\mathsf{T})$ and $\mathbf{Mod}(\mathbf{T})$ of algebras or models are the respective categories of $\mathcal{F}$-functors from T or $\mathbf{T}$ into $\mathbf{Set}$. Since the $\mathcal{F}$-category of sets is representable, the universal property of the representable reflection asserts the equivalence

$$\mathbf{Alg}(\mathsf{T}) = \mathcal{F}\text{-}\mathbf{CAT}(\mathsf{T}, \mathbf{Set}) \simeq \mathcal{F}\text{-}\mathbf{CAT}(\mathbf{T}, \mathbf{Set}) = \mathbf{Mod}(\mathbf{T})$$

of the algebras of the monad with the models of the theory.

We will account for two further phenomena from the enriched-categorical perspective. The first is the possibility of taking models in categories other than $\mathbf{Set}$. This is most easily understood in the formulation using Lawvere theories: a model of a Lawvere theory $\mathbf{T}$ can be defined in any category $\mathcal{C}$ with finite powers as a finite-power-preserving functor $\mathbf{T} \to \mathcal{C}$. On the face of it, it is less clear how to take algebras in $\mathcal{C}$ of a finitary monad T in a way that is functorial in T and $\mathcal{C}$. This is where the enriched perspective is superior: both models of a Lawvere theory and algebras of a finitary monad in $\mathcal{C}$ may be defined with equal simplicity as $\mathcal{F}$-functors from the theory or the monad into $\mathcal{C}$, seen as an $\mathcal{F}$-category.

The second further point we consider is the construction of left adjoints to algebraic functors. Taking the Lawvere theory perspective, an *algebraic functor* is a functor $\mathbf{Mod}(\mathbf{T}, \mathcal{C}) \to \mathbf{Mod}(\mathbf{S}, \mathcal{C})$ induced by composition with a map $\mathbf{S} \to \mathbf{T}$ of Lawvere theories. It is known that such functors have left adjoints under rather general circumstances (see [12], for example); we will consider when they may be constructed by $\mathcal{F}$-enriched left Kan extension. It turns out that this is the case just when unenriched left Kan extensions along the ordinary functor $\mathbf{S} \to \mathbf{T}$ exist and distribute appropriately over finite powers.

In this article, we have only considered the classical correspondence between finitary monads and Lawvere theories; but each of the generalisations of the monad–theory correspondence listed above should also arise in this manner on replacing $\mathcal{F}$ by some other appropriate monoidal category or bicategory of endofunctors; in future work with Hyland, we will study generalised monad–theory correspondences using enrichment over the Kleisli bicategories of [5,8].

2. Finitary monads and their algebras via enriched categories

2.1. Finitary monads as $\mathcal{F}$-categories

In this section, we describe finitary monads on $\mathbf{Set}$ and their algebras from the perspective of enriched category theory. In the introduction, we stated that our base $\mathcal{F}$ for enrichment would be the category of finitary endofunctors of $\mathbf{Set}$, seen as monoidal under composition; however, following [14, Section 4], we will find it convenient to work not with $\mathbf{End}_f(\mathbf{Set})$ itself, but with an equivalent and more elementary category.

Let $\mathbf{F}$ denote the full subcategory of $\mathbf{Set}$ spanned by the finite cardinals. $\mathbf{F}$ is in fact the free category with finite colimits on 1, and so by [13, Proposition 5.41] the inclusion $I : \mathbf{F} \to \mathbf{Set}$ exhibits $\mathbf{Set}$ as the free completion of $\mathbf{F}$ under filtered colimits. Restriction and left Kan extension along I thus exhibits $\mathbf{End}_f(\mathbf{Set})$ as equivalent to the functor category $[\mathbf{F}, \mathbf{Set}]$. The compositional monoidal structure of $\mathbf{End}_f(\mathbf{Set})$ transports across the equivalence to yield a monoidal structure on $[\mathbf{F}, \mathbf{Set}]$ whose unit object is the inclusion I, and whose tensor product is defined as on the left in:

$$(A \otimes B)(n) = \int^{m \in \mathbf{F}} Am \times (Bn)^m \qquad [B, C](m) = \int_{n \in \mathbf{F}} \big[(Bn)^m, Cn\big].$$

Henceforth, we shall write $\mathcal{F}$ to denote $[\mathbf{F}, \mathbf{Set}]$ equipped with this monoidal structure. We record for future use that the tensor product is non-symmetric and *right closed*, meaning that each $(-) \otimes B : \mathcal{F} \to \mathcal{F}$ admits a right adjoint $[B, -]$, defined as on the right above. The category of monoids in $\mathcal{F}$ is, of course, equivalent to the category of monoids in $\mathbf{End}_f(\mathbf{Set})$ and so to the category of finitary monads on $\mathbf{Set}$, and so we have:

2.2. Proposition. *The category* $\mathbf{Mnd}_f(\mathbf{Set})$ *of finitary monads on* $\mathbf{Set}$ *is equivalent to the category of one-object* $\mathcal{F}$*-categories.*

2.3. Monad algebras as $\mathcal{F}$-functors

We now describe algebras for finitary monads and the maps between them in terms of $\mathcal{F}$-enriched functors and transformations. For this, we use an analysis which appears in can be traced back to [11, Section 3]; it is based on certain general considerations concerning monoidal actions, which may be found, for example, in [10].

Suppose that $\mathcal{V}$ is a monoidal category. By a *monoidal action* of $\mathcal{V}$ on a category $\mathcal{W}$, we mean a functor $\diamond : \mathcal{V} \times \mathcal{W} \to \mathcal{W}$ together with a strong monoidal structure on its transpose $\mathcal{V} \to [\mathcal{W}, \mathcal{W}]$ (viewing $[\mathcal{W}, \mathcal{W}]$ as strict monoidal under composition); to give this strong monoidal structure is equally to give natural isomorphisms $I \diamond X \cong X$ and $(A \otimes A') \diamond X \cong A \diamond (A' \diamond X)$ satisfying the evident coherence laws. The action is said to be *right closed* if each $(-) \diamond X : \mathcal{V} \to \mathcal{W}$ admits a right adjoint $\langle X, - \rangle : \mathcal{W} \to \mathcal{V}$, with counit components $\varepsilon : \langle X, Y \rangle \diamond X \to Y$, say. In these circumstances, the category $\mathcal{W}$ acquires a $\mathcal{V}$-enrichment, with hom-objects the $\langle X, Y \rangle$'s and identities and composition $I \to \langle X, X \rangle$ and $\langle Y, Z \rangle \otimes \langle X, Y \rangle \to \langle X, Z \rangle$ obtained by transposing the respective morphisms $I \diamond X \cong X$ and

$$\big(\langle Y, Z \rangle \otimes \langle X, Y \rangle\big) \diamond X \xrightarrow{\cong} \langle Y, Z \rangle \diamond \big(\langle X, Y \rangle \diamond X\big) \xrightarrow{1 \diamond \varepsilon} \langle Y, Z \rangle \diamond Y \xrightarrow{\varepsilon} Z$$

of $\mathcal{W}$ under the right closure adjunctions. Observe that, as in [10, Lemma 2.1], this structure makes $\mathcal{W}$ into a *tensored* $\mathcal{V}$-category in the sense of Section 3.4 below: the tensor of $X \in \mathcal{W}$ by $A \in \mathcal{V}$ is given by $A \diamond X$.

Suppose now that A is a monoid in $\mathcal{V}$; its image under the strong monoidal functor $\mathcal{V} \to [\mathcal{W}, \mathcal{W}]$ is then a monoid in $[\mathcal{W}, \mathcal{W}]$, hence a monad $A \diamond (-)$ on $\mathcal{W}$.

