1. Find solutions of the following differential equation using the Frobenius method, identifying where possible the series as expansions of known functions.

\[ xy'' + (1 - 2x)y' + (x - 1)y = 0. \]

2. Various differential equations can be reduced to Bessel equation. Find, where possible, the general solution of the following equations, in terms of \( J_{\nu} \) and \( J_{-\nu} \), or indicate why these functions do not give a general solution. [In parts (b) and (c), use the substitutions as indicated.]

(a) \( x^2y'' + xy' + (x^2 - \frac{1}{9})y = 0; \)

(b) \( xy'' + 5y' + xy = 0 \quad (y = u/x^2); \)

(c) \( 9x^2y'' + 9xy' + (36x^4 - 16)y = 0 \quad (x^2 = z). \)

(a) Using the general series representation for the Bessel functions \( J_{\nu}(x) \), compute and plot the first two Bessel functions \( J_0(x) \) and \( J_1(x) \) in the interval \( 0 \leq x \leq 20. \)

(b) Use the recursion formula to compute and plot \( J_2(x) \), \( J_3(x) \) and \( J_4(x) \) in the interval \( 0 \leq x \leq 20. \)

(c) Observe that these functions \( J_0(x), \ldots, J_4(x) \) look similar to \( \sin x \) and \( \cos x \), but their zeros are not completely regularly spaced and the height of the ‘waves’ decreases with increasing \( x \). Heuristically, the terms \( \frac{\nu^2}{x^2} \) and \( y'/x \) occurring in

\[ y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right) = 0 \]

are small in absolute value for large \( x \), so that the Bessel equation comes close to \( y'' + y = 0 \), the defining equation for \( \cos x \) and \( \sin x \); also, \( y'/x \) acts as a "damping term", in part responsible for the decrease in height. Verify graphically that for large \( x \), the following asymptotic formula

\[ J_{\nu}(x) \approx \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\nu\pi}{2} - \frac{\pi}{4} \right) \]  \hspace{1cm} (1)

is valid for \( \nu = 3 \) and \( \nu = 4 \): plot, on the same figure, your previously computed (exact) values of \( J_3(x) \) and \( J_4(x) \) and the approximations given by (1) in the interval \( 1 \leq x \leq 20. \)
3. Find the asymptotic expression for $J_{\nu}(x)$ when $\nu$ is fixed and $x \to 0$, where $\nu \neq -1, -2, -3, ...$ (use the series representation of $J_{\nu}(x)$).

4. Show that:
   
   (a) $J'_0(x) = -J_1(x)$;
   
   (b) $J'_2(x) = \frac{1}{2} [J_1(x) - J_3(x)]$;
   
   (c) $J'_1(x) = J_0(x) - \frac{1}{x} J_1(x)$.

   (a) Show that (where $C$ denotes an arbitrary constant):
   
   1. $\int x^{\nu} J_{\nu-1}(x) dx = x^{\nu} J_{\nu}(x) + C$;
   2. $\int x^{-\nu} J_{\nu+1}(x) dx = -x^{-\nu} J_{\nu}(x) + C$;
   3. $\int J_{\nu+1}(x) dx = \int J_{\nu-1}(x) dx - 2J_{\nu}(x) + C$.

   (b) Use part (a) to evaluate:
   
   1. $\int J_5(x) dx$;
   2. $\int x^3 J_0(x) dx$. 