Cofibrantly generated natural weak factorisation systems

Richard Garner*
Department of Mathematics, Uppsala University,
Box 480, S-751 06 Uppsala, Sweden

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Abstract

There is an "algebraisation" of the notion of weak factorisation system (w.f.s.) known as a *natural weak factorisation system*. In it, the two classes of maps of a w.f.s. are replaced by two categories of maps-with-structure, where the extra structure on a map now encodes a *choice* of liftings with respect to the other class. This extra structure has pleasant consequences: for example, a natural w.f.s. on $\mathcal{C}$ induces a canonical natural w.f.s. structure on any functor category $[\mathcal{A}, \mathcal{C}]$.

In this paper, we define cofibrantly generated natural weak factorisation systems by analogy with cofibrantly generated w.f.s.'s. We then construct them by a method which is reminiscent of Quillen's small object argument but produces factorisations which are much smaller and easier to handle, and show that the resultant natural w.f.s. is, in a suitable sense, *freely* generated by its generating cofibrations. Finally, we show that the two categories of maps-with-structure for a natural w.f.s. are closed under all the constructions we would expect of them: (co)limits, pushouts / pullbacks, transfinite composition, and so on.

1 Introduction

A *weak factorisation system* on a category is given by two classes of maps $\mathcal{L}$ and $\mathcal{R}$ which are related by a "lifting-extension" property guaranteeing the existence of fill-ins for certain commuting squares, along with a way of factorising an arbitrary map as $f = pi$, where the maps i and p lie in the respective classes $\mathcal{L}$ and $\mathcal{R}$. In typical examples, these two classes of maps have distinctive feels to them: an $\mathcal{L}$-map is given by freely "glueing" structure onto the source of the map to obtain the target, whilst an $\mathcal{R}$-map allows one to lift structure from the target to its source.

The most common place where weak factorisation systems (henceforth w.f.s.'s) arise is in *Quillen model structures* [16] on a category: here one has two weak factorisation systems (trivial cofibration, fibration) and (cofibration, trivial fibration) which interact in a pleasant way, providing a powerful framework within which one can do a lot of abstract homotopy theory: one obtains formal notions of homotopy category, homotopy equivalence, homotopy limits and colimits, simplicial resolution, and so on.

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The definition of weak factorisation system is, as the name suggests, a weakening of the older categorical notion of *orthogonal factorisation system* [8], in which the "existence" in the lifting-extension property becomes *unique* existence. This makes the theory of orthogonal factorisation systems cleaner than that of their weak cousins: for example, their factorisations can always be chosen in a functorial way; their "$\mathcal{R}$-maps" are closed under limits and their "$\mathcal{L}$-maps" under colimits; and they can be lifted with no effort to functor categories. However, one also has that the factorisations themselves must be (essentially) unique, making them ill-suited to homotopy-theoretic ends.

However, it turns out that if one is slightly more subtle about the way in which one defines a w.f.s., one can have both non-uniqueness of factorisations and also many of the pleasant properties of orthogonal factorisation systems. The *natural weak factorisation systems* (henceforth n.w.f.s.'s) of [9] are an "algebraisation" of the concept of w.f.s. We review their formal definition in Section 2, but the intuition can be quickly illustrated by an analogy with the notion of *Grothendieck fibration*.

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, it may or may not have the *property* of being a Grothendieck fibration, namely that every arrow of $\mathcal{D}$ should have a cartesian lifting to $\mathcal{C}$. However, if one asks for $F\colon \mathcal{C} \to \mathcal{D}$ to be a *cloven fibration*, that is, to be equipped with an explicitly given choice of cartesian lifting for every arrow of $\mathcal{D}$, then this is no longer a property of F but *extra structure* borne by F. And in fact, this extra structure is algebraic: there is a monad on $\mathbf{Cat}^{\mathbf{2}}$ – the category of arrows in $\mathbf{Cat}$ – whose algebras are precisely the cloven fibrations.

Likewise, for a w.f.s. on $\mathcal{C}$, we speak of a map having the *property* of being an $\mathcal{L}$-map or a $\mathcal{R}$-map, whilst in a n.w.f.s. on $\mathcal{C}$, we speak instead of equipping a map with the *structure* of an $\mathcal{L}$-map or a $\mathcal{R}$-map. And again, this extra structure is (co)algebraic: there is a monad R on $\mathcal{C}^{\mathbf{2}}$ whose algebras are precisely the $\mathcal{R}$-maps in this new sense; and dually, there is a comonad L on $\mathcal{C}^{\mathbf{2}}$ whose coalgebras are the $\mathcal{L}$-maps.

In the language of the first paragraph, an $\mathcal{L}$-map now becomes an arrow *together with an explicit description* of how one should glue structure onto the source to obtain the target; and a $\mathcal{R}$-map becomes an arrow *together with an explicit description* of how one should lift structure from the target to the source. These explicit descriptions conspire to give one a *canonical* choice of fill-ins for the "lifting-extension" property, whilst the one remaining ingredient in a w.f.s., namely factorisation, is already encoded in the comonad-monad pair on $\mathcal{C}^{\mathbf{2}}$: the functor parts of L and R simply send an arrow to the first and second halves of its factorisation.

Natural w.f.s.'s have certain advantages over plain w.f.s.'s: for instance, the category of $\mathcal{L}$-maps for a n.w.f.s. is closed under *all* colimits, and the category of $\mathcal{R}$-maps under *all* limits; moreover, n.w.f.s. structures on $\mathcal{C}$ induce n.w.f.s. structures on each functor category $[\mathcal{A}, \mathcal{C}]$ in a completely canonical way. However, with this greater power comes greater complexity, and thus one needs to do a good deal of groundwork to obtain a useful computational tool.

For example, one knows that the $\mathcal{L}$-maps (or dually, the $\mathcal{R}$-maps) for a w.f.s. are closed under constructions like pushout, retracts, fibre coproducts and transfinite composition: we would obviously like the same to be true for n.w.f.s.'s, and this is what we show in Section 6. Because the $\mathcal{L}$-maps and $\mathcal{R}$-maps now carry extra structure, giving a precise meaning to "the pushout of an $\mathcal{L}$-map" is a little more subtle, and showing that it exists a little more involved: but beyond this, we find that we are able to proceed essentially as before.

The main meat of this paper, however, is Sections 4 and 5, where we define and construct *cofibrantly generated* n.w.f.s.'s. The definition generalises the notion of a cofibrantly generated w.f.s., a notion which describes almost every w.f.s. found in nature; whilst the construction is both an adaptation of Quillen's *small object argument* and an example of the sort of *free monoid* construction studied by Kelly in [13]. In fact, we see that a cofibrantly generated n.w.f.s. is, in a suitable sense, *freely* generated by its generating cofibrations. There are ramifications for the study of plain w.f.s.'s as well, since our method gives a recipe for the construction of functorial factorisations which are much less redundant than Quillen's original argument, and which in many cases can be easily calculated by hand.

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2 Natural weak factorisation systems

Let us start by recalling the notion of a **weak factorisation system** $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$. This is given by two classes of maps $\mathcal{L}$ and $\mathcal{R}$ in $\mathcal{C}$ which are closed under retracts in the arrow category of $\mathcal{C}$, and which satisfy both

(lifting) Whenever we are given a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "p"] \\ B \arrow[r, "g"'] & D \end{tikzcd} \tag{1}$$

in $\mathcal{C}$, where $i \in \mathcal{L}$ and $p \in \mathcal{R}$, we can find a fill-in $j \colon B \to C$ such that $pj = g$ and $ji = f$; and

(factorisation) Every map $f \colon X \to Y$ in $\mathcal{C}$ can be factorised as $f = pi$, where $i \in \mathcal{L}$ and $p \in \mathcal{R}$.

In general, given maps $i \colon A \to B$ and $p \colon C \to D$, we call a square like (1) an (i, p)*-lifting problem*. If every such square has a fill-in (or "solution") then we say that i has the *left lifting property* (llp) with respect to p and that p has the *right lifting property* (rlp) with respect to i. Thus we can restate the lifting axiom as: every $\mathcal{R}$-map has the rlp with respect to every $\mathcal{L}$-map, and vice versa. In fact, the $\mathcal{R}$-maps are *precisely* the maps with the rlp with respect to every $\mathcal{L}$-map, and vice versa, so that the classes $\mathcal{L}$ and $\mathcal{R}$ determine each other.

One frequently requires that the factorisations for a w.f.s. should be functorial in the following sense. Let us write $\mathcal{C}^{\mathbf{2}}$ for the arrow category of $\mathcal{C}$; we have two functors $\mathrm{dom}, \mathrm{cod} \colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ and a natural transformation $\kappa \colon \mathrm{dom} \Rightarrow \mathrm{cod}$ whose component κ_f is the map f. By a **functorial factorisation** (E, λ, ρ), we now mean a functor $E \colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ together with natural transformations

$$\mathrm{dom} \xRightarrow{\lambda} E \xRightarrow{\rho} \mathrm{cod}$$

satisfying $\kappa = \rho \cdot \lambda$. A **functorial weak factorisation system** is given by a w.f.s. $(\mathcal{L}, \mathcal{R})$ together with a functorial factorisation (E, λ, ρ) such that each λ_f is in $\mathcal{L}$ and each ρ_f is in $\mathcal{R}$. This notion is stronger than a plain w.f.s., but technically more convenient.

By strengthening a functorial w.f.s. further still, one arrives at the notion of a natural w.f.s., which as explained in the introduction, consists of a comonad L and a monad R on $\mathcal{C}^{\mathbf{2}}$, interacting in a certain way. The definition we give is essentially that of [9], with the only novelty being the addition of a *distributive law* of the comonad over the monad. This is a natural transformation $\Delta\colon LR \Rightarrow RL$ satisfying axioms expressing a form of compatibility between the monad and the comonad; more precisely, it encodes a way of lifting the comonad L to the category of free algebras for the monad R, and vice versa. We won't spell out the details here, since equation (3) below tells us everything we need to know about the distributive law; but the reader may like to consult [4] for further details.

Definition 1. A **natural weak factorisation system** $(\mathsf{L}, \mathsf{R}, \Delta)$ on a category $\mathcal{C}$ is given by:

- A comonad $\mathsf{L} = (L, \Phi, \Sigma)$ on $\mathcal{C}^{\mathbf{2}}$,
- A monad $\mathsf{R} = (R, \Lambda, \Pi)$ on $\mathcal{C}^{\mathbf{2}}$, and
- A distributive law $\Delta\colon LR \Rightarrow RL$,

satisfying the following equalities:

$$\begin{array}{lllll} & \mathrm{dom} \cdot L = \mathrm{dom}, & \mathrm{cod} \cdot L = \mathrm{dom} \cdot R, & \mathrm{cod} \cdot R = \mathrm{cod}; & \\ & \mathrm{dom} \cdot \Phi = 1_{\mathrm{dom}}, & \mathrm{cod} \cdot \Phi = \kappa \cdot R, & \mathrm{dom} \cdot \Lambda = \kappa \cdot L, & \mathrm{cod} \cdot \Lambda = 1_{\mathrm{cod}}; \\ \text{and} & \mathrm{dom} \cdot \Sigma = 1_{\mathrm{dom}}, & \mathrm{cod} \cdot \Sigma = \mathrm{dom} \cdot \Delta, & \mathrm{dom} \cdot \Pi = \mathrm{cod} \cdot \Delta, & \mathrm{cod} \cdot \Pi = 1_{\mathrm{cod}}. \end{array}$$

We will shortly see that there is a good deal of redundancy in this definition: this is unavoidable if we want to capture the (co)algebraic aspects of the system, but on the plus side means that we can unravel the definition and give a much more compact description of a n.w.f.s. Firstly, we have functors $L, R\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ satisfying $\mathrm{dom} \cdot L = \mathrm{dom}$, $\mathrm{cod} \cdot L = \mathrm{dom} \cdot R$ and $\mathrm{cod} \cdot R = \mathrm{cod}$, which we write as:

$$L\left(\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}\right) = \begin{tikzcd} X \arrow[d, "\lambda_f"] \\ Ef, \end{tikzcd} \qquad R\left(\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}\right) = \begin{tikzcd} Ef \arrow[d, "\rho_f"] \\ Y, \end{tikzcd}$$

$$L\left(\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & W \arrow[d, "g"] \\ Y \arrow[r, "k"'] & Z \end{tikzcd}\right) = \begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\lambda_f"'] & W \arrow[d, "\lambda_g"] \\ Ef \arrow[r, "E(h,k)"'] & Eg, \end{tikzcd} \qquad R\left(\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & W \arrow[d, "g"] \\ Y \arrow[r, "k"'] & Z \end{tikzcd}\right) = \begin{tikzcd} Ef \arrow[r, "E(h,k)"] \arrow[d, "\rho_f"'] & Eg \arrow[d, "\rho_g"] \\ Y \arrow[r, "k"'] & Z. \end{tikzcd}$$

This diagram should illustrate our conventions for working with the category $\mathcal{C}^{\mathbf{2}}$. When we view a morphism $f\colon X \to Y$ of $\mathcal{C}$ as an object of $\mathcal{C}^{\mathbf{2}}$ we will draw it *vertically*; thus a morphism of $\mathcal{C}^{\mathbf{2}}$ from f to g is a commutative square, bounded by f and g vertically and by two maps h and k horizontally, which we write as $(h, k)\colon f \to g$.

Next in the definition of n.w.f.s. we have natural transformations $\Phi\colon L \Rightarrow \mathrm{id}$ and $\Lambda\colon \mathrm{id} \Rightarrow R$ satisfying $\mathrm{dom} \cdot \Phi = 1_{\mathrm{dom}}$, $\mathrm{cod} \cdot \Phi = \kappa \cdot R$, $\mathrm{dom} \cdot \Lambda = \kappa \cdot L$ and $\mathrm{cod} \cdot \Lambda = 1_{\mathrm{cod}}$.

These conditions completely determine the components of Φ and Λ as being:

$$\Phi_f \;=\; \begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\lambda_f"'] & X \arrow[d, "f"] \\ Ef \arrow[r, "\rho_f"'] & Y \end{tikzcd} \quad \text{and} \quad \Lambda_f \;=\; \begin{tikzcd} X \arrow[r, "\lambda_f"] \arrow[d, "f"'] & Ef \arrow[d, "\rho_f"] \\ Y \arrow[r, "\mathrm{id}_Y"'] & Y. \end{tikzcd}$$

However, the existence of Φ and Λ is not without force, since the above squares must commute, which tells us that $f = \rho_f \cdot \lambda_f$ for each $f \in \mathcal{C}^{\mathbf{2}}$. Thus what we have so far is precisely a functorial factorisation (E, λ, ρ): we can think of a n.w.f.s. as a "functorial-factorisation-with-structure", a viewpoint we will espouse more comprehensively in the next section.

Continuing, we have the natural transformations $\Sigma\colon L \Rightarrow LL$, $\Delta\colon LR \Rightarrow RL$ and $\Pi\colon RR \Rightarrow R$, satisfying $\mathrm{dom} \cdot \Sigma = 1_{\mathrm{dom}}$, $\mathrm{cod} \cdot \Sigma = \mathrm{dom} \cdot \Delta$, $\mathrm{cod} \cdot \Delta = \mathrm{dom} \cdot \Pi$, and $\mathrm{cod} \cdot \Pi = 1_{\mathrm{cod}}$, and thus we have

$$\Sigma_f \;=\; \begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\lambda_f"'] & X \arrow[d, "\lambda_{Lf}"] \\ Ef \arrow[r, "\sigma_f"'] & ELf, \end{tikzcd} \quad \Delta_f \;=\; \begin{tikzcd} Ef \arrow[r, "\sigma_f"] \arrow[d, "\lambda_{Rf}"'] & ELf \arrow[d, "\rho_{Lf}"] \\ ERf \arrow[r, "\pi_f"'] & Ef \end{tikzcd} \quad \text{and} \quad \Pi_f \;=\; \begin{tikzcd} ERf \arrow[r, "\pi_f"] \arrow[d, "\rho_{Rf}"'] & Ef \arrow[d, "\rho_f"] \\ Y \arrow[r, "\mathrm{id}_Y"'] & Y. \end{tikzcd}$$

The intuition behind these maps is as follows. If we were dealing with a functorial w.f.s., then the λ's would have the left lifting property with respect to the ρ's, and so we would have fill-ins for squares like this:

$$\begin{tikzcd} X \arrow[r, "\lambda_{Lf}"] \arrow[d, "\lambda_f"'] & ELf \arrow[d, "\rho_{Lf}"] \\ Ef \arrow[r, "\mathrm{id}"'] \arrow[ur, dotted] & Ef \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Ef \arrow[r, "\mathrm{id}"] \arrow[d, "\lambda_{Rf}"'] & Ef \arrow[d, "\rho_f"] \\ ERf \arrow[r, "\rho_{Rf}"'] \arrow[ur, dotted] & Y. \end{tikzcd}$$

This is what σ_f and π_f provide us with, in a uniform way. Indeed, we already know that σ_f and π_f make the upper left and lower right triangles in the displayed squares commute; that the same is true for the lower left and upper right follows from the comonad and monad identities for L and R. Explicitly, these identities assert that:

$$\begin{aligned} \rho_{Lf} \cdot \sigma_f &= \mathrm{id}_{Ef}, & \pi_f \cdot \lambda_{Rf} &= \mathrm{id}_{Ef}, \\ E(1_X, \rho_f) \cdot \sigma_f &= \mathrm{id}_{Ef}, & \pi_f \cdot E(\lambda_f, 1_Y) &= \mathrm{id}_{Ef}, \\ E(1_X, \sigma_f) \cdot \sigma_f &= \sigma_{Lf} \cdot \sigma_f & \text{and } \pi_f \cdot E(\pi_f, 1_Y) &= \pi_f \cdot \pi_{Rf}. \end{aligned} \tag{2}$$

All that remains to account for are the axioms for the distributive law Δ. Most of these just repeat things we already know, and the only new equality we obtain is:

$$\sigma_f \cdot \pi_f = \pi_{Lf} \cdot E(\sigma_f, \pi_f) \cdot \sigma_{Rf}. \tag{3}$$

The equations of (2) and (3) may seem rather puzzling at first; a reasonable intuition is that they can be viewed as ensuring that every possible way of constructing a lifting from the λ's, ρ's, σ's and π's will give the same result. We can now give the promised "more compact" version of the definition of n.w.f.s.

Definition 2. A **reduced n.w.f.s.** $(E, \lambda, \rho, \sigma, \pi)$ on $\mathcal{C}$ is given by:

- A functorial factorisation (E, λ, ρ) on $\mathcal{C}$;
- Natural transformations $\sigma\colon E \Rightarrow EL$ and $\pi\colon ER \Rightarrow E$, where L and R are the unique functors $\mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ satisfying $\kappa \cdot L = \lambda$ and $\kappa \cdot R = \rho$,

such that $\sigma \cdot \lambda = \lambda L$ and $\rho \cdot \pi = \rho R$, and such that the equations of (2) and (3) hold.

From the preceding discussion, we see that n.w.f.s.'s on $\mathcal{C}$ are in bijection with reduced natural w.f.s.'s on $\mathcal{C}$ and thus we will pass between the two views without further comment.

Let us now examine the manner in which a n.w.f.s. generalises a plain w.f.s. As explained in the Introduction, we capture the "$\mathcal{L}$-maps" and "$\mathcal{R}$-maps" for a n.w.f.s. by means of the categories of (co)algebras for the (co)monad part of the n.w.f.s. So given given a n.w.f.s. $(\mathsf{L}, \mathsf{R}, \Delta)$, let us write L-**Map** for the category of coalgebras for L, and call it the **category of L-maps**; and similarly write R-**Map** for the category of algebras for R and call it the **category of R-maps**. Explicitly, L-**Map** has

- **Objects** (f, s) being arrows $f\colon X \to Y$ and $s\colon Y \to Ef$ of $\mathcal{C}$ satisfying $s \cdot f = \lambda_f$, $\rho_f \cdot s = \mathrm{id}_Y$ and $\sigma_f \cdot s = E(1_X, s) \cdot s$, and
- **Morphisms** $(h, k)\colon (f, s) \to (g, t)$ being morphisms $(h, k)\colon f \to g$ in $\mathcal{C}^{\mathbf{2}}$ such that $t \cdot k = E(h, k) \cdot s$,

whilst R-**Map** has

- **Objects** (f, p) being arrows $f\colon X \to Y$ and $p\colon Ef \to X$ of $\mathcal{C}$ satisfying $f \cdot p = \rho_f$, $p \cdot \lambda_f = \mathrm{id}_X$ and $p \cdot \pi_f = p \cdot E(p, 1_Y)$, and
- **Morphisms** $(h, k)\colon (f, p) \to (g, q)$ being morphisms $(h, k)\colon f \to g$ in $\mathcal{C}^{\mathbf{2}}$ such that $h \cdot p = q \cdot E(h, k)$.

We will sometimes abuse notation slightly, and write $(f, s)\colon X \to Y$ to signify an L-map or R-map for which $f\colon X \to Y$; this emphasises the idea that an L-map or R-map is just a map of $\mathcal{C}$ with extra structure. Now, to see that these definitions make sense, consider the case where we have a mere functorial w.f.s. $(\mathcal{L}, \mathcal{R})$ and are given an $\mathcal{L}$-map $f\colon A \to B$. If we take the factorisation of f as $f = \rho_f \cdot \lambda_f$, then, since every ρ is an $\mathcal{R}$-map, we will have a solution s to the lifting problem

$$\begin{tikzcd} A \arrow[r, "\lambda_f"] \arrow[d, "f"'] & Ef \arrow[d, "\rho_f"] \\ B \arrow[r, equal] \arrow[ur, dotted, "s"'] & B. \end{tikzcd}$$

It is lifting data of this form which accompanies the L-maps and R-maps for a n.w.f.s.; moreover, this extra data is sufficient to give us canonical solutions to *all* "(L, R)-lifting problems". More precisely, if we are given an L-map (f, s), an R-map (g, p), and an (f, g)-lifting problem

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] & D, \end{tikzcd}$$

then we have a canonical choice of lifting $j\colon B \to C$ given by

$$j := B \xrightarrow{s} Ff \xrightarrow{F(h,k)} Fg \xrightarrow{p} C,$$

which is now *natural*, in that it is stable under composition with morphisms of L-**Map** on the left and morphisms of R-**Map** on the right.

The remaining ingredient in a n.w.f.s. is of course *factorisation*. Because L-**Map** and R-**Map** are categories of (co)algebras, we have adjunctions:

$$\mathsf{L}\text{-}\mathbf{Map} \underset{F_{\mathsf{L}}}{\overset{U_{\mathsf{L}}}{\underset{\perp}{\rightleftarrows}}} \mathcal{C}^{\mathbf{2}} \qquad \text{and} \qquad \mathsf{R}\text{-}\mathbf{Map} \underset{F_{\mathsf{R}}}{\overset{U_{\mathsf{R}}}{\underset{\top}{\rightleftarrows}}} \mathcal{C}^{\mathbf{2}}.$$

The forgetful functors U_{L} and U_{R} send an L-map (f, s) or R-map (f, p) to its underlying $\mathcal{C}$-map f, whilst the free[1] functors F_{L} and F_{R} respectively send a map $f\colon A \to B$ of $\mathcal{C}$ to the L-map (λ_f, σ_f) and the R-map (ρ_f, π_f). Thus F_{L} and F_{R} give us a functorial factorisation of any map of $\mathcal{C}$ into an L-map followed by an R-map:

$$f\colon A \to B = A \xrightarrow{(\lambda_f,\sigma_f)} Ef \xrightarrow{(\rho_f,\pi_f)} B.$$

Now, underlying each n.w.f.s. $(\mathsf{L}, \mathsf{R}, \Delta)$ is a functorial w.f.s.: if we let $\mathcal{L}$ be the class of maps in $\mathcal{C}$ which admit some L-coalgebra structure, and $\mathcal{R}$ be the class of maps admitting some R-algebra structure, then the pair $(\mathcal{L}, \mathcal{R})$ satisfy all the conditions for a functorial w.f.s. except, possibly, closure of $\mathcal{L}$ and $\mathcal{R}$ under retracts. So if we write $\overline{\mathcal{L}}$ and $\overline{\mathcal{R}}$ for the respective retract-closures, we obtain a functorial w.f.s. $(\overline{\mathcal{L}}, \overline{\mathcal{R}})$, with the property that the given factorisations land inside the smaller classes $\mathcal{L}$ and $\mathcal{R}$.

Examples 3.

- If $(\mathsf{L}, \mathsf{R}, \Delta)$ is a n.w.f.s. on $\mathcal{C}$, then $(\mathsf{R}^{\mathrm{op}}, \mathsf{L}^{\mathrm{op}}, \Delta^{\mathrm{op}})$ is a n.w.f.s. on $\mathcal{C}^{\mathrm{op}}$, and so the notion of n.w.f.s. is self-dual.
- If $(\mathsf{L}, \mathsf{R}, \Delta)$ is a n.w.f.s. on $\mathcal{C}$, then for any object $X \in \mathcal{C}$ we induce a n.w.f.s. of the same name on the slice category $\mathcal{C}/X$ and the coslice category $X/\mathcal{C}$.
- If $(\mathsf{L}, \mathsf{R}, \Delta)$ is a n.w.f.s. on $\mathcal{C}$, then for any other category $\mathcal{A}$ we induce a n.w.f.s. calculated pointwise on $[\mathcal{A}, \mathcal{C}]$. This stands in strong contrast to the situation with w.f.s.'s, where there is *no* canonical lifting to functor categories.