2.4. Proposition. *Given a right closed monoidal action $\diamond : \mathcal{V} \times \mathcal{W} \to \mathcal{W}$ and a monoid $A \in \mathcal{V}$, we have, on viewing A as one-object $\mathcal{V}$-category ΣA and equipping $\mathcal{W}$ with the $\mathcal{V}$-enrichment derived from the action, an isomorphism of categories $\mathcal{V}\text{-}\mathbf{CAT}(\Sigma A, \mathcal{W}) \cong (A \diamond -)\text{-}\mathbf{Alg}$.*

Proof. To give a $\mathcal{V}$-functor $\Sigma A \to \mathcal{W}$ is to give an object $X \in \mathcal{W}$ and a map $x : A \to \langle X, X \rangle$ in $\mathcal{V}$ making the diagrams

$$\begin{tikzcd} & I \arrow[dl, "j"'] \arrow[dr, "j"] & \\ A \arrow[rr, "x"'] & & \langle X, X \rangle \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A \otimes A \arrow[r, "x \otimes x"] \arrow[d, "m"'] & \langle X, X \rangle \otimes \langle X, X \rangle \arrow[d, "m"] \\ A \arrow[r, "x"'] & \langle X, X \rangle \end{tikzcd}$$

commute. Transposing under adjunction, this is equally to give an object $X \in \mathcal{W}$ and a map $A \diamond X \to X$ satisfying the two axioms for an $A \diamond (-)$-algebra. To give a $\mathcal{V}$-natural transformation $F \Rightarrow G : \Sigma A \to \mathcal{W}$ is to give a map $\varphi : I \to \langle X, Y \rangle$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "y"] \arrow[d, "x"'] & \langle Y, Y \rangle \arrow[r, "\langle Y, Y \rangle \otimes \varphi"] & \langle Y, Y \rangle \otimes \langle X, Y \rangle \arrow[d, "m"] \\ \langle X, X \rangle \arrow[r, "\varphi \otimes \langle X, X \rangle"'] & \langle X, Y \rangle \otimes \langle X, X \rangle \arrow[r, "m"'] & \langle X, Y \rangle \end{tikzcd}$$

commutes; which, transposing under adjunction and using the coherence constraint $I \diamond X \cong X$, is equally to give a map $X \to Y$ commuting with the $A \diamond (-)$-actions. This gives the isomorphism $\mathcal{V}\text{-}\mathbf{CAT}(\Sigma A, \mathcal{W}) \cong (A \diamond -)\text{-}\mathbf{Alg}$; naturality in A is easily verified. $\square$

We now apply the preceding generalities to the case $\mathcal{V} = \mathcal{F}$. Transporting the evident monoidal action of $\mathbf{End}_f(\mathbf{Set})$ on $\mathbf{Set}$ across the monoidal equivalence $\mathbf{End}_f(\mathbf{Set}) \simeq \mathcal{F}$ yields a monoidal action $\diamond : \mathcal{F} \times \mathbf{Set} \to \mathbf{Set}$, given as on the left in:

$$A \diamond X = \int^{n \in \mathbf{F}} An \times X^n \qquad \langle X, Y\rangle(n) = \mathbf{Set}\big(X^n, Y\big).$$

This action is right closed, with the right adjoints $\langle X, – \rangle$ being defined as on the right above. We thus obtain a canonical enrichment of the category of sets to an $\mathcal{F}$-category $\mathcal{S}$ with hom-objects given by $\mathcal{S}(X, Y) = \mathbf{Set}(X^{(-)}, Y)$; and by the preceding result, we conclude that:

2.5. Proposition. *The embedding of finitary monads into one-object $\mathcal{F}$-categories fits into a pseudocommuting triangle of functors*

$$\begin{tikzcd}
\mathbf{Mnd}_f(\mathbf{Set}) \arrow[rr] \arrow[dr, "(-)\text{-}\mathbf{Alg}"'] & \arrow[d, phantom, "\simeq"] & \mathcal{F}\text{-}\mathbf{CAT}. \arrow[dl, "\mathcal{F}\text{-}\mathbf{CAT}(-{,}\mathcal{S})"] \\
& \mathbf{CAT}^{\mathrm{op}} &
\end{tikzcd}$$

3. Representable $\mathcal{F}$-categories

We have now described finitary monads and their algebras in terms of $\mathcal{F}$-category theory, and in the next section, we will do the same for Lawvere theories and their models. We now set up the results that will be necessary to do this; as anticipated in the introduction, this will involve showing that ordinary categories admitting finite powers may identified with those $\mathcal{F}$-categories admitting a certain class of absolute colimits.

3.1. $\mathcal{F}$-categories, functors and transformations

We begin by unfolding the basic notions of $\mathcal{F}$-category theory. An $\mathcal{F}$-category $\mathcal{M}$ is given by the following data:

(i) A set of objects $\mathrm{ob}\,\mathcal{M}$;
(ii) For all $X, Y \in \mathrm{ob}\,\mathcal{M}$ and $n \in \mathbf{F}$, a homset $\mathcal{M}_n(X, Y)$;
(iii) For all $X, Y \in \mathrm{ob}\,\mathcal{M}$ and $\varphi : n \to m \in \mathbf{F}$, functorial reindexing operations

$$\varphi_* : \mathcal{M}_n(X, Y) \to \mathcal{M}_m(X, Y);$$

(iv) For all $X \in \mathrm{ob}\,\mathcal{M}$, an identity map $1_X \in \mathcal{M}_1(X, X)$; and
(v) For all $X, Y, Z \in \mathrm{ob}\,\mathcal{M}$, composition operations, natural in n and m:

$$\mathcal{M}_m(Y, Z) \times \mathcal{M}_n(X, Y)^m \to \mathcal{M}_n(X, Z)$$
$$(g, f_1, \ldots, f_m) \mapsto g \circ (f_1, \ldots, f_m),$$

obeying the following axioms, wherein we write $\pi_1, \ldots, \pi_n \in \mathcal{M}_n(X, X)$ for the images of the element $1_A \in \mathcal{M}_1(X, X)$ under the n distinct maps $1 \to n$ in $\mathbf{F}$:

(vi) $\pi_i \circ (f_1, \ldots, f_n) = f_i$;
(vii) $g \circ (\pi_1, \pi_2, \ldots, \pi_n) = g$;
(viii) $h \circ (g_1 \circ (f_1, \ldots, f_k), \ldots, g_j \circ (f_1, \ldots, f_k)) = (h \circ (g_1, \ldots, g_j)) \circ (f_1, \ldots, f_k)$.

An $\mathcal{F}$-functor $F : \mathcal{M} \to \mathcal{N}$ is given by an assignation on objects and assignations on homsets $\mathcal{M}_n(X, Y) \to \mathcal{N}_n(FX, FY)$ which are natural in n and preserve composition and identities, whilst an $\mathcal{F}$-transformation $\alpha : F \Rightarrow G$ is given by elements $\alpha_X \in \mathcal{N}_1(FX, GX)$ such that

$$\alpha_Y \circ Ff = Gf \circ (\alpha_X \circ \pi_1, \dots, \alpha_X \circ \pi_n) \tag{3.1}$$

for all $f \in \mathcal{M}_n(X,Y)$. Every $\mathcal{F}$-category $\mathcal{M}$ has an underlying ordinary category $V\mathcal{M}$ with objects those of $\mathcal{M}$ and homsets $V\mathcal{M}(X,Y) = \mathcal{M}_1(X,Y)$; the evident extension to 1- and 2-cells yields a forgetful 2-functor $V : \mathcal{F}\text{-}\mathbf{CAT} \to \mathbf{CAT}$.

3.2. Remark. Axioms (v) and (vii) in the definition of $\mathcal{F}$-category force the maps in (iii) to be given by $\varphi_*(g) = g \circ (\pi_{\varphi(1)}, \dots, \pi_{\varphi(n)})$; and in fact, this leads to an alternative axiomatisation of $\mathcal{F}$-categories. Suppose we are given:

(i) A set of objects $\mathrm{ob}\,\mathcal{M}$;
(ii) For all $X, Y \in \mathrm{ob}\,\mathcal{M}$ and $n \in \mathbf{F}$, a homset $\mathcal{M}_n(X,Y)$;
(iv′) For all $X \in \mathrm{ob}\,\mathcal{M}$ and $n \in \mathbf{F}$, projection maps $\pi_1, \dots, \pi_n \in \mathcal{M}_n(X,X)$;
(v′) For all $X, Y, Z \in \mathrm{ob}\,\mathcal{M}$, composition operations:

$$\begin{aligned} \mathcal{M}_m(Y,Z) \times \mathcal{M}_n(X,Y)^m &\to \mathcal{M}_n(X,Z) \\ (g, f_1, \dots, f_m) &\mapsto g \circ (f_1, \dots, f_m), \end{aligned}$$

such that axioms (vi)–(viii) are verified. Then we define identities as in (iv) by $1_A = \pi_1 \in \mathcal{M}_1(A,A)$, and functorial reindexing maps as in (iii) by the above formula; on doing so, (v′) becomes natural in n and m, so yielding (v). This alternative axiomatisation is a many-object version of the universal algebraists' notion of *abstract clone* [6]. In terms of this axiomatisation, an $\mathcal{F}$-functor $\mathcal{M} \to \mathcal{N}$ is given by assignations on objects and on homs which preserve composition and the projection maps.

