Quantum particles are localized in quantum space.

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A. Abstract

The spatial continuum for quantum phenomena is usually taken to be classical Euclidean space. However both the two slit experiments and experiments to test Bell’s inequalities and quantum teleportation reveal non-local behaviour of quantum systems in classical space. In the former a single quantum particle is non-localized (wave-like) because it can pass simultaneously through two spatially separated slits, in the latter two particle systems appear to contravene the Einstein separability principle that spatially distant objects exist independently of each other.

I claim that the standard descriptions of these experiments use the wrong spatial continuum, that there is a quantum space, different from classical space, in which quantum particles are located. Both the metric and the topology of the quantum space of a particle are determined when its locations are assumed to be in one-to-one correspondence with triplets of quantum real numbers (qr-numbers), the values of its position operators. I will discuss the structure of the quantum space for systems of Galilean relativistic quantum particles. The talk is based on two recent papers with Thomas Durt, [5, 6], as well as the earlier work in [4]. These papers contain more of the mathematical details.

B. The quantum real number interpretation of quantum mechanics.

The quantum real number (qr-number) interpretation of quantum mechanics is based on the claim that the ontological properties of microscopic entities differ from those of classical objects in that their attributes, which are represented by operators, always have values not as standard real numbers but as topos real numbers called qr-numbers. In place of the standard quantum states it uses conditions that are open subsets of quantum states. The qr-numbers, having extents as well as values, behave like functions whose domains are the extents. If a particle is in the condition \( W \) then the qr-number value of its attribute, given by the operator \( \hat{A} \), is the continuous functions \( a_Q(A) \) defined on \( W \) by the formula \( a_Q(A) = Tr\rho \hat{A} \) for all states \( \rho \in W \). The functions \( a_Q(A) \) are locally linear qr-numbers. All qr-numbers are obtained as continuous functions of locally linear qr-numbers, they are sections of the sheaf of Dedekind reals \( \mathbb{D}(E_S) \cong C(E_S) \) in the topos of sheaves on the particle’s quantum state space \( E_S \).

The fact that no experiment has infinitely accurate outcomes means that each experiment has a level of accuracy, \( \epsilon > 0 \), which places limitations on the condition of the system. For example; preparing attributes to have certain ranges of numerical values determines a condition and hence exact qr-number values of the attributes, or when measuring an attribute the condition of the system is restricted so that a standard real number is obtainable as an approximation to its exact qr-number value. This latter process is called \( \epsilon \) sharp collimation. Technically \( \epsilon \) sharp collimation of \( \hat{A} \) in \( I = [a, b] \) occurs on \( W \) if both \( a_Q(A) \in ]a, b[ \) and \( a \leq a_Q(W) - \sqrt{\frac{s(a)W}{a}} < a_Q(W) + \sqrt{\frac{s(a)W}{a}} \leq b \) where \( s(a) = \sqrt{(a^2_Q(W) - a_Q(W)^2)} \). It follows that \( a_Q(W) \in I \) and \( 4[\frac{s(a)W}{a}] \leq \epsilon \).

Finally we note the difference between the condition, called epistemic, of a prepared particle that an experimentalist determines from the parameters of the experimental set up and the condition, called ontic, that the particle may actually have. If a particle is prepared with \( W \) as its epistemic condition, its ontic condition can be any non-empty open subset \( V \) of \( W \).

C. The experiments

1. The single-electron interference experiment

The Bologna (1974)[8] and Hitachi (1989)[9] double slit experiments show single electrons building up an interference pattern. The standard quantum mechanical descriptions, such as Feynman’s, do not describe the observable build up of single events, but this build up is describable using the qr-number model. In its quantum space a single massive particle can have a trajectory that passes through the two slits without it being detectable at either slit even though
it is detected at the screen. Moreover the interference pattern emerges on the screen when sufficiently many single events have been observed that the union of the initial ontic conditions of the electrons covers the initial epistemic condition of the experiment.

The qr-number description answers the following questions:
A. Why can particles prepared in the same manner be detected at different spots on $\Sigma_2$? Because the same preparation refers to the initial epistemic condition but different particles can have different ontic conditions. Different ontic conditions determine different trajectories in qr-number space.

B. How does the interference pattern emerge as the accumulated effect of many single particle events? Because the probability function for the ensemble is obtained as the collation of the probability functions for all the individual particles in ontic conditions that are compatible with the prepared epistemic condition.