We will not give any substantial examples now, as these will arise in due course from the theory of *cofibrantly generated* n.w.f.s.'s that we develop in Sections 4 and 5: thus the reader may like to look ahead to these examples, or to look at those given in [9].

3 An alternative view of natural weak factorisation systems

We observed in the previous section that every n.w.f.s. has an underlying functorial factorisation. In this section we shall go in the other direction, and characterise n.w.f.s.'s as functorial factorisations equipped with a *bialgebra* structure.

Classically, a bialgebra is a vector space A equipped with both an algebra and a coalgebra structure, such that the coalgebra maps $\Delta\colon A \to A \otimes A$ and $\epsilon\colon A \to k$ are algebra homomorphisms; the notion of bialgebra that we deploy to characterise n.w.f.s.'s

[1] We should call F_{L} "cofree", but we won't labour the point.

is a mild generalisation of this. This is an intuitively plausible idea, since both bialgebras and n.w.f.s.'s have a "multiplicative" and a "comultiplicative" part satisfying compatibility conditions. However, to go from *plausible* to *precise* will require a little work.

3.1 Bialgebras in 2-fold monoidal categories

One obvious way to generalise the notion of bialgebra is to restate its definition of in an arbitrary symmetric (or braided) monoidal category; however, we will need something slightly more general still, namely bialgebra objects where the "algebra" and the "coalgebra" parts are given with respect to two *different* monoidal structures on the same category. In order to express the compatibility of the algebra and coalgebra parts, we first need a higher-level compatibility between the two monoidal category structures with respect to which they are taken. This compatibility is captured by the concept of a *2-fold monoidal category*. In fact, we only really need a *strict* 2-fold monoidal category in this case, which makes the definition a little simpler.

We recall first that a *lax monoidal functor* between strict monoidal categories $\mathcal{V}$ and $\mathcal{W}$ is given by a functor $F\colon \mathcal{V} \to \mathcal{W}$ together with a natural family of (not-necessarily-invertible) maps $m_{a,b}\colon Fa \otimes Fb \to F(a \otimes b)$ and a map $m_I\colon I \to FI$, satisfying two coherence axioms: the first equates the two obvious ways of getting from $Fa \otimes Fb \otimes Fc$ to $F(a \otimes b \otimes c)$, and the second says that $m_{a,I} = m_{I,a} = \mathrm{id}_{Fa}$ for all $a \in \mathcal{V}$. One can compose lax monoidal functors to obtain a category $\mathbf{StrMonCat}_{lax}$ of strict monoidal categories and lax monoidal functors; and since $\mathbf{StrMonCat}_{lax}$ has finite products, we can consider monoids in it.

Definition 4. A **strict 2-fold monoidal category** is a monoid in $\mathbf{StrMonCat}_{lax}$.

If we expand this definition, a strict 2-fold monoidal category consists of a category $\mathcal{V}$, two strict monoidal structures $(\otimes, I)$ and $(\odot, \perp)$ on it, maps $m\colon \perp \otimes \perp \to \perp$, $c\colon I \to I \odot I$ and $j\colon I \to \perp$ making $(\perp, j, m)$ into a $\otimes$-monoid and (I, j, c) into a $\odot$-comonoid, and a natural family of maps

$$z_{A,B,C,D}\colon (A \odot B) \otimes (C \odot D) \to (A \otimes C) \odot (B \otimes D)$$

obeying six coherence laws, which equate, respectively, the two possible ways of getting from:

$$\begin{aligned}
(A \odot B \odot C) \otimes (A' \odot B' \odot C') \quad &\text{to} \quad (A \otimes A') \odot (B \otimes B') \odot (C \otimes C'),\\
(A \odot A') \otimes (B \odot B') \otimes (C \odot C') \quad &\text{to} \quad (A \otimes B \otimes C) \odot (A' \otimes B' \otimes C'),\\
(A \odot B) \otimes I \quad &\text{to} \quad (A \otimes I) \odot (B \otimes I),\\
I \otimes (A \odot B) \quad &\text{to} \quad (I \otimes A) \odot (I \otimes B),\\
(\perp \odot A) \otimes (\perp \odot B) \quad &\text{to} \quad \perp \odot (A \otimes B),\\
\text{and} \quad (A \odot \perp) \otimes (B \odot \perp) \quad &\text{to} \quad (A \otimes B) \odot \perp.
\end{aligned}$$

We will write such a 2-fold monoidal category as $(\mathcal{V}, \otimes, I, \odot, \perp)$. A 2-fold monoidal category is the simplest example of an *iterated* monoidal category, in the sense of [2, 7], to which we refer the reader for further examples and applications. Note that, in one aspect, our definition is slightly more general than those of the above-cited papers, since it does not assume that the units I and $\perp$ coincide.

Let us now see why a 2-fold monoidal category is a suitable environment for defining a notion of bialgebra. What we want to say is the following: a bialgebra in $(\mathcal{V}, \otimes, I, \odot, \perp)$

is an object A together with maps $\mu\colon A \otimes A \to A$, $\eta\colon I \to A$, $\Delta\colon A \to A \odot A$ and $\epsilon\colon A \to \perp$ such that (A, η, μ) is a monoid, (A, ϵ, Δ) is a comonoid, and such that the maps $\Delta\colon A \to A \odot A$ and $\epsilon\colon A \to \perp$ are monoid homomorphisms.

For this last clause to make sense, we need $\otimes$-monoid structures on $A \odot A$ and on $\perp$; and one way of obtaining these is by lifting the $(\odot, \perp)$ monoidal structure on $\mathcal{V}$ to the category $\mathbf{Mon}_\otimes(\mathcal{V})$ of $\otimes$-monoid objects in $\mathcal{V}$. But this is precisely what the 2-fold monoidal structure allows us to do. The unit for this lifted monoidal structure is the $\otimes$-monoid $(\perp, j, m)$, whilst the tensor product of two $\otimes$-monoids (A, η^A, μ^A) and (B, η^B, μ^B) is given by $(A \odot B, \eta^{A\odot B}, \mu^{A\odot B})$, where

$$\eta^{A\odot B} = I \xrightarrow{c} I \odot I \xrightarrow{\eta^A \odot \eta^B} A \odot B$$

and

$$\mu^{A\odot B} = (A \odot B) \otimes (A \odot B) \xrightarrow{z_{A,B,A,B}} (A \otimes A) \odot (B \otimes B) \xrightarrow{\mu^A \odot \mu^B} A \odot B.$$

This lifting process can be seen more abstractly by noting that the operation that assigns to each strict monoidal category the category of monoids in it extends to a finite-product preserving functor $\mathbf{Mon}(-)\colon \mathbf{StrMonCat}_{lax} \to \mathbf{Cat}$. Thus monoids in $\mathbf{StrMonCat}_{lax}$ – which are 2-fold monoidal categories – are sent to monoids in $\mathbf{Cat}$ – which are strict monoidal categories. Regardless of how we obtain it, this lifting allows us to define:

Definition 5. Let $(\mathcal{V}, \otimes, I, \odot, \perp)$ be a 2-fold monoidal category. The category $\mathbf{Bialg}(\mathcal{V})$ of **bialgebras in** $\mathcal{V}$ is given by $\mathbf{Comon}_\odot(\mathbf{Mon}_\otimes(\mathcal{V}))$, the category of $\odot$-comonoid objects in $\mathbf{Mon}_\otimes(\mathcal{V})$.

Explicitly, such a bialgebra is given by a quintuple $(A, \eta, \mu, \epsilon, \Delta)$ as above, such that such that (A, η, μ) is a $\otimes$-monoid, (A, ϵ, Δ) is a $\odot$-comonoid, and such the following four diagrams commute:

$$\begin{tikzcd}
I \arrow[r, "\eta"] \arrow[d, "c"'] & A \arrow[d, "\Delta"] \\
I \odot I \arrow[r, "\eta\odot\eta"'] & A \odot A,
\end{tikzcd}
\quad
\begin{tikzcd}
A \otimes A \arrow[r, "\mu"] \arrow[d, "\epsilon\otimes\epsilon"'] & A \arrow[d, "\epsilon"] \\
\perp \otimes \perp \arrow[r, "m"'] & \perp,
\end{tikzcd}
\quad
\begin{tikzcd}
& A \arrow[dr, "\epsilon"] & \\
I \arrow[ur, "\eta"] \arrow[rr, "j"'] & & \perp,
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\mu"] \arrow[d, "\Delta\otimes\Delta"'] & & A \arrow[d, "\Delta"] \\
(A \odot A) \otimes (A \odot A) \arrow[r, "z_{A,A,A,A}"'] & (A \otimes A) \odot (A \otimes A) \arrow[r, "\mu\odot\mu"'] & A \odot A,
\end{tikzcd} \tag{4}$$

whilst a bialgebra homomorphism is a morphism of $\mathcal{V}$ which is simultaneously a monoid homomorphism and a comonoid homomorphism.

Remark 6. Before returning to our pursuit of n.w.f.s.'s, we note that we can obtain the notion of bialgebra in a 2-fold monoidal category in a dual way: it is not only a "$\odot$-comonoid in the category of $\otimes$-monoids" but also a "$\otimes$-monoid in the category of $\odot$-comonoids". Indeed, we can lift the $(\otimes, I)$ monoidal structure on $\mathcal{V}$ to the category $\mathbf{Comon}_\odot(\mathcal{V})$ of $\odot$-comonoids: the unit object is (I, j, c) whilst the tensor product of two $\odot$-comonoids $(A, \epsilon^A, \Delta^A)$ and $(B, \epsilon^B, \Delta^B)$ is given by $(A \otimes B, \epsilon^{A\otimes B}, \Delta^{A\otimes B})$, where

$$\epsilon^{A\otimes B} = A \otimes B \xrightarrow{\epsilon^A \otimes \epsilon^B} \perp \otimes \perp \xrightarrow{m} \perp$$

and

$$\Delta^{A\otimes B} = A \otimes B \xrightarrow{\Delta^A \otimes \Delta^B} (A \odot A) \otimes (B \odot B) \xrightarrow{z_{A,A,B,B}} (A \otimes B) \odot (A \otimes B).$$

And a $\otimes$-monoid in $\mathbf{Comon}_{\odot}(\mathcal{V})$ is once again a bialgebra in $\mathcal{V}$. To see this abstractly, observe that if $(\mathcal{V}, \otimes, I, \odot, \perp)$ is a 2-fold monoidal category, then so is $(\mathcal{V}^{\mathrm{op}}, \odot, \perp, \otimes, I)$; and from the explicit definition of bialgebras given above, we can easily see that $\mathbf{Bialg}(\mathcal{V}^{\mathrm{op}}) \cong \mathbf{Bialg}(\mathcal{V})^{\mathrm{op}}$. So now

$$\begin{aligned}
\mathbf{Bialg}(\mathcal{V})^{\mathrm{op}} &\cong \mathbf{Bialg}(\mathcal{V}^{\mathrm{op}}) \\
&= \mathbf{Comon}_{\otimes}(\mathbf{Mon}_{\odot}(\mathcal{V}^{\mathrm{op}})) \\
&\cong \mathbf{Comon}_{\otimes}(\mathbf{Comon}_{\odot}(\mathcal{V})^{\mathrm{op}}) \\
&\cong \mathbf{Mon}_{\otimes}(\mathbf{Comon}_{\odot}(\mathcal{V}))^{\mathrm{op}}
\end{aligned}$$

so that $\mathbf{Bialg}(\mathcal{V}) \cong \mathbf{Mon}_{\otimes}(\mathbf{Comon}_{\odot}(\mathcal{V}))$ as claimed. This second characterisation of bialgebras will be the most useful to us when we are working with n.w.f.s.'s.

3.2 Natural weak factorisation systems as bialgebras

We are now ready to characterise n.w.f.s.'s on a category $\mathcal{C}$, which as we have already suggested, will arise as functorial factorisations on $\mathcal{C}$ bearing a bialgebra structure. We now know that for this to make sense, we need to organise functorial factorisations on $\mathcal{C}$ into a 2-fold monoidal category.

Making them form a category – let us call it $\mathbf{Ff}_{\mathcal{C}}$ – is easy enough: objects are functorial factorisations (E, λ, ρ), and morphisms $\alpha \colon (E, \lambda, \rho) \to (E', \lambda', \rho')$ are natural transformations $\alpha \colon E \Rightarrow E'$ making the diagram

$$\begin{tikzcd}
& \mathrm{dom} \arrow[dl, Rightarrow, "\lambda"'] \arrow[dr, Rightarrow, "\lambda'"] & \\
E \arrow[rr, Rightarrow, "\alpha"'] \arrow[dr, Rightarrow, "\rho"'] & & E' \arrow[dl, Rightarrow, "\rho'"] \\
& \mathrm{cod} &
\end{tikzcd}$$

commute. What remains is to give the two interacting monoidal structures $(\otimes, I)$ and $(\odot, \perp)$ on $\mathbf{Ff}_{\mathcal{C}}$: and these arise very naturally from two ways of combining functorial factorisations. In the first such, the tensor product $(E', \lambda', \rho') \otimes (E, \lambda, \rho)$ factorises $f \colon X \to Y$ by applying E to it, then applying E' to the right half of this factorisation:

$$X \xrightarrow{f} Y \quad \mapsto \quad X \xrightarrow{\lambda_f} Ef \xrightarrow{\rho_f} Y \quad \mapsto \quad X \xrightarrow{\lambda_f} Ef \xrightarrow{\lambda'_{Rf}} E'Rf \xrightarrow{\rho'_{Rf}} Y,$$[2]

and finally composing together the two "left" parts, λ_f and λ'_{Rf} to obtain the factorisation

$$X \xrightarrow{f} Y \quad \mapsto \quad X \xrightarrow{\lambda'_{Rf} \cdot \lambda_f} E'Rf \xrightarrow{\rho'_{Rf}} Y.$$

[2]Note that, as in section 2, we write R for the functor $\mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ corresponding to the natural transformation $\rho \colon E \Rightarrow \mathrm{cod}$, and so on. We shall continue to do this without further note throughout the paper.

The unit I for this tensor product is also the initial object of $\mathbf{Ff}_{\mathcal{C}}$, namely $(\mathrm{dom}, 1_{\mathrm{dom}}, \kappa)$ which factorises $f\colon X \to Y$ as

$$X \xrightarrow{1_X} X \xrightarrow{f} Y.$$

The second monoidal structure is completely dual to the first; so $(E', \lambda', \rho') \odot (E, \lambda, \rho)$ factorises $f\colon X \to Y$ by applying E to it, then applying E' to the *left* half of this factorisation:

$$X \xrightarrow{f} Y \quad \mapsto \quad X \xrightarrow{\lambda_f} Ef \xrightarrow{\rho_f} Y \quad \mapsto \quad X \xrightarrow{\lambda'_{Lf}} E'Lf \xrightarrow{\rho'_{Lf}} Ef \xrightarrow{\rho_f} Y,$$

and finally composing together the two "right" parts, ρ_f and ρ'_{Lf} to obtain the factorisation

$$X \xrightarrow{f} Y \quad \mapsto \quad X \xrightarrow{\lambda'_{Lf}} E'Lf \xrightarrow{\rho_f \cdot \rho'_{Lf}} Y.$$

The unit $\perp$ for this tensor product is now the terminal object of $\mathbf{Ff}_{\mathcal{C}}$, namely $(\mathrm{cod}, \kappa, 1_{\mathrm{cod}})$ which factorises $f\colon X \to Y$ as

$$X \xrightarrow{f} Y \xrightarrow{1_Y} Y.$$

One can easily check directly that these operations yield two strict monoidal structures on $\mathbf{Ff}_{\mathcal{C}}$, but it will be more illuminating to see how we can deduce their existence indirectly. Let us say that a functor $F\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ is *over* $\mathrm{cod}\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ if $\mathrm{cod} \cdot F = \mathrm{cod}\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$, and likewise, that a natural transformation $\alpha\colon F \Rightarrow G\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ is over cod if $\mathrm{cod} \cdot \alpha = \mathrm{id}_{\mathrm{cod}}$. Now it's easy to show that $\mathbf{Ff}_{\mathcal{C}}$ is isomorphic to the category $\mathbf{Ff}'_{\mathcal{C}}$ with:

- **Objects** being pairs (R, Λ) where $R\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ and $\Lambda\colon \mathrm{id}_{\mathcal{C}^{\mathbf{2}}} \Rightarrow R$ over cod.
- **Morphisms** $\Gamma\colon (R, \Lambda) \to (R', \Lambda')$ being commutative triangles over cod:

$$\begin{tikzcd} & \mathrm{id}_{\mathcal{C}^{\mathbf{2}}} \arrow[dl, Rightarrow, "\Lambda"'] \arrow[dr, Rightarrow, "\Lambda'"] & \\ R \arrow[rr, Rightarrow, "\Gamma"'] & & R'. \end{tikzcd}$$

Now $\mathbf{Ff}'_{\mathcal{C}}$ has a strict monoidal structure on it:

$$I = \begin{tikzcd} \mathrm{id}_{\mathcal{C}^{\mathbf{2}}} \arrow[d, Rightarrow, "\mathrm{id}"] \\ \mathrm{id}_{\mathcal{C}^{\mathbf{2}}} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathrm{id}_{\mathcal{C}^{\mathbf{2}}} \arrow[d, Rightarrow, "\Lambda"] \\ R \end{tikzcd} \otimes \begin{tikzcd} \mathrm{id}_{\mathcal{C}^{\mathbf{2}}} \arrow[d, Rightarrow, "\Lambda'"] \\ R' \end{tikzcd} = \begin{tikzcd} \mathrm{id}_{\mathcal{C}^{\mathbf{2}}} \arrow[d, Rightarrow, "\Lambda\Lambda'"] \\ RR', \end{tikzcd}$$

which transfers back to $\mathbf{Ff}_{\mathcal{C}}$ to give us the same-named structure there: and so we deduce the associativity and unitality of the latter from that of the former. Moreover, we can now easily classify $\otimes$-monoids in $\mathbf{Ff}_{\mathcal{C}}$, since they correspond to $\otimes$-monoids in $\mathbf{Ff}'_{\mathcal{C}}$; and giving a monoid structure on $(R, \Lambda) \in \mathbf{Ff}'_{\mathcal{C}}$ is the same as giving a natural transformation $\Pi\colon RR \Rightarrow R$ making (R, Λ, Π) into a monad over cod.

We can argue dually for the $(\odot, \perp)$ monoidal structure, where the first step is now the observation that $\mathbf{Ff}_{\mathcal{C}}$ is isomorphic to the category with:

- **Objects** being pairs (L, Φ) where $L\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ and $\Phi\colon L \Rightarrow \mathrm{id}_{\mathcal{C}^{\mathbf{2}}}$ over dom;
- **Morphisms** $\Gamma\colon (L, \Phi) \to (L', \Phi')$ being commutative triangles over dom.

Thus we have proved:

Proposition 7. *There is a strict monoidal structure $(\otimes, I)$ on $\mathbf{Ff}_{\mathcal{C}}$ such that an $\otimes$-monoid structure on (E, λ, ρ) is the same as an extension of the corresponding pair (R, Λ) to a monad over* cod. *Dually, there is a strict monoidal structure $(\odot, \bot)$ on $\mathbf{Ff}_{\mathcal{C}}$ such that a $\odot$-comonoid structure on (E, λ, ρ) is the same as an extension of the corresponding pair (L, Φ) to a comonad over* dom.

To relate this to n.w.f.s.'s, observe that if we take only the data which concerns R in Definition 1 then what we have is a monad over cod on $\mathcal{C}^{\mathbf{2}}$; and likewise, taking only the data relating to L gives us a comonad over dom. So we can think of $\mathbf{Mon}_{\otimes}(\mathbf{Ff}_{\mathcal{C}})$ as the category of "right halves of n.w.f.s.'s" and $\mathbf{Comon}_{\odot}(\mathbf{Ff}_{\mathcal{C}})$ as the category of "left halves of n.w.f.s.'s". Moreover, if we combine the two monoidal structures in a simple-minded way, we nearly get enough to capture a full n.w.f.s.:

Proposition 8. *To give an object (E, λ, ρ) of $\mathbf{Ff}_{\mathcal{C}}$ which is simultaneously a $\otimes$-monoid and a $\odot$-comonoid is to give:*

- *A comonad $\mathsf{L} = (L, \Phi, \Sigma)$ on $\mathcal{C}^{\mathbf{2}}$,*
- *A monad $\mathsf{R} = (R, \Lambda, \Pi)$ on $\mathcal{C}^{\mathbf{2}}$, and*
- *A natural transformation $\Delta \colon LR \Rightarrow RL$,*

satisfying the following equalities:

$$\begin{array}{llll} & \mathrm{dom} \cdot L = \mathrm{dom}, & \mathrm{cod} \cdot L = \mathrm{dom} \cdot R, & \mathrm{cod} \cdot R = \mathrm{cod}; \\ & \mathrm{dom} \cdot \Phi = 1_{\mathrm{dom}}, & \mathrm{cod} \cdot \Phi = \kappa \cdot R, & \mathrm{dom} \cdot \Lambda = \kappa \cdot L, \quad \mathrm{cod} \cdot \Lambda = 1_{\mathrm{cod}}; \\ \textit{and} & \mathrm{dom} \cdot \Sigma = 1_{\mathrm{dom}}, & \mathrm{cod} \cdot \Sigma = \mathrm{dom} \cdot \Delta, & \mathrm{dom} \cdot \Pi = \mathrm{cod} \cdot \Delta, \quad \mathrm{cod} \cdot \Pi = 1_{\mathrm{cod}}. \end{array}$$

Proof. Just as in Section 2, the functor $\Delta \colon LR \Rightarrow RL$ is completely determined by the other data and the requirements $\mathrm{cod} \cdot \Sigma = \mathrm{dom} \cdot \Delta$ and $\mathrm{dom} \cdot \Pi = \mathrm{cod} \cdot \Delta$. Everything else follows immediately from Proposition 7. ☐

Comparing this Corollary with Definition 1, we see that the only thing missing is the stipulation that Δ should be not only a natural transformation, but also a distributive law; and this extra requirement amounts to requiring that what we actually have is a *bialgebra* in $\mathbf{Ff}_{\mathcal{C}}$. For this to make sense, we first need to show that the two monoidal structures on $\mathbf{Ff}_{\mathcal{C}}$ interact properly:

Proposition 9. $(\mathbf{Ff}_{\mathcal{C}}, \otimes, I, \odot, \bot)$ *is a strict 2-fold monoidal category.*

Proof. Recall that this amounts to giving maps $m \colon \bot \otimes \bot \to \bot$, $c \colon I \to I \odot I$ and $j \colon I \to \bot$ and a natural family of maps

$$z_{A,B,C,D} \colon (A \odot B) \otimes (C \odot D) \to (A \otimes C) \odot (B \otimes D)$$

obeying laws. Since I is initial and $\bot$ is terminal in $\mathbf{Ff}_{\mathcal{C}}$, the maps m, j and c are uniquely determined, and so we need only give the maps $z_{A,B,C,D}$, which we do directly. Suppose that we have:

$$A = (E^1, \lambda^1, \rho^1), \quad B = (E^2, \lambda^2, \rho^2), \quad C = (E^3, \lambda^3, \rho^3) \quad \text{and} \quad D = (E^4, \lambda^4, \rho^4);$$

then the factorisation $(A \odot B) \otimes (C \odot D)$ sends the map $f \colon X \to Y$ to

$$X \xrightarrow{\lambda^1_{L^2 R^{3 \odot 4} f} \cdot \lambda^3_{L^4 f}} E^1 L^2 R^{3 \odot 4} f \xrightarrow{\rho^2_{R^{3 \odot 4} f} \cdot \rho^1_{L^2 R^{3 \odot 4} f}} Y,$$

where $R^{3\odot 4}$ corresponds to the right-hand part of the factorisation $(E^3,\lambda^3,\rho^3)\odot(E^4,\lambda^4,\rho^4)$. Likewise, the factorisation $(A\otimes C)\odot(B\otimes D)$ sends f to

$$X \xrightarrow{\lambda^1_{R^3L^{2\otimes 4}f}\cdot\lambda^3_{L^{2\otimes 4}f}} E^1R^3L^{2\otimes 4}f \xrightarrow{\rho^2_{R^4f}\cdot\rho^1_{R^3L^{2\otimes 4}f}} Y,$$

where a similar meaning is attached to $L^{2\otimes 4}$. Now, to give $z_{A,B,C,D}$ we must give, for each such f, a map $E^1L^2R^{3\odot 4}f\to E^1R^3L^{2\otimes 4}f$, compatible with the maps from X and to Y and natural in f. To do this, consider the following diagram:

$$\begin{tikzcd}
E^3L^4f \arrow[r, "\lambda^2_{R^{3\odot 4}f}"] \arrow[d, "E^3(\mathrm{id}_X,\lambda^2_{R^4f})"'] & E^2R^{3\odot 4}f \arrow[d, "E^2(\rho^3_{L^4f},\mathrm{id}_Y)"] \\
E^3L^{2\otimes 4}f \arrow[r, "\rho^3_{L^{2\otimes 4}f}"'] & E^2R^4f.
\end{tikzcd}$$

This commutes, with both sides equal to $E^3L^4f \xrightarrow{\rho^3_{L^4f}} E^4f \xrightarrow{\lambda^2_{R^4f}} E^2R^4f$. Applying E^1, we obtain the map

$$E^1\big(E^3(\mathrm{id}_X,\lambda^2_{R^4f}),E^2(\rho^3_{L^4f},\mathrm{id}_Y)\big)\colon E^1L^2R^{3\odot 4}f\to E^1R^3L^{2\otimes 4}f,$$

which we take to be the component of $z_{A,B,C,D}$ at f. The remaining (extensive) details are routine. □

So, since our category $\mathbf{Ff}_{\mathcal{C}}$ bears the structure of a strict 2-fold monoidal category, we can consider bialgebras in it, in the sense of Definition 5; and as we might hope, we have:

Proposition 10. *Natural weak factorisation systems on $\mathcal{C}$ are in bijection with bialgebras in the strict 2-fold monoidal category $\mathbf{Ff}_{\mathcal{C}}$.*

Proof. By Proposition 8, it suffices to show that, given an object (E,λ,ρ) of $\mathbf{Ff}_{\mathcal{C}}$ which is both a $\otimes$-monoid and a $\odot$-comonoid, the bialgebra axioms (4) hold just when the equation (3) does. Now, since I is an initial object and $\perp$ is a terminal object, the first three bialgebra axioms will always hold; whilst the fourth holds just when the following two composites are equal for all $f\colon X\to Y$:

$$ERf \xrightarrow{\pi_f} Ef \xrightarrow{\sigma_f} ELf$$

and

$$\begin{tikzcd}
ERf \arrow[r, "\sigma_{Rf}"] & ELRf \arrow[r, "E(\sigma_f,E(\sigma_f,1_Y))"] & ELR^{E\odot E}f \arrow[d, "E(E(1_X,\lambda_{Rf}),E(\rho_{Lf},1_Y))"] \\
ELf & ERLf \arrow[l, "\pi_{Lf}"] & ERL^{E\otimes E}f \arrow[l, "E(E(1_X,\pi_f),\pi_f)"]
\end{tikzcd}$$

(where the meaning of $R^{E\odot E}$ and $L^{E\otimes E}$ is as in the proof of Proposition 9). Now, considering the central three maps in the latter composite, we calculate:

$$
\begin{aligned}
&E\big(E(1_X,\pi_f),\pi_f\big)\cdot E\big(E(1_X,\lambda_{Rf}),E(\rho_{Lf},1_Y)\big)\cdot E\big(\sigma_f,E(\sigma_f,1_Y)\big)\\
&= E\big(E(1_X,\pi_f)\cdot E(1_X,\lambda_{Rf})\cdot\sigma_f,\ \pi_f\cdot E(\rho_{Lf},1_Y)\cdot E(\sigma_f,1_Y)\big)\\
&= E\big(E(1_X,\pi_f\cdot\lambda_{Rf})\cdot\sigma_f,\ \pi_f\cdot E(\rho_{Lf}\cdot\sigma_f,1_Y)\big)\\
&= E\big(\sigma_f,\pi_f\big).
\end{aligned}
$$

Thus the fourth bialgebra axiom holds just when $\sigma_f\cdot\pi_f=\pi_{Lf}\cdot E(\sigma_f,\pi_f)\cdot\sigma_{Rf}$ for all f, as required. □

We note in passing that we can use this characterisation theorem to read off the correct notion of *morphism* between n.w.f.s.'s on $\mathcal{C}$: it is simply a map of functorial factorisations which respects both the monad and the comonad structure. We will not make direct use of this notion in the current paper, but it is undoubtedly rather important, since once would expect a putative "algebraic" version of a *Quillen model structure* to contain (amongst other data) two n.w.f.s.'s, (trivial cofibration, fibration) and (cofibration, trivial fibration), together with a morphism of n.w.f.s.'s from the former to the latter.