3.3. Remark. An $\mathcal{F}$-category also admits a *linear composition* operation

$$\begin{aligned} \mathcal{M}_m(Y,Z) \times \prod_{i=1}^m \mathcal{M}_{n_i}(X,Y) &\to \mathcal{M}_{\sum_i n_i}(X,Z) \\ (g, f_1, \dots, f_m) &\mapsto g \otimes (f_1, \dots, f_m) \end{aligned} \tag{3.2}$$

given by $g \otimes (f_1, \dots, f_n) = g \circ ((\iota_1)_*(f_1), \dots, (\iota_n)_*(f_n))$ with $(\iota_j : n_j \to \sum_i n_i)_{j=1}^m$ the coproduct injections. This composition, together with the identity elements in (iv) and reindexing maps in (iii), makes $\mathcal{M}$ into a cartesian multicategory (called a *Gentzen multicategory* in [16]), where we take the set of multimaps $X_1, \dots, X_n \to Y$ to be empty unless $X_1 = \dots = X_n = X$, in which case we take it to be $\mathcal{M}_n(X,Y)$. This in fact gives a further alternative axiomatisation[1] of $\mathcal{F}$-categories: they are precisely the cartesian multicategories in which every multimap $X_1, \dots, X_n \to Y$ has $X_1 = \dots = X_n$; the key point is that composition (v) is definable in terms of (3.2) and (iii) as:

$$g \circ (f_1, \dots, f_m) = (\pi_1)_*\big(g \otimes (f_1, \dots, f_m)\big) \quad \text{(with } \pi_1 : n \times m \to n \text{ the projection).}$$

3.4. Tensors in $\mathcal{F}$-categories

The relevant colimits for enriched category theory are the *weighted* (there called *indexed*) limits of [13, Chapter 3]. For the moment, we shall need only the following case of the general notion. Given $\mathcal{V}$ a right-

[1] The correspondence between this axiomatisation and the original one corresponds to the correspondence between the "multiplicative" and "additive" treatment of contexts in the classical sequent calculus. The basic calculation underlying these correspondences is that, for cartesian monoidal $\mathcal{C}$, the convolution monoidal structure on $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is again cartesian monoidal.

closed[2] monoidal category and $\mathcal{C}$ a $\mathcal{V}$-category, a *tensor* of $X \in \mathcal{C}$ by $A \in \mathcal{V}$ is an object Z of $\mathcal{C}$ and map $i : A \to \mathcal{C}(X, Z)$ in $\mathcal{V}$ such that for all $Y \in \mathcal{C}$, the composite

$$\mathcal{C}(Z,Y) \xrightarrow{\mathcal{C}(X,-)} \big[\mathcal{C}(X,Z), \mathcal{C}(X,Y)\big] \xrightarrow{[i,1]} \big[A, \mathcal{C}(X,Y)\big] \tag{3.3}$$

is invertible in $\mathcal{V}$; we may sometimes write Z as $A \otimes X$, or say that i *exhibits* Z *as* $A \otimes X$. Taking now $\mathcal{V} = \mathcal{F}$ and $A = y_n = \mathbf{F}(n, -)$, we see that, for an $\mathcal{F}$-category $\mathcal{M}$ and an object $X \in \mathcal{M}$, a tensor of X by y_n is given by an object Z and map $i : y_n \to \mathcal{C}(X, Z)$—which by the Yoneda lemma is equally an element $i \in \mathcal{C}_n(X, Z)$—such that for all $Y \in \mathcal{C}$, the map (3.3) is invertible. Unfolding the definitions, this says that for all $Y \in \mathcal{C}$ and $k \in \mathbf{F}$, the function

$$\begin{aligned} \mathcal{C}_k(Z,Y) &\to \mathcal{C}_{kn}(X,Y) \\ g &\mapsto g \otimes (i, \ldots, i) \end{aligned} \tag{3.4}$$

is invertible; here we use the linear composition operation of (3.2).

3.5. Proposition. *Let $\mathcal{M}$ be an $\mathcal{F}$-category. For all $X, Z \in \mathcal{M}$, the following data are equivalent*:

(a) *An element $i \in \mathcal{M}_n(X, Z)$ exhibiting Z as $y_n \otimes X$;*
(b) *Elements $p_1, \ldots, p_n \in \mathcal{M}_1(Z, X)$ exhibiting Z as the (enriched) power X^n;*
(c) *Elements i and $p_1, \ldots, p_n$ as above such that*

$$i \circ (p_1, \ldots, p_n) = 1_Z \quad \textit{and} \quad p_k \circ i = \pi_k \quad \textit{for all } 1 \leqslant k \leqslant n. \tag{3.5}$$

It follows that, in an $\mathcal{F}$-category, tensors by representables y_n are absolute colimits *in the sense of being preserved by any $\mathcal{F}$-functor.*

The universal property asserted in (b) of the maps $p_1, \ldots, p_n$ is that, for all $Y \in \mathcal{C}$ and $k \in \mathbf{F}$, the map of homsets $\mathcal{C}_k(Y, Z) \to \mathcal{C}_k(Y, X)^n$ given by postcomposition with $p_1, \ldots, p_n$ is invertible. In particular, this implies that Z is the power X^n in the underlying ordinary category $V\mathcal{M}$.

Proof. Given i as in (a), we define $p_1, \ldots, p_n$ as in (c) by the universal property of the tensor applied to the maps $\pi_1, \ldots, \pi_n \in \mathcal{M}_n(X, X)$; then the right-hand equalities in (3.5) are immediate, and the left-hand one follows on precomposing with i and applying the universal property. Conversely, given (c), we obtain the inverse to (3.4) required for (a) by sending $h \in \mathcal{C}_{kn}(X, Y)$ to the composite $h \circ (p_1 \circ \pi_1, \ldots, p_1 \circ \pi_k, \ldots, p_n \circ \pi_1, \ldots, p_n \circ \pi_k) \in \mathcal{C}_k(Z, Y)$.

On the other hand, given $p_1, \ldots, p_n$ as in (b), we define i as in (c) by the universal property of the power applied to the family $(\pi_1, \ldots, \pi_n) \in \mathcal{C}_n(X, X)^n$; then the right-hand equalities in (3.5) are immediate, and the left-hand one follows on postcomposing with $p_1, \ldots, p_n$ and applying the universal property. Conversely, given (c), we obtain an inverse $\mathcal{C}_k(Y, X)^n \to \mathcal{C}_k(Y, Z)$ for postcomposition with $p_1, \ldots, p_n$, as required for (b), by the mapping $(g_1, \ldots, g_n) \mapsto i \circ (g_1, \ldots, g_n)$.

Finally, since tensors by representables admit the equational reformulation in (c), they are clearly preserved by any $\mathcal{F}$-functor. □

3.6. Remark. The above direct proof could also be deduced from the general considerations of [26] on absolute colimits. Applied to the case of $\mathcal{V}$-enriched tensors, the main theorem of ibid. states that tensors

[2] Actually, [13] assumes a *symmetric* monoidal closed base $\mathcal{V}$, but the definition extends without fuss to the non-symmetric, right-closed case.

by $A \in \mathcal{V}$ are absolute just when A admits a *left dual* A^o in $\mathcal{V}$, meaning that we have maps $\eta : I \to A \otimes A^o$ and $\varepsilon : A^o \otimes A \to I$ satisfying the usual triangle identities for an adjunction; moreover, tensors by A may then be identified with *cotensors* (the dual limit notion) by A^o. Specialising to the situation at hand, the object $A = y_n$ of $\mathcal{F}$ has left dual $A^o = h_n = n \times (-)$, since h_n and y_n correspond to the adjoint finitary endofunctors $(-) \times n \dashv (-)^n$ of **Set**; and so we conclude that tensors by y_n are absolute and correspond to cotensors by h_n. Since h_n is isomorphic to the n-fold coproduct $I + \cdots + I$ of the unit object, a cotensor of X by h_n is equally well an $\mathcal{F}$-enriched power X^n, which gives the equivalence (a) $\Leftrightarrow$ (b) of Proposition 3.5; a more refined analysis of the general case also yields the formulation in (c).

3.7. Representable $\mathcal{F}$-categories

We define an $\mathcal{F}$-category $\mathcal{M}$ to be *representable* if it admits all tensors by y_n's. The terminology is motivated by the observation in Remark 3.3 that an $\mathcal{F}$-category can be seen as a particular kind of multicategory; when seen in this way, our notion of representability reduces to the standard notion of representability (as given, for example, in [19, Definition 3.3.1]) for the non-empty homs of this multicategory.

Let us write $\mathcal{F}\text{-}\mathbf{CAT}_{\mathrm{rep}}$ for the full sub-2-category on the representable $\mathcal{F}$-categories. By (a) $\Rightarrow$ (b) in Proposition 3.5, the underlying category $V\mathcal{M}$ of a representable $\mathcal{F}$-category admits all finite powers; and by the absoluteness of tensors by representables, the underlying functor VF of any $\mathcal{F}$-functor $F : \mathcal{M} \to \mathcal{N}$ between representable $\mathcal{F}$-categories preserves finite powers. Thus the restriction of the underlying category 2-functor to representable $\mathcal{F}$-categories factors through $\mathbf{CAT}_{\mathrm{fp}}$, the 2-category of categories with finite powers and finite-power-preserving functors.