C. How can a single particle pass through both slits and subsequently interfere with itself? Because in its qr-number space, a particle may be at a single qr-number spatial location that corresponds to two or more locations in classical space.

2. The Einstein separability principle experiments.

The EPR-type experiments contravene the the Einstein separability principle, needed for the integrity of laboratory experiments, that spatially distant objects exist independently of each other.

In the two particle experiment two quantum particles are prepared with total momentum zero at a source $S$.

They move freely away from $S$ towards two apparatuses, labeled $L$ and $R$, placed at significant distances to the left and right of $S$ along the $z$-axis. Measurements are carried out at $L$ and $R$. It is observed that an experimenter at $L$ can change the outcomes measured at $R$ by changing the attribute that is measured at $L$.

This raises the questions of how the outcome of the measurement of an attribute of the particle at $L$ depend upon the outcome of the measurement of the attribute of the particle at $R$ when the distance between the apparatuses $L$ and $R$ prevents any interaction between them?

The qr-number description provides the response; the quantum particles can always interact with each other because they remain close to each other in their joint qr-number space.

When the particles are prepared in an open neighbourhood of the standard entangled state [6] then the two particles move freely in such a way that the qr-number distance between them always remains close to zero. Moreover each quantum particle moves freely to both the left and right to non-empty single particle extents that are common when the particles are identical or differ only in their masses.

D. What is the quantum space of a particle

The quantum space is a continuum that is defined by its open sets, not by its points. We call the open sets the particles in ontic conditions that are compatible with the prepared epistemic condition.

The quantum space of a Galilean relativistic massive particle is the vector space $R^3_D(\mathcal{E}_S)$ whose coordinate axes are parametrized by the locally linear qr-numbers, $A^{aq}(\mathcal{E}_S)$ for $a = x, y, z$, generated by the position operators $\hat{X}, \hat{Y}$ and $\hat{Z}$ of the particle.

The locations of a particle in quantum space are given by triplets of qr-numbers $\vec{x}_Q(W) = (x_Q(W), y_Q(W), z_Q(W))$ for $W$ an open subset of $\mathcal{E}_S$. They are not classical points because they have inner structure. They are open sets similar to Russell’s "events"[10] because by construction any section $a_Q(U)$ for $U \in \mathcal{O}(\mathcal{E}_S)$ is open, for more details see [7], Proposition 1, §II.6.

For any open set $U$, the qr-number distance between locations $P, Q$ with coordinates $\vec{x}_Q(W), \vec{x}_Q(V)$ is $d(P,Q)(U) = (|x_Q(W \cap U) - x_Q(V \cap U)|^2 + |y_Q(W \cap U) - y_Q(V \cap U)|^2 + |z_Q(W \cap U) - z_Q(V \cap U)|^2)^{1/2}$ to extent $U$. If $W \cap V = \emptyset$ then $d(P,Q)(U) > 0$, $\forall U \in \mathcal{O}(V \cup W)$ but if $W \cap V \neq \emptyset$ then $d(P,Q)(U) = 0$, $\forall U \in \mathcal{O}(V \cap W)$. The quantum number distance function defines a pseudo-metric on the space which is a metric when restricted to those locations whose qr-number coordinates are apart. Aparnerness is stronger than not equal to, as $\vec{x}_Q(W) \neq \vec{x}_Q(V)$ iff $W \neq V$, but $\vec{x}_Q(W)$ is apart from $\vec{x}_Q(V)$ iff $W \cap V = \emptyset$. The quantum continuum differs from the usual real continuum in that not all pairs of qr-numbers are apart so that the corresponding locations in quantum space are not disjoint.

On measurement, an attribute will record a standard real number value to an accuracy $\epsilon$ only when the extent of its qr-number value contains an (approximate) eigenstate of the self adjoint operator that represents the attribute, then the particle is $\epsilon$ located. Therefore the qr-number coordinates of a quantum particle may lie in an interval to a non-empty extent without the particle being observed in the corresponding slit.
E. One particle - two slit experiment

The particles move in the \(yz\)-plane. \((0, z_m)\) is the centre of the exit to \(S\). The screens \(\Sigma_j\) lie on \(y = y_j, j = 1, 2\). \(I_S = [z_m - \delta x, z_m + \delta x]\) for small \(\delta x\) is the exit to \(S\), and the momenta of the emitted particles are limited to \(J_2 = [p_y - \delta p_2, p_y + \delta p_2]\) in the \(y\)-direction and \(J_3 = [- \delta p_3, \delta p_3]\) in the \(z\)-direction. Both \(\delta p_2\) and \(\delta p_3\) are small.