4 Cofibrantly generated n.w.f.s.'s: definition

We are now ready to tackle the main topic of this paper, the definition and construction of *cofibrantly generated* n.w.f.s.'s. Recall first that a plain w.f.s. $(\mathcal{L},\mathcal{R})$ on a category $\mathcal{C}$ is said to be **cofibrantly generated** if there is a set J of $\mathcal{L}$-maps, called the **generating cofibrations**, such that $\mathcal{R}$ is precisely the class of maps with the right lifting property with respect to each of the maps in J; it then follows that $\mathcal{L}$ is the class of maps with the left lifting property with respect to each of the maps in $\mathcal{R}$, and so the set J completely determines the factorisation system.

For example, there is a cofibrantly generated w.f.s. on the category of topological spaces whose generating cofibrations J are the inclusions $S_{n-1}\to D_n$ of the $(n-1)$-sphere into the n-disc. It typifies a certain "topological" kind type of w.f.s., where one thinks of each generating cofibration $f\colon A\to B$ as specifying a *shape* or *cell* B together with the inclusion of its *boundary* A.

When the generating cofibrations are viewed in this way, one arrives at very natural interpretations of the two classes of the resultant w.f.s. The left class consists of retracts of *cell complexes*, which are maps $X\to Y$ obtained by a transfinite process which, starting with X, iteratively picks out boundaries along which to glue in cells until arriving at Y. In the example of the previous paragraph, these cell complexes directly generalise the topologist's *CW-complexes*. The right class is, of course, still determined by the right lifting property; but given a generating cofibration f, we might suggestively call a lifting problem like

$$
\begin{array}{ccc}
A & \xrightarrow{h} & C\\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g}\\
B & \xrightarrow[k]{} & D
\end{array}
\qquad (5)
$$

a *relative horn of shape f in g*, and call a solution for this lifting problem a *filler*. Then the right class consists of those maps g such that *every relative horn in g has a filler.*

The purpose of this section is to develop a corresponding notion of cofibrantly generated *natural* w.f.s.: roughly speaking, we will say that a n.w.f.s. $(\mathsf{L}, \mathsf{R}, \Delta)$ on $\mathcal{C}$ is cofibrantly generated by J if its R-maps are arrows $g\colon X \to Y$ in $\mathcal{C}$ equipped with a *choice* of filler for every relative horn.

So, suppose we are given a category $\mathcal{C}$ and a set of maps J in it. Then we can "algebraise" the notion of "having the right lifting property with respect to J": given $g\colon C \to D$ in $\mathcal{C}$, we write S_g for the set whose elements are (f, g)-lifting problems as in (5) as f ranges over J, and define *right lifting data for g w.r.t. J* to be a function δ assigning to each lifting problem $x \in S_g$ as in (5) a chosen fill-in $\delta(x)\colon B \to C$ with $\delta(x) \cdot f = h$ and $g \cdot \delta(x) = k$. We can form such right lifting data into a category $J^{\square}$, with

- **Objects** being pairs (g, δ) where $g\colon C \to D$ and δ is right lifting data for g with respect to J, and
- **Morphisms** $(g, \delta) \to (g', \delta')$ being commutative squares

$$\begin{tikzcd} C \arrow[r, "m"] \arrow[d, "g"'] & C' \arrow[d, "g'"] \\ D \arrow[r, "n"'] & D' \end{tikzcd}$$

  which commute with the right lifting data for g and g': that is, given an element of S_g, we should get the same result from first applying δ and then postcomposing with m, or from first composing on the right with the square (m, n) and then applying δ'.

This category comes equipped with an obvious forgetful functor to $\mathcal{C}^{\mathbf{2}}$ which we will denote by $U_J\colon J^{\square} \to \mathcal{C}^{\mathbf{2}}$.

Examples 11.

- Let $\mathcal{C} = \mathbf{Set}$ and let $J = \{0 \to 1\}$. Then for any map $g\colon C \to D$, we have $S_g = D$; a typical object of $J^{\square}$ is a map $g\colon C \to D$ together with a map $i\colon D \to C$ satisfying $gi = \mathrm{id}_D$; and a typical morphism $(g, i) \to (g', i')$ of $J^{\square}$ is a map $(h, k)\colon g \to g'$ of $\mathcal{C}^{\mathbf{2}}$ such that $i'k = hi$.
- Let $\mathcal{C} = \mathbf{Set}$ and let $J = \{\mathsf{in}_1\colon 1 \to 1 + 1\}$. Then for any map $g\colon C \to D$, we have $S_g = C \times D$; a typical object of $J^{\square}$ is a map $g\colon C \to D$ together with a function $\theta\colon C \times D \to C$ satisfying $g(\theta(c, d)) = d$ for all $c, d \in D$; and a typical morphism $(g, \theta) \to (g', \theta')$ of $J^{\square}$ is a map $(h, k)\colon g \to g'$ satisfying $h(\theta(c, d)) = \theta'(h(c), k(d))$ for all $c \in C$ and $d \in D$.
- Combining the previous two, if $\mathcal{C} = \mathbf{Set}$ and $J = \{!\colon 0 \to 1, \mathsf{in}_1\colon 1 \to 1 + 1\}$, then for any map $g\colon C \to D$, we have $S_g = D + C \times D$; a typical object of $J^{\square}$ is a map $g\colon C \to D$ together with a both a map $i\colon D \to C$ satisfying $gi = \mathrm{id}_D$ and a map $\theta\colon C \times D \to C$ satisfying $g(\theta(c, d)) = d$ for all $c, d \in D$.
- Let $\mathcal{C} = R\text{-}\mathbf{Mod}$ be the category of modules over a commutative ring R, and let $J = \{0 \to R\}$. Then for any map $g\colon M \to N$, we have $S_g = |N|$, the underlying set of the R-module N, whilst a typical object of $J^{\square}$ is a map $g\colon M \to N$ together with a mere *function* $k\colon |N| \to |M|$ such that $gk(n) = n$ for all $n \in N$.

- Let $\mathcal{C}$ be the category of directed multigraphs, i.e., the functor category $[\cdot \rightrightarrows \cdot, \mathbf{Set}]$. We write a typical object of $\mathcal{C}$ as

$$X = X_a \overset{s}{\underset{t}{\rightrightarrows}} X_v$$

(for *a*rrows, *v*ertices, *s*ource and *t*arget), and a typical map as $f = (f_a \colon X_a \to Y_a, f_v \colon X_v \to Y_v)$. Let $J = \{(\bullet) \to (\bullet \to \bullet)\}$ consist of the inclusion of the graph with one vertex as the source of the graph with one arrow and two vertices; i.e., the following map:

$$\begin{tikzcd} 0 \arrow[r, shift left, "!"] \arrow[r, shift right, "!"'] \arrow[d, "!"'] & 1 \arrow[d, "\mathsf{in}_1"] \\ 1 \arrow[r, shift left, "\mathsf{in}_1"] \arrow[r, shift right, "\mathsf{in}_2"'] & 2. \end{tikzcd}$$

Now, given a map $f \colon X \to Y$ in $\mathcal{C}$, we have S_f given by the pullback

$$\begin{tikzcd} S_f \arrow[r, "\overline{s}"] \arrow[d, "\overline{f}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_v \arrow[d, "f_v"] \\ Y_a \arrow[r, "s"'] & Y_v; \end{tikzcd}$$

thus a typical element (a, x) of S_f is given by an arrow a in Y together with a vertex x of X lying over its source. A typical object of $J^\square$ is given by a map $f \colon X \to Y$ equipped with a lifting of $\overline{f}$ through f_a:

$$\begin{tikzcd} & A_x \arrow[d, "f_a"] \\ S_f \arrow[ur, "p"] \arrow[r, "\overline{f}"'] & A_y, \end{tikzcd}$$

which is to give, for each element (a, x) of S_f, an arrow $p(a, x)$ of X lying over a.

[All of these may look like toy examples: but we have chosen them as such to give us something to play with. If we look at something more substantial – $\mathcal{C}$ the category of simplicial sets and J the set of horn inclusions, for example – then the most explicit we can be is that an element of $J^\square$ as a map $g \colon C \to D$ equipped with a chosen filler for every relative horn: which is fine but doesn't really give us anything to get our hands on.]

Now suppose that as well as a set of maps J, we are also given a n.w.f.s. $(\mathsf{L}, \mathsf{R}, \Delta)$ on $\mathcal{C}$. To say that this is cofibrantly generated by J should mean that its category of R-maps is isomorphic to the category $J^\square$ we have just defined; however, if we take this as our definition then we have missed out on an important subtlety: R-**Map** should not be isomorphic to $J^\square$ in any old way, but in a *canonical* way.

To make sense of this, we need some extra data. Observe that for a cofibrantly generated w.f.s., each map of J is an $\mathcal{L}$-map: so for a cofibrantly generated n.w.f.s., we expect each map of J to be an L-map. We could take this to mean that each map of J admits at least one L-map structure, but if we are going to be consistent about our philosophy of "algebraisation", we should surely take it to mean that each element of J

comes equipped with a *chosen* L-map structure. Thus our additional piece of data is a factorisation

$$
\begin{tikzcd}
J \arrow[rr, "\alpha"] \arrow[dr, "\iota"'] & & \mathsf{L}\text{-}\mathbf{Map}. \arrow[dl, "U_{\mathsf{L}}"] \\
& \mathcal{C}^{\mathbf{2}} &
\end{tikzcd}
$$

(Here we view J as a discrete subcategory of $\mathcal{C}^{\mathbf{2}}$). Using this data, we now have a *canonical* way of obtaining right lifting data w.r.t. J from any R-map (g, s). Indeed, α equips each element $f \in J$ with an L-map structure (f, α_f), and so we can solve (f, g)-lifting problems like (5) using the liftings from the n.w.f.s. between the L-map (f, α_f) and the R-map (g, s). This assignation extends to a functor θ:

$$
\begin{tikzcd}
\mathsf{R}\text{-}\mathbf{Map} \arrow[rr, "\theta"] \arrow[dr, "U_{\mathsf{R}}"'] & & J^{\square}. \arrow[dl, "U_J"] \\
& \mathcal{C}^{\mathbf{2}} &
\end{tikzcd}
$$

We now say that the n.w.f.s. $(\mathsf{L}, \mathsf{R}, \Delta)$ is **cofibrantly generated** by (J, α) if the functor θ so defined is an isomorphism of categories: in other words, if the R-maps are completely determined by the lifting data that they give with respect to the generating cofibrations J.

In order for this to be a sensible definition, it should be conservative over the corresponding definition for plain w.f.s.'s. To see this, suppose that $(\mathsf{L}, \mathsf{R}, \Delta)$ is cofibrantly generated by J and consider its underlying functorial w.f.s. $(\overline{\mathcal{L}}, \overline{\mathcal{R}})$. We recall that $\overline{\mathcal{L}}$ and $\overline{\mathcal{R}}$ are the respective closures under retracts of $\mathcal{L}$, the class of maps in $\mathcal{C}$ admitting some L-coalgebra structure, and $\mathcal{R}$, the class of maps admitting some R-algebra structure. In this case the category of R-maps is isomorphic to the category $J^{\square}$ of right lifting data with respect to J, and so a map of $\mathcal{C}$ lies in $\mathcal{R}$ precisely when it has the right lifting property with respect to J. Thus the underlying functorial w.f.s. $(\overline{\mathcal{L}}, \overline{\mathcal{R}})$ is precisely the w.f.s. cofibrantly generated by J.

In particular, the functorial factorisation that we construct for a cofibrantly generated n.w.f.s. in the next section gives rise to a functorial factorisation for the underlying plain w.f.s.: and it is a much smaller and more tractable factorisation than one generally obtains for cofibrantly generated w.f.s.'s. For this reason, even the reader who feels that natural w.f.s.'s have nothing much to commend them over functorial w.f.s.'s should find the following results of interest.

5 Cofibrantly generated n.w.f.s.'s: construction

5.1 Introduction

Now we know what a cofibrantly generated n.w.f.s. *is*, we can begin to investigate the circumstances under which we can build one. The method we use will be familiar both to topologists, who will recognise it as a variant of Quillen's *small object argument* (for a modern account of which, see [11] or [10], for example), and to category theorists, who will recognise it as an example of the construction of the *free monad on a pointed endofunctor*, a subject treated in detail by Kelly [13].

So, suppose that we are given a category $\mathcal{C}$ and a set J of maps in it; let us work backwards from the definition of a cofibrantly generated n.w.f.s. and see if we can build one which is generated by J. Our starting point is the observation that, if $\mathcal{C}$ is cocomplete, we can greatly simplify the definition of our category $J^{\square}$. Fix a map $g\colon C \to D$ of $\mathcal{C}$, and consider again the set S_g of commutative squares as in (5). We can view each $x \in S_g$ as a morphism $(h_x, k_x)\colon f_x \to g$ in $\mathcal{C}^{\mathbf{2}}$, and thus we can combine them into a map

$$\langle (h_x, k_x) \rangle_{x \in S_g} \colon \sum_{x \in S_g} f_x \to g$$

of $\mathcal{C}^{\mathbf{2}}$; that is, a diagram

$$\begin{tikzcd} \sum A_x \arrow[r, "\langle h_x \rangle"] \arrow[d, "\sum_x f_x"'] & C \arrow[d, "g"] \\ \sum B_x \arrow[r, "\langle k_x \rangle"'] & D. \end{tikzcd} \tag{6}$$

Now, to give right lifting data for g w.r.t. J is equivalent to giving a diagonal fill-in for this single square. We take this process one stage further by observing that (6) factorises as

$$\begin{tikzcd} \sum A_x \arrow[r, "\langle h_x \rangle"] \arrow[d, "\sum_x f_x"'] & C \arrow[r, equal] \arrow[d, "\lambda^1_g"] & C \arrow[d, "g"] \\ \sum B_x \arrow[r] & E^1 g \arrow[r, "\rho^1_g"'] & D, \end{tikzcd} \tag{7}$$

where the left-hand square is a pushout, and that giving a fill-in for (6) is equivalent to giving a fill-in for the right-hand square of (7): that is, a map $k\colon E^1 g \to C$ satisfying the two equalities $g \cdot k = \rho^1_g$ and $k \cdot \lambda^1_g = \mathrm{id}_C$. The first of these says that there is a commutative square

$$\begin{tikzcd} E^1 g \arrow[r, "k"] \arrow[d, "\rho^1_g"'] & C \arrow[d, "g"] \\ D \arrow[r, equal] & D, \end{tikzcd} \tag{8}$$

which, if we write $R^1\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ for the functor[3] sending g to ρ^1_g, corresponds to a map $\gamma = (k, 1_D)\colon R^1 g \to g$. The second equality says that

$$\begin{tikzcd} C \arrow[r, "\lambda^1_g"] \arrow[d, "g"'] & E^1 g \arrow[r, "k"] \arrow[d, "\rho^1_g"'] & C \arrow[d, "g"] \\ D \arrow[r, equal] & D \arrow[r, equal] & D \end{tikzcd} \quad = \quad \begin{tikzcd} C \arrow[r, equal] \arrow[d, "g"'] & C \arrow[d, "g"] \\ D \arrow[r, equal] & D; \end{tikzcd} \tag{9}$$

which, writing $\Lambda^1\colon \mathrm{id}_{\mathcal{C}^{\mathbf{2}}} \Rightarrow R^1$ for the natural transformation[4] whose component at g is $\Lambda^1_g = (\lambda^1_g, 1_D)$, corresponds to the assertion that $\gamma \cdot \Lambda^1_g = \mathrm{id}_g$. Moreover, *every* map $\gamma\colon R^1 g \to g$ in $\mathcal{C}^{\mathbf{2}}$ satisfying this equality must arise in this way, since the equality $\gamma \cdot \Lambda^1_g = \mathrm{id}_g$ forces γ to be of the form $(k, 1_D)$. Thus we have proven:

Proposition 12. *Giving right lifting data w.r.t. J for g is equivalent to giving a map $\gamma\colon R^1 g \to g$ satisfying $\gamma \cdot \Lambda^1_g = \mathrm{id}_g$.*

[3] We will see that this operation really is a functor in the next section.

[4] Ditto.

We can see this as an example of a more general concept: a **pointed endofunctor** (T, τ) on a category $\mathcal{K}$ is a functor $T \colon \mathcal{K} \to \mathcal{K}$ together with a natural transformation $\tau \colon \mathrm{id}_{\mathcal{K}} \Rightarrow T$. So (T, τ) is a "monad without the multiplication", and like a monad, it gives rise to a **category of algebras**, T-**Alg**, with:

- **Objects** being pairs (X, x) where $X \in \mathcal{K}$ and $x \colon TX \to X$, satisfying the unit condition $x \cdot \tau_X = \mathrm{id}_X$;
- **Morphisms** $(X, x) \to (Y, y)$ being maps $f \colon X \to Y$ such that $y \cdot Tf = f \cdot x$.

In particular, we can consider the category R^1-**Alg** for the pointed endofunctor (R^1, Λ^1) above; and in this language, Proposition 12 says that objects of R^1-**Alg** are the same as objects of $J^\square$. As one would hope, this correspondence extends to morphisms:

Proposition 13. *There is an isomorphism, commuting with the forgetful functors to $\mathcal{C}^{\mathbf{2}}$, between the category $J^\square$ of right lifting data w.r.t. J and the category R^1-**Alg** of algebras for the pointed endofunctor (R^1, Λ^1) on $\mathcal{C}^{\mathbf{2}}$.*

Now, according to the definition in the previous section, a n.w.f.s. $(\mathsf{L}, \mathsf{R}, \Delta)$ is cofibrantly generated by J if $J^\square$ is isomorphic to R-**Map** in a canonical way. Leaving aside the "in a canonical way" part for the moment, and using the characterisation of the previous Proposition, this means that the category R^1-**Alg** we have just defined must be isomorphic to the category of algebras for some monad R on $\mathcal{C}^{\mathbf{2}}$; and this monad R will provide us with the right-hand side of our n.w.f.s.

We thus are led to ask: when does such an isomorphism exist? Questions such as this are dealt with comprehensively in [13], and in this case the answer is very simple.

Proposition 14. *Let (T, τ) be a pointed endofunctor on a category $\mathcal{K}$. Then T-**Alg** is isomorphic to the category of algebras for a monad R on $\mathcal{K}$ just when the forgetful functor $U \colon T$-**Alg** $\to \mathcal{K}$ has a left adjoint. In this case, R is called the **algebraically-free monad** on the pointed endofunctor (T, τ) and is isomorphic to the monad generated by the adjunction $F \dashv U \colon T$-**Alg** $\to \mathcal{K}$.*

So the obvious next question is, when does this left adjoint exist? Again, [13] provides an answer: if $\mathcal{K}$ is cocomplete and the functor T is suitably "small", we can construct the desired left adjoint as the colimit of a transfinite sequence. Here, "small" can mean something very general, but we will only need the following two cases of it. The first is very familiar:

Definition 15. Let α be a cardinal. We say that a limit ordinal β is **α-filtered** if, for every subset $A \subset \beta$ of cardinality $\leqslant \alpha$, we have $\sup A < \beta$. We say that a functor $T \colon \mathcal{K} \to \mathcal{L}$ is **α-small** if it preserves colimits of chains indexed by α-filtered ordinals.

The second is a slight refinement of the first, and requires the notion of an **orthogonal factorisation system** [8] on our category $\mathcal{K}$; as mentioned in the introduction, this is given by two classes of maps $(\mathcal{E}, \mathcal{M})$ satisfying the same axioms as a weak factorisation system, except we strengthen the "lifting" property to

(unique lifting) Whenever we are given a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "e"'] & C \arrow[d, "m"] \\ B \arrow[r, "g"'] & D \end{tikzcd}$$

in $\mathcal{K}$, where $e \in \mathcal{E}$ and $m \in \mathcal{M}$, we can find a *unique* fill-in $j \colon B \to C$ such that $mj = g$ and $je = f$.

Typical examples are (epi, mono) factorisations in **Set**; and either (surjection, embedding) or (quotient, injection) factorisations in **Top**. Given an orthogonal factorisation system $(\mathcal{E}, \mathcal{M})$ on a category $\mathcal{K}$, we shall say that it is **cowellpowered** if every object X of $\mathcal{K}$ possesses a mere set of isomorphism classes of $\mathcal{E}$-maps with domain X: for example, each of the three factorisation systems just cited are cowellpowered.

Definition 16. Let $\mathcal{K}$ be a category equipped with a cowellpowered factorisation system $(\mathcal{E}, \mathcal{M})$. We say that a functor $T \colon \mathcal{K} \to \mathcal{L}$ is **α-small relative to $\mathcal{M}$** if it preserves colimits of chains of $\mathcal{M}$-maps indexed by α-filtered ordinals.

Proposition 17. *Let (T, τ) be a pointed endofunctor on a cocomplete category $\mathcal{K}$, such that T is either α-small or α-small relative to $\mathcal{M}$ for some cowellpowered $(\mathcal{E}, \mathcal{M})$. Then the forgetful functor $U \colon T$-**Alg** $\to \mathcal{K}$ has a left adjoint.*

Applying this result to our pointed endofunctor (R^1, Λ^1), we see that, as long as R^1 is α-small – which amounts to requiring our set of generating maps J to be α-small in a suitable sense – we can build the algebraically-free monad R on (R^1, Λ^1). However, we are not out of the woods yet: this approach builds a monad R for the right-hand side of our putative n.w.f.s., but does not produce a corresponding comonad L. Given how intertwined the two parts of a n.w.f.s. are, it may appear that we are in a somewhat hopeless situation.

This is where the view of n.w.f.s.'s as bialgebras comes into play. The pointed endofunctor (R^1, Λ^1) is really another presentation of the functorial factorisation $T = (E^1, \lambda^1, \rho^1)$. This is an object of $\mathbf{Ff}_{\mathcal{C}}$, and in fact a *pointed object* $\tau \colon I \to T$, where the map $\tau \colon I \to T$ is the unique map from the initial object I. From this perspective, building the free monad on the pointed endofunctor (R^1, Λ^1) is more-or-less the same thing as building the free $\otimes$-monoid on the pointed object (T, τ) of $\mathbf{Ff}_{\mathcal{C}}$.