3.8. Proposition. *The* 2*-functor* $V : \mathcal{F}\text{-}\mathbf{CAT}_{\mathrm{rep}} \to \mathbf{CAT}_{\mathrm{fp}}$ *is an equivalence of* 2*-categories.*

Bearing in mind the identification of $\mathcal{F}$-categories with particular cartesian multicategories, this result can be seen as part of the equivalence between cartesian multicategories and categories with finite products (cf. [16]).

Proof. We exhibit a pseudoinverse 2-functor $R : \mathbf{CAT}_{\mathrm{fp}} \to \mathcal{F}\text{-}\mathbf{CAT}_{\mathrm{rep}}$. For a category $\mathcal{C}$ with finite powers, we take $R\mathcal{C}$ to have the same collection of objects, hom-sets $R\mathcal{C}_n(X, Y) = \mathcal{C}(X^n, Y)$, composition given by:

$$\mathcal{C}(Y^m, Z) \times \mathcal{C}(X^n, Y)^m \to \mathcal{C}(X^n, Z)$$
$$(g, f_1, \ldots, f_m) \mapsto g \circ \langle f_1, \ldots, f_m \rangle$$

and projections $\pi_1, \ldots, \pi_n \in R\mathcal{C}_n(X, X) = \mathcal{C}(X^n, X)$ given by product projections. Note that $R\mathcal{C}$ is representable, since for any $X \in R\mathcal{C}$ and $n \in \mathbf{F}$, the elements $1_{X^n} \in R\mathcal{C}_n(X, X^n)$ and $\pi_1, \ldots, \pi_n \in R\mathcal{C}_1(X^n, X)$ satisfy (3.5) and so exhibit X^n as a tensor of X by y_n. Given next a functor $F : \mathcal{C} \to \mathcal{D}$ in $\mathbf{CAT}_{\mathrm{fp}}$, the $\mathcal{F}$-functor RF has the same action on objects and action on homs:

$$\mathcal{C}(X^n, Y) \xrightarrow{F} \mathcal{D}(F(X^n), FY) \xrightarrow{\cong} \mathcal{D}((FX)^n, FY).$$

Finally, for any $\alpha : F \Rightarrow G$ in $\mathbf{CAT}_{\mathrm{fp}}$ we take $R\alpha$ to be the $\mathcal{F}$-transformation with the same components; $\mathcal{F}$-naturality is easily verified. It is clear that the 2-functor R so defined satisfies $VR \cong 1$, and it remains to show that $RV \cong 1$.

Given $\mathcal{M}$ a representable $\mathcal{F}$-category, $RV\mathcal{M}$ has the same objects and hom-sets $(RV\mathcal{M})_n(X, Y) = \mathcal{M}_1(X^n, Y)$, where X^n is a chosen tensor of X by y_n. The projection maps in $RV\mathcal{M}_n(X, X) = \mathcal{M}_1(X^n, X)$ are the maps $p_1, \ldots, p_n$ exhibiting X^n as the n-fold power of X; whilst composition is given by

$$\begin{aligned}\mathcal{M}_1\big(Y^m, Z\big) \times \mathcal{M}_1\big(X^n, Y\big)^m &\to \mathcal{M}_1\big(X^n, Z\big)\\ (g, f_1, \ldots, f_m) &\mapsto g \circ i \circ (f_1, \ldots, f_m)\end{aligned}$$

where $i \in \mathcal{M}_m(Y, Y^m)$ exhibits Y^m as the tensor of Y by y_m. We define an identity-on-objects $\mathcal{F}$-functor $RV\mathcal{M} \to \mathcal{M}$ with action on homs given by

$$\begin{aligned}\mathcal{M}_1\big(X^n, Y\big) &\to \mathcal{M}_n(X, Y)\\ g &\mapsto g \circ i\end{aligned}$$

where $i \in \mathcal{M}_n(X, X^n)$ exhibits X^n as the tensor of X by y_n. The universal property implies that the actions on homs are invertible, and it is immediate from the definitions that composition and projections are preserved. We thus have an isomorphism $RV\mathcal{M} \to \mathcal{M}$; the naturality of these isomorphisms in $\mathcal{M}$ is now easily verified. $\square$

4. Lawvere theories and their models

4.1. Lawvere theories as $\mathcal{F}$-categories

As in the introduction, a *Lawvere theory* is a category $\mathbf{T}$ whose objects are the distinct finite powers X^n of a distinguished object X, whilst a morphism of Lawvere theories is a functor $\mathbf{T} \to \mathbf{T}'$ strictly preserving finite powers and the distinguished object. These definitions immediately translate via Proposition 3.8 into the language of $\mathcal{F}$-category theory. We call a representable $\mathcal{F}$-category $\mathcal{M}$ *Lawvere* if its objects are the tensors $y_n \otimes X$ by distinct representables of a distinguished object X, and call a functor between two such categories *Lawvere* if it strictly preserves the distinguished object and its chosen tensors. It is now immediate from Proposition 3.8 that:

4.2. Proposition. *The category of Lawvere theories is equivalent to the category of Lawvere $\mathcal{F}$-categories and Lawvere functors.*

4.3. Models of Lawvere theories as $\mathcal{F}$-functors

A *model* of a Lawvere theory $\mathbf{T}$ is a finite-power-preserving functor $\mathbf{T} \to \mathbf{Set}$; which by Proposition 3.8, is equally an $\mathcal{F}$-functor $R\mathbf{T} \to R(\mathbf{Set})$. Note that $R(\mathbf{Set})$ is precisely the $\mathcal{F}$-category $\mathcal{S}$ defined before Proposition 2.5, and so we have:

4.4. Proposition. *The functor which views a Lawvere theory as an $[\mathbf{F}, \mathbf{Set}]$-category fits into a pseudocommuting triangle*

$$\begin{tikzcd} \mathbf{Law} \arrow[rr] \arrow[dr, "(-)\text{-}\mathbf{Mod}"'] & \arrow[d, phantom, "\simeq"] & \mathcal{F}\text{-}\mathbf{CAT}. \arrow[dl, "\mathcal{F}\text{-}\mathbf{CAT}(-{,}\mathcal{S})"] \\ & \mathbf{CAT}^{\mathrm{op}} & \end{tikzcd}$$

5. The equivalence of finitary monads and Lawvere theories

5.1. The representable completion

Having described both finitary monads on $\mathbf{Set}$ and Lawvere theories in terms of $\mathcal{F}$-category theory, we now describe their equivalence in the same terms. The following result is the key to doing so.

5.2. Proposition. *The inclusion 2-functor $\mathcal{F}$-$\mathbf{CAT}_{\mathrm{rep}} \to \mathcal{F}$-$\mathbf{CAT}$ admits a left biadjoint L.*

As with Proposition 3.8, this result is an essentially standard one about representability in multicategories; see, for instance, [7, Section 7]. The point is not that the result is new, but rather that the proof we give involves only standard enriched-categorical notions.

Proof. An $\mathcal{F}$-category is representable just when it admits certain absolute colimits, namely tensors by representables; so L must be given by completion under these colimits. By [13, Proposition 5.62] and the absoluteness of the colimits at issue, the unit $J : \mathcal{M} \to L\mathcal{M}$ of the biadjunction at $\mathcal{M}$ is characterised by three properties: (i) J is fully faithful; (ii) $L\mathcal{M}$ is representable; (iii) every object of $L\mathcal{M}$ is a tensor by some y_n of an object of $\mathcal{M}$. We may thus obtain $L\mathcal{M}$ by first forming the *Cauchy completion* $Q\mathcal{M}$ of $\mathcal{M}$—its completion under all absolute colimits, described in [3, Section 1]—and then taking $L\mathcal{M}$ to be the closure of $\mathcal{M}$ in $Q\mathcal{M}$ under tensors by representables.