On \(\Sigma_1, (y_1, z_m)\) is the midpoint between the slits. \(I_+\) and \(I_-\) are both \(2\delta_1\) wide with midpoints \((y_1, z_{\pm})\), \(z_{\pm} = z_m \mp (\delta_1 + \delta_2)\), so that \(I_{\pm} = [z_{\pm} - \delta_1, z_{\pm} + \delta_1]\). The strip between is \(I_0 = [z_m - \delta_2, z_m + \delta_2]\), \(\delta_2\) is tiny [8, 9] .

I. A QR-NUMBER DESCRIPTION OF THE 2-SLIT EXPERIMENT

Assume that \(0 < \epsilon < \frac{1}{10}\). Each particle will be described with its own time-line. At \(S\), the epistemic condition of the prepared particles is \(W_S(\epsilon)\) and the \(\alpha\)th particle has an ontic condition \(V_\alpha \subseteq W_S(\epsilon)\), e.g. \(V_\alpha = \nu(\rho_\alpha, \delta_\alpha)\) with \(\rho_\alpha \in W_S(\epsilon)\) and \(\delta_\alpha > 0\) sufficiently small. At \(t = 0\) the particle leaves \(S\) and moves freely to \(\Sigma_1\) reaching it at \(t = t_1\).

Passage through the slits is controlled by three disjoint open subsets of \(W_S(\epsilon)\) for \(\epsilon < \frac{1}{10}\).

If \(V_\alpha \subseteq W_+(\epsilon)\), the particle passes through \(I_+\),
if \(V_\alpha \subseteq W_-(\epsilon)\), it passes through \(I_-\) and
if \(V_\alpha \subseteq W_m(\epsilon)\), it passes through the double slit \(I_+ \cup I_-\) but not separately through either \(I_+\) or \(I_-\).

1. Construction of \(W_r(\epsilon), r = +, -, m\)

The spectral projection operators of \(\hat{X}_3(t_1)\) for \(r = +, -, 0\), \(\hat{P}_r(t_1) = \hat{P}^{{\hat{X}_3}(t_1)}(I_r)\). They are orthogonal, \(\hat{P}_r(t_1) \perp \hat{P}_s(t_1)\) as \(I_r \cap I_s = \emptyset\) if \(r \neq s\). \(\hat{P}^{{\hat{X}_3}(t_1)}(I_+ \cup I_-) = \hat{P}_+(t_1) + \hat{P}_-(t_1)\) is the projection operator for the double slit.

The projection operator \(\hat{P}_m(t_1) = \frac{1}{2}[(\hat{P}_+(t_1) + \hat{P}_-(t_1)) + T_{(+,-)}(t_1) + T_{(-,+)}(t_1)]\), where \(T_{(+,-)}(t_1)\) is a partial isometry with initial subspace \(\hat{P}_-(t_1)\) and final subspace \(\hat{P}_+(t_1)\) and \(T_{(-,+)}(t_1)\) is its adjoint. \(\hat{P}_m(t_1)\) is not in the spectral family of \(\hat{X}_3(t_1)\) but it bisects the angle between \(\hat{P}_+(t_1)\) and \(\hat{P}_-(t_1)\). This means that for each \(\phi_m \in \hat{P}_m(t_1)\) there is a pair of orthogonal vectors \(\phi_+ \in \hat{P}_+(t_1)\) and \(\phi_- \in \hat{P}_-(t_1)\) such that \(\phi_m = \frac{1}{\sqrt{2}}(\phi_+ + \phi_-)\).

Definition 1. Let \(\mathcal{A}P^{P_2}(\lambda; \epsilon) = \{\rho_\alpha : |\text{Tr} \rho_\alpha \hat{P}_2 - \lambda| < \epsilon\}\) denote the set of approximate eigenstates \(\rho_\alpha = |\psi_\alpha\rangle \langle \psi_\alpha|\) of the self-adjoint operator \(\hat{P}_2\) for \(\lambda \in \sigma_{c}(\hat{P}_2)\), its continuous spectrum.