What we shall shortly see is that the functorial factorisation $T = (E^1, \lambda^1, \rho^1)$ – or rather, its alternative presentation as a pair (L^1, Φ^1) – already admits a comonad structure: so by Proposition 7, we can lift T from an object of $\mathbf{Ff}_{\mathcal{C}}$ to an object of $\mathbf{Comon}_{\odot}(\mathbf{Ff}_{\mathcal{C}})$. Moreover, because $\mathbf{Ff}_{\mathcal{C}}$ is a 2-fold monoidal category, the $(\otimes, I)$ monoidal structure also lifts from $\mathbf{Ff}_{\mathcal{C}}$ to $\mathbf{Comon}_{\odot}(\mathbf{Ff}_{\mathcal{C}})$.

But now we can try to lift the free monoid construction for (T, τ) from $\mathbf{Ff}_{\mathcal{C}}$ to $\mathbf{Comon}_{\odot}(\mathbf{Ff}_{\mathcal{C}})$, thereby obtaining a $\otimes$-monoid in $\mathbf{Comon}_{\odot}(\mathbf{Ff}_{\mathcal{C}})$, which is a bialgebra in $\mathbf{Ff}_{\mathcal{C}}$, which, by Proposition 10, is a n.w.f.s. on $\mathcal{C}$. Moreover, the monad for this n.w.f.s. will be the right thing – the algebraically-free monad R on (R^1, Λ^1) – because the construction we used is just a lifting of this free-monad construction.

Our plan is now as follows: first we show that our functorial factorisation (E^1, λ^1, ρ^1) admits a comonad structure, and thus lifts from $\mathbf{Ff}_{\mathcal{C}}$ to $\mathbf{Comon}_{\odot}(\mathbf{Ff}_{\mathcal{C}})$. We then give an explicit description of the construction of the free $\otimes$-monoid on our lifted functorial factorisation. Finally, we show that the resultant n.w.f.s. really is cofibrantly generated by J: which is where the "in a canonical way" which we laid aside earlier will be picked back up again.

5.2 The one-step comonad

Our task in this section is to take the assignation

$$g \colon C \to D \qquad \mapsto \qquad C \xrightarrow{\lambda^1_g} E^1 g \xrightarrow{\rho^1_g} D \tag{10}$$

of the previous section and show that it gives us a functorial factorisation (E^1, λ^1, ρ^1) on $\mathcal{C}$ for which the corresponding pair (L^1, Φ^1) has a natural extension to a comonad $\mathsf{L}^1 = (L^1, \Phi^1, \Sigma^1)$. This functorial factorisation will be very familiar to readers who know Quillen's small object argument for w.f.s.'s: it provides the "iterative step" by which one transfinitely constructs factorisations. The comonad $\mathsf{L}^1 = (L^1, \Phi^1, \Sigma^1)$ extending it will play a similar role in the construction of natural w.f.s.'s, and thus we christen it the "one-step comonad". It turns out to have a very satisfactory universal property:

Proposition 18. L^1 *is the free "comonad over* dom*" generated by J, in that there are bijections, natural in* $\mathsf{L} \in \mathbf{Comon}_{\odot}(\mathbf{Ff}_{\mathcal{C}})$*, between morphisms* $\mathsf{L}^1 \to \mathsf{L}$ *of* $\mathbf{Comon}_{\odot}(\mathbf{Ff}_{\mathcal{C}})$ *and morphisms*

$$\begin{tikzcd} J \arrow[rr] \arrow[dr, "\iota"'] & & \mathsf{L}\text{-}\mathbf{Map} \arrow[dl, "U_{\mathsf{L}}"] \\ & \mathcal{C}^{\mathbf{2}} & \end{tikzcd}$$

of $\mathbf{Cat}/\mathcal{C}^{\mathbf{2}}$.

Both this Proposition and the construction of L^1 which we are about to given can be deduced from the fact that L^1 is a *density comonad* in a certain 2-category. The notion of density comonad embodies the idea of a comonad being "freely generated" by an arrow: in this case, by the arrow $\iota\colon J \to \mathcal{C}^{\mathbf{2}}$ exhibiting J as a discrete subcategory of $\mathcal{C}^{\mathbf{2}}$. Setting up the theory to explain this here would lead us too far afield, and instead we defer this task to the Appendix. What we give in the remainder of this section is the explicit description of what this abstract framework yields.

So let us return to our contemplation of equation (10), which we recall arose from the following process:

$$\begin{tikzcd} \sum_{x\in S_g} A_x \arrow[r, "\langle h_x\rangle"] \arrow[d, "\sum_x f_x"'] & C \arrow[d, "g"] \\ \sum_{x\in S_g} B_x \arrow[r, "\langle k_x\rangle"'] & D \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \sum_{x\in S_g} A_x \arrow[r, "\langle h_x\rangle"] \arrow[d, "\sum_x f_x"'] & C \arrow[r, equal] \arrow[d, "\lambda^1_g"] & C \arrow[d, "g"] \\ \sum_{x\in S_g} B_x \arrow[r] & E^1 g \arrow[r, "\rho^1_g"'] \arrow[ul, phantom, "\ulcorner", very near start] & D. \end{tikzcd}$$

We want to make the assignation $g \mapsto L^1 g$ into the object part of a functor, for which we must give the value of L^1 on a morphism $\gamma = (h, k)\colon g \to g'$ of $\mathcal{C}^{\mathbf{2}}$. We do this by first making the assignation $g \mapsto Kg := \sum_{x\in S_g} f_x$ into the object part of a functor, whose value on a map $\gamma\colon g \to g'$ of $\mathcal{C}^{\mathbf{2}}$ is given as follows. Observe that postcomposition with γ induces a function

$$\begin{aligned} S_\gamma\colon S_g &\to S_{g'} \\ (f \xrightarrow{\delta} g) &\mapsto (f \xrightarrow{\delta} g \xrightarrow{\gamma} g'). \end{aligned}$$

And thus we take $K\gamma$ to be:

$$K\gamma = \left\langle \mathsf{in}_{S_\gamma(x)} \right\rangle\colon \sum_{x\in S_g} f_x \to \sum_{y\in S_{g'}} f_y.$$

Thus we have a functor $K\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ for which the maps

$$\phi_g = \langle (h_x, k_x) \rangle_{x\in S_g}\colon \sum_{x\in S_g} f_x = Kg \to g$$

become the components of a natural transformation $\phi\colon K \Rightarrow \mathrm{id}_{\mathcal{C}^{\mathbf{2}}}$. We will now use the functor K to give the value of L^1 on morphisms of $\mathcal{C}^{\mathbf{2}}$. Indeed, given such a morphism $\gamma = (h,k)\colon g \to g'$, we have the following diagram, whose left-hand face is $K\gamma$ and whose top face is the domain part of a naturality square for ϕ:

$$
\begin{tikzcd}
\sum_{x\in S_g} A_x \arrow[rr, "\langle h_x\rangle"] \arrow[dr] \arrow[dd, "Kg"'] & & C \arrow[dr, "h"] & \\
& \sum_{y\in S_{g'}} A_y \arrow[rr, "\langle h_y\rangle"] \arrow[dd, "Kg'"] & & C' \\
\sum_{x\in S_g} B_x \arrow[dr] & & & \\
& \sum_{y\in S_{g'}} B_y & &
\end{tikzcd}
$$

Pushing out the rear face gives us $L^1 g$, pushing out the front face gives us $L^1 g'$, and so we take $L^1\gamma$ to be the induced map from the rear to the front of the right-hand face. Now we see that the diagram

$$
\begin{tikzcd}
\sum_{x\in S_g} A_x \arrow[r, "\langle h_x\rangle"] \arrow[d, "\sum_x f_x"'] & C \arrow[r, equal] \arrow[d, "\lambda^1_g"] & C \arrow[d, "g"] \\
\sum_{x\in S_g} B_x \arrow[r] & E^1 g \arrow[r, "\rho^1_g"'] \arrow[ul, phantom, "\ulcorner", very near start] & D,
\end{tikzcd}
\tag{11}
$$

viewed as a pair of maps $\epsilon_g\colon Kg \to L^1 g$ and $\Phi^1_g\colon L^1 g \to g$ in $\mathcal{C}^{\mathbf{2}}$, gives us the components of natural transformations $\epsilon\colon K \Rightarrow L^1$ and $\Phi^1\colon L^1 \Rightarrow \mathrm{id}_{\mathcal{C}^{\mathbf{2}}}$, satisfying $\Phi^1 \cdot \epsilon = \phi$. In particular, we have a copointed endofunctor (L^1, Φ^1) over dom which corresponds to a functorial factorisation (E^1, λ^1, ρ^1), as claimed.

We now show that (L^1, Φ^1) can be extended to a comonad $\mathsf{L}^1 = (L^1, \Phi^1, \Sigma^1)$, for which we must give maps $\Sigma^1_g\colon L^1 g \to L^1 L^1 g$ over dom, which we write as:

$$
\begin{tikzcd}
C \arrow[r, equal] \arrow[d, "\lambda^1_g"'] & C \arrow[d, "\lambda^1_{L^1 g}"] \\
E^1 g \arrow[r, "\sigma^1_g"'] & E^1 L^1 g.
\end{tikzcd}
$$

We obtain these maps as follows: we have a function

$$
\begin{aligned}
\psi_g\colon S_g &\to S_{L^1 g} \\
x &\mapsto (f_x \xrightarrow{\mathsf{in}_x} Kg \xrightarrow{\epsilon_g} L^1 g),
\end{aligned}
$$

and so can form the map

$$
\delta_g = \big\langle \mathsf{in}_{\psi_g(x)} \big\rangle\colon \sum_{x\in S_g} f_x \to \sum_{y\in S_{L^1 g}} f_y.
$$

Now we have the following diagram, whose left-hand face is δ_g:

$$\begin{tikzcd}
\sum_{x \in S_g} A_x \arrow[rr, "\langle h_x \rangle"] \arrow[dr] \arrow[dd, "Kg"'] & & C \arrow[dr, equal] & \\
& \sum_{y \in S_{L^1 g}} A_y \arrow[rr, "\langle h_y \rangle"] \arrow[dd, "KL^1 g"] & & C \\
\sum_{x \in S_g} B_x \arrow[dr] & & & \\
& \sum_{y \in S_{L^1 g}} B_y & &
\end{tikzcd}$$

Pushing out the rear face gives us $E^1 g$, pushing out the front face gives us $E^1 L^1 g$, and the induced map along the bottom-right diagonal we take to be the value of σ^1_g.

This completes our description of the one-step comonad L^1: but before moving on to consider how we can use it to build a n.w.f.s., we should discuss what L^1-coalgebras are. Let us write L^1-**Map** for the category of such, and call its objects L^1-maps; as in Section 2, we write them as pairs (f, s) where $f\colon X \to Y$ and $s\colon Y \to E^1 f$. Now, every L^1-map will induce an L-map in the n.w.f.s. generated by J, and the intuition is that they should be just those L-maps which can be obtained using only one step's worth of "glueing on cells". We will make this intuition precise in Proposition 47, where we will characterise L^1-maps as (certain) retracts of pushouts of coproducts of the generating cofibrations; but for now the following examples should give a good feel for what happens.

Examples 19.

- When $\mathcal{C} = \mathbf{Set}$ and $J = \{0 \to 1\}$, we obtain the one-step factorisation

$$g\colon X \to Y \qquad \mapsto \qquad X \xrightarrow{\mathsf{in}_1} X + Y \xrightarrow{\langle g, \mathrm{id} \rangle} Y,$$

  and $\sigma^1_g = \langle \mathsf{in}_1, \mathsf{in}_3 \rangle\colon X + Y \to X + (X + Y)$. In this case, an L^1-map (f, s) is a map $f\colon X \to Y$ which is an injection: this comonad is "property-like" in that any map can carry at most *one* coalgebra structure. A morphism of L^1-coalgebras $(f, s) \to (f', s')$ is given by a map

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\
Y \arrow[r, "k"'] & Y'
\end{tikzcd} \tag{12}$$

  such that k maps $Y \setminus f(X)$ into $Y' \setminus f'(X')$.

- When $\mathcal{C} = \mathbf{Set}$ and $J = \{\mathsf{in}_1\colon 1 \to 1 + 1\}$, we obtain the one-step factorisation

$$g\colon X \to Y \qquad \mapsto \qquad X \xrightarrow{\mathsf{in}_1} X + X \times Y \xrightarrow{\langle g, \pi_2 \rangle} Y,$$

  and $\sigma^1_g = \langle \mathsf{in}_1, \psi \rangle\colon X + X \times Y \to X + X \times (X + X \times Y)$, where $\psi(c, d) = (c, (c, d))$. An L^1-map (f, s) is given by an injection $f\colon X \to Y$ together with a map $i\colon Y \setminus f(X) \to X$ (saying "where the extra elements were attached").

A morphism of L^1-coalgebras $(h,k)\colon f \to f'$ is a map as in (12) for which the following diagram commutes:

$$\begin{tikzcd}
Y \setminus f(X) \arrow[r, "i"] \arrow[d, "k"'] & X \arrow[d, "h"] \\
Y' \setminus f'(X') \arrow[r, "i'"'] & X'.
\end{tikzcd}$$

- When $\mathcal{C} = \mathbf{Set}$ and $J = \{!\colon 0 \to 1, \mathsf{in}_1\colon 1 \to 1+1\}$, we obtain the one-step factorisation

$$g\colon X \to Y \qquad \mapsto \qquad X \xrightarrow{\mathsf{in}_1} X + Y + X \times Y \xrightarrow{\langle g, \mathrm{id}_Y, \pi_2\rangle} Y,$$

and we leave the description of σ^1_g to the reader. An L^1-map (f,s) is given by an injection $f\colon X \to Y$, a partition of $Y \setminus f(X)$ into disjoint subsets Y_1 and Y_2, and a function $i\colon Y_2 \to X$. The elements of Y_1 correspond to elements attached via $!\colon 0 \to 1$, whilst the elements of Y_2 correspond to elements attached via $\mathsf{in}_1\colon 1 \to 1+1$, and i tells us how these elements were attached.

- When $\mathcal{C} = R$-**Mod** and $J = \{0 \to R\}$, we obtain the one-step factorisation

$$g\colon M \to N \qquad \mapsto \qquad M \xrightarrow{\mathsf{in}_1} M \oplus FN \xrightarrow{\langle g, \mathsf{ev}\rangle} N.$$

Here, FN is the free R-module on the underlying set of N and ev is the obvious map from there to N. If we write the generators of FN as $\{x_n\}_{n \in |N|}$, then the comultiplication map is given by

$$\begin{aligned}
\sigma^1_g\colon M \oplus FN &\to M \oplus F(M \oplus FN) \\
(m,0) &\mapsto (m,0) \\
(0,x_n) &\mapsto (0, x_{x_n}) \quad \text{for } n \in |N|
\end{aligned}$$

extended linearly. An L^1-coalgebra (f,s) is a map $f\colon M \to N$ together with a subset $X \subset N$ such that the canonical map $M \oplus FX \to N$ is an isomorphism of R-modules. A map of L^1-coalgebras is a morphism $(h,k)\colon g \to g'$ such that $k(X) \subset X'$, and such that the diagram

$$\begin{tikzcd}
M \oplus FX \arrow[r, "h \oplus Fk"] \arrow[d, "\mathrm{can}"'] & M' \oplus FX' \arrow[d, "\mathrm{can}"] \\
N \arrow[r, "k"'] & N'
\end{tikzcd}$$

commutes.

- When $\mathcal{C}$ is the category of directed graphs and $J = \{(\bullet) \to (\bullet \to \bullet)\}$, the one-step factorisation is given by:

$$\begin{tikzcd}
X_a \arrow[r, "f_a"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & Y_a \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
X_v \arrow[r, "f_v"'] & Y_v
\end{tikzcd}
\qquad \mapsto \qquad
\begin{tikzcd}
X_a \arrow[r, "\mathsf{in}_1"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & X_a + S_f \arrow[r, "\langle f_a, \overline{f}\rangle"] \arrow[d, shift right, "\langle \mathsf{in}_1 \cdot s, \mathsf{in}_1 \cdot \overline{s}\rangle"'] \arrow[d, shift left, "\langle \mathsf{in}_1 \cdot t, \mathsf{in}_2\rangle"] & Y_a \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
X_v \arrow[r, "\mathsf{in}_1"'] & X_v + S_f \arrow[r, "\langle f_v, t\overline{f}\rangle"'] & Y_v.
\end{tikzcd}$$

Again, we omit the description of σ^1. In this case, an L^1-map (f, s) is given by a map $f\colon X \to Y$ such that $f_a\colon X_a \to Y_a$ and $f_v\colon X_v \to Y_v$ are injections and such that $t\big(Y_a \setminus f_a(X_a)\big) = Y_v \setminus f_v(X_v)$. This is another "property-like" comonad, and comparison with the second example is instructive: despite the similarities, that example was not "property-like", and it is the extra structure borne by the category of directed graphs relative to the category of sets which is responsible for this difference.

5.3 Iterating the one-step monad

Now that we have defined and described the one-step comonad, we are ready to use it to build a n.w.f.s.

Definition 20. Let $\mathcal{C}$ be a cocomplete category, J a set of maps in $\mathcal{C}$, and L^1 the one-step comonad corresponding to J. As in the discussion at the end of Section 5.1, we may view L^1 as a pointed object (L^1, τ) of $\mathbf{Comon} := \mathbf{Comon}_{\odot}(\mathbf{Ff}_{\mathcal{C}})$, and we write (L, η, μ) for the free $\otimes$-monoid[5] on the pointed object (L^1, τ) – if it exists – and call the corresponding n.w.f.s. $(\mathsf{L}, \mathsf{R}, \Delta)$ the **n.w.f.s. generated by** J.

In terms of the R-maps, this process is a suitably refined way of forming the *free monad* on the pointed endofunctor (R^1, Λ^1) which corresponds to L^1, which is the right thing to do because it yields a category of R-maps which is consistent with the requirement that our n.w.f.s. should be generated by J. However, we can also give a natural interpretation of what we are doing in terms of the L-maps.

Indeed, we shall see in the next section that L^1-maps are closed under every possible operation we might like them to be closed under *except composition*: in which terms, we can see the process of constructing our n.w.f.s. from the one-step comonad as "closing off L^1 under composition". To make this intuition slightly less vague, we must examine in more detail the process which assigns to an object $\mathsf{L} \in \mathbf{Comon}$ – viewed as a comonad over dom – its category of coalgebras; in keeping with our previous notation, we will write this assignation as $\mathsf{L} \mapsto \mathsf{L}\text{-}\mathbf{Map}$.

The first observation is that we can make this into a functor $\mathcal{G}\colon \mathbf{Comon} \to \mathbf{Cat}/\mathcal{C}^{\mathbf{2}}$: indeed, a morphism $\alpha\colon \mathsf{L} \to \mathsf{L}'$ in $\mathbf{Comon}$ is a map of the underlying functorial factorisations $\alpha\colon (E, \lambda, \rho) \to (E', \lambda', \rho')$ which is also a map of comonads $\mathsf{L} \to \mathsf{L}'$; thus it induces a morphism

$$\begin{tikzcd} \mathsf{L}\text{-}\mathbf{Map} \arrow[rr, "\alpha_*"] \arrow[dr, "U_{\mathsf{L}}"'] & & \mathsf{L}'\text{-}\mathbf{Map}. \arrow[dl, "U_{\mathsf{L}'}"] \\ & \mathcal{C}^{\mathbf{2}} & \end{tikzcd}$$

Explicitly, this sends an L-algebra (f, s) to the L'-algebra $(f, \alpha_f \cdot s)$: so α_* witnesses that "every L-map is an L'-map".[6] We can now ask how this functor $\mathcal{G}$ interacts with the monoidal structure on $\mathbf{Comon}$. The unit is straightforward: $I \in \mathbf{Comon}$ corresponds to the comonad which sends a map $f\colon X \to Y$ to $\mathrm{id}_X\colon X \to X$, and its category of coalgebras is precisely the full subcategory of $\mathcal{C}^{\mathbf{2}}$ whose objects are the isomorphisms.

The multiplication $\otimes$ is more interesting: from an L-map $(f, s)\colon X \to Y$ and an L'-map $(g, t)\colon Y \to Z$, we can obtain an $(\mathsf{L}' \otimes \mathsf{L})$-map structure on $gf\colon X \to Z$. We

[5] For a formal definition of which, see Definition 22.

[6] The sharp-eyed reader will have spotted that we implicitly used this functor $\mathcal{G}$ in the statement of Proposition 18.

will prove this formally in Section 6.3, where it becomes part of the statement that the functor $\mathcal{G}$ is *lax monoidal* with respect to a suitably defined "compositional" monoidal structure on $\mathbf{Cat}/\mathcal{C}^{\mathbf{2}}$.

Given this, we can see that if we have a *monoid* (L, η, μ) in **Comon**, then its category of coalgebras will be closed under composition: since from a pair of L-maps $f\colon X \to Y$ and $g\colon Y \to Z$, we obtain an $\mathsf{L}\otimes\mathsf{L}$-map $gf\colon X \to Z$; applying $\mu_*\colon (\mathsf{L}\otimes\mathsf{L})\text{-}\mathbf{Map} \to \mathsf{L}\text{-}\mathbf{Map}$ to which gives us an L-map structure on gf. In particular, forming the free monoid on a pointed object of **Comon** can be seen as freely closing off its category of coalgebras under composition.

A final perspective on what we are doing, and perhaps the most convincing, comes from the combination of Proposition 18 and Definition 20:

Proposition 21. *If the n.w.f.s.* $(\mathsf{L}, \mathsf{R}, \Delta)$ *generated by a set of maps* J *exists, then it is the free n.w.f.s. on* J*, in the sense that there are bijections, natural in* $(\mathsf{L}', \mathsf{R}', \Delta')$*, between morphisms* $(\mathsf{L}, \mathsf{R}, \Delta) \to (\mathsf{L}', \mathsf{R}', \Delta')$ *of* $\mathbf{Bialg}(\mathbf{Ff}_{\mathcal{C}})$ *and morphisms*

$$\begin{tikzcd} J \arrow[rr] \arrow[dr, "\iota"'] & & \mathsf{L}'\text{-}\mathbf{Map} \arrow[dl, "U_{\mathsf{L}}"] \\ & \mathcal{C}^{\mathbf{2}} & \end{tikzcd}$$

of $\mathbf{Cat}/\mathcal{C}^{\mathbf{2}}$.

Before we examine the details of the free monoid construction (drawing on [13] once more), let us motivate why it takes the form it does by answering the following question: if L^1-maps corresponds to doing one step's worth of glueing, then for what comonad L^2 do L^2-maps correspond to doing *two* step's worth of glueing? The obvious first guess, $\mathsf{L}^1 \otimes \mathsf{L}^1$, turns out to be not quite right. For we observe that there are *two* copies of L^1 embedded inside $\mathsf{L}^1 \otimes \mathsf{L}^1$, via the maps:

$$\mathsf{L}^1 \xrightarrow{\tau\otimes\mathsf{L}^1} \mathsf{L}^1 \otimes \mathsf{L}^1 \xleftarrow{\mathsf{L}^1\otimes\tau} \mathsf{L}^1.$$

These two embeddings correspond to taking one step's worth of glueing on cells and either prepending or postpending it with one step's worth of doing nothing. But surely we would like to identify these two: we do not really want to record how long we waited around before glueing some cells on, after all. Thus, more correctly, we should take L^2 to be the coequaliser:

$$\begin{tikzcd} \mathsf{L}^1 \arrow[r, shift left, "\tau\otimes\mathsf{L}^1"] \arrow[r, shift right, "\mathsf{L}^1\otimes\tau"'] & \mathsf{L}^1 \otimes \mathsf{L}^1 \arrow[r] & \mathsf{L}^2. \end{tikzcd}$$

This is exactly what we will do, forming each L^{α^+} as a suitable coequaliser of $\mathsf{L}^1 \otimes \mathsf{L}^\alpha$ and then taking L to be the colimit of this sequence. This is somewhat different from a simple-minded generalisation of the small object argument, which would correspond to forming the colimit of a suitably long sequence of the form:

$$\mathsf{L}^1 \xrightarrow{\tau\otimes\mathsf{L}^1} \mathsf{L}^1 \otimes \mathsf{L}^1 \xrightarrow{\tau\otimes\mathsf{L}^1\otimes\mathsf{L}^1} \mathsf{L}^1 \otimes \mathsf{L}^1 \otimes \mathsf{L}^1 \to \cdots$$

in **Comon**. Here, the problem we have just described with respect to $\mathsf{L}^1 \otimes \mathsf{L}^1$ is present and, indeed, drastically multiplied, giving us a plethora of different ways of glueing on the same cells depending on how much waiting around we choose to do; and this is surely not what we want.

5.3.1 The theory

In this section, we give a brief summary of the material we need from [13] pertaining to the construction of a free monoid on a pointed object; except where noted, everything in this section can be found in that paper. We will site our summary in an arbitrary cocomplete[7] monoidal category $(\mathcal{V}, \otimes, I)$, because the degeneracy of the particular example we are interested in (where the unit I is also the initial object) sometimes makes it harder to see what is going on.

We have, of course, the familiar notions of *monoid* and *monoid map* in $\mathcal{V}$, whilst, as we have mentioned before, a *pointed object* (T, τ) of $\mathcal{V}$ is an object $T \in \mathcal{V}$ equipped with a map $\tau \colon I \to T$; finally, by a *map of pointed objects* $\alpha \colon (S, \sigma) \to (T, \tau)$ we mean a map $\alpha \colon S \to T$ satisfying $\alpha\sigma = \tau$.