Since tensors by representables satisfy $y_1 \otimes X \cong X$ and $y_n \otimes (y_m \otimes X) \cong (y_n \otimes y_m) \otimes X \cong y_{nm} \otimes X$, this closure process converges after one step, and so we may as well take the objects of $L\mathcal{M}$ to be of the form $X^{(n)}$, representing the tensor of $X \in \mathcal{M}$ by y_n. Now from the description of $Q\mathcal{M}$ given in [3], the hom-objects of $L\mathcal{M}$ are given by

$$L\mathcal{M}\big(X^{(n)}, Y^{(m)}\big) = Q\mathcal{M}(y_n \otimes X, y_m \otimes Y) = y_m \otimes \mathcal{M}(X, Y) \otimes h_n,$$

where as in Remark 3.6, $h_n = (-) \times n$ is the left dual of y_n in $\mathcal{F}$. Identities and composition in $L\mathcal{M}$ are obtained from those of $\mathcal{M}$ together with the unit maps $I \to y_n \otimes h_n$ (for the identities) and counit maps $h_m \otimes y_m \to I$ (for the composition). Spelling this out explicitly, we have that

$$L\mathcal{M}_k\big(X^{(n)}, Y^{(m)}\big) = \mathcal{M}_{nk}(X, Y)^m$$

with identities given by $(\pi_1, \ldots, \pi_n) \in L\mathcal{M}_1(X^{(n)}, X^{(n)}) = \mathcal{M}_n(X, X)^n$, and composition

$$\mathcal{L}\mathcal{M}_p\big(Y^{(m)}, Z^{(k)}\big) \times \mathcal{L}\mathcal{M}_q\big(X^{(n)}, Y^{(m)}\big)^p \to L\mathcal{M}_q\big(X^{(n)}, Z^{(k)}\big)$$

by

$$\begin{aligned} \mathcal{M}_{mp}(Y, Z)^k \times \mathcal{M}_{nq}(X, Y)^{mp} &\to \mathcal{M}_{nq}(X, Z)^k \\ (f_1, \ldots, f_k, \vec{g}) &\mapsto (f_1 \circ \vec{g}, \ldots, f_k \circ \vec{g}). \end{aligned}$$

The tensor of $X^{(n)}$ by y_m is $X^{(nm)}$, as witnessed by the element $i = (\pi_1, \ldots, \pi_{nm}) \in L\mathcal{M}_m(X^{(n)}, X^{(nm)}) = \mathcal{M}_{nm}(X, X)^{nm}$. Finally, the reflection map $\mathcal{M} \to L\mathcal{M}$ sends X to $X^{(1)}$ and is the identity on homs. □

5.3. Remark. A priori, the universal property of biadjunction only makes L pseudofunctorial in $\mathcal{M}$; however, it is easy to see that we may make it strictly 2-functorial, by defining its action $LF : L\mathcal{M} \to L\mathcal{N}$ on morphisms by $(LF)(X^{(n)}) = (FX)^{(n)}$ and with the evident action on homs.

We have now developed enough $\mathcal{F}$-category theory to prove:

5.4. Proposition. *There is an equivalence $\mathbf{Mnd}_f(\mathbf{Set}) \simeq \mathbf{Law}$, fitting into a pseudocommuting triangle of functors*

$$\begin{tikzcd} \mathbf{Mnd}_f(\mathbf{Set}) \arrow[rr] \arrow[dr, "(-)\text{-}\mathbf{Alg}"'] & \arrow[d, phantom, "\simeq"] & \mathbf{Law}. \arrow[dl, "(-)\text{-}\mathbf{Mod}"] \\ & \mathbf{CAT}^{\mathrm{op}} & \end{tikzcd} \tag{5.1}$$

Proof. To show that $\mathbf{Mnd}_f(\mathbf{Set}) \simeq \mathbf{Law}$, it suffices by Propositions 2.2 and 4.2 to exhibit an equivalence between the category $\mathcal{A}$ of one-object $\mathcal{F}$-categories and the category $\mathcal{B}$ of Lawvere $\mathcal{F}$-categories and Lawvere functors. In one direction, there is a functor $\mathcal{B} \to \mathcal{A}$ sending each Lawvere $\mathcal{M}$ to the one-object sub-$\mathcal{F}$-category $\mathcal{M}_X$ on the distinguished object X. In the other, if $\mathcal{M}$ is an $\mathcal{F}$-category with unique object X, then $L\mathcal{M}$ becomes Lawvere when equipped with the distinguished object $X^{(1)}$; and so we have a functor $\mathcal{A} \to \mathcal{B}$. If $\mathcal{M}$ has one object, then clearly $(L\mathcal{M})_{X^{(1)}} \cong \mathcal{M}$; so the composite $\mathcal{A} \to \mathcal{B} \to \mathcal{A}$ is isomorphic to the identity. On the other hand, if $\mathcal{M}$ is Lawvere, then the inclusion $\mathcal{M}_X \to \mathcal{M}$ satisfies conditions (i)–(iii) from the proof of Proposition 5.2, and so exhibits $\mathcal{M}$ as the free representable $\mathcal{F}$-category on $\mathcal{M}_X$; thus $L\mathcal{M}_X \simeq \mathcal{M}$. This equivalence is in fact bijective on objects, so that $L\mathcal{M}_X \cong \mathcal{M}$ and the composite $\mathcal{B} \to \mathcal{A} \to \mathcal{B}$ is isomorphic to the identity, as required.

Finally, we must show that the triangle (5.1) commutes to within pseudonatural equivalence. Consider the diagram

$$\begin{tikzcd}
\mathbf{Mnd}_f(\mathbf{Set}) \arrow[rr] \arrow[d] \arrow[drr, phantom, "\cong"] & & \mathbf{Law}. \arrow[d] \\
\mathcal{F}\text{-}\mathbf{CAT} \arrow[rr, "L"] \arrow[dr, "\mathcal{F}\text{-}\mathbf{CAT}(-,\mathcal{S})"'] & \simeq & \mathcal{F}\text{-}\mathbf{CAT}_{\mathrm{rep}} \arrow[dl, "\mathcal{F}\text{-}\mathbf{CAT}(-,\mathcal{S})"] \\
& \mathbf{CAT}^{\mathrm{op}} &
\end{tikzcd}$$

The top square commutes to within isomorphism by our construction of the equivalence $\mathbf{Mnd}_f(\mathbf{Set}) \simeq \mathbf{Law}$; whilst the lower triangle commutes to within pseudonatural equivalence because L is a bireflector into representable $\mathcal{F}$-categories and $\mathcal{S}$ is representable. Finally, by Propositions 2.5 and 4.4, the composites down the left and the right are pseudonaturally equivalent to $(-)$-**Alg** and $(-)$-**Mod** respectively; whence the result. □

6. Functorial semantics

One advantage of Lawvere theories over finitary monads is the relative ease with which we may consider models in categories other than **Set**. A model of a Lawvere theory $\mathbf{T}$ in a category $\mathcal{C}$ with finite powers is simply a finite-power-preserving functor $\mathbf{T} \to \mathcal{C}$, and the formulation makes it apparent that any finite-power-preserving functor between theories $\mathbf{T} \to \mathbf{S}$ or semantic domains $\mathcal{C} \to \mathcal{D}$ induces a functor $\mathbf{Mod}(\mathbf{S}, \mathcal{C}) \to \mathbf{Mod}(\mathbf{T}, \mathcal{C})$ or $\mathbf{Mod}(\mathbf{T}, \mathcal{C}) \to \mathbf{Mod}(\mathbf{T}, \mathcal{D})$ by pre- or postcomposition, respectively. For a finitary monad on **Set**, by contrast, it requires work to define algebras in other categories, and further work to verify the functoriality of such a definition in the monad T and the semantic domain $\mathcal{C}$.

The perspective of $\mathcal{F}$-category theory dissolves this apparent distinction. We retain the functorial semantics for Lawvere theories by defining $\mathbf{Mod}(\mathbf{T}, \mathcal{C}) = \mathcal{F}\text{-}\mathbf{CAT}(R\mathbf{T}, R\mathcal{C})$, but now have an equally clear functorial semantics for monads on taking $\mathbf{Alg}(\mathsf{T}, \mathcal{C}) = \mathcal{F}\text{-}\mathbf{CAT}(\Sigma\mathsf{T}, R\mathcal{C})$. Moreover, the two kinds of semantics are equivalent: if $\mathbf{T}$ is the Lawvere theory corresponding to T, then $R\mathbf{T}$ is a representable completion of ΣT, so that by the universal property of such completion,

$$\mathbf{Alg}(\mathsf{T}, \mathcal{C}) = \mathcal{F}\text{-}\mathbf{CAT}(\Sigma\mathsf{T}, R\mathcal{C}) \simeq \mathcal{F}\text{-}\mathbf{CAT}(R\mathbf{T}, R\mathcal{C}) = \mathbf{Mod}(\mathbf{T}, \mathcal{C})$$

pseudonaturally in the representable $\mathcal{F}$-category $\mathcal{C}$.