Definition 2. For \(r = +, -, m\), let \(\mathcal{P}S_r(t_1)\) be the set of pure states \(\{\rho_\alpha = |\phi_\alpha\rangle \langle \phi_\alpha| : \phi_\alpha \in \hat{P}_r(t_1)\}\) satisfying \(\mathcal{A}P^{P_2}(\rho_\alpha; \epsilon)\). Then \(\bigcup_{\rho_\alpha \in \mathcal{P}S_r(t_1)} \nu(\rho_\alpha; \epsilon)\) is the condition for the passage through \(I_r\) and \(W_r(\epsilon) = \bigcup_{\rho_\alpha \in \mathcal{P}S_r(t_1)} \nu(\rho_\alpha; \epsilon) \cap W_S(\epsilon)\) is the extent to which an electron prepared at \(S\) passes through \(I_r\).
In the qr-numbers framework, a particle is said to pass through a single slit if at some time it is \( \epsilon \)-located in the slit and hence could be observed in the slit with accuracy \( \epsilon \). A particle in condition \( V_a \) is \( \epsilon \) located in \( J \) at \( t_1 \), if the qr-number value \( \pi_J(t_1)(V_a) \) of \( \hat{P} \hat{X}_{\epsilon}(t_1)(J) \) satisfies \( \pi_J(t_1)(V_a) > 1 - \epsilon \). It is a quantum phenomenon that the position coordinate of a quantum particle can have a qr-number value that lies in an interval to a non-empty extent without it being \( \epsilon \)-located in the slit. Hence it will not be observed in the slit at the required accuracy \( \epsilon \). This is what happens to the particle with condition \( V_a \subseteq W_m(\epsilon) \) that cannot be observed as passing separately through either \( I_+ \) or \( I_- \).

After leaving \( \Sigma_1 \) the electron moves freely to \( \Sigma_2 \) which it reaches at \( t = t_2 \). Particles that have passed through at least one slit at accuracy \( \epsilon < 1/4 \) have an ontological condition \( V_a \subset W_+(\epsilon) \cup W_-(\epsilon) \cup W_m(\epsilon) \subset W_S(\epsilon) \).

3. Detection on \( \Sigma_2 \)

At time \( t_2 \) the qr-number probability that an electron in the condition \( V_a \) is detected at the \( d^{th} \) detector, with aperture \( I_d \), is \( \pi_d(t_2)(V_a) \), the qr-number value of the projection operator \( \hat{P}_d(t_2) = \hat{P} \hat{Z}_{\epsilon}(I_d) \).\(^4\) Hence a particle with ontic condition \( V_a \) is detected in the \( d^{th} \) detector if \( V_a \subseteq \hat{V}_d \), where \( V_d \) is the condition for being \( \epsilon \)-located in \( I_d \) because then \( \pi_d(t_2)(V_a) > 1 - \epsilon \).

The probability of any particle in the ensemble passing into the \( d^{th} \) detector is obtained by collating the functions \( \pi_d(t_2)(V_a) \) for all the possible ontic conditions \( V_a \) of particles in the ensemble. When sufficiently many particles have been prepared that the union of their ontic conditions covers \( W_m(\epsilon) \), the qr-number probability that a particle, prepared in the epistemic condition \( W_m(\epsilon) \), is detected in the \( d^{th} \) slit is given by \( \pi_d(t_2)(W_m(\epsilon)) = \pi_d(t_2)(\cup_{\alpha} V_a) \).

4. The standard formula for the interference pattern

The standard quantum mechanical formula is obtained when the ranges of each of \( \hat{P}_+ \) and \( \hat{P}_- \) are one dimensional and hence the range of \( \hat{P}_m \) is spanned by \( \psi_m = \frac{1}{\sqrt{2}} (\psi_+ + \psi_-) \), \( \psi_\pm \in \hat{P}_\pm \mathcal{H} \). Let \( W_m(\epsilon)(\rho_m, \epsilon) \) for the state \( \rho_m = |\psi_m\rangle \langle \psi_m| \). If \( W_m(\epsilon) \) is covered by the union of the ontic conditions \( V_a = \nu(\rho_a, \delta_a) \) then when sufficiently many different conditions have been prepared the qr-number probability of detecting a particle at the \( d^{th} \) detector is \( Tr \rho_m \hat{P}_d(t_2) \) to within \( \epsilon \), by Theorem 5 of \([4]\). But

\[
Tr \rho_m \hat{P}_d(t_2) = \frac{1}{2} (|\langle \psi_+ | \hat{P}_d(t_2) \psi_+ \rangle + |\langle \psi_+ | \hat{P}_d(t_2) \psi_- \rangle + |\langle \psi_- | \hat{P}_d(t_2) \psi_+ \rangle + |\langle \psi_- | \hat{P}_d(t_2) \psi_- \rangle |)
\]

is the standard expression for the probability that a particle is detected at the \( d^{th} \) detector.