Definition 22. Let (T, τ) be a pointed object of $\mathcal{V}$: then the **free monoid** on (T, τ) is a monoid (U, η, μ) together with a map of pointed objects $\chi \colon (T, \tau) \to (U, \eta)$ such that precomposition with χ induces a isomorphism, natural in V, between maps of monoids $(U, \eta, \mu) \to (V, \eta', \mu')$ and maps of pointed objects $(T, \tau) \to (V, \eta')$.

Now, to build the free monoid on (T, τ), it often suffices to construct the "free object with an action by T". To make this precise, we consider the category T-**Mod** of **modules** for a pointed object (T, τ), with

- **Objects** being pairs $(X,\ x \colon T \otimes X \to X)$ in $\mathcal{V}$ satisfying $x \cdot (\tau \otimes X) = 1_X$;
- **Morphisms** $f \colon (X, x) \to (Y, y)$ being maps $f \colon X \to Y$ satisfying $f \cdot x = y \cdot (T \otimes f)$.

Observe that if (X, x) is a T-module and $A \in \mathcal{V}$, then so is $(X \otimes A, x \otimes A)$, and that this assignation extends to a "right action" of the monoidal category $\mathcal{V}$ on T-**Mod**; that is, a functor

$$\begin{aligned} \star \colon T\text{-}\mathbf{Mod} \times \mathcal{V} &\to T\text{-}\mathbf{Mod} \\ \big((X, x), A\big) &\mapsto (X \otimes A, x \otimes A) \end{aligned}$$

satisfying the two usual laws for a right action, but weakened up to coherent isomorphism.

Proposition 23. *Let (T, τ) be a pointed object in $\mathcal{V}$. If there is a T-module (X, x) such that $(X, x) \star (–) \colon \mathcal{V} \to T$-**Mod** is left adjoint to the forgetful functor $U \colon T$-**Mod** $\to \mathcal{V}$, then X is the underlying object of the free monoid on (T, τ).*

Proof. The isomorphism $(X \otimes I, x \otimes I) = (X, x) \star I \to (X, x)$ in T-**Mod** corresponds under adjunction to a map $\eta \colon I \to X$, whilst the map $1_X \colon X \to X$ corresponds under adjunction to a map $(X, x) \star X = (X \otimes X, x \otimes X) \to (X, x)$ in T-**Mod** underlying which is a map $\mu \colon X \otimes X \to X$ in $\mathcal{V}$. We refer the reader to [13] for the remaining details. □

The hypotheses of this Proposition are equivalent to saying that, firstly, the free T-module on I exists and is given by (X, x), and secondly, that the free T-module on any other $W \in \mathcal{V}$ can be obtained "pointwise" from this as $(W \otimes X, x \otimes X)$. Our route to satisfying these hypotheses will be to attempt to construct a left adjoint to $U \colon T$-**Mod** $\to \mathcal{V}$ using a certain transfinite construction, which we describe in Definition 26. This process may or not converge when applied to an object X; if it does, we say

[7] For the moment, we make no assumptions about the preservation of colimits in $\mathcal{V}$ by any of the functors $A \otimes (–)$ or $(–) \otimes A$; in particular, we do not assume that $\mathcal{V}$ is either left or right closed.

that the free T-module on X *exists constructively*. All we need to know about this construction for the moment is that it is obtained as the colimit of a certain transfinite sequence – the *free module sequence* for X – each stage of which is built using tensor products and connected colimits of the previous stages.

Proposition 24. *Let (T, τ) be a pointed object in $\mathcal{V}$ and suppose that the free T-module on I exists constructively and is given by (X, x). If each functor $(–) \otimes A\colon \mathcal{V} \to \mathcal{V}$ preserves connected colimits then the forgetful functor T-$\mathbf{Alg} \to \mathcal{V}$ has a left adjoint given by $(X, x) \star (–)\colon \mathcal{V} \to T$-$\mathbf{Alg}$.*

Proof. Because the functor $(–)\otimes A\colon \mathcal{V} \to \mathcal{V}$ preserves connected colimits, the free algebra sequence for A is obtained, up to isomorphism, as $(–) \otimes A$ of the free algebra sequence for I. In particular, we can take the free algebra on A to be $(X \otimes A, x \otimes A)$. ☐

Thus, under the assumption that each functor $(–) \otimes A\colon \mathcal{V} \to \mathcal{V}$ preserves connected colimits, we can build the free monoid on (T, τ) whenever the free T-module on I exists constructively; the only thing remaining is to describe what "exists constructively" means. We first fix some notation concerning transfinite sequences:

Definition 25. Let κ be a regular inaccessible cardinal. We write $\mathbf{On}$ for the well-ordered set of ordinals smaller than κ, viewed as a posetal category. By a *transfinite sequence* X in $\mathcal{V}$, we mean a functor $X\colon \mathbf{On} \to \mathcal{V}$; we write the image of an ordinal α as X_α and the image of the inequality $\alpha \leqslant \beta$ as $X_{\alpha,\beta}\colon X_\alpha \to X_\beta$.

And now:

Definition 26. Let (T, τ) be a pointed object of $\mathcal{V}$, and let $A \in \mathcal{V}$. By the **free T-module sequence** for A, we mean the following transfinite sequence $X\colon \mathbf{On} \to \mathcal{V}$, which we will construct simultaneously with a family of maps $\sigma_\alpha\colon T \otimes X_\alpha \to X_{\alpha^+}$, natural in α and satisfying $\sigma_\alpha \cdot (\tau \otimes X_{\alpha^+}) = X_{\alpha,\alpha^+}$. Note that this last condition determines the value of the connecting maps X_{α,α^+} for any ordinal $\alpha \in \mathbf{On}$, and thus we need only give $X_{\alpha,\beta}$ when β is a limit ordinal.

- $X_0 = A$, $X_1 = T \otimes A$, $\sigma_0 = 1_{T\otimes A}$;
- For a successor ordinal α^+, we give $X_{\alpha^{++}}$ and σ_{α^+} by the following coequaliser diagram:

$$\begin{tikzcd}
 & X_{\alpha^+} \arrow[dr, bend left, "\tau\otimes X_{\alpha^+}"] & & \\
T \otimes X_\alpha \arrow[ur, bend left, "\sigma_\alpha"] \arrow[dr, bend right, "T\otimes\tau\otimes X_\alpha"'] & & T \otimes X_{\alpha^+} \arrow[r, "\sigma_{\alpha^+}"] & X_{\alpha^{++}}. \\
 & T \otimes T \otimes X_\alpha \arrow[ur, bend right, "T\otimes\sigma_\alpha"'] & &
\end{tikzcd}$$

- For a non-zero limit ordinal γ, we give X_γ by $\operatorname{colim}_{\alpha<\gamma} X_\alpha$, with connecting maps $X_{\alpha,\gamma}$ given by the injections into the colimit. We give X_{γ^+} and σ_γ by the following coequaliser diagram:

$$\begin{tikzcd}
 & \operatorname{colim} X_{\alpha^+} = X_\gamma \arrow[dr, "\tau\otimes X_\gamma"] & & \\
\operatorname{colim}(T \otimes X_\alpha) \arrow[ur, "\operatorname{colim}\sigma_\alpha"] \arrow[rr, "\mathrm{can}"'] & & T \otimes \operatorname{colim} X_\alpha = T \otimes X_\gamma \arrow[r, "\sigma_\gamma"'] & X_{\gamma^+}
\end{tikzcd}$$

where "can" is the canonical map induced by the cocone $T \otimes X_\alpha \to T \otimes \operatorname{colim} X_\alpha$.

We say that this sequence *converges at* α if $X_{\alpha,\alpha^+} : X_\alpha \to X_{\alpha^+}$ is invertible for some $\alpha \in \mathbf{On}$; it then follows that $X_{\alpha,\beta}$ is invertible for *every* $\beta > \alpha$.

Proposition 27. *Let* (T, τ) *be a pointed object of* $\mathcal{V}$ *and* $A \in \mathcal{V}$ *for which the free* T*-module sequence for* A *converges at* α*. Then the free* T*-module on* A *exists constructively, and is given by* X_α *equipped with the algebra map*

$$T \otimes X_\alpha \xrightarrow{\theta_\alpha} X_{\alpha^+} \xrightarrow{X^{-1}_{\alpha,\alpha^+}} X_\alpha .$$

The universal map from A *is given by* $X_{0,\alpha} \colon A \to X_\alpha$*.*

To ensure that T-module sequences do converge, we require, as in Proposition 17, some smallness assumption on T. Recall that we gave two such notions in Definitions 15 and 16, which we can reuse by stipulating that an object $T \in \mathcal{V}$ is **α-small** or **α-small relative to $\mathcal{M}$** just when the corresponding endofunctor $T \otimes (–) \colon \mathcal{V} \to \mathcal{V}$ is so. Kelly shows that if T is α-small, then every free T-module sequence will converge at α, whilst if T is α-small relative to some $\mathcal{M}$, then every free T-module sequence will converge, though not necessarily at α. Combining all of the above, we have the following result:

Corollary 28. *Let* $\mathcal{V}$ *be a cocomplete monoidal category such that each functor* $(–) \otimes A \colon \mathcal{V} \to \mathcal{V}$ *preserves connected colimits, and let* (T, τ) *be a pointed object of* $\mathcal{V}$ *such that* T *is either* α*-small or* α*-small relative to* $\mathcal{M}$ *for some cowellpowered* $(\mathcal{E}, \mathcal{M})$*. Then the free monoid on* (T, τ) *exists: its underlying object is given by the free* T*-module on* I*, as constructed in Definition 26, and its multiplication and unit are given as in Proposition 23.*

5.3.2 The practice

We are now ready to apply this machinery to the case of interest to us. The result we are aiming for is the following:

Proposition 29. *Let* $\{f_j \colon A_j \to B_j\}_{j \in J}$ *be a set of maps in a cocomplete category* $\mathcal{C}$ *such that either every* $\mathcal{C}(A_j, –) \colon \mathcal{C} \to \mathbf{Set}$ *is* α_j*-small or there is a cowellpowered factorisation system* $(\mathcal{E}, \mathcal{M})$ *on* $\mathcal{C}$ *such that every* $\mathcal{C}(A_j, –) \colon \mathcal{C} \to \mathbf{Set}$ *is* α_j*-small with respect to* $\mathcal{M}$*. Then the n.w.f.s. generated by* J *exists.*

Our method, of course, will be to apply Corollary 28, and so we will obtain the n.w.f.s. generated by a set of maps J as the colimiting value L of the free L^1-module sequence on I in **Comon**. Thus we will have:

- For each $\alpha \in \mathbf{On}$, an object $\mathsf{L}^\alpha \in \mathbf{Comon}$ (where $\mathsf{L}^0 = I$ and L^1 is itself);
- For each $\alpha \in \mathbf{On}$, an "action morphism" $\theta_\alpha \colon \mathsf{L}^1 \otimes \mathsf{L}^\alpha \to \mathsf{L}^{\alpha^+}$;
- For each $\alpha < \beta \in \mathbf{On}$, a connecting morphism $X_{\alpha,\beta} \colon \mathsf{L}^\alpha \to \mathsf{L}^\beta$,

In particular, the connecting map from L^1 into the colimiting value L is the "universal map of pointed objects" $\chi \colon \mathsf{L}^1 \to \mathsf{L}$ of Definition 22; we will make use of this map in the next subsection.

In terms of the intuitive description given at the start of Section 5.3, the L^α-maps correspond to maps given by "at most α steps of glueing on cells"; the morphisms $X_{\alpha,\beta}$

witness the fact that every L^α-map is an L^β-map when $\alpha < \beta$; and the morphisms θ_α attest to the fact that anything we can do in most α steps of glueing followed by a single further step of glueing, we can do in at most α^+ steps of glueing.

Now, although the hypotheses of Proposition 29 should be fairly unsurprising to anyone used to the small object argument for plain w.f.s.'s, the proof that they are sufficient is surprisingly technical. Before we give it, let us see how widely these hypotheses are satisfied. Firstly, they hold for any set of maps J whatsoever in a *locally presentable* category $\mathcal{C}$. We recall that a category $\mathcal{C}$ is **locally κ-presentable** for some regular cardinal κ if it is cocomplete and has a set $\mathcal{S}$ of objects such that we have both

(density) Every $X \in \mathcal{C}$ is a canonical colimit of elements of $\mathcal{S}$; and

(smallness) For each $A \in \mathcal{S}$, $\mathcal{C}(A, \mathord{-}) \colon \mathcal{C} \to \mathbf{Set}$ preserves κ-filtered colimits,

and that $\mathcal{C}$ is **locally presentable** if it is locally κ-presentable for some κ. Now, for any object X in a locally presentable category, there exists some cardinal α such that $\mathcal{C}(X, \mathord{-})$ is α-small, and thus the hypotheses of the Proposition will always hold.

Any sufficiently "algebraic" category is locally presentable: for example, **Set**, **Ab**, R-**Mod** (modules over a ring R), $\mathbf{Ch}(R)$ (chain complexes over a ring R) or any presheaf category, in particular the category **SSet** of simplicial sets, are all locally presentable and in fact locally *finitely* presentable, i.e., locally ω-presentable. Examples of locally presentable categories that are not locally finitely presentable include the category $\mathbf{Shv}(\mathcal{C})$ of sheaves for a site $\mathcal{C}$ and the category ω-**CPO** of ω-complete partially ordered sets. For more on the theory of locally presentable categories, one might refer to [5] or [1].

There are also important examples of non-locally presentable categories in which our Proposition can be applied: for example, when J is any set of maps whatsoever in the category **Top** of topological spaces or the category **Haus** of compact Hausdorff topological spaces. In these cases, we need our more refined notion of smallness: namely, relative to a cowellpowered orthogonal factorisation system $(\mathcal{E}, \mathcal{M})$, which for both of the named categories, we can take to be given by $\mathcal{E}$ = projections and $\mathcal{M}$ = subspace embeddings. We have that *for any topological space X of cardinality $< \alpha$, $\mathbf{Top}(X, \mathord{-})$ is α-small relative to the subspace embeddings*: see, for example, [11, Section 2.4] for a proof. The same holds when **Top** is replaced by **Haus**, and thus the hypotheses of the Proposition will be satisfied, for any set J of maps whatsoever, in either of these two categories.

The rest of this section will be devoted to proving Proposition 29. Our method, of course, will be to apply Corollary 28, and so we need to check that all the relevant hypotheses are satisfied. Apart from the smallness condition, this amounts to showing that:

Proposition 30. *The category* **Comon** *is cocomplete, and for each $A \in$* **Comon***, the functor $(\mathord{-}) \otimes A \colon$* **Comon** $\to$ **Comon** *preserves connected colimits.*

Proof. We know that $\mathcal{C}$ is cocomplete, and thus so also is $[\mathcal{C}^\mathbf{2}, \mathcal{C}]$. Now the category $\mathbf{Ff}_\mathcal{C}$ can be obtained by slicing $[\mathcal{C}^\mathbf{2}, \mathcal{C}]$ over the object cod, and then coslicing this under the object $(\kappa \colon \mathrm{dom} \Rightarrow \mathrm{cod})$: thus $\mathbf{Ff}_\mathcal{C}$ is cocomplete. Moreover, the forgetful functor $\mathbf{Comon}_\odot(\mathbf{Ff}_\mathcal{C}) \to \mathbf{Ff}_\mathcal{C}$ creates colimits, and so $\mathbf{Comon}_\odot(\mathbf{Ff}_\mathcal{C})$ is also cocomplete. For the second part, consider the composite forgetful functor

$$U \colon \mathbf{Comon}_\odot(\mathbf{Ff}_\mathcal{C}) \to \mathbf{Ff}_\mathcal{C} \to [\mathcal{C}^\mathbf{2}, \mathcal{C}^\mathbf{2}],$$

where the second arrow sends a functorial factorisation (F, λ, ρ) to the corresponding functor $R \colon \mathcal{C}^\mathbf{2} \to \mathcal{C}^\mathbf{2}$. Now, the first part creates all colimits whilst the second creates connected colimits, and thus U creates connected colimits. Moreover, it sends the $(\otimes, I)$ monoidal structure on **Comon** to the monoidal structure on $[\mathcal{C}^{\mathbf{2}}, \mathcal{C}^{\mathbf{2}}]$ given by composition; and so, given $A \in \mathbf{Comon}$ with underlying object (F, λ, ρ) in $\mathbf{Ff}_{\mathcal{C}}$, the following diagram commutes:

$$\begin{tikzcd} \mathbf{Comon} \arrow[r, "(-)\otimes A"] \arrow[d, "U"'] & \mathbf{Comon} \arrow[d, "U"] \\ {[\mathcal{C}^{\mathbf{2}}, \mathcal{C}^{\mathbf{2}}]} \arrow[r, "(-)\cdot R"'] & {[\mathcal{C}^{\mathbf{2}}, \mathcal{C}^{\mathbf{2}}]}. \end{tikzcd}$$

But $(-) \cdot R$ preserves connected colimits (indeed, all colimits), and so the result follows. $\square$

We will not directly satisfy the smallness condition for L^1, but will give sufficient conditions for free L^1-module sequences to converge nonetheless:

Proposition 31. *Let $\mathsf{L}^1 \in$ **Comon** be the one-step comonad generated by a set of maps J in $\mathcal{C}$, and let (R^1, Λ^1) be the corresponding pointed endofunctor of $\mathcal{C}^{\mathbf{2}}$. Then free L^1-modules exist constructively whenever the functor R^1 is either α-small or α-small with respect to $\mathcal{M}$ for some cowellpowered $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}^{\mathbf{2}}$.*

Proof. It suffices to show that the free L^1-module on I exists constructively, since all other free L^1-modules are computed "pointwise" from this one. We do this by considering the forgetful functor $U \colon \mathbf{Comon} \to [\mathcal{C}^{\mathbf{2}}, \mathcal{C}^{\mathbf{2}}]$ from the proof of the previous Proposition. Observe first that U sends the pointed object (L^1, τ) to the pointed endofunctor (R^1, Λ^1). Moreover, because U both preserves the monoidal structure and creates connected colimits, it sends the free L^1-module sequence on I to the *free monad* sequence for the pointed endofunctor (R^1, Λ^1); we will not define this formally – the reader can find it in [13] – but it should suffice if we say that it looks like Definition 26 with the tensor product symbols removed.

We now observe that U reflects isomorphisms, so that the free L^1-module sequence on I will converge whenever the free monad sequence for (R^1, Λ^1) does, and so in particular whenever R^1 satisfies either of the given smallness hypotheses. $\square$

Note that it also follows from this proof that the underlying monad of the n.w.f.s. generated from L^1 will be the algebraically-free monad on the pointed endofunctor (R^1, Λ^1), which is what we wanted. All that remains is to show that under the hypotheses of Proposition 29, the corresponding R^1 is suitably small.

Before we do so, we note that we can weaken the smallness requirement on R^1 slightly: indeed, if we take the free monad sequence for R^1 – which is given by a chain of endofunctors of $\mathcal{C}^{\mathbf{2}}$ and natural transformations between them – and evaluate it at any object $f \in \mathcal{C}^{\mathbf{2}}$, then the resulting chain in $\mathcal{C}^{\mathbf{2}}$ will be *constant* in its codomain part, so that it suffices for R^1 to be small with respect to chains of this sort. To put this another way, observe that, since R^1 is a functor over cod, it restricts to functors $R^1\big|_X : \mathcal{C}/X \to \mathcal{C}/X$ on each slice category of $\mathcal{C}$, and it is sufficient for each of these functors to be small.

Proposition 32. *Suppose that $\mathsf{L}^1 \in$ **Comon** is generated by a set of morphisms $\{f_j \colon A_j \to B_j\}_{j \in J}$ such that each $\mathcal{C}(A_j, -) \colon \mathcal{C} \to$ **Set** is α_j-small. Then there is a regular cardinal α such that each functor $R^1\big|_X : \mathcal{C}/X \to \mathcal{C}/X$ is α-small.*

Proof. Take α to be a regular cardinal which is larger than all of the α_j, so that each $\mathcal{C}(A_j,\text{–})\colon \mathcal{C} \to \mathbf{Set}$ is α-small. We aim to show that each $R^1\big|_X : \mathcal{C}/X \to \mathcal{C}/X$ is α-small. Our first observation is that, because the functor $\Sigma_X : \mathcal{C}/X \to \mathcal{C}$ which forgets the projection onto X creates colimits, it will suffice to show that each composite functor $\Sigma_X \cdot R^1\big|_X : \mathcal{C}/X \to \mathcal{C}$ is α-small. Now, $\Sigma_X \cdot R^1\big|_X$ is just the restriction of $F^1 \colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ along the inclusion $i\colon \mathcal{C}/X \to \mathcal{C}^{\mathbf{2}}$, which we can write as the composite:

$$\mathcal{C}/X \xrightarrow{i} \mathcal{C}^{\mathbf{2}} \xrightarrow{L^1} \mathcal{C}^{\mathbf{2}} \xrightarrow{\mathrm{cod}} \mathcal{C}.$$

So, since $\mathrm{cod}\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ preserves all colimits, we will be done if we can prove that the composite $L^1 \cdot i\colon \mathcal{C}/X \to \mathcal{C}^{\mathbf{2}}$ is α-small. To do this, we first show that the composite $K \cdot i\colon \mathcal{C}/X \to \mathcal{C}^{\mathbf{2}}$ is α-small, where we recall from Section 5.2 that $K\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ is the functor defined by

$$Kg = \sum_{x \in S_g} f_x.$$

So let β be an α-filtered ordinal and let $g_{(-)}\colon \beta \to \mathcal{C}/X$: we must show that $K \cdot i$ preserves the colimit of this sequence, or equivalently, since i preserves connected colimits, that K preserves the colimit of $i \cdot g_{(-)}\colon \beta \to \mathcal{C}^{\mathbf{2}}$. To do this, observe first that for any $f\colon A \to B$ and $g\colon C \to X$ in $\mathcal{C}$, we have

$$\mathcal{C}^{\mathbf{2}}(f,g) \cong \sum_{k \in \mathcal{C}(B,X)} \mathcal{C}/X(\Sigma_k f, g)$$

where $\Sigma_k\colon \mathcal{C}/B \to \mathcal{C}/X$ is the functor given by postcomposition with k. Thus for any $f\colon A \to B$ such that that the functor $\mathcal{C}(A,\text{–})\colon \mathcal{C} \to \mathbf{Set}$ is α-small, we have:

$$\begin{aligned}
\mathcal{C}^{\mathbf{2}}(f, \mathrm{colim}_i\, g_i) &\cong \sum_{k \in \mathcal{C}(B,X)} \mathcal{C}/X(\Sigma_k f, \mathrm{colim}_i\, g_i) \\
&\cong \sum_{k \in \mathcal{C}(B,X)} \mathrm{colim}_i\, \mathcal{C}/X(\Sigma_k f, g_i) \\
&\cong \mathrm{colim}_i \sum_{k \in \mathcal{C}(B,X)} \mathcal{C}/X(\Sigma_k f, g_i) \\
&= \mathrm{colim}_i\, \mathcal{C}^{\mathbf{2}}(f, g_i),
\end{aligned}$$

where the step from the first to the second line follows from the smallness of $\mathcal{C}(A,\text{–})$ and the fact that the forgetful functor $\mathcal{C}/X \to \mathcal{C}$ creates colimits. Since for each $f_j \in J$, we know that $\mathcal{C}(A_j,\text{–})$ is α-small, we now deduce:

$$\begin{aligned}
S_{\mathrm{colim}_i\, g_i} &= \sum_{j \in J} \mathcal{C}^{\mathbf{2}}(f_j, \mathrm{colim}_i\, g_i) \\
&\cong \sum_{j \in J} \mathrm{colim}_i\, \mathcal{C}^{\mathbf{2}}(f_j, g_i) \\
&\cong \mathrm{colim}_i \sum_{j \in J} \mathcal{C}^{\mathbf{2}}(f_j, g_i) = \mathrm{colim}_i\, S_{g_i}
\end{aligned}$$

And so we have:

$$\begin{aligned}
K(\operatorname{colim}_i g_i) &= \sum_{x \in S_{\operatorname{colim}_i g_i}} f_x \\
&\cong \sum_{x \in \operatorname{colim}_i S_{g_i}} f_x \\
&\cong \operatorname{colim}_i \sum_{x \in S_{g_i}} f_x = \operatorname{colim}_i K g_i.
\end{aligned}$$

Thus $K \cdot i \colon \mathcal{C}/X \to \mathcal{C}^{\mathbf{2}}$ is α-small; it remains to deduce that the same is true of $L^1 \cdot i$. What we will in fact prove is the stronger statement that L^1 preserves any colimit which K does. We first recall that $L^1 g$ is obtained from Kg by pushing out along the domain of the canonical map $\phi_g \colon Kg \to g$ given by

$$\phi_g = \langle (h_x, k_x) \rangle_{x \in S_g} \colon \sum_{x \in S_g} f_x \to g,$$

which we shall write as $L^1 g = (h_g)_*(Kg)$. Now suppose that K preserves the colimit of some $F \colon \mathcal{A} \to \mathcal{C}^{\mathbf{2}}$. Then on the one hand, we have

$$\operatorname{colim}\big(L^1(Fa)\big) = \operatorname{colim}\big((h_{Fa})_*(KFa)\big) \cong (\operatorname{colim} h_{Fa})_*(\operatorname{colim} KFa)$$

since colimits commute with pushouts. On the other, we have

$$L^1(\operatorname{colim} Fa) = (h_{\operatorname{colim} Fa})_*(K \operatorname{colim} Fa).$$

But we have the following commutative diagram:

$$\begin{tikzcd}
\operatorname{colim} KFa \arrow[r, "\operatorname{colim} \phi_{Fa}"] \arrow[d, "\mathrm{can}"'] & \operatorname{colim} Fa \arrow[d, equal] \\
K \operatorname{colim} Fa \arrow[r, "\phi_{\operatorname{colim} Fa}"'] & \operatorname{colim} Fa,
\end{tikzcd}$$

and thus, since can is an isomorphism and hence already a pushout, we deduce that

$$\begin{aligned}
\operatorname{colim}\big(L^1(Fa)\big) &= (\operatorname{colim} h_{Fa})_*(\operatorname{colim} KFa) \\
&\cong (h_{\operatorname{colim} Fa})_*(\mathrm{dom\ can})_*(\operatorname{colim} KFa) \\
&\cong (h_{\operatorname{colim} Fa})_*(K \operatorname{colim} Fa) \\
&= L^1(\operatorname{colim} Fa)
\end{aligned}$$

as desired. □

In order to state the corresponding result for our more refined form of smallness, we first need the following straightforward fact: if a category $\mathcal{C}$ comes equipped with a cowellpowered orthogonal factorisation system $(\mathcal{E}, \mathcal{M})$ then there is a corresponding cowellpowered factorisation system of the same name on each slice category $\mathcal{C}/X$, whose $\mathcal{E}$-maps and $\mathcal{M}$-maps are precisely those which become so when one forgets the projection down to X.

