1. Consider automobile traffic moving along a section of road with no entrances or exits; let \( u(x,t) \) denote the density of cars (number of cars per kilometre) at position \( x \) and time \( t \); at position \( x \), they move with a speed \( c = c(x) \) that is position-dependent.

(a) Show that \( u \) satisfies
\[
  u_t + (c(x)u)_x = 0. \tag{1}
\]

(b) Suppose the road narrows in a region near the point \( x = 0 \), so the speed of the traffic is modelled by
\[
  c(x) = \frac{c_1(x^2 + \varepsilon^2)}{x^2 + 1}
\]
where \( c_1 \) and \( \varepsilon \) are positive constants, with \( \varepsilon < 1 \). Explain the significance of these constants, using a graph.

(c) Determine the partial differential equation satisfied by the function \( v(x,t) = c(x)u(x,t) \). Use the method of characteristic curves to solve this equation and show that the general solution for the equation (1) is given by
\[
  u = \frac{1}{c(x)} g\left(c_1 t - x - \frac{1 - \varepsilon^2}{\varepsilon} \tan^{-1}\left(\frac{x}{\varepsilon}\right)\right)
\]
where \( g \) is an arbitrary differentiable function of one variable. Discuss the nature of the solution once traffic is well clear of the region in which the road narrows, i.e., as \( x \to \infty \).

2. Consider automobile traffic moving at constant speed \( c \) along a tunnel with no entrances or exits; let \( u(x,t) \) denote the density of cars (number of cars per kilometre) at position \( x \) and time \( t \). The cars emit exhaust fumes: let \( v(x,t) \) be the concentration of fumes at position \( x \) and time \( t \). The rate of increase of fume concentration due to traffic flow is proportional to \( u(x,t) \); ventilation and filtration plant in the tunnel reduce the fume concentration \( v(x,t) \) at a rate proportional to \( v \) itself.

(a) Show that \( u \) and \( v \) satisfy the following equations of form
\[
  u_t + cu_x = 0 \]
\[
  v_t + \alpha v = \beta u, \tag{2}
\]
explaining how the constants \( \alpha \) and \( \beta \) arise. Find a second order PDE satisfied by \( v \).

(b) How must this be modified to take account of gaseous diffusion in the tunnel (assuming a Fickian process)?

(c) Solve the system (2) given the initial conditions that, at time \( t = 0 \), (i) \( u(x,0) = f(x) \), where \( f \) is a given differentiable function of one variable, and (ii) the concentration of fumes in the tunnel is zero. Explain your answer in terms of \( f \).
3. (a) A string of length \( l \) occupies the interval \( 0 \leq x \leq l \) and has non-uniform density \( \rho(x) \). The string remains in a plane while it vibrates; let \( u(x, t) \) denote its displacement from the rest position. The tension \( T(x) \) in the string is assumed to be tangent to the string at each point \( x \), but varies in magnitude along the string. Ignoring all other forces except tension, and supposing that \( |u_x| \ll 1 \) for all \( x \) ("small vibrations"), show that

\[
\rho(x) u_{tt} = (T(x) u_x)_x .
\]

Explain the relevance of the assumptions at each point in your derivation.

(b) A flexible chain of length \( l \) and uniform density \( \rho \) hangs down from its attachment at one end \( (x = 0) \) and oscillates horizontally, remaining in a plane as it does. Let the \( x \)-axis point downwards; let \( u(x, t) \) denote the chain’s displacement from the rest position. Assume the force of gravity at each point \( x \) of the chain equals the weight of the part of the chain below the point, is directed tangentially to the chain at each point \( x \), and is balanced exactly by the tension \( T(x) \) in the chain. Ignoring all other forces except tension, and supposing that \( |u_x| \ll 1 \) for all \( x \) ("small oscillations"), use part (a) to derive an equation for the motion of the chain. (The force of gravity on a chain segment of length \( \Delta x \) is \( \rho g \Delta x \) where \( g \) is the acceleration due to gravity.)

4. A string of uniform density \( \rho \) and length \( l \) occupies the interval \( 0 \leq x \leq l \). It remains in a plane while it vibrates; let \( u(x, t) \) denote its displacement from the rest position. The tension \( T \) in the string is assumed to be uniform and tangent to the string at each point \( x \). As well as tension, two other forces influence the motion of the string: the friction experienced by a small segment \([x, x + \Delta x]\) has magnitude proportional to its velocity but oppositely directed (to the velocity), and a restoring force due to the elasticity of the string has magnitude proportional to its displacement \( u \), also oppositely directed (to \( u \)); the constants of proportionality do not depend upon position (or time). Supposing that \( |u_x| \ll 1 \) for all \( x \) ("small vibrations"), show that the displacement satisfies an equation of the form

\[
\rho u_{tt} = T u_{xx} - F u_t - R u
\]

for suitable positive constants \( F \) and \( R \), explaining how these two constants arise in your derivation.