5. qr-number trajectories

The trajectory of a freely moving quantum particle of mass \( m \) is given by \( \bar{x}_Q(V_a)(t) = \bar{x}_Q(V_a)(0) + \frac{1}{m} \int_0^t \bar{p}_Q(V_a)(0) \, dt \), \( 0 < t \in \mathbb{R} \), if its condition is \( V_a \in \mathcal{O}(\mathcal{E}_x) \). The electron leaves \( \Sigma_1 \) at \( t_1 \) with \( x_2(t_1)(V_a) \equiv y_1 Q(V_a) \). Since \( y_2 - y_1 \) is large assume that it arrives on \( \Sigma_2 \) at \( t_2 \) given by \( t_2 = t_1 + \frac{m(y_2 - y_1)}{p_y} \).

A natural question is whether the interference pattern is built up because more particles whose ontic conditions \( V_a \subset W_m(\epsilon) \) have qr-number trajectories detected on \( \Sigma_2 \) near points of "destructive interference" than are detected near points of "constructive interference". Even though the particle is not \( \epsilon \) located in either of the slits \( I_+ \) and \( I_- \), we can restrict its position and momentum operators to the slits; \( \hat{X}_+^{\pm}(t_1) = \hat{P}_x(t_1) \hat{X}_3(t_1) \hat{P}_x(t_1) \) and \( \hat{P}_x^{\pm}(t_1) = \hat{P}_x^{\pm}(t_1) \hat{P}_x(t_1) \hat{P}_x^{\pm}(t_1) \). Then the qr-number trajectory of an electron whose condition \( V_a \subset W_m(\epsilon) \) can be split into two parts, one starting at \((y_1, (x_3^\pm)(Q(V_a))) \) with momentum \((p_y, (p_{z_3}^\pm)(Q(V_a))) \), the other at \((y_1, (x_3^\pm)(Q(V_a))) \) with momentum \((p_y, (p_{z_3}^\pm)(Q(V_a))) \). The parts end at the same point on \( \Sigma_2 \) only when \( V_a \) is such that

\[
(x_3^+)(Q(V_a)) - (x_3^-)(Q(V_a)) = ((p_{z_3}^+)(Q(V_a)) - (p_{z_3}^-)(Q(V_a))) \frac{(y_2 - y_1)}{p_y}.
\]

If \( p_y \gg (p_{z_3}^\pm)(Q(V_a)) \), the lengths of the two parts differ by the wavelength \( \lambda = \frac{h}{p_y} \) at points whose \( z \) coordinates are

\[
\left| (x_3^+)(Q(V_a)) + (x_3^-)(Q(V_a)) \right| = \frac{h(y_2 - y_1)}{(p_{z_3}^+)(Q(V_a)) - (p_{z_3}^-)(Q(V_a))} \approx z_m = \frac{h(y_2 - y_1)}{(z_+ - z_-)p_y}.
\]
These locations are consistent with experimental results in which the fringe distance between the central and first maxima is approximately \( \frac{K_0}{L} = \frac{K_1}{L} \) and in accordance with the de Broglie relation \( p_x = \frac{\hbar}{\lambda} \).

6. Conclusions for the double slit experiment.

In the qr-number description of the double slit experiment each particle that passes through the slits moves freely to the detecting screen where it is detected as a single particle. The “wave-like” interference pattern only emerges after many particles have been detected. A quantum particle always behaves as a quantum particle. Each qr-number trajectory is well defined in qr-number space.

There is a less stringent form of complementarity. Bohr’s insistence that macroscopic experimental set-ups define the conditions of a quantum system only refers to epistemological and not to ontological conditions. Many different ontic conditions are compatible with each epistemic condition. The set-up of the double slit experiment is compatible with the ontic condition of a particle that passes through one slit or with that of a particle that passes through both slits.

The qr-number description is realist in the sense that it postulates the existence of entities possessing properties corresponding to qualities such as the position or momentum of a particle but does not identify the ontological quantitative values of these qualities with their observed numerical values. The ontological properties are related to the observed properties but are not identical with them.