Proposition 33. *Let $(\mathcal{E}, \mathcal{M})$ be a cowellpowered orthogonal factorisation system on $\mathcal{C}$, and let $\mathsf{L}^1 \in$ **Comon** be generated as before by a set of morphisms $\{f_j \colon A_j \to B_j\}_{j \in J}$ such that each $\mathcal{C}(A_j, \text{–}) \colon \mathcal{C} \to$ **Set** is α_j-small with respect to $\mathcal{M}$. Then there is a regular cardinal α such that each functor $R^1\big|_X : \mathcal{C}/X \to \mathcal{C}/X$ is α-small with respect to $\mathcal{M}$.*

The proof of the following is identical to the proof of the previous Proposition; and with it we have the final ingredient to complete the proof of Proposition 29.

Examples 34.

- When $\mathcal{C} = \mathbf{Set}$ and $J = \{0 \to 1\}$, the n.w.f.s. generated by J is the same as the one-step factorisation L^1 we constructed before. Essentially, this is because there is no *need* for more than one step's worth of "glueing on cells", because such glueings do not create any new boundaries into which further cells can be glued. So the underlying functorial factorisation is

$$g \colon X \to Y \qquad \mapsto \qquad X \xrightarrow{\mathsf{in}_1} X + Y \xrightarrow{\langle g, \mathrm{id} \rangle} Y;$$

  as before, and we have seen that the L-coalgebras are precisely the injections. For the monad part R, we have $\pi_g = \langle \mathsf{in}_1, \mathsf{in}_2, \mathsf{in}_2 \rangle \colon X + Y + Y \to X + Y$, but we do not need to describe the R-algebras, for this example or any of the following ones, because they are precisely the elements of $J^{\square}$ that we described in Examples 11: which is as we would hope, since this was the whole point of setting up all this machinery!

- When $\mathcal{C} = \mathbf{Set}$ and $J = \{\mathsf{in}_1 \colon 1 \to 1 + 1\}$, the n.w.f.s. generated by J has the underlying functorial factorisation

$$g \colon X \to Y \qquad \mapsto \qquad X \xrightarrow{\lambda_g} X \times Y^* \xrightarrow{\rho_g} Y,$$

  where Y^* is the free monoid on Y, whose elements are (possibly empty) lists $(y_1, \dots, y_n)$ of elements of Y; we thus write elements of $X \times Y^*$ as $(x, y_1, \dots, y_n)$ for some $n \geqslant 0$. Now λ_g sends x to (x) and ρ_g is given by:

$$\begin{aligned} \rho_g(x) &= g(x) \\ \rho_g(x, y_1, \dots, y_n) &= y_n \qquad \text{for all } n > 0. \end{aligned}$$

  The map $\sigma_g \colon X \times Y^* \to X \times (X \times Y^*)^*$ sends $(x, y_1, \dots, y_n)$ to the element

$$\big(x, (x, y_1), (x, y_1, y_2), \dots, (x, y_1, \dots, y_n)\big)$$

  of $X \times (X \times Y^*)^*$, whilst the map $\pi_g \colon X \times Y^* \times Y^* \to X \times Y^*$ is given by

$$\pi_g(x, (y_1, \dots, y_k), (y_{k+1}, \dots, y_n)) = (x, y_1, \dots, y_n).$$

  An L-map is given by an injection $f \colon X \to Y$, together with a partition of Y into disjoint subsets $Y_0, Y_1, Y_2, \dots$, where $Y_0 = f(X)$, and for each $n \geqslant 0$ a map $i_n \colon Y_{n+1} \to Y_n$. We can view such maps as specifying an X-indexed family of well-founded trees, whose roots are labelled by the elements of X and whose other nodes are labelled by the elements of $Y \setminus f(X)$. A morphism of L-maps is given by a map $(h, k) \colon f \to f'$ which respects the partitions of Y and Y' and commutes with the attaching maps i_n and i'_n; in terms of trees, this amounts to giving a function $h \colon X \to X'$ together with a X-indexed family of height-preserving morphisms from the tree labeled by x to the tree labelled by $h(x)$.

- When $\mathcal{C} = \mathbf{Set}$ and $J = \{!\colon 0 \to 1, \mathsf{in}_1\colon 1 \to 1+1\}$, the n.w.f.s. generated by J has functorial factorisation

$$g\colon X \to Y \qquad \mapsto \qquad X \xrightarrow{\lambda_g} (X+Y) \times Y^* \xrightarrow{\rho_g} Y;$$

  where with the same conventions as before, λ_g sends x to (x) whilst ρ_g is given by:

$$\begin{aligned} \rho_g(x) &= g(x) && \text{for } x \in X, \\ \rho_g(y) &= y && \text{for } y \in Y, \\ \rho_g(\star, y_1, \dots, y_n) &= y_n && \text{for all } n > 0 \text{ and } \star \in X+Y. \end{aligned}$$

  An L-map is given by an injection $f\colon X \to Y$, a partition of Y into disjoint subsets $Y_0, Z_0, Y_1, Z_1, \dots$, where $Y_0 = f(X)$, and for each $n \geqslant 0$, functions $i_n\colon Y_{n+1} \to Y_n$ and $j_n\colon Z_{n+1} \to Z_n$. In terms of trees, we can see this as specifying a family of trees, such that every node is labelled by an element of Y and such that elements of $f(X)$ only ever label the roots of trees. We can give a description of morphisms of L-maps in similar terms.
- When $\mathcal{C} = R\text{-}\mathbf{Mod}$ and $J = \{0 \to R\}$, we are in the same situation as in the first example: the n.w.f.s. generated by J coincides with the one-step comonad, and for the same reason – that we can always glue on all the cells we want in only one step.
- When $\mathcal{C}$ is the category of directed graphs and $J = \{(\bullet) \to (\bullet \to \bullet)\}$, the n.w.f.s. generated by J has the following functorial factorisation. Given a map $g\colon X \to Y$, the directed graph Eg has vertices of two sorts:
  - Vertices $x \in X_v$, and
  - Sequences $(x, b_1, y_1, b_2, y_2, \dots, b_n, y_n)$, where x is a vertex of X, each y_i is a vertex of Y and each b_i is an arrow of Y, satisfying $s(b_i) = y_{i-1}$ and $t(b_i) = y_i$ for each $i \geqslant 0$ (with the convention that $y_0 = f(x)$),

  and arrows of two sorts:
  - Arrows $a \in X_v$, and
  - Sequences $(x, b_1, y_1, b_2, y_2, \dots, y_{n-1}, b_n)$ as above, but omitting the final y_n.

  The source and target of an arrow $a \in X_v$ are given by $s(a)$ and $t(a)$, whilst the source and target of an arrow $(x, b_1, y_1, \dots, y_{n-1}, b_n)$ are given by:

$$\begin{aligned} s(x, b_1, y_1, \dots, y_{n-1}, b_n) &= (x, b_1, y_1, \dots, b_{n-1}, y_{n-1}) \\ t(x, b_1, y_1, \dots, y_{n-1}, b_n) &= \big(x, b_1, y_1, \dots, b_n, t(b_n)\big). \end{aligned}$$

  The map $\lambda_g\colon X \to Eg$ is the obvious inclusion, whilst the map $\rho_g\colon Eg \to Y$ is given by

$$\begin{aligned} \rho_g(x) &= g(x) \text{ for } x \text{ a vertex of } X; \\ \rho_g(x, b_1, y_1, b_2, y_2, \dots, b_n, y_n) &= y_n; \\ \rho_g(a) &= g(a) \text{ for } a \text{ an arrow of } X, \text{ and} \\ \rho_g(x, b_1, y_1, \dots, y_{n-1}, b_n) &= b_n. \end{aligned}$$

  Skipping over the description of σ and π, which the reader should be able to figure out by now, we observe that once again the comonad L is "property-like." This time, a map $f\colon X \to Y$ is an L-map just when both f_a and f_v are injections and we can partition Y_a into sets $A_0, A_1, A_2, \ldots$ and Y_v into sets $V_0, V_1, \ldots$ satisfying the following properties:

- $A_0 = f_a(X_a)$ and $V_0 = f_v(X_v)$;
- $s(A_i) \subset V_{i-1}$ for $i \geqslant 1$, and
- $t(A_i) = V_i$ for $i \geqslant 1$.

5.4 Cofibrant generation

We have one final loose end to tie up in this section: we must show that the n.w.f.s. $(\mathsf{L}, \mathsf{R}, \Delta)$ generated by a set of maps J is in fact *cofibrantly generated* by J, in the sense of Section 4.

In order to do this, we need to specify one addition piece of data, namely how we want to view our generating cofibrations J as L-maps. The map by which we do this is, in the language of Proposition 21, the universal map exhibiting $(\mathsf{L}, \mathsf{R}, \Delta)$ as the free n.w.f.s. on J. To construct it explicitly, we first lift through the category of L^1-maps:

Proposition 35. *Every generating cofibration $f \in J$ carries a canonical structure L^1-map structure; in other words, we have a lifting*

$$\begin{tikzcd} J \arrow[rr, "\alpha"] \arrow[dr, "\iota"'] & & \mathsf{L}^1\text{-}\mathbf{Map}. \arrow[dl, "U_{\mathsf{L}^1}"] \\ & \mathcal{C}^{\mathbf{2}} & \end{tikzcd}$$

Proof. If f is a generating cofibration, then we have the element $i = (\mathrm{id}_f\colon f \to f)$ in S_f, and so can make f into an L^1-coalgebra (f, α_f) by taking α_f to be the codomain part of the morphism

$$f \xrightarrow{\mathsf{in}_i} Kf \xrightarrow{\epsilon_f} L^1 f$$

of $\mathcal{C}^{\mathbf{2}}$. The "canonicity" of this lifting amounts to the fact that it is *another* universal map, this time the one exhibiting L^1 as the free "comonad over dom" on J in the sense of Proposition 18. $\square$

We now use the fact that "every L^1-map is an L-map": more formally, we obtain from the universal map $\chi\colon \mathsf{L}^1 \to \mathsf{L}$ of Definition 22 a functor $\mathsf{L}^1\text{-}\mathbf{Map} \to \mathsf{L}\text{-}\mathbf{Map}$, and hence a lifting of $\beta\colon J \to \mathsf{L}\text{-}\mathbf{Map}$ given by the following composite:

$$\begin{tikzcd} J \arrow[r, "\alpha"] \arrow[dr, "\iota"'] & \mathsf{L}^1\text{-}\mathbf{Map} \arrow[r, "\chi_*"] \arrow[d, "U_{\mathsf{L}^1}"] & \mathsf{L}\text{-}\mathbf{Map} \arrow[dl, "U_{\mathsf{L}}"] \\ & \mathcal{C}^{\mathbf{2}}. & \end{tikzcd}$$

Concretely, if we write $\chi_f\colon E^1 f \to Ef$ for the underlying components of $\chi\colon \mathsf{L}^1 \to \mathsf{L}$, then we have $\beta(f) = (f, \chi_f \cdot \alpha_f)$. Our goal now is the following result:

Proposition 36. *Let $(\mathsf{L}, \mathsf{R}, \Delta)$ be the n.w.f.s. generated by a set of maps J. Then $(\mathsf{L}, \mathsf{R}, \Delta)$ is cofibrantly generated by (J, β).*

Proof. Recall that this means that there is a canonical isomorphism between the category $J^\square$ of right lifting data with respect to J and the category R^1-**Alg** of algebras for the pointed endofunctor (R^1, Λ^1) corresponding to L^1. In order to show this, we must examine the relationship between the categories R^1-**Alg** and R-**Map**. First we note that underlying the universal map of pointed objects $\chi\colon \mathsf{L}^1 \to \mathsf{L}$ is a map of pointed endofunctors $(R^1, \Lambda^1) \to (R, \Lambda)$, which induces a functor

$$\chi^*\colon \mathsf{R}\text{-}\mathbf{Map} \to R^1\text{-}\mathbf{Alg}$$
$$(f\colon X \to Y, s\colon Ef \to X) \mapsto (f\colon X \to Y, s \cdot \chi_f\colon E^1 f \to X).$$

But by the proof of Proposition 31, we know that R is the algebraically-free monad on (R^1, Λ^1), and thus that R-**Map** is isomorphic to R^1-**Alg**; which stated more carefully says that the functor χ^* is an isomorphism of categories. We now have the following situation:

$$\begin{tikzcd} \mathsf{R}\text{-}\mathbf{Map} \arrow[rr, "\theta"] \arrow[dr, "\chi^*"'] & & J^\square. \\ & R^1\text{-}\mathbf{Alg} \arrow[ur, "\psi"'] & \end{tikzcd}$$

where ψ is the isomorphism of categories of Proposition 13 and where θ is the canonical map of Section 4. We know that both the diagonal arrows are isomorphisms, and want to conclude that the horizontal arrow is an isomorphism: so if we can show that the diagram commutes, we will be done, and we can do this by direct calculation. First, the upper side. If we are given an R-map $(g\colon C \to D, s\colon Eg \to C)$ and an element $x \in S_g$: that is, an (f, g)-lifting problem

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] & D, \end{tikzcd}$$

for some $f \in J$, then $\theta(g, s)$ solves it by taking the codomain part of the morphism

$$f \xrightarrow{\mathsf{in}_i} Kf \xrightarrow{\epsilon_f} L^1 f,$$

which is a map $B \to E^1 f$, and composing it with the morphism

$$E^1 f \xrightarrow{\chi_f} Ef \xrightarrow{E(h,k)} Eg \xrightarrow{s} C$$

to obtain a morphism $j\colon B \to C$. For the lower side, if we are given the same R-map (g, s) and lifting problem $x \in S_g$, then $\psi\chi^*(g, s)$ solves it by taking the codomain part of the morphism

$$f \xrightarrow{\mathsf{in}_x} Kg \xrightarrow{\epsilon_g} L^1 g,$$

which is a map $B \to E^1 f$, and composing it with the morphism

$$E^1 g \xrightarrow{\chi_g} Eg \xrightarrow{s} C.$$

But since the following diagram commutes:

$$\begin{tikzcd} & Kf \arrow[r, "\epsilon_f"] \arrow[d, "K(h,k)"] & L^1 f \arrow[d, "L^1(h,k)"] \\ f \arrow[ur, "\mathsf{in}_i"] \arrow[r, "\mathsf{in}_x"'] & Kg \arrow[r, "\epsilon_g"'] & L^1 g, \end{tikzcd}$$

this latter is the same as taking the codomain part of $\epsilon_f \cdot \mathsf{in}_i$ and composing it with the morphism

$$E^1 f \xrightarrow{E^1(h,k)} E^1 g \xrightarrow{\chi_g} Eg \xrightarrow{s} C,$$

which by naturality of χ is the same as $s \cdot E(h,k) \cdot \chi_f$. Thus we have $\theta = \psi\chi^*$ as claimed, and so θ is an isomorphism as desired. ☐

6 Properties of L-maps and R-maps for n.w.f.s.'s

In this section, we broaden our attention from cofibrantly generated n.w.f.s.'s to n.w.f.s.'s in general. Our concern will be to enumerate the closure properties that the categories of L-maps and R-maps for a n.w.f.s. have. As we go along, we will apply our results in the cofibrantly generated case and see to what extent they allow us to give a characterisation of the L-maps. The answer is not wholly satisfactory: we do not achieve such a neat result as we have for plain w.f.s.'s, but we can come close.

In fact, most of the properties we are about to exhibit do not even require a full n.w.f.s., but only *half* of one: either a comonad over dom or a monad over cod. The particular example we should bear in mind is the one-step comonad L^1 for a cofibrantly generated n.w.f.s., and for this reason it is the comonad case that we will consider, though of course everything we do can be straightforwardly dualised.

6.1 Basic properties

Suppose we are given a comonad $\mathsf{L} = (L, \Phi, \Sigma)$ over cod on $\mathcal{C}^{\mathbf{2}}$, to which we apply our usual conventions, writing $L(f\colon X \to Y)$ as $\lambda_f\colon X \to Ef$, and so on. We begin simply:

Proposition 37. *The category* L-$\mathbf{Map}$ *is* ***closed under colimits*** *in* $\mathcal{C}^{\mathbf{2}}$*, in that the forgetful functor* $U_{\mathsf{L}}\colon \mathsf{L}\text{-}\mathbf{Map} \to \mathcal{C}^{\mathbf{2}}$ *creates colimits.*

Proof. Because L-$\mathbf{Map}$ is the category of coalgebras for a comonad on $\mathcal{C}^{\mathbf{2}}$. ☐

Proposition 38. *The category* L-$\mathbf{Map}$ ***contains the isomorphisms****, in that every isomorphism* $f\colon X \to Y$ *in* $\mathcal{C}$ *can be equipped with a unique* L*-coalgebra structure.*

Proof. Given an isomorphism $f\colon X \to Y$, we make it into an L-coalgebra (f, s) by taking $s = \lambda_f \cdot f^{-1}$; easy verification shows that this satisfies the coalgebra axioms. Conversely, if $s\colon Y \to Ef$ makes (f, s) into an L-coalgebra, then from the coalgebra axiom $s \cdot f = \lambda_f$ we deduce that $s = \lambda_f \cdot f^{-1}$. ☐

A more abstract view of this last Proposition is available: viewing L as an object of the category $\mathbf{Comon} = \mathbf{Comon}_{\odot}(\mathbf{Ff}_{\mathcal{C}})$, as in the previous section, we know that there is a unique map $\tau\colon I \to \mathsf{L}$ from the initial object inducing a functor I-$\mathbf{Map} \to \mathsf{L}$-$\mathbf{Map}$. So "every I-map is an L-map", and the I-maps are precisely the isomorphisms.

Corollary 39. *Suppose we are given* L*-coalgebras* (f, s) *and* (g, t)*, where* $f\colon X \to Y$ *an isomorphism. Then every map* $(h, k)\colon f \to g$ *of* $\mathcal{C}^{\mathbf{2}}$ *lifts to a map of* L*-coalgebras* $(f, s) \to (g, t)$*.*

Proof. We must verify that $tk = E(h,k)s$. By the previous Proposition, $s = \lambda_f f^{-1}$ so that $E(h,k)s = E(h,k)\lambda_f f^{-1} = \lambda_g h f^{-1} = tghf^{-1} = tkff^{-1} = tk$ as required. ☐

Though this last result might seem somewhat technical, when combined with Proposition 37 it already implies that L-maps are closed under pushout. To give a precise meaning to "closure under pushout", we need the notion of a *cocartesian lifting*.

Suppose we are given an arbitrary functor $U\colon \mathcal{A} \to \mathcal{C}$, an arrow $f\colon C \to D$ in $\mathcal{C}$ and an object $X \in \mathcal{A}$ lying over C. Then a **cocartesian lifting of f at X** is a universal way of turning X into an object f_*X lying over D: it is given by such an object together with a "push forward" map $\vec{f}\colon X \to f_*X$ in $\mathcal{A}$ lying over f:

$$\begin{tikzcd} X \arrow[r, "\vec{f}"] \arrow[d, phantom, "U"] & f_*X \\ C \arrow[r, "f"'] & D. \end{tikzcd}$$

The universality of this lifting amounts to saying that whenever we are given a map $g\colon X \to Z$ in $\mathcal{A}$ whose underlying map in $\mathcal{C}$ factors through $f\colon C \to D$, there is a unique lifting to a factorisation of g through $\vec{f}$ in $\mathcal{A}$. If every cocartesian lifting exists we call $U\colon \mathcal{A} \to \mathcal{C}$ an **opfibration**; in this case we can think of U as manifesting $\mathcal{A}$ as a category "indexed over $\mathcal{C}$". There is no encyclopaedic reference dealing with (op)fibrations, but one might consult [5], for example.

Example 40. For the domain functor $\mathrm{dom}\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$, a cocartesian lifting of a map $h\colon A \to C$ of $\mathcal{C}$ at an object $f\colon A \to B$ of $\mathcal{C}^{\mathbf{2}}$ is given by a *pushout* of f along h:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"] & D \arrow[ul, phantom, "\ulcorner", very near start] \\ A \arrow[r, "h"'] & C. \end{tikzcd}$$

cod

In particular, if $\mathcal{C}$ has all pushouts, then $\mathrm{dom}\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ is an opfibration.

Suppose that as well as $U\colon \mathcal{A} \to \mathcal{C}$, we have a further functor $U'\colon \mathcal{B} \to \mathcal{C}$ together with a morphism $F\colon \mathcal{A} \to \mathcal{B}$ commuting with the projections to $\mathcal{C}$. Then we say that F **creates cocartesian liftings over** $\mathcal{C}$ if, for every cocartesian map $g\colon X \to Y$ of $\mathcal{B}$ and object A of $\mathcal{A}$ lying over X, there exists a unique cocartesian map $\overline{g}\colon A \to B$ of $\mathcal{A}$ satisfying $F\overline{g} = g$.

Example 41. Suppose we are given a class of maps $\mathcal{E}$ in a category $\mathcal{C}$ which are *stable under pushout*, in the weak sense that whenever a pushout of an $\mathcal{E}$-map exists, it is another $\mathcal{E}$-map; then, viewing $\mathcal{E}$ as a full subcategory of $\mathcal{C}^{\mathbf{2}}$, we can express this by saying that the inclusion functor $\mathcal{E} \to \mathcal{C}^{\mathbf{2}}$ creates cocartesian liftings over $\mathcal{C}$.

The generalisation of this last example to the present situation is now immediate:

Proposition 42. *The category of* L*-maps is* ***stable under pushout*** *along arbitrary maps of* $\mathcal{C}$*, in the sense that in the diagram*

$$\begin{tikzcd} \mathsf{L}\text{-}\mathbf{Map} \arrow[r, "U_{\mathsf{L}}"] \arrow[dr, "\mathrm{dom}\cdot U_{\mathsf{L}}"'] & \mathcal{C}^{\mathbf{2}} \arrow[d, "\mathrm{dom}"] \\ & \mathcal{C}, \end{tikzcd}$$

the horizontal arrow creates cocartesian liftings over $\mathcal{C}$.

In other words, we must show that given a pushout square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] & D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

in $\mathcal{C}$, together with an L-map structure (f, s) on f, there is a unique lift of g to an L-map (g, t) with respect to which (h, k) is a cartesian arrow.