In elementary terms, a T-algebra $\Sigma\mathsf{T} \to R\mathcal{C}$ is given by an object $X \in \mathcal{C}$ and a monad morphism $\mathsf{T} \to \mathsf{End}(X)$; here, $\mathsf{End}(X) = R\mathcal{C}(X, X)$ is the finitary monad on **Set** with action on finite sets $n \mapsto \mathcal{C}(X^n, X)$. A map between T-algebras is a morphism $f : X \to Y$ of $\mathcal{C}$ making the square of finitary endofunctors

$$\begin{tikzcd} T \arrow[r] \arrow[d] & R\mathcal{C}(X,X) \arrow[d, "R\mathcal{C}(X,f)"] \\ R\mathcal{C}(Y,Y) \arrow[r, "R\mathcal{C}(f,X)"] & R\mathcal{C}(X,Y) \end{tikzcd}$$

commute; the functor $\mathbf{Alg}(\mathsf{T},\mathcal{C}) \to \mathbf{Alg}(\mathsf{S},\mathcal{C})$ induced by a monad map $\mathsf{S} \to \mathsf{T}$ sends $\mathsf{T} \to \mathsf{End}(X)$ to $\mathsf{S} \to \mathsf{T} \to \mathsf{End}(X)$; and the functor $\mathbf{Alg}(\mathsf{T},\mathcal{C}) \to \mathbf{Alg}(\mathsf{T},\mathcal{D})$ induced by a finite-power-preserving $F : \mathcal{C} \to \mathcal{D}$ sends $\mathsf{T} \to \mathsf{End}(X)$ to $\mathsf{T} \to \mathsf{End}(X) \to \mathsf{End}(FX)$, where $\mathsf{End}(X) \to \mathsf{End}(FX)$ is the finitary monad map defined at n by $\mathcal{C}(X^n, X) \to \mathcal{D}(F(X^n), FX) \cong \mathcal{D}((FX)^n, FX)$. These definitions are by no means new[3]; the point is that, from the $\mathcal{F}$-enriched viewpoint, they are entirely self-evident.

7. Left adjoints to algebraic functors

A functor $\mathbf{Alg}(\mathsf{T},\mathcal{C}) \to \mathbf{Alg}(\mathsf{S},\mathcal{C})$ or $\mathbf{Mod}(\mathbf{T},\mathcal{C}) \to \mathbf{Mod}(\mathbf{S},\mathcal{C})$ induced by precomposition with a map of finitary monads or of Lawvere theories is called an *algebraic functor*. Under reasonable hypotheses on $\mathcal{C}$, such functors have left adjoints; in this final section, we consider the case where such adjoints can be constructed from $\mathcal{F}$-enriched left Kan extensions in the sense of [13].

7.1. Left Kan extensions

Given $\mathcal{V}$-functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{A} \to \mathcal{D}$, the *left Kan extension*[4] of G along F is the $\mathcal{V}$-functor $\mathrm{Lan}_F G : \mathcal{B} \to \mathcal{D}$ defined by the colimit formula $(\mathrm{Lan}_F G)(B) = \mathcal{B}(F–, B) \star G$. If $\mathrm{Lan}_F G$ exists for all $G : \mathcal{A} \to \mathcal{D}$, then by [13, Theorem 4.43] it provides the values of an (ordinary) functor

$$\mathrm{Lan}_F : \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A},\mathcal{D}) \to \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{B},\mathcal{D}),$$

left adjoint to precomposition with F. When $\mathcal{V}$ is $\mathcal{F}$, $\mathcal{D} = R\mathcal{C}$ is a representable $\mathcal{F}$-category, and $F : \Sigma\mathsf{S} \to \Sigma\mathsf{T}$ is the $\mathcal{F}$-functor induced by a map of finitary monads, we see that Lan_F must provide a left adjoint for the algebraic functor $\mathbf{Alg}(\mathsf{T},\mathcal{C}) \to \mathbf{Alg}(\mathsf{S},\mathcal{C})$; similarly, when F is the $\mathcal{F}$-functor $R\mathbf{S} \to R\mathbf{T}$ induced by a map of Lawvere theories, Lan_F must provide a left adjoint for the algebraic functor $\mathbf{Mod}(\mathbf{T},\mathcal{C}) \to \mathbf{Mod}(\mathbf{S},\mathcal{C})$.

7.2. Remark. Note that a left adjoint to an algebraic functor may exist without being computed by $\mathcal{F}$-enriched Kan extension. An example may help to clarify this point. Take T to be the monoid monad and S the identity monad on **Set**; then $\mathbf{Alg}(\mathsf{T},\mathcal{C}) \to \mathbf{Alg}(\mathsf{S},\mathcal{C})$ is the forgetful functor $\mathbf{Mon}(\mathcal{C}) \to \mathcal{C}$ from the category of monoids in $\mathcal{C}$ with respect to cartesian product. This has a left adjoint given by $\mathcal{F}$-enriched Kan extension just when free monoids in $\mathcal{C}$ can be constructed by the geometric series formula $\sum_{n=0}^{\infty} X^n$; which requires the countable coproducts at issue to exist in $\mathcal{C}$ and to distribute over finite powers. There are much more general conditions under which free monoids exist in $\mathcal{C}$—see [12]—but these require properties of $\mathcal{C}$ which are not intrinsically $\mathcal{F}$-categorical, for example local presentability; which is reflected in the fact that the left adjoint in such cases is *not* computed by $\mathcal{F}$-enriched Kan extension.

7.3. Relative tensors

In calculating the $\mathcal{F}$-enriched left Kan extensions that yield left adjoints to algebraic functors, we will need a more general kind of weighted colimit than the tensors introduced previously. The following definitions

[3] The construction of $\mathsf{End}(X)$ is in [25, Section 2] but dates back to Lawvere's thesis [17]; the analysis in the form just given is essentially in [11, Section 3].

[4] We follow Kelly in reserving the name "left Kan extension" for what [4,22] call a *pointwise* left Kan extension: one computed at each object by a weighted colimit in the codomain category.

are special cases of ones in [27]; note that the material of [13] is not applicable, as it assumes that $\mathcal{V}$ is biclosed symmetric, whereas we assume only right closure without symmetry.

Let $\mathcal{S} = \Sigma S$ be a one-object $\mathcal{V}$-category; by a *right $\mathcal{S}$-module*, we mean a right module for the underlying monoid S. If $b : B \otimes S \to B$ is a right $\mathcal{S}$-module, then so too is $A \otimes b : A \otimes B \otimes S \to A \otimes B$, and the assignation $(A, b) \mapsto A \otimes b$ underlies a monoidal action $\mathcal{V} \times \mathcal{S}\text{-}\mathbf{Mod} \to \mathcal{S}\text{-}\mathbf{Mod}$. The action is right closed, with $\langle B, C\rangle$ being defined as the equaliser of the two maps

$$[B, C] \xrightarrow{[b,1]} [B \otimes S, C] \quad \text{and} \quad [B, C] \xrightarrow{-\otimes S} [B \otimes S, C \otimes S] \xrightarrow{[1,c]} [B \otimes S, C],$$

so that, by the argument of Section 2.3, we have an enrichment of $\mathcal{S}$-$\mathbf{Mod}$ to a tensored $\mathcal{V}$-category; when seen in this way, we write it as $\mathcal{PS}$.

Given a $\mathcal{V}$-functor $\mathcal{S} \to \mathcal{C}$, comprising an object $X \in \mathcal{C}$ and a monoid morphism $x : S \to \mathcal{C}(X, X)$ which we might think of as a *left $\mathcal{S}$-action on X*, we have a lifting of the hom-functor $\mathcal{C}(X, –) : \mathcal{C} \to \mathcal{V}$ through the forgetful $\mathcal{PS} \to \mathcal{V}$, obtained by equipping each $\mathcal{C}(X, Y)$ with the right $\mathcal{S}$-action

$$\mathcal{C}(X, Y) \otimes S \xrightarrow{1\otimes x} \mathcal{C}(X, Y) \otimes \mathcal{C}(X, X) \xrightarrow{m} \mathcal{C}(X, Y).$$

Now given X with its left $\mathcal{S}$-action and a right $\mathcal{S}$-module A, the *relative tensor of X by B over $\mathcal{S}$* is given by an object $A \otimes_{\mathcal{S}} X \in \mathcal{C}$ and a map $i : A \to \mathcal{C}(X, A \otimes_{\mathcal{S}} X)$ of right $\mathcal{S}$-modules such that, for every $Y \in \mathcal{C}$, the map

$$\mathcal{C}(A \otimes_{\mathcal{S}} X, Y) \xrightarrow{\mathcal{C}(X,–)} \mathcal{PS}\big(\mathcal{C}(X, A \otimes_{\mathcal{S}} X), \mathcal{C}(X, Y)\big) \xrightarrow{\mathcal{PS}(i,1)} \mathcal{PS}\big(A, \mathcal{C}(X, Y)\big)$$

is invertible in $\mathcal{V}$.