II. QR-NUMBER MODEL OF A TWO PARTICLE SYSTEM

As in standard theory, \( \mathcal{H}(1,2) = \mathcal{H}(1) \otimes \mathcal{H}(2) \) the tensor product of the Hilbert spaces \( \mathcal{H}(1) \) and \( \mathcal{H}(2) \) that carry irreducible projective unitary representations of the symmetry group \( G \). For \( j = 1,2 \), the physical qualities associated with particle \( j \) are operators in an algebra \( A(j) \), the representation of the enveloping algebra of the Lie algebra of \( G \), that are essentially self-adjoint on subspaces \( D(j) \) dense in \( \mathcal{H}(j) \). \( D(1,2) = D(1) \otimes D(2) \) is dense in \( \mathcal{H}(1,2) \). The algebra of physical qualities for the combined system is \( A(1,2) \), constructed from elements of the algebras \( A(j) \), \( j = 1,2 \) and closed in an operator topology, c.f. Weyl’s prescription for measurable quantities.

Qualities of the combined system of the form \( A(1) \otimes B(2) \in A(1,2) \) have qr-number values \( (a(1) \otimes b(2))_{Q}(W(1,2)) \) for \( W(1,2) \in \mathcal{O}(\mathbb{E}(1,2)) \). As part of the combined system, the qualities of particle 1 are represented by operators of the form \( A(1) \otimes I(2) \) and those of particle 2 by operators of the form \( I(1) \otimes B(2) \), with \( I(j) = I_j \), the identity operator on \( \mathcal{H}(j) \). \( j = 1,2 \). \( (1(1) \otimes b(2))_{Q}(W(1,2)) = b(2)_{Q}(W(2)) \) where the reduced condition \( W(2) \) is the open subset of \( \mathbb{E}(2) \) is obtained by partial tracing \( W(1,2) \) over an orthonormal basis in \( \mathcal{H}(1) \). Similarly \( (a(1) \otimes 1(2))_{Q}(W(1,2)) = a(1)_{Q}(W(1)) \) where the reduced condition \( W(1) \) is an open subset of \( \mathbb{E}(1) \). Identical particles are discussed in [6].

7. Two particle entanglement

In the qr-number interpretation a two particle condition \( W(1,2) \in \mathcal{O}(\mathbb{E}(M(1,2))) \) reduces by partial tracing to the single particle conditions \( W(j) \in \mathcal{O}(\mathbb{E}(M(j))) \) for \( j = 1,2 \). The reduction maps, being continuous, define a pair of covariant functors \( f_j \) and a pair of contravariant functors \( (f_j)^* \). These relate the qr-number values in \( \mathbb{R}_{D}(\mathbb{E}(M(j))) \) of the single particles to qr-number values in \( \mathbb{R}_{D}(\mathbb{E}(M(1,2))) \) of the joint two particle system.

The wave-function \( \Psi(1,2) = \frac{1}{\sqrt{2}}(\phi_{R}(1)\phi_{L}(2) + \phi_{L}(1)\phi_{R}(2)) \) with orthogonal single particle wave functions corresponds to the entangled pure state \( \rho_{0}(1,2) = \hat{P}_{\Psi(1,2)} \), its reduced single particle states are the mixed states \( \rho_{0}(j) = \frac{1}{2}(\hat{P}_{\phi_{R}(j)} + \hat{P}_{\phi_{L}(j)}) \) for \( j = 1,2 \).

When \( \rho_{0}(1,2) \) is the entangled pure state, \( W_{0}(1,2) = \nu(\rho_{0}(1,2);\epsilon) \) contains many pure states \( \sigma_{\delta}(1,2) = \hat{P}_{\chi_{\delta}(1,2)} \), e.g., any unit vector \( \chi_{\delta}(1,2) \in \mathcal{H}(1,2) \) with \( |\langle \chi_{\delta}(1,2), \Psi(1,2) \rangle|^{2} = 1 - (1/4)\delta^{2} \) and \( \delta \leq \epsilon \). For each such \( \sigma_{\delta}(1,2) \), \( \nu(\sigma_{\delta}(1,2);\alpha) \) is an open subset of \( W_{0}(1,2) \) when \( \alpha \leq (\epsilon - \delta)/2 \).