Proof. The argument we give here is due to [12]. We have the following situation:

$$\begin{tikzcd}[column sep=small] & (\mathrm{id}_A, \star) \arrow[dl, "{(h,h)}"'] \arrow[dr, "{(\mathrm{id}_A, f)}"] & & & & \mathrm{id}_A \arrow[dl, "{(h,h)}"'] \arrow[dr, "{(\mathrm{id}_A, f)}"] & \\ (\mathrm{id}_C, \star) & & (f, s) \arrow[r, dotted, "U_{\mathsf{L}}"] & {} & \mathrm{id}_C \arrow[dr, "{(\mathrm{id}_C, g)}"'] & & f \arrow[dl, "{(h,k)}"] \\ & & & & & g, & \end{tikzcd}$$

where $\star$ represents the unique L-coalgebra structures on the isomorphisms id_A and id_C, and we are applying Corollary 39 to deduce the existence of the arrows on the left-hand side. Now, the right-hand diagram is a pushout in $\mathcal{C}^{\mathbf{2}}$ and so, since U_{L} creates colimits, there has a unique lifting of it to a pushout diagram in L-$\mathbf{Map}$. Thus we have a lifting of g to some $(g, t) \in \mathsf{L}$-$\mathbf{Map}$ together with an arrow $(f, s) \to (g, t)$; and the universal property of pushout says that this arrow is cocartesian. ☐

It is quite useful to have a concrete description of the induced L-algebra structure on g in the situation of the previous proof, and a short calculation shows that it is given by the induced map $t \colon D \to Eg$ in the following diagram:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \arrow[ddr, bend left, "\lambda_g"] & \\ B \arrow[r, "k"'] \arrow[drr, bend right, "E(h,k) \cdot s"'] & D \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "t"] & \\ & & Eg. \end{tikzcd}$$

The final thing we wish to consider in this section is the issue of *closure under retracts*. In the case of a plain w.f.s., the class of $\mathcal{L}$-maps is closed under all retracts. The same is not true here: indeed, if we are given an L-map (g, s) and a retract diagram

$$\begin{tikzcd} U \arrow[r, tail, "i_1"] \arrow[d, "f"'] & V \arrow[r, two heads, "p_1"] \arrow[d, "g"] & U \arrow[d, "f"] \\ X \arrow[r, tail, "i_2"'] & Y \arrow[r, two heads, "p_2"'] & X \end{tikzcd}$$

in $\mathcal{C}^{\mathbf{2}}$ (so $p_1 i_1 = \mathrm{id}_U$ and $p_2 i_2 = \mathrm{id}_X$) and want to make f into an L-map, then the only logical way to do so is via the map

$$r := X \xrightarrow{i_1} Y \xrightarrow{s} Eg \xrightarrow{E(p_1,p_2)} Ef, \tag{13}$$

and, although this satisfies the first two axioms for an L-map, it need not satisfy the third. However, category theory provides us with conditions under which the above procedure will work: roughly speaking, if we can find a further L-map (h,t) which "measures" (g,s) in a suitable sense. Explicitly, we have:

Definition 43. A **contractible equaliser** in $\mathcal{C}^{\mathbf{2}}$ is a diagram

$$\begin{tikzcd} f \arrow[r, tail, "i"] & g \arrow[l, bend left, "p"] \arrow[r, tail, shift left, "k"] \arrow[r, tail, shift right, "j"'] & h \arrow[l, bend left=60, "q"] \end{tikzcd}$$

such that (i,p) and (j,q) are retracts satisfying $ji = ki$ and $qk = ip$. A **contractible pair** in L-**Map** is given by a contractible equaliser in $\mathcal{C}^{\mathbf{2}}$ together with a lifting of k and j to morphisms of L-**Map**:

$$\begin{tikzcd} (g,s) \arrow[r, shift left, "k"] \arrow[r, shift right, "j"'] & (h,t). \end{tikzcd}$$

In this richer setting, where the retract (i,p) forms part of a contractible pair, (13) *does* give a valid L-map structure on f, and moreover one for which (i_1,i_2) – though not necessarily (p_1,p_2) – becomes a morphism of L-**Map**. This is a consequence of the standard result that *the forgetful functor $U_{\mathsf{L}} \colon \mathsf{L}\text{-}\mathbf{Map} \to \mathcal{C}^{\mathbf{2}}$ creates equalisers for contractible pairs*; this is part of the proof of Beck's monadicity theorem which can be found in any good book on category theory: [3], for example. In such a situation, we shall call (f,r) a **retract equaliser** of (g,s). To summarise, we have that:

Proposition 44. *The category of L-maps is* ***closed under retract equalisers****, in that the forgetful functor $U_{\mathsf{L}} \colon \mathsf{L}\text{-}\mathbf{Map} \to \mathcal{C}^{\mathbf{2}}$ creates equalisers for contractible pairs.*

We can now use this to give the analogue for n.w.f.s.'s of the so-called "retract argument" for plain w.f.s.'s.

Proposition 45. *Every L-map (f,r) is a retract equaliser of a cofree one: that is, one of the form (λ_f, σ_f).*

Proof. This is another standard part of the monadicity theorem. The retract in question is

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d, "\lambda_f"] & X \arrow[d, "f"] \\ Y \arrow[r, tail, "r"'] & Ef \arrow[r, two heads, "\rho_f"'] & Y \end{tikzcd}$$

which appears in the following contractible pair:

$$\begin{tikzcd} F_{\mathsf{L}} f \arrow[r, shift left, "\Sigma_f"] \arrow[r, shift right, "Lr"'] & F_{\mathsf{L}} L f \end{tikzcd} \quad \dashrightarrow \quad \begin{tikzcd} f \arrow[r, tail, "r"] & Lf \arrow[l, bend left, "\Phi_f"] \arrow[r, tail, shift left, "\Sigma_f"] \arrow[r, tail, shift right, "Lr"'] & LLf. \arrow[l, bend left=60, "\Phi_{Lf}"] \end{tikzcd}$$ ◻

6.2 Characterising L^1-maps for cofibrantly generated n.w.f.s.'s

We now apply the results we have just proved to give a characterisation of the L^1-maps for the one-step comonad generated by a set of maps in a cocomplete category $\mathcal{C}$. We know from Proposition 35 that every generating cofibration is an L^1-map; we know from Proposition 37 that every coproduct of L^1-maps is an L^1-map; and we know from Proposition 42 that every pushout of an L^1-map is an L^1-map. So in particular, *every pushout of a coproduct of generating cofibrations is an* L^1*-map.*

There now arises a very natural question: observe that the functor $L^1 \colon \mathcal{C}^\mathbf{2} \to \mathcal{C}^\mathbf{2}$ sends a map g to a pushout of a coproduct of generating cofibrations, so that if we equip each of these generating cofibrations with its canonical L^1-map structure we obtain a lifting of L^1 through L^1-**Map**:

$$\begin{tikzcd} \mathcal{C}^\mathbf{2} \arrow[rr, "\overline{L^1}"] \arrow[dr, "L^1"'] & & \mathsf{L}^1\text{-}\mathbf{Map} \arrow[dl, "U_{\mathsf{L}^1}"] \\ & \mathcal{C}^\mathbf{2} & \end{tikzcd} \qquad \text{where} \qquad \overline{L^1}g = \langle h_x \rangle_* \Big(\sum_{x \in S_g} (f_s, \alpha_{f_s}) \Big).$$

Now, we have another lifting of L^1 through L-**Map** – namely, the free functor $F_\mathsf{L} \colon \mathcal{C}^\mathbf{2} \to \mathsf{L}$-**Map** – and since, morally, both of these liftings do exactly the same thing, we might wonder if they are naturally isomorphic. In fact, the result is even stronger:

Proposition 46. *With the notation of the previous discussion, we have* $\overline{L^1} = F_{\mathsf{L}^1}$.

Proof. Either a somewhat fiddly calculation, or, using the more abstract language of the Appendix, a straightforward manipulation with universal properties; it can be found as Proposition II.4.2 of [6]. ☐

This result implies, in particular, that *every cofree* L^1*-map* – i.e., one of the form $(\lambda^1_g, \sigma^1_g)$ – *is a pushout of a coproduct of generating cofibrations.* This is almost a complete characterisation of the L^1-maps, and the final step, as in the case of plain w.f.s.'s, is to apply the retract argument. From Proposition 45 we now deduce:

Proposition 47. *The coalgebras for the one-step comonad* L^1 *generated by a set* J *are precisely the retract equalisers of pushouts of coproducts of generating cofibrations.*

6.3 Compositional properties

There is one further property which we expect of the L-maps and R-maps for a n.w.f.s.: namely, that they should be closed under composition, and even *transfinite* composition. Importantly, these properties do *not* hold in general for a mere comonad over dom or monad over cod; nonetheless, we can still frame our results in these broader settings. As before, we prefer the comonadic version, but note that everything we do in this section applies equally well on the monadic side.

To express the notion of being closed under composition, we will use a suitable monoidal structure $(\mathcal{I}, \bullet)$ on the category $\mathbf{Cat}/\mathcal{C}^\mathbf{2}$. The idea is that, if we view objects of $\mathbf{Cat}/\mathcal{C}^\mathbf{2}$ as being abstract categories of "structured maps of $\mathcal{C}$", then $\bullet$-monoid structures on an object should correspond to ways of *composing* these structured maps. The monoidal structure in question has unit $\mathcal{I}$ given by:

$$\mathcal{I} = (\mathcal{C} \xrightarrow{\mathrm{id}_{(-)}} \mathcal{C}^\mathbf{2});$$

whilst the tensor product of $U_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{C}^{\mathbf{2}}$ and $U_{\mathcal{A}} \colon \mathcal{B} \to \mathcal{C}^{\mathbf{2}}$ is given as follows. First we form the diagram

$$\begin{tikzcd}
\mathcal{A} \bullet \mathcal{B} \arrow[rr] \arrow[ddd] \arrow[dr, dotted, "U_{\mathcal{A},\mathcal{B}}"] & & \mathcal{A} \arrow[d, "U_{\mathcal{A}}"] \\
& \mathcal{C}^{\mathbf{3}} \arrow[r] \arrow[dd] & \mathcal{C}^{\mathbf{2}} \arrow[dd, "\mathrm{dom}"] \\
& & \\
\mathcal{B} \arrow[r, "U_{\mathcal{B}}"'] & \mathcal{C}^{\mathbf{2}} \arrow[r, "\mathrm{cod}"'] & \mathcal{C}
\end{tikzcd}$$

in which both squares are pullbacks. $\mathcal{C}^{\mathbf{3}}$ is the functor category $[\cdot \to \cdot \to \cdot, \mathcal{C}]$, whose objects are pairs of composable arrows in $\mathcal{C}$, and so we can consider the evident "composition" functor $\mathrm{cmp} \colon \mathcal{C}^{\mathbf{3}} \to \mathcal{C}^{\mathbf{2}}$. We now define the tensor product of $(\mathcal{A}, U_{\mathcal{A}})$ and $(\mathcal{B}, U_{\mathcal{B}})$ to be $(\mathcal{A} \bullet \mathcal{B}, U_{\mathcal{A} \bullet \mathcal{B}})$, where

$$U_{\mathcal{A} \bullet \mathcal{B}} = \mathcal{A} \bullet \mathcal{B} \xrightarrow{U_{\mathcal{A},\mathcal{B}}} \mathcal{C}^{\mathbf{3}} \xrightarrow{\mathrm{cmp}} \mathcal{C}^{\mathbf{2}}.$$

So a typical element of $(\mathcal{A} \bullet \mathcal{B})$ has the form (σ, τ), where $\sigma \in \mathcal{B}$ lies over $f \colon X \to Y$ and $\tau \in \mathcal{A}$ lies over $g \colon Y \to Z$, whilst the projection onto $\mathcal{C}^{\mathbf{2}}$ is given by $U_{\mathcal{A} \bullet \mathcal{B}}(\sigma, \tau) = gf$.

The result we aiming for is:

Proposition 48. *The category of* L*-maps for a n.w.f.s. is* ***closed under composition****, in that* $U_{\mathsf{L}} \colon \mathsf{L}\text{-}\mathbf{Map} \to \mathcal{C}^{\mathbf{2}}$ *is a* $\bullet$*-monoid in* $\mathbf{Cat}/\mathcal{C}^{\mathbf{2}}$.

Note that this statement says not only that L-maps are closed under composition, but also that this composition is associative and unital. As foreshadowed in Section 5.3, our method for proving it will be to show that the functor $\mathcal{G} \colon \mathbf{Comon} \to \mathbf{Cat}/\mathcal{C}^{\mathbf{2}}$ which sends a comonad over dom to its category of coalgebras is *lax monoidal*. In particular, $\mathcal{G}$ sends monoids to monoids, so that if we have a $\otimes$-monoid (L, η, μ) in $\mathbf{Comon}$ – i.e., a n.w.f.s. on $\mathcal{C}$ – then we induce a $\bullet$-monoid structure on $\mathsf{L}\text{-}\mathbf{Map}$ in $\mathbf{Cat}/\mathcal{C}^{\mathbf{2}}$. Thus we will have proved the previous Proposition if we can prove:

Proposition 49. $\mathcal{G}$ *is a lax monoidal functor* $(\mathbf{Comon}, \otimes, I) \to (\mathbf{Cat}/\mathcal{C}^{\mathbf{2}}, \bullet, \mathcal{I})$.

In order to do this, we need to provide an explicit description of the monoidal structure $(\otimes, I)$ on $\mathbf{Comon}$. The unit I, of course, we understand: it is the comonad on $\mathcal{C}^{\mathbf{2}}$ which sends a map $f \colon X \to Y$ to $\mathrm{id}_X \colon X \to X$. The tensor product $\mathsf{L}^2 \otimes \mathsf{L}^1$ of two comonads is somewhat trickier to describe, primarily for notational reasons. Its underlying functorial factorisation is:

$$X \xrightarrow{f} Y \quad \mapsto \quad X \xrightarrow{\lambda^2_{R^1 f} \cdot \lambda^1_f} E^2 R^1 f \xrightarrow{\rho^2_{R^1 f}} Y;$$

and to make it into a comonad over dom, we must give maps

$$\sigma^{2 \otimes 1}_f \colon E^2 R^1 f \to E^2 R^1 L^{2 \otimes 1} f,$$

where we write $L^{2 \otimes 1}$ for the functor sending f to $\lambda^2_{R^1 f} \cdot \lambda^1_f$. If we extract the relevant data from Remark 6 and Proposition 9, we find that $\sigma^{2 \otimes 1}_f$ is given by:

$$E^2 R^1 f \xrightarrow{\sigma^2_{R^1 f}} E^2 L^2 R^1 f \xrightarrow{E^2(\sigma^1_f, 1)} E^2(\lambda^2_{R^1 f} \cdot \rho^1_{L^1 f}) \xrightarrow{E^2\left(E^1(1, \lambda^2_{R^1 f}), 1\right)} E^2 R^1 L^{2 \otimes 1} f.$$

With this in place, we can now prove Proposition 49. For $\mathcal{G}$ to be a lax functor, we first need to provide a unit comparison map

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "m_I"] \arrow[dr, "\mathrm{id}_{(-)}"'] & & I\text{-}\mathbf{Map} \arrow[dl, "U_I"] \\ & \mathcal{C}^{\mathbf{2}}. & \end{tikzcd}$$

But I-**Map** is just the full subcategory of $\mathcal{C}^{\mathbf{2}}$ whose objects are the isomorphisms, and so we can take the obvious factorisation of $\mathrm{id}_{(-)}$ through I-**Map**. We also need to provide comparison maps

$$\begin{tikzcd} \mathsf{L}^2\text{-}\mathbf{Map} \bullet \mathsf{L}^1\text{-}\mathbf{Map} \arrow[rr, "m_{\mathsf{L}^2,\mathsf{L}^1}"] \arrow[dr, "U_{\mathsf{L}^2 \bullet \mathsf{L}^1}"'] & & (\mathsf{L}^2 \otimes \mathsf{L}^1)\text{-}\mathbf{Map} \arrow[dl, "U_{\mathsf{L}^2 \otimes \mathsf{L}^1}"] \\ & \mathcal{C}^{\mathbf{2}}. & \end{tikzcd}$$

On objects $m_{\mathsf{L}^1,\mathsf{L}^2}$ is given by specifying, for every L^1-map $(f,s)\colon X \to Y$ and L^2-map $(g,t)\colon Y \to Z$, an $(\mathsf{L}^2 \otimes \mathsf{L}^1)$-map structure on $gf\colon X \to Z$. We take this to be the following map $u\colon Z \to E^2R^1(gf)$:

$$Z \xrightarrow{t} E^2 g \xrightarrow{E^2(s,1)} E^2(g \cdot \rho^1_f) \xrightarrow{E^2(E^1(1,g),1)} E^2R^1(gf).$$

Verifying the coalgebra axioms is routine. On morphisms, we have no real choice: a morphism on the left hand side is a triple (h,k,l) where $(h,k)\colon (f,s) \to (f',s')$ is a morphism L^1-**Map** and $(k,l)\colon (g,t) \to (g',t')$ is a morphism of L^2-**Map**, and we are forced to send this to the morphism $(h,l)\colon (gf,u) \to (g'f',u')$ of $(\mathsf{L}^2 \otimes \mathsf{L}^1)$-**Map**. Of course, we have to check that this morphism is a coalgebra morphism; but this is again routine, as are the remaining details of the proof, which amount to nothing more than checking a large number of coherence axioms. □

Thus we conclude that the category of L-maps for a n.w.f.s. is closed under composition; explicitly, if we are given two L-maps $(f,s)\colon X \to Y$ and $(g,t)\colon Y \to Z$, then their composite is given by $(gf,u)\colon X \to Z$, where $u\colon Z \to E(gf)$ is given by:

$$Z \xrightarrow{t} Eg \xrightarrow{E(s,1)} E(g \cdot \rho_f) \xrightarrow{E(E(1,g),1)} ER(gf) \xrightarrow{\pi_{gf}} E(gf). \tag{14}$$

We have entirely dual results for the R-maps, but it might be worth spelling these out, since a little care is required. In this case, we consider the category $\mathbf{Mon} := \mathbf{Mon}_\otimes(\mathbf{Ff}_C)$ of $\otimes$-monoids in $\mathbf{Ff}_\mathcal{C}$, which we can view as the category of "monads over cod". Once more we have a functor into $\mathbf{Cat}/\mathcal{C}^{\mathbf{2}}$, assigning to each such monad its category of algebras, but it is now a *contravariant* functor

$$\mathcal{H}\colon \mathbf{Mon}^{\mathrm{op}} \to \mathbf{Cat}/\mathcal{C}^{\mathbf{2}}.$$

We can show as before that $\mathcal{H}$ is lax monoidal, where the monoidal structure on the left-hand side is now the $(\odot, \perp)$ monoidal structure. In particular, $\mathcal{H}$ sends $\odot$-monoids in $\mathbf{Mon}^{\mathrm{op}}$, which are $\odot$-*co*monoids in **Mon**, to $\bullet$-monoids in $\mathbf{Cat}/\mathcal{C}^{\mathbf{2}}$; and since $\odot$-comonoids in **Mon** are precisely n.w.f.s.'s, we deduce that the category of R-maps for a n.w.f.s. is closed under composition.

We turn now from finite to transfinite composition of maps. In general, if γ is an ordinal and X is a γ-indexed chain in some category $\mathcal{D}$ – in other words, a functor $X\colon \gamma \to \mathcal{D}$ – then the **transfinite composite** of X exists just when the colimit of X does, and is given by the injection of X_0 into the colimit. Thus to say that a category admits transfinite composition is simply to say that it admits colimits of chains.

In order to apply this notion to the L-maps for a n.w.f.s., we need to form them into a category which is different from the category L-**Map**: namely, the category whose objects are those of $\mathcal{C}$ and whose set of morphisms from X to Y is the set of L-maps $(f, s)\colon X \to Y$. By Proposition 48, this does gives us a category, whose composition law is given by (14) and whose identity at X is given by the unique lifting of id_X to an L-map. We shall denote this category by $\mathcal{C}_\mathsf{L}$ (and correspondingly $\mathcal{C}_\mathsf{R}$); observe that whereas L-**Map** had a forgetful functor to $\mathcal{C}^\mathbf{2}$, the category $\mathcal{C}_\mathsf{L}$ has a forgetful functor $U\colon \mathcal{C}_\mathsf{L} \to \mathcal{C}$.

Proposition 50. *The category of* L*-maps for a n.w.f.s. is* ***closed under transfinite composition****, in that the forgetful functor* $U\colon \mathcal{C}_\mathsf{L} \to \mathcal{C}$ *creates colimits of chains.*

The proof is surprisingly complex. First we need two lemmas:

Lemma 51. *Colimits in* L-**Map** *commute with composition in the sense that the functors* $i\colon \mathcal{C} \to \mathsf{L}$-**Map** *and* $m\colon \mathsf{L}$-**Map**$\bullet\mathsf{L}$-**Map** $\to \mathsf{L}$-**Map** *exhibiting* L-**Map** *as a* $\bullet$*-monoid preserve colimits strictly.*

Proof. This is clear for $i\colon \mathcal{C} \to \mathsf{L}$-**Map**, which sends $X \in \mathcal{C}$ to the unique L-map structure $(\mathrm{id}_X, \star)$ on id_X. To show the same for m, suppose that we are given a diagram $F\colon \mathcal{A} \to$ L-**Map** $\bullet$ L-**Map** such that $\operatorname{colim} F$ exists. In particular, this implies that the colimit of the underlying diagram $U_{\mathsf{L}\bullet\mathsf{L}}F\colon \mathcal{A} \to \mathcal{C}^\mathbf{2}$ exists. But this is also the underlying diagram of $mF\colon \mathcal{A} \to \mathsf{L}$-**Map**, and so because U_L creates colimits, mF also has a colimit, which moreover has the same underlying object in $\mathcal{C}^\mathbf{2}$ as $m \operatorname{colim} F$. All that remains to do is to check that the L-coalgebra structures on $\operatorname{colim} mF$ and $m \operatorname{colim} F$ agree. So suppose that each $F(a)$ is given by an L-map $(f_a, s_a)\colon X_a \to Y_a$ and an L-map $(g_a, t_a)\colon Y_a \to Z_a$, and consider the following diagram:

$$\begin{tikzcd}
\overrightarrow{Z_a} \arrow[r, "\overrightarrow{t_a}"] & \overrightarrow{Eg_a} \arrow[r, "\overrightarrow{E(s_a,1)}"] \arrow[d, "\mathrm{can}"'] & \overrightarrow{E(g_a \cdot \rho_{f_a})} \arrow[r, "\overrightarrow{E(E(1,g_a),1)}"] \arrow[d, "\mathrm{can}"'] & \overrightarrow{ER(g_a f_a)} \arrow[r, "\overrightarrow{\pi_{g_a f_a}}"] \arrow[d, "\mathrm{can}"] & \overrightarrow{E(g_a f_a)} \arrow[dd, "\mathrm{can}"] \\
& E\overrightarrow{g_a} \arrow[r, "E(\overrightarrow{s_a},1)"'] & E(\overrightarrow{g_a \cdot \rho_{f_a}}) \arrow[r, "E(\overrightarrow{E(1,g_a)},1)"'] \arrow[d, "E(\mathrm{can},1)"'] & E\overrightarrow{R(g_a f_a)} \arrow[d, "\mathrm{can}"] & \\
& & E(\overrightarrow{g_a} \cdot \rho_{\overrightarrow{f_a}}) \arrow[r, "E(E(1,\overrightarrow{g_a}),1)"'] & ER(\overrightarrow{g_a f_a}) \arrow[r, "\pi_{\overrightarrow{g_a f_a}}"'] & E(\overrightarrow{g_a f_a}).
\end{tikzcd}$$

Here, $\overrightarrow{Z_a}$ is shorthand for $\operatorname{colim}_a Z_a$, and so on. Each small square commutes, and thus the two extremal routes around the big diagram are the same: but the upper of these is the L-coalgebra structure on $\operatorname{colim} mF$, whilst the lower is the coalgebra structure on $m \operatorname{colim} F$. $\square$

Lemma 52. *Suppose that* $(f, s)\colon X \to Y$ *and* $(g, t)\colon Y \to Z$ *are* L*-maps, and that* (gf, u) *is their composite according to* (14)*. Then the morphism* $(1_X, g)\colon f \to gf$ *of* $\mathcal{C}^\mathbf{2}$ *is a morphism of* L*-maps* $(1_X, g)\colon (f, s) \to (gf, u)$*.*

Proof. We must show that $ug = E(1,g)s$, and so calculate

$$\begin{aligned} ug &= \pi_{gf} \cdot E(E(1,g),1) \cdot E(s,1) \cdot t \cdot g \\ &= \pi_{gf} \cdot E(E(1,g),1) \cdot E(s,1) \cdot \lambda_g \\ &= \pi_{gf} \cdot E(E(1,g),1) \cdot \lambda_{g \cdot \rho_f} \cdot s \\ &= \pi_{gf} \cdot \lambda_{R(gf)} \cdot E(1,g) \cdot s \\ &= E(1,g) \cdot s \end{aligned}$$

as required. ☐

Proof of Proposition 50. Suppose that we are given a γ-chain $X \colon \gamma \to \mathcal{C}_{\mathsf{L}}$: so for each $\alpha < \gamma$ we have an object X_α of $\mathcal{C}$ and for each $\alpha < \beta < \gamma$ we have an L-map $(f_{\alpha,\beta}, s_{\alpha,\beta}) \colon X_\alpha \to X_\beta$. Suppose also that we have a colimit for the underlying chain $UX \colon \gamma \to \mathcal{C}$; so we have a colimiting object Y of $\mathcal{C}$ and maps $g_\alpha \colon X_\alpha \to Y$ commuting with the $f_{\alpha,\beta}$'s. We must show that this can be lifted to a colimit for X, for which we must equip each g_α with an L-map structure (g_α, t_α) compatible with the $s_{\alpha,\beta}$'s.

So given $\delta < \gamma$, we equip g_δ with an L-map structure by considering the chain $Z^\delta \colon \gamma \to \mathsf{L}$-**Map** given as follows. For $\alpha \leqslant \delta$, we take Z^δ to be constant with value $(\mathrm{id}_{X_\delta}, \star)$, and for $\alpha > \delta$ we take $Z^\delta_\alpha = (f_{\delta,\alpha}, s_{\delta,\alpha})$ with connecting maps given by $Z^\delta_{\alpha,\beta} = (\mathrm{id}_{X_0}, f_{\alpha,\beta})$: by the second of our two lemmas this connecting map is a valid morphism of L-maps.

Now, from the given colimit for UX we obtain a colimit for the underlying chain $U_{\mathsf{L}} \cdot Z^\delta \colon \gamma \to \mathcal{C}^{\mathbf{2}}$, namely the object $g_\delta \colon X_\delta \to Y$ of $\mathcal{C}^{\mathbf{2}}$; and because the forgetful functor U_{L} creates colimits, we obtain from this a colimit for the chain $Z^\delta \colon \gamma \to \mathsf{L}$-**Map**, which gives us the required L-map structure (g_δ, t_δ) on g_δ.