7.4. Relative tensors in $\mathcal{F}$-categories

The key to describing relative tensors in $\mathcal{F}$-categories is the following result, a standard part of the folklore on algebraic theories:

7.5. Proposition. *If S is a finitary monad on $\mathbf{Set}$ and $\mathbf{S}$ the corresponding Lawvere theory, then $\mathcal{P}(\Sigma\mathsf{S})$ is the representable $\mathcal{F}$-category on $[\mathbf{S}^{\mathrm{op}}, \mathbf{Set}]$.*

Proof. By its construction, $\mathcal{P}(\Sigma\mathsf{S})$ is a tensored $\mathcal{F}$-category, and so in particular representable; it thus suffices to show that its underlying ordinary category is isomorphic to $[\mathbf{S}^{\mathrm{op}}, \mathbf{Set}]$. Now, for any $A \in \mathcal{F}$, to give a map $A \otimes S \to A$ is to give maps $An \times (Sm)^n \to Am$ or equally maps $(Sm)^n \to (Am)^{An}$, natural in n and m. Since $\mathbf{S}(n, m) = (Sn)^m$, this is to give a graph morphism $\mathbf{S}^{\mathrm{op}} \to \mathbf{Set}$ which (by naturality in n and m) restricts along $\mathbf{F} \to \mathbf{S}^{\mathrm{op}}$ to give back A. Imposing the requirement that $A \otimes S \to A$ satisfy the unit and associativity conditions for a right module now forces the graph morphism $\mathbf{S}^{\mathrm{op}} \to \mathbf{Set}$ to be a functor, so that, in sum, a right $\Sigma\mathsf{S}$-module is equally a pair of $A : \mathbf{F} \to \mathbf{Set}$ together with an extension of A through $\mathbf{F} \to \mathbf{S}^{\mathrm{op}}$; which is equally just a functor $\mathbf{S}^{\mathrm{op}} \to \mathbf{Set}$. Arguing similarly for the morphisms, we conclude that the underlying category of $\mathcal{P}(\Sigma\mathsf{S})$ is isomorphic to $[\mathbf{S}^{\mathrm{op}}, \mathbf{Set}]$, as claimed. $\square$

We now characterise relative tensors in a representable $\mathcal{F}$-category in terms of colimits in the underlying ordinary category which distributive over finite powers. First we make the sense of this distributivity precise. Let $\mathcal{C}$ be a category with finite powers, and $D : \mathcal{A} \to \mathcal{C}$ a functor whose values are taken in powers X^n of some fixed object X of $\mathcal{C}$. Suppose that $i : D \Rightarrow \Delta Z$ is a colimiting cocone for D. For each $k \in \mathbb{N}$, write $D^k : \mathcal{A}^k \to \mathcal{C}$ for the functor $(a_1, \ldots, a_k) \mapsto D(a_1) \times \cdots \times D(a_k)$ (note these products will exist by the assumption on D), and $i^k : D^k \Rightarrow \Delta(Z^k)$ for the induced cocone with components $i_{a_1} \times \cdots \times i_{a_k}$. If the cocone i^k is colimiting for each k, we say that the colimit of D *distributes over finite powers*.

7.6. Proposition. *Let $\mathcal{C}$ be a category with finite powers, and S a finitary monad on* **Set**. *The relative tensor of $X : \Sigma\mathsf{S} \to R\mathcal{C}$ by $A \in \mathcal{P}\Sigma\mathsf{S}$ exists if and only if the composite ordinary functor*

$$D = \mathrm{el}\,\tilde{A} \to \mathbf{S} \xrightarrow{\tilde{X}} \mathcal{C} \tag{7.1}$$

admits a colimit which distributes over finite powers; here, $\mathbf{S}$ *is the Lawvere theory associated to* S*, the presheaf $\tilde{A} \in [\mathbf{S}^{\mathrm{op}}, \mathbf{Set}]$ corresponds to $A \in \mathcal{P}\Sigma\mathsf{S}$ under Proposition 7.5, and $\tilde{X} : \mathbf{S} \to \mathcal{C}$ is the essentially-unique finite-power-preserving functor whose restriction $\Sigma\mathsf{S} \to R\mathbf{S} \to R\mathcal{C}$ is isomorphic to X.*

The colimit of D is thus the (unenriched) weighted colimit $\tilde{A} \star \tilde{X}$, given in coend notation[5] by $\int^{n \in \mathbf{S}} \tilde{A}n \cdot \tilde{X}n$. The distributivity of the colimit over finite powers is the requirement that, for all $k \in \mathbb{N}$, we have a canonical isomorphism

$$\Big(\int^{n} \tilde{A}n \cdot \tilde{X}n\Big)^k \cong \int^{n_1,\ldots,n_k} (\tilde{A}n_1 \times \cdots \times \tilde{A}n_k) \cdot \tilde{X}(n_1 + \cdots + n_k),$$

in the sense that the evident cocone of maps exhibits the left-hand side as the colimit on the right. Note that if $\mathcal{C}$ is a cartesian closed category, then any colimit in $\mathcal{C}$ distributes over finite powers; thus if $\mathcal{C}$ is cartesian closed and cocomplete (in particular, if $\mathcal{C} = \mathbf{Set}$) then $R\mathcal{C}$ admits all $\mathcal{F}$-enriched relative tensors.

Proof. For any $Y \in \mathcal{C}$, we induce as in Section 7.3 a right $\Sigma\mathsf{S}$-module structure on the hom-object $R\mathcal{C}(X, Y)$, which under the isomorphism of Proposition 7.5 is easily identified with the presheaf $\mathcal{C}(\tilde{X}, Y) \in [\mathbf{S}^{\mathrm{op}}, \mathbf{Set}]$. In these terms, the universal property of the tensor $Z = A \otimes_{\mathsf{S}} X$ mandates isomorphisms

$$R\mathcal{C}(Z, Y) \cong R\big[\mathbf{S}^{\mathrm{op}}, \mathbf{Set}\big]\big(\tilde{A}, \mathcal{C}(\tilde{X}, Y)\big) \tag{7.2}$$

in $\mathcal{F}$, induced by composition with a universal map $i : \tilde{A} \to \mathcal{C}(\tilde{X}, Z)$ in $[\mathbf{S}^{\mathrm{op}}, \mathbf{Set}]$. To give i is to give functions $\tilde{A}n \to \mathcal{C}(\tilde{X}n, Z)$ natural in $n \in \mathbf{S}$, thus a cocone $i : D \Rightarrow \Delta Z$ under (7.1) with vertex Z. Evaluating (7.2) at $1 \in \mathbf{F}$, we see that this cocone must be colimiting; evaluating at $k \in \mathbf{F}$, we find that composition with i^k induces isomorphisms $\mathcal{C}(Z^k, Y) \cong [\mathbf{S}^{\mathrm{op}}, \mathbf{Set}](\tilde{A}^k, \mathcal{C}(\tilde{X}, Y))$. Since $[\mathbf{S}^{\mathrm{op}}, \mathbf{Set}]$ is cartesian closed, we calculate that

$$\begin{aligned} \tilde{A}^k = \Big(\int^{n} \tilde{A}n \cdot y_n\Big)^k &\cong \int^{n_1,\ldots,n_k} (\tilde{A}n_1 \times \cdots \times \tilde{A}n_k) \cdot y_{n_1} \times \cdots \times y_{n_k} \\ &\cong \int^{n_1,\ldots,n_k} (\tilde{A}n_1 \times \cdots \times \tilde{A}n_k) \cdot y_{n_1 + \cdots + n_k}, \end{aligned} \tag{7.3}$$

so that $[\mathbf{S}^{\mathrm{op}}, \mathbf{Set}](\tilde{A}^k, \mathcal{C}_1(\tilde{X}, Y))$ is equally the set of cocones $D^k \Rightarrow \Delta Y$; the natural isomorphism of this with $\mathcal{C}(Z^k, Y)$ specified by (7.2) now asserts that $i^k : D^k \to \Delta(Z^k)$ is a colimiting cocone, as required. $\square$

[5] This is merely notation: we are only asserting the existence of the colimit whose universal property is expressed by this coend, and not that of the copowers $\tilde{A}n \cdot \tilde{X}n$ constituting it.

7.7. Left adjoints to algebraic functors

We are now finally in a position to describe when left adjoints to algebraic functors can be obtained by $\mathcal{F}$-enriched Kan extension. The construction given in the following result is once again not new, at least when stated in the form stated in (iii) and (iv); what is new is its abstract justification via the universal property in (i) and (ii).