If \( \epsilon > \frac{1}{2} \) the product states \( \hat{P}_{\phi_{R}(1)\phi_{L}(2)} \) and \( \hat{P}_{\phi_{L}(1)\phi_{R}(2)} \) do not belong to \( W_{0}(1,2) \), but the mixed states \( \tau_{1,2} = \lambda_{1}\hat{P}_{\phi_{R}(1)\phi_{L}(2)} + \lambda_{2}\hat{P}_{\phi_{L}(1)\phi_{R}(2)} + \sum_{n=3}^{\infty} \lambda_{n}\hat{P}_{\phi_{R}(1)\phi_{L}(2)} \) do if \( \lambda_{n} \geq 0 \), \( \sum_{n=1}^{\infty} \lambda_{n} = 1 \) and \( \lambda_{1} + \lambda_{2} > 1 - \epsilon \), when \{ \phi_{L}(j), \phi_{R}(j), \phi_{3}(j), ... \} \) are orthonormal sets in \( \mathcal{H}(j) \) for \( j = 1,2 \).
8. Single particle conditions reduced from $W_0(1,2)$

$W_0(1,2)$ has reduced single particle conditions $\tilde{W}_0(j) = \nu(\rho_0(j); \delta_j)$, $j = 1, 2$, for $\rho_0(j) = \frac{1}{2}(\hat{P}_{\phi_R(j)} + \hat{P}_{\phi_L(j)})$ for $j = 1, 2$ and $\delta_j \leq \epsilon$.

**Theorem 1.** For both $j = 1, 2$, if $\delta_j \leq \epsilon < \frac{1}{2}$ there are no pure states in $\tilde{W}_0(j)$, in fact if $\dim \mathcal{H}(j) = \infty$, every state in $\tilde{W}_0(j)$ is of the form $\lambda_1 \hat{P}_{\phi_R(j)} + \lambda_2 \hat{P}_{\phi_L(j)} + \sum_{n=3}^{\infty} \lambda_n \hat{P}_{\phi_n(j)}$ with $\sum_{n=3}^{\infty} \lambda_n = 1 - \lambda_1 - \lambda_2 < \delta_j/2$ when $\{\phi_L(j), \phi_R(j), \phi_3(j), \ldots\}$ is an orthonormal set and $0 \leq \lambda_n < 1$ for all $n$.

This means that the particle $j$ in the condition $\tilde{W}_0(j)$ cannot be $\epsilon$ sharp collimated in any interval for any quality and so it is not $\epsilon$ located in any region in space nor would it yield any definite measured value for any quality. Furthermore, any product state in $W_0(1,2)$ would reduce to pure states in both $W_0(j)$, $j = 1, 2$, therefore:

**Corollary 3.** There are no product states in $W_0(1,2)$ if $\epsilon < \frac{1}{2}$ so that $W_0(1,2)$ is an entangled condition.

The open sets $W_K(j) = \nu(P_{\phi_K(j)}; \delta_j)$, $K = R, L$ are not subsets of $\tilde{W}_0(j)$ because they contain pure states, however:

**Theorem 2.** If $\delta_j \leq \epsilon \ll \frac{1}{2}$ and $\rho(j) \in \nu(\rho_0(j); \delta_j)$ then there exist states $\rho_K(j) \in W_K(j)$ for $K = R, L$ such that $\|\rho(j) - \frac{1}{2}(\rho_R(j) + \rho_L(j))\| < \delta_j$. That is the set of states of the form $\frac{1}{2}(\rho_R(j) + \rho_L(j))$, with $\rho_K(j) \in W_K(j)$, $K = R, L$, is dense in $\nu(\rho_0(j); \delta_j)$ in the trace norm topology.

**Corollary 4.** Locally linear qr-numbers $a_Q(\tilde{W}_0(j))$ are well approximated by convex combinations $\frac{1}{2}(a_R(W_R(j)) + a_L(W_L(j)))$.

In particular, if $\delta_j \leq \epsilon \ll \frac{1}{2}$, the qr-number value of the attribute $\hat{A}(j)$ of particle $j$ in the condition $\tilde{W}_0(j)$ is

$$a(j)_{Q(\tilde{W}_0(j))} \approx \frac{1}{2} a_R L Q(\nu(P_{\phi_R(j)}; \delta_j)) + \frac{1}{2} a_R L Q(\nu(P_{\phi_L(j)}; \delta_j)).$$

(4)

where $a_K = Tr\hat{A} \hat{P}_{\phi_K(j)}$. This means that for $\delta_j$ sufficiently small the qr-number $a_Q(\tilde{W}_0(j))$ is well approximated by a locally constant function taking the values $\frac{1}{2}a_R$ on $\nu(\hat{P}_{\phi_R(j)}; \delta_j)$, $\frac{1}{2}a_L$ on $\nu(\hat{P}_{\phi_L(j)}; \delta_j)$ and zero elsewhere.