It remains to show that these L-map structures (g_α, t_α) are compatible with the $s_{\alpha,\beta}$'s; explicitly, we need to show that $(g_\beta, t_\beta) \circ (f_{\alpha,\beta}, s_{\alpha,\beta}) = (g_\alpha, t_\alpha)$. But by our first lemma, we know that precomposing $\operatorname{colim} Z^\beta$ with $(f_{\alpha,\beta}, s_{\alpha,\beta})$ will give us the same result as precomposing every element of Z^β with $(f_{\alpha,\beta}, s_{\alpha,\beta})$ and taking the colimit of the resultant chain Z'. But it is easy to see that $\operatorname{colim} Z'$ is precisely $\operatorname{colim} Z^\alpha$ and so the result follows. ☐

6.4 Characterising L-maps for cofibrantly generated n.w.f.s.'s

We would now like to use the results of the previous section to give a characterisation of the L-maps for a cofibrantly generated n.w.f.s. In one direction, this is straightforward: by Section 5.4 we know that every generating cofibration is an L-map, and by Propositions 37, 42, 44 and 50, we know that the L-maps are closed under colimits, pushouts, retract equalisers and transfinite composition. So in particular, *every retract equaliser of a transfinite composition of pushouts of coproducts of generating cofibrations is an L-map.*

What is harder to come by is a result in the other direction, saying that *every* L-map is of this form. We would like to mimic the argument we gave for the L^1-maps, where we first characterised the "cofree" L^1-maps – that is, those of the form $(\lambda^1_f, \sigma^1_f)$ – and then applied Proposition 45, the "generalised retract argument", to produce a characterisation of an arbitrary L^1-map.

The problem is that we have been unable to find a simple characterisation of the "cofree" L-maps. The reason for this is the way in which these cofree maps are constructed: we perform a transfinite induction, at each stage of which we first glue some extra cells on, and then coequalise away the ones that we shouldn't have added because they were there already. If we could show that this was equivalent to simply glueing slightly fewer cells on in the first place, then we could conclude that every cofree map was just given by transfinitely glueing on cells, and then our characterisation result would follow easily.

However, we can see no way of proving this statement, and so the best we can do for the moment is to refer back to the plain w.f.s. underlying our cofibrantly generated n.w.f.s. We know that this is a cofibrantly generated w.f.s., and there *is* a characterisation of its left class of maps: they are precisely the retracts of transfinite composites of pushouts of coproducts of generating cofibrations.

Proposition 53. *Every retract equaliser of a transfinite composition of pushouts of coproducts of generating cofibrations is an* L*-map, whilst the underlying morphism in* $\mathcal{C}$ *of any* L*-map is a retract of a transfinite composition of pushouts of coproducts of generating cofibrations.*

Appendix: universality of the one-step comonad

The purpose of this appendix is to expand upon the abstract description of the one-step comonad L^1 generated by a set of maps J which we hinted at in Section 5.2, and to use it to give a proof of Proposition 18, which, we recall, told us that that L^1 is *freely* generated by J.

We will do this by using a certain amount of the theory of *2-categories*. Now, a 2-category is a category which has not only objects $X, Y, Z, \ldots$ and morphisms $f, g, h, \ldots$ but also 2-cells $\alpha\colon f \Rightarrow g$ between these 1-cells which can be composed together in various ways, subject to axioms which make any multiple composites we might form unambiguous; for a good introduction to the subject the reader might refer to [15]. The ur-2-category is **Cat** whose objects, 1-cells and 2-cells are respectively, (large) categories, functors and natural transformations; and with this in mind, one can crudely think of two-dimensional category theory as being *abstract category theory*, in the same way that topos theory is *abstract set theory* and model category theory is *abstract homotopy theory*. We will be using it to "do category theory" in one particular 2-category, which is a close relative of **Cat**:

Definition 54. For $\mathcal{C}$ a category, the 2-category $\mathbf{Cat}/\mathcal{C}$ of "categories over $\mathcal{C}$" has:

- **Objects** $(\mathcal{A}, U_\mathcal{A})$ being categories $\mathcal{A}$ together with a functor $U_\mathcal{A}\colon \mathcal{A} \to \mathcal{C}$;
- **1-cells** $F\colon (\mathcal{A}, U_\mathcal{A}) \to (B, U_\mathcal{B})$ being functors $F\colon \mathcal{A} \to \mathcal{B}$ such that

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "F"] \arrow[dr, "U_\mathcal{A}"'] & & \mathcal{B} \arrow[dl, "U_\mathcal{B}"] \\ & \mathcal{C} & \end{tikzcd}$$

  commutes;
- **2-cells** $\alpha\colon F \Rightarrow G$ being natural transformations $\alpha\colon F \Rightarrow G$ such that $U_\mathcal{B} \cdot \alpha =$

$\mathrm{id}_{U_\mathcal{A}}$; or diagramatically

$$\begin{tikzcd} \mathcal{A} \arrow[rr, bend left, "F"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\Downarrow\alpha"] \arrow[dr, "U_\mathcal{A}"'] & & \mathcal{B} \arrow[dl, "U_\mathcal{B}"] \\ & \mathcal{C}. & \end{tikzcd}$$

The way one "does category theory" in a 2-category is by recognising that concepts which we are familiar with in **Cat** are definable purely in terms of diagrams of objects, 1-cells and 2-cells and so can be defined in an arbitrary 2-category. For example, we can define a **comonad** in a 2-category $\mathcal{K}$ to be an object X, together with a 1-cell $t\colon X \to X$ and a pair of 2-cells $\epsilon\colon t \Rightarrow \mathrm{id}_X$ and $\Delta\colon t \Rightarrow tt$ making the following two diagrams commute:

$$\begin{tikzcd} & t \arrow[dl, Rightarrow, "\Delta"'] \arrow[d, Rightarrow, "\mathrm{id}_t"] \arrow[dr, Rightarrow, "\Delta"] & \\ t & tt \arrow[l, Rightarrow, "t\epsilon"] \arrow[r, Rightarrow, "\epsilon t"'] & t, \end{tikzcd} \qquad \begin{tikzcd} t \arrow[r, Rightarrow, "\Delta"] \arrow[d, Rightarrow, "\Delta"'] & tt \arrow[d, Rightarrow, "t\Delta"] \\ tt \arrow[r, Rightarrow, "\Delta t"'] & ttt. \end{tikzcd}$$

In **Cat**, this reduces to the usual notion of comonad; whilst a comonad in the 2-category $\mathbf{Cat}/\mathcal{C}$ consists of:

- An object $(U_\mathcal{A}\colon \mathcal{A} \to \mathcal{C})$;
- A functor $T\colon \mathcal{A} \to \mathcal{A}$ satisfying $U_\mathcal{A} \cdot T = T$, and
- Natural transformations $\epsilon\colon T \Rightarrow \mathrm{id}_\mathcal{A}$ and $\Delta\colon T \Rightarrow TT$ satisfying $U_\mathcal{A} \cdot T = \mathrm{id}_{U_\mathcal{A}}$,

and such that (T, ϵ, Δ) is a comonad in the usual sense. The relevance of this notion becomes manifest when we observe that a comonad on the object $(\mathrm{dom}\colon \mathcal{C}^\mathbf{2} \to \mathcal{C})$ is precisely what we have been calling a "comonad over dom": that is, a functorial factorisation (E, λ, ρ) together with a comonad structure on the associated pair (L, Φ).

As a further instance of this process of abstraction, consider the notion of a *left Kan extension*. We can define an analogous notion, which is usually known just as *left extension*, in an arbitrary 2-category $\mathcal{K}$: given 1-cells $f\colon X \to Y$ and $g\colon X \to Z$, we say that a pair $(h\colon Y \to Z, \theta\colon g \Rightarrow hf)$, as in

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\Leftarrow\theta"] & \\ Y \arrow[rr, "h"'] & & Z \end{tikzcd}$$

exhibits h as the[8] **left extension** of g along f if, given any diagram

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\Leftarrow\psi"] & \\ Y \arrow[rr, "k"'] & & Z \end{tikzcd}$$

[8] Note that a left extension, if it exists, is unique up to unique isomorphism, and so we can speak with justifiable looseness of *the* left extension.

there is a unique 2-cell $\phi\colon h \Rightarrow k$ such that

$$
\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\overset{\Leftarrow}{\theta}"] & \\
Y \arrow[rr, "h"] \arrow[rr, bend right=40, "k"'] \arrow[rr, phantom, bend right=20, "\Downarrow\phi"] & & Z
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\overset{\Leftarrow}{\psi}"] & \\
Y \arrow[rr, "k"'] & & Z.
\end{tikzcd}
$$

In the 2-category **Cat**, the notion of left extension is the familiar notion of left Kan extension[9]; and what we will be interested in is left extensions in the 2-category $\mathbf{Cat}/\mathcal{C}$. The reason for this is that we can use left extensions to construct comonads:

Proposition 55. *Let $\mathcal{K}$ be a 2-category, and suppose that $f\colon X \to Y$ is a 1-cell of $\mathcal{K}$ such that the left extension of f along itself exists, and is given by:*

$$
\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[dr, "f"] \arrow[d, phantom, "\overset{\Leftarrow}{\theta}"] & \\
Y \arrow[rr, "t"'] & & Y.
\end{tikzcd}
$$

Then t can be made into a comonad for which $\theta\colon f \Rightarrow tf$ is a "coaction" of t on f.

Proof. This is essentially an exercise in the universal property of the left extension. Since we have a diagram

$$
\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[dr, "f"] \arrow[d, phantom, "\overset{\Leftarrow}{\mathrm{id}_f}"] & \\
Y \arrow[rr, "\mathrm{id}_Y"'] & & Y,
\end{tikzcd}
$$

we induce, by the universal property of left extension, a 2-cell $\epsilon\colon t \Rightarrow \mathrm{id}_Y$ such that $\epsilon \cdot \theta = \mathrm{id}_f$. Moreover, we have a diagram

$$
\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[d, "f"] \arrow[dr, "f"] & \\
Y \arrow[r, "t"'] \arrow[ur, phantom, near end, "\overset{\Leftarrow}{\theta}"] & Y \arrow[r, "t"'] \arrow[ur, phantom, near start, "\overset{\Leftarrow}{\theta}"] & Y,
\end{tikzcd}
$$

and so, again by the universal property, we induce a 2-cell $\Delta\colon t \Rightarrow hh$ such that $\Delta \cdot \theta = t\theta \cdot \theta$. Applying the universal property again (the "uniqueness" part this time), we deduce that (t, ϵ, Δ) satisfies the axioms for a comonad in our 2-category $\mathcal{K}$. Finally, to say that $\theta\colon f \Rightarrow tf$ is a "coaction" of t on f is to say that the following diagrams commute:

$$
\begin{tikzcd}
f \arrow[d, Rightarrow, "\theta"'] \arrow[dr, Rightarrow, "\mathrm{id}_f"] & \\
tf \arrow[r, Rightarrow, "\epsilon f"'] & f,
\end{tikzcd}
\qquad
\begin{tikzcd}
f \arrow[r, Rightarrow, "\theta"] \arrow[d, Rightarrow, "\theta"'] & tf \arrow[d, Rightarrow, "t\theta"] \\
tf \arrow[r, Rightarrow, "\Delta f"'] & ttf,
\end{tikzcd}
$$

which follows from yet another application of the universal property. □

[9]Though some authors reserve the name left Kan extension for a slightly stronger notion: see the discussion in Section 4.3 of [14].

This comonad is known as the *density comonad* of $f\colon X \to Y$: the most comprehensive source of information on density comonads – or rather the dual *codensity monads* – is also the place where they were first introduced, namely the thesis of Dubuc [6]. One important property demonstrated by Dubuc is that the coaction $\theta\colon f \Rightarrow tf$ induced by a density comonad is the *universal* coaction on f. To make this precise, we first define a **morphism of comonads** $\alpha\colon (t, \epsilon, \Delta) \to (t', \epsilon', \Delta')$ on Y to be given by a 2-cell $\alpha\colon t \Rightarrow t'$ in $\mathcal{K}$ which is compatible with the comonad structures in that the two diagrams

$$\begin{tikzcd} t \arrow[d, Rightarrow, "\alpha"'] \arrow[dr, Rightarrow, "\epsilon"] & \\ t' \arrow[r, Rightarrow, "\epsilon'"'] & \mathrm{id}_t, \end{tikzcd} \qquad \begin{tikzcd} t \arrow[r, Rightarrow, "\Delta"] \arrow[d, Rightarrow, "\alpha"'] & tt \arrow[d, Rightarrow, "\alpha\alpha"] \\ t' \arrow[r, Rightarrow, "\Delta'"'] & t't' \end{tikzcd}$$

commute. Now the universality of the coaction $\theta\colon f \Rightarrow tf$ given above can be expressed by saying that, for any other comonad (t', ϵ', Δ') on Y and coaction $\theta'\colon f \Rightarrow t'f$, there is a unique comonad morphism $\alpha\colon (t, \epsilon, \Delta) \to (t', \epsilon', \Delta')$ for which $\theta' = \alpha\theta$.

Let us now see what this universal property says in the 2-category $\mathbf{Cat}/\mathcal{C}$ we are interested in. Observe first that to give a coaction

$$\begin{tikzcd} & (\mathcal{A}, U_{\mathcal{A}}) \arrow[dl, "F"'] \arrow[dr, "F"] \arrow[d, phantom, "\overset{\Leftarrow}{\theta}"] & \\ (\mathcal{B}, U_{\mathcal{B}}) \arrow[rr, "T"'] & & (\mathcal{B}, U_{\mathcal{B}}). \end{tikzcd}$$

of a comonad $\mathsf{T} = (T, \epsilon, \Delta)$ on a 1-cell F is to give, for each $A \in \mathcal{A}$, a morphism $\theta_A\colon FA \to TFA$ in $\mathcal{B}$ making FA into a T-coalgebra in a manner that is natural in morphisms of $\mathcal{A}$. We deduce[10] that coactions of T on F are in bijection with liftings of $F\colon \mathcal{A} \to \mathcal{B}$ through the category of coalgebras for T:

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\overline{F}"] \arrow[dr, "F"'] & & \mathsf{T}\text{-}\mathbf{Coalg} \arrow[dl, "U_{\mathsf{T}}"] \\ & \mathcal{B}, & \end{tikzcd}$$

in which terms the above universal property can be restated as:

Proposition 56. *Let $F\colon (\mathcal{A}, U_{\mathcal{A}}) \to (\mathcal{B}, U_{\mathcal{B}})$ be a 1-cell of $\mathbf{Cat}/\mathcal{C}$ which admits a codensity monad $\mathsf{T} = (T, \epsilon, \Delta)$. Then there is a bijection, natural in T', between morphisms of comonads $\mathsf{T} \to \mathsf{T}'$ on $(\mathcal{B}, U_{\mathcal{B}})$ and liftings*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\overline{F}"] \arrow[dr, "F"'] & & \mathsf{T}'\text{-}\mathbf{Coalg} \arrow[dl, "U_{\mathsf{T}'}"] \\ & \mathcal{B}, & \end{tikzcd}$$

[10] One can use this universal property to define the concept of "category of algebras" in an arbitrary 2-category, where it is known as an **Eilenberg-Moore object**: this is the *formal theory of monads* [17].

of F through T'-**Coalg**.

In particular, let us consider the case where F is the following 1-cell of $\mathbf{Cat}/\mathcal{C}$:

$$\begin{tikzcd} J \arrow[rr, "\iota"] \arrow[dr, "\mathrm{dom}\cdot\iota"'] & & \mathcal{C}^{\mathbf{2}} \arrow[dl, "\mathrm{dom}"] \\ & \mathcal{C}. & \end{tikzcd}$$

Suppose for a moment that this 1-cell admits a left extension along itself; now if we suggestively write the corresponding comonad on $(\mathcal{C}^{\mathbf{2}}, \mathrm{dom})$ as L^1, then the above Proposition reduces to precisely Proposition 18. Therefore we will have proved this latter Proposition, and given our promised abstract description of the comonad L^1, if we can show that the left extension of ι along itself exists. To do this, we use the fact that the 2-category $\mathbf{Cat}/\mathcal{C}$ that we are working in is very closely related to $\mathbf{Cat}$, so that under suitable circumstances, we can build left extensions in $\mathbf{Cat}/\mathcal{C}$ from left extensions in $\mathbf{Cat}$.

Proposition 57. *Let F and G be 1-cells*

$$\begin{tikzcd} & (\mathcal{A}, U_{\mathcal{A}}) \arrow[dl, "F"'] \arrow[dr, "G"] & \\ (\mathcal{B}, U_{\mathcal{B}}) & & (\mathcal{D}, U_{\mathcal{D}}) \end{tikzcd}$$

in $\mathbf{Cat}/\mathcal{C}$, where $\mathcal{A}$ is a small category, $\mathcal{D}$ has colimits preserved by $U_{\mathcal{D}}$, and $U_{\mathcal{D}} \colon \mathcal{D} \to \mathcal{C}$ is an opfibration. Then the left extension of G along F exists.

This is not the most general result possible: it suffices that $\mathcal{D}$ be *fibre-cocomplete* – so that every fibre category of $\mathcal{D}$ has colimits which every cocartesian map preserves. However, the proof becomes much more intricate if we do so, and so we shall content ourselves with this slightly weaker result.

Proof. As outlined above, we will construct the required left extension using left Kan extensions in $\mathbf{Cat}$ together with the opfibration structure on $U_{\mathcal{D}}$. We begin by taking the left Kan extension (K, ψ) of G along F, which we can do because $\mathcal{A}$ is small and $\mathcal{D}$ is cocomplete:

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "F"'] \arrow[dr, "G"] \arrow[d, phantom, "\Leftarrow\psi"] & \\ \mathcal{B} \arrow[rr, "K"'] & {} & \mathcal{D}. \end{tikzcd}$$

The problem is that this Kan extension does not respect the functors down to $\mathcal{C}$: there is no reason for us to have $U_{\mathcal{D}}K = U_{\mathcal{B}}$. However, if we can produce a natural family of maps

$$\phi_b \colon U_{\mathcal{D}}Kb \to U_{\mathcal{B}}b$$

then we can use the opfibration structure on $U_{\mathcal{D}}$ to "correct" the element Kb to an element $Hb = (\phi_b)_*(Kb)$ which lies in the right fibre: in other words, we obtain a new functor $H \colon \mathcal{B} \to \mathcal{D}$ that *does* satisfy $U_{\mathcal{D}}H = U_{\mathcal{B}}$ as required. So all we need now is a suitable natural transformation $\phi\colon U_{\mathcal{D}}K \Rightarrow U_{\mathcal{B}}$. To get this, observe that because $U_{\mathcal{D}}$ preserves colimits, the diagram

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "F"'] \arrow[dr, "U_{\mathcal{D}}G"] \arrow[d, phantom, "\overset{\Leftarrow}{U_{\mathcal{D}}\psi}"] & \\ \mathcal{B} \arrow[rr, "U_{\mathcal{D}}K"'] & {} & \mathcal{C} \end{tikzcd}$$

is also a left Kan extension. Now, because $U_{\mathcal{B}}F = U_{\mathcal{D}}G = U_{\mathcal{A}}$, we have the diagram

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "F"'] \arrow[dr, "U_{\mathcal{D}}G"] \arrow[d, phantom, "\overset{\Leftarrow}{\mathrm{id}_{U_{\mathcal{A}}}}"] & \\ \mathcal{B} \arrow[rr, "U_{\mathcal{B}}"'] & {} & \mathcal{C}, \end{tikzcd}$$

and so by the universal property of Kan extension, we induce a natural transformation $\phi\colon U_{\mathcal{D}}K \Rightarrow U_{\mathcal{B}}$ satisfying $F\phi \cdot U_{\mathcal{D}}\psi = \mathrm{id}_{U_{\mathcal{A}}}$. Now we can use ϕ to "correct" K as outlined above. Formally, we say that because $U_{\mathcal{D}}$ is an opfibration, so is the functor $[\mathcal{B}, U_{\mathcal{D}}]\colon [\mathcal{B}, \mathcal{D}] \to [\mathcal{B}, \mathcal{C}]$, and so the natural transformation ϕ – seen as a morphism of $[\mathcal{B}, \mathcal{C}]$ – has a cocartesian lifting at K:

$$\begin{tikzcd} K \arrow[rr, "\vec{\phi}"] & {} \arrow[d, dotted, "U_{\mathcal{D}}\cdot(-)"'] & \phi_* K \\ U_{\mathcal{D}}K \arrow[rr, "\phi"'] & {} & U_{\mathcal{B}}. \end{tikzcd}$$

This yields our "corrected" functor $H = \phi_* K\colon \mathcal{B} \to \mathcal{D}$ satisfying $U_{\mathcal{D}}H = U_{\mathcal{B}}$, together with a natural transformation $\chi = \vec{\phi}\colon K \Rightarrow H$ satisfying $U_{\mathcal{D}}\chi = \phi$. Since $U_{\mathcal{D}}H = U_{\mathcal{B}}$, we have a 1-cell $H\colon (\mathcal{B}, U_{\mathcal{B}}) \to (\mathcal{D}, U_{\mathcal{D}})$ in $\mathbf{Cat}/\mathcal{C}$; whilst if we define the natural transformation $\theta\colon G \Rightarrow HF$ to be the composite

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "F"'] \arrow[dr, "G"] \arrow[d, phantom, "\overset{\Leftarrow}{\psi}"] & \\ \mathcal{B} \arrow[rr, "K" description] \arrow[rr, bend right=50, "H"'] & {} \arrow[d, phantom, "\Downarrow\chi"] & \mathcal{D} \\ & {} & \end{tikzcd}$$

then we have $U_{\mathcal{D}}\theta = U_{\mathcal{D}}\chi F \cdot U_{\mathcal{D}}\psi = \phi F \cdot U_{\mathcal{D}}\psi = \mathrm{id}_{U_{\mathcal{A}}}$ so that θ gives a 2-cell of $\mathbf{Cat}/\mathcal{C}$, which exhibits H as the left extension of G along F in $\mathbf{Cat}/\mathcal{C}$:

$$\begin{tikzcd} & (\mathcal{A}, U_{\mathcal{A}}) \arrow[dl, "F"'] \arrow[dr, "G"] \arrow[d, phantom, "\overset{\Leftarrow}{\theta}"] & \\ (\mathcal{B}, U_{\mathcal{B}}) \arrow[rr, "H"'] & {} & (\mathcal{D}, U_{\mathcal{D}}). \end{tikzcd}$$

Verification of the universal property is left as an exercise to the reader. $\square$

Corollary 58. *Let $\mathcal{C}$ be a cocomplete category, let J be a set of maps in $\mathcal{C}$ and let $\iota\colon J \to \mathcal{C}^{\mathbf{2}}$ be the inclusion of the discrete subcategory J into $\mathcal{C}^{\mathbf{2}}$. Then the left extension*

$$\begin{tikzcd} & (J, \mathrm{dom}\cdot\iota) \arrow[dl, "\iota"'] \arrow[dr, "\iota"] \arrow[d, phantom, "\Leftarrow \theta"] & \\ (\mathcal{C}^{\mathbf{2}}, \mathrm{dom}) \arrow[rr, "L^1"'] & & (\mathcal{C}^{\mathbf{2}}, \mathrm{dom}) \end{tikzcd}$$

exists in $\mathbf{Cat}/\mathcal{C}$*, and thus generates a comonad over* $\mathrm{dom}\ \mathsf{L}^1$ *satisfying Proposition 18.*

This completes our abstract description of the comonad L^1. Our final task is to calculate explicitly what our machinery gives us, and check that it tallies with the comonad we gave in Section 5.2. Since the description of the comultiplication that we gave there was secretly derived from the results of this Appendix, the only thing we have to check is that our machinery gives the right underlying functorial factorisation. Now, the first thing the construction of Proposition 57 tells us to do is to form the following left Kan extension:

$$\begin{tikzcd} & J \arrow[dl, "\iota"'] \arrow[dr, "\iota"] \arrow[d, phantom, "\Leftarrow \psi"] & \\ \mathcal{C}^{\mathbf{2}} \arrow[rr, "K"'] & & \mathcal{C}^{\mathbf{2}}, \end{tikzcd}$$

the value of which at an object $g\colon C \to D \in \mathcal{C}^{\mathbf{2}}$ is given by the colimit

$$Kg = \int^{(f,\gamma)\in\iota\downarrow g} \iota(f).$$

Because J is discrete, the indexing category $\iota \downarrow g$ degenerates to a set, an element of which is a pair (f, γ) where $f \in J$ and $\gamma\colon f \to g$ in $\mathcal{C}^{\mathbf{2}}$. So $\iota \downarrow g$ is the set S_g that we considered before, and

$$Kg = \sum_{x\in S_g} f_x.$$

Having formed K, the next step is to "correct" it to a functor over dom, which we do by pushing out along the components of a natural transformation $\phi\colon \mathrm{dom}\cdot K \Rightarrow \mathrm{dom}\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ obtained from the universal property of Kan extension. We find that in this case, ϕ_g is the map:

$$\phi_g = \langle h_x \rangle_{x\in S_g}\colon \sum_x A_x \to C$$

(where $g\colon C \to D$ as before). Therefore $L^1 g$ is given by the right-hand map in the pushout diagram

$$\begin{tikzcd} \sum A_x \arrow[r, "\langle h_x\rangle"] \arrow[d, "\sum_x f_x"'] & C \arrow[d, "\lambda^1_g"] \\ \sum B_x \arrow[r] & E^1 g, \end{tikzcd}$$

which is precisely what we had in Section 5.2.

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