7.8. Proposition. *Let $F : \mathsf{S} \to \mathsf{T}$ be a map of finitary monads on* **Set** *and $G : \mathbf{S} \to \mathbf{T}$ the associated map of Lawvere theories, inducing the left-hand square of $\mathcal{F}$-functors in*:

$$\begin{tikzcd} \Sigma\mathsf{S} \arrow[r, "\eta"] \arrow[d, "\Sigma F"'] & R\mathbf{S} \arrow[rr, "X"] \arrow[d, "RG"] & & R\mathcal{C}. \\ \Sigma\mathsf{T} \arrow[r, "\eta"'] & R\mathbf{T} \end{tikzcd}$$

Here, the maps labelled η exhibit $R\mathbf{S}$ and $R\mathbf{T}$ as $L\Sigma\mathsf{S}$ and $L\Sigma\mathsf{T}$. Let $\mathcal{C}$ be a category with finite powers, and let X as displayed above be an $\mathbf{S}$-model in $\mathcal{C}$, with $X\eta : \Sigma\mathsf{S} \to R\mathcal{C}$ the corresponding S-algebra. Then the following are equivalent:

(i) *The Kan extension $\mathrm{Lan}_{RG}(X) : R\mathbf{T} \to R\mathcal{C}$ exists*;
(ii) *The Kan extension $\mathrm{Lan}_{\Sigma F}(X\eta) : R\mathsf{T} \to R\mathcal{C}$ exists*;
(iii) *The ordinary functor*

$$D : G \downarrow A \xrightarrow{\pi_1} \mathbf{S} \xrightarrow{VX} \mathcal{C}$$

(*where A is the distinguished object of* $\mathbf{T}$) *admits a colimit which distributes over finite powers;*
(iv) *The unenriched Kan extension $\mathrm{Lan}_G(VX) : \mathbf{T} \to \mathcal{C}$ exists and is a finite-power-preserving functor.*

Once again, the hypotheses of this proposition will always be satisfied when $\mathcal{C}$ is cartesian closed and cocomplete, so in particular when $\mathcal{C} = \mathbf{Set}$.

Proof. Assume (i). By [13, Theorem 5.35], we have $X \cong \mathrm{Lan}_\eta(X\eta)$, and so $\mathrm{Lan}_{RG}(X) \cong \mathrm{Lan}_{RG.\eta}(X\eta) \cong \mathrm{Lan}_{\eta.\Sigma F}(X\eta)$. As $\eta : \Sigma\mathsf{T} \to R\mathbf{T}$ is fully faithful, it follows that $(\mathrm{Lan}_{RG}X).\eta \cong \mathrm{Lan}_{\Sigma F}(X\eta)$, as required for (ii). Conversely, given (ii), the left Kan extension of $\mathrm{Lan}_{\Sigma F}(X\eta)$ along $\eta : \Sigma\mathsf{T} \to R\mathbf{T}$ exists, again by [13, Theorem 5.35], and is isomorphic to $\mathrm{Lan}_{RG}(X)$ as above, giving (i).

We next show that (ii) $\Leftrightarrow$ (iii). By definition, $\mathrm{Lan}_{\Sigma F}(X\eta)$ has its value at the unique object of $\Sigma\mathsf{T}$ given by the relative tensor $T \otimes_{\Sigma\mathsf{S}} (X\eta)$; here T is regarded as a right $\Sigma\mathsf{S}$-module via the action

$$T \otimes S \xrightarrow{1 \otimes F} T \otimes T \xrightarrow{\mu} T.$$

Under the isomorphism of Proposition 7.5, this right module is easily seen to correspond to the presheaf $\mathbf{T}(G, A) \in [\mathbf{S}^{\mathrm{op}}, \mathbf{Set}]$ (with A the distinguished object of $\mathbf{T}$); whence, by Proposition 7.6, the relative tensor $T \otimes_{\Sigma\mathsf{S}} (X\eta)$ exists if and only if the conditions in (iii) hold.

Finally, we show (iii) $\Leftrightarrow$ (iv). The value at A^n of $\mathrm{Lan}_G(VX)$ is given by the (unenriched) weighted colimit $\mathbf{T}(G, A^n) \star VX$; but $\mathbf{T}(G, A^n) \cong (\mathbf{T}(G, A))^n$ and so, repeating the calculation in (7.3), this weighted colimit can be computed as the conical colimit of $D^n : (G \downarrow A)^n \to \mathcal{C}$. Thus the existence and finite-power-preservation of $\mathrm{Lan}_G(VX)$ is equivalent to the existence and distributivity of colim D, as required. $\square$

References

[1] M. Barr, C. Wells, Toposes, Triples and Theories, Grundlehren Math. Wiss., vol. 278, Springer, 1985.
[2] C. Berger, P.-A. Melliès, M. Weber, Monads with arities and their associated theories, J. Pure Appl. Algebra 216 (2012) 2029–2048.
[3] R. Betti, A. Carboni, Cauchy-completion and the associated sheaf, Cah. Topol. Géom. Différ. Catég. 23 (3) (1982) 243–256.
[4] E.J. Dubuc, Kan Extensions in Enriched Category Theory, Lect. Notes Math., vol. 145, Springer, 1970.
[5] M. Fiore, N. Gambino, M. Hyland, G. Winskel, The cartesian closed bicategory of generalised species of structures, J. Lond. Math. Soc. 77 (1) (2008) 203–220.
[6] P. Hall, Some word-problems, J. Lond. Math. Soc. 33 (1958) 482–496.
[7] C. Hermida, Representable multicategories, Adv. Math. 151 (2) (2000) 164–225.
[8] M. Hyland, Elements of a theory of algebraic theories, Theor. Comput. Sci. (2014), in press, http://dx.doi.org/10.1016/j.tcs.2014.03.005.
[9] M. Hyland, J. Power, The category theoretic understanding of universal algebra: Lawvere theories and monads, Electron. Notes Theor. Comput. Sci. 172 (2007) 437–458.
[10] G. Janelidze, G.M. Kelly, A note on actions of a monoidal category, Theory Appl. Categ. 9 (2001) 61–91.
[11] G.M. Kelly, Coherence theorems for lax algebras and for distributive laws, in: Category Seminar, Sydney, 1972/1973, in: Lect. Notes Math., vol. 420, Springer, 1974, pp. 281–375.
[12] G.M. Kelly, A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on, Bull. Aust. Math. Soc. 22 (1) (1980) 1–83.
[13] G.M. Kelly, Basic Concepts of Enriched Category Theory, Lond. Math. Soc. Lect. Note Ser., vol. 64, Cambridge University Press, 1982, republished as: Repr. Theory Appl. Categ., vol. 10, 2005.
[14] G.M. Kelly, A.J. Power, Adjunctions whose counits are coequalizers, and presentations of finitary enriched monads, J. Pure Appl. Algebra 89 (1–2) (1993) 163–179.
[15] S. Lack, J. Rosický, Notions of Lawvere theory, Appl. Categ. Struct. 19 (2011) 363–391.
[16] J. Lambek, Multicategories revisited, in: Categories in Computer Science and Logic, Boulder, 1987, in: Contemp. Math., vol. 92, American Mathematical Society, 1989, pp. 217–239.
[17] F.W. Lawvere, Functorial semantics of algebraic theories, PhD thesis, Columbia University, 1963, republished as: Repr. Theory Appl. Categ., vol. 5, 2004.
[18] F.W. Lawvere, Metric spaces, generalized logic, and closed categories, Rend. Sem. Mat. Fis. Milano 43 (1973) 135–166, republished as: Repr. Theory Appl. Categ., vol. 1, 2002.
[19] T. Leinster, Higher Operads, Higher Categories, Lond. Math. Soc. Lect. Note Ser., vol. 298, Cambridge University Press, 2004.
[20] F.E.J. Linton, Some aspects of equational categories, in: Conference on Categorical Algebra, La Jolla, 1965, Springer, 1966, pp. 84–94.
[21] S. Mac Lane, Categorical algebra, Bull. Am. Math. Soc. 71 (1965) 40–106.
[22] S. Mac Lane, Categories for the Working Mathematician, Grad. Texts Math., vol. 5, Springer, 1971.
[23] K. Nishizawa, J. Power, Lawvere theories enriched over a general base, J. Pure Appl. Algebra 213 (3) (2009) 377–386.
[24] J. Power, Enriched, Lawvere theories, Theory Appl. Categ. 6 (1999) 83–93.
[25] R. Street, The formal theory of monads, J. Pure Appl. Algebra 2 (2) (1972) 149–168.
[26] R. Street, Absolute colimits in enriched categories, Cah. Topol. Géom. Différ. Catég. 24 (4) (1983) 377–379.
[27] R. Street, Enriched categories and cohomology, Quaest. Math. 6 (1983) 265–283, republished as: Repr. Theory Appl. Categ., vol. 14, 2005.
[28] R. Walters, Sheaves on sites as Cauchy-complete categories, J. Pure Appl. Algebra 24 (1) (1982) 95–102.