1. Let \( n \) be a positive integer and subdivide the interval \([0, 1]\) into \( n \) subintervals \([x_0, x_1], [x_1, x_2], \ldots, [x_{n-2}, x_{n-1}], [x_{n-1}, x_n]\) of equal width \( h = n^{-1} \), where \( x_r = rh \) \((0 \leq r \leq n)\).

Let \( y_r = y(x_r) \) denote the value of the function \( y(x) \) at the grid point \( x_r \). Let

\[
f(x) = \begin{cases} 
1 + x - e^x \Big/ x, & 0 < x \leq 1, \\
0, & x = 0.
\end{cases}
\]  

Note that \( f \) is continuous at \( x = 0 \): \( \lim_{x \to 0} f(x) = 0 \).

(a) Consider the integral equation

\[
y(x) = f(x) + \int_0^1 e^{xt}y(t) \, dt, \quad 0 \leq x \leq 1.
\]

Check that the exact solution to the integral equation is the constant function

\[
y_{\text{exact}}(x) = 1, \quad 0 \leq x \leq 1.
\]

Use the results of lectures that gives the trapezoidal rule approximation for the solution to the integral equation in the form of a matrix equation

\[
y = f + hADy
\]

where \( y = (y_0, y_1, \ldots, y_{n-1}, y_n)^t \), \( f \) is a suitable vector, \( D \) is a diagonal matrix with components \( \left( \frac{1}{2}, 1, 1, \ldots, 1, 1, \frac{1}{2} \right) \) and \( A \) is a suitable \((n + 1)\) by \((n + 1)\) matrix.

(b) Use a MATLAB program to generate the approximate solution to the integral equation based upon (2), for arbitrary values of the parameter \( n \). Plot the approximate solution and exact solution for values of \( n \) equal to 4, 8 and 16, calculating in each case

\[
\max_{0 \leq r \leq n} |y_{\text{exact}}(x_r) - y_r|.
\]

(c) Consider the integral equation

\[
\int_0^1 e^{xt}y(t) \, dt = 1 - f(x), \quad 0 \leq x \leq 1
\]

Check that the exact solution to the integral equation is the constant function

\[
y_{\text{exact}}(x) = 1, \quad 0 \leq x \leq 1.
\]

The trapezoidal rule approximation developed in part (a) for the matrix equation solution to the integral equation now takes form

\[
hADy = f_1
\]
where \( y \) and \( A \) are as defined above, and \( f_1 \) is a suitable vector. Modify the MATLAB program developed in part (b) to generate the approximate solution to the integral equation based upon (3), for arbitrary values of the parameter \( n \). Plot the approximate solution and exact solution for values of \( n \) equal to 4, 8 and 16, calculating in each case

\[
\max_{0 \leq r \leq n} |y_{\text{exact}}(x_r) - y_r|.
\]

(d) Comment on the apparent differences between the two integral equations.

2. (a) Compute the Legendre polynomials \( P_n(x) \) of degrees 0, 1, \ldots, 7 in the interval \(-1 \leq x \leq 1\) using the recursion formula

\[
(n + 1)P_{n+1}(x) - x(2n + 1)P_n(x) + nP_{n-1}(x) = 0
\]

and the starting values \( P_0(x) = 1, P_1(x) = x \). [Computationally it may be convenient to use a vector \( x = (-1 : 0.01 : 1) \) and store the values of the polynomials in a matrix \( P(n, x) \).]

Plot the polynomials of degree 0, 1, \ldots, 4 as functions of \( x \) on one figure and those of degree 5, 6 and 7 on another figure.

Verify that your computed values of the Legendre polynomials \( P_n(x) \) are in agreement with the theory:

\[
P_n(1) = 1, \quad P_n(-1) = (-1)^n, \quad P_{2n+1}(0) = 0, \quad P_{2n}(0) \neq 0;
\]

verify that your computed polynomial \( P_n(x) \) of degree \( n \) has \( n \) zeros in the interval \(-1 \leq x \leq 1\).

(b) Use the same recursion formula, but with the starting values

\[
Q_0(x) = \frac{1}{2} \log \frac{1 + x}{1 - x}, \quad Q_1(x) = x \frac{1}{2} \log \frac{1 + x}{1 - x} - 1,
\]

to compute the Legendre functions \( Q_n(x) \) of degrees 0, 1, \ldots, 5 in the interval \(-0.95 \leq x \leq 0.95\). Plot them on two figures as functions of \( x \). [Again, it is convenient to define \( x \) as a vector \( x = (-0.95 : 0.01 : 0.95) \) and store the values of \( Q_n(x) \) as a matrix \( Q(n, x) \).]