9. An EPR-type condition for a pair of particles

The epistemic condition of the pair is $W_0(1,2) = \nu(\hat{P}_{\psi(1,2)}; \epsilon)$ with $\epsilon \ll \frac{1}{2}$, their individual initial positions and momenta are $\langle \phi_R/L(\epsilon), \hat{X}(\epsilon) \rangle \approx (0,0,\pm p_z)$ for $p_z > 0$ and $\langle \phi_R/L(\epsilon), \hat{X}(\epsilon) \rangle \approx 0$. They have the same qr-number value for their positions, $\langle \hat{X}(1) \hat{I}(2) \rangle_Q(W_0) = \langle \hat{X}(1) \hat{I}(2) \rangle_Q(W_0) \approx 0$, but equal and opposite qr-number values for their momenta, $\langle \hat{p}(1) \hat{I}(2) \rangle_Q(W_0) = \langle \hat{p}(1) \hat{I}(2) \rangle_Q(W_0) \approx 0$.

The two particles move freely on qr-number trajectories in such a way that they remain close to each other in their qr-number space, because at any time $t > 0$ both the centre of mass and the relative qr-number position vectors remain near the origin, $\tilde{x}_C(t)_Q(W_0) \approx 0$ and $\tilde{x}_r(t)_Q(W_0) \approx 0$ so that the qr-number distance between the particles remains approximately zero.

**Lemma 5.** If the system is prepared in the condition $W_0(1,2)$ with $\epsilon \ll \frac{1}{2}$, then the $i$th particle has a continuous qr-number trajectory along the z-axis, going to the right to extent $\tilde{W}_R(i) = \nu(\hat{P}_{\phi_R(i)}; \epsilon)$ and to the left to extent $\tilde{W}_L(i) = \nu(\hat{P}_{\phi_L(i)}; \epsilon)$. Therefore, $\mathcal{H}(1) = \mathcal{H}(2) = L_2(\mathbb{R}^3)$ and the operators $\hat{X}_v(1) = \hat{X}_R(2) = \hat{X}_L$ and $\hat{P}_i(1) = \hat{P}_i(2) = \hat{P}_i$ therefore, if $\phi_R(1) = \phi_R(2)$ and $\phi_L(1) = \phi_L(2)$, then $\tilde{W}_R(1) = \tilde{W}_R(2) = \tilde{W}_R$ and $\tilde{W}_L(1) = \tilde{W}_L(2) = \tilde{W}_L$.

**Proposition 6.** If the particles are identical or only differ in their masses, $m_1 \neq m_2$, then at time $t$ to extent $\tilde{W}_R$ a particle with mass $\frac{m_1}{2}$ can be detected near $z = p_z \frac{m_1}{m_2}$ and to extent $\tilde{W}_L$ can be detected near $z = -p_z \frac{m_2}{m_1}$.

This is supported by two particle trajectory matrix whose $ij$th component is the qr-number $\langle x_i(1) \hat{x}_j(2) \rangle_Q(W_0)(t)$ if $i \neq j$ then $(x_i(1) \hat{x}_j(2))(W_0)(t) \approx 0$ and the only non-zero diagonal term is $\langle x_i(1) \hat{x}_i(2) \rangle_Q(W_0)(t) \approx -p_z \frac{m_2}{m_1}$, that is, the two particles move in qr-number space so that the product of their displacements along the z-axis is always negative.
At a source $S$, two massive spin-1/2 quantum particles are prepared with both their total momentum and total spin zero. They then move freely from $S$ towards two apparatuses, labeled $L$ and $R$, placed to the left and right of $S$ along the $z$-axis. Spin correlations are obtained using Stern-Gerlach devices with magnetic fields orthogonal to the $z$-axis by measuring the positions of the particles after their passage through the devices. The magnetic field at $L/R$ is in the $\tilde{u}_L/\tilde{u}_R$ direction.

By changing $\tilde{u}_L$ the outcome at $R$ can be changed in such a way that in a long run of experiments the statistical distribution of outcomes violates Bell’s inequality (which was obtained using classical hidden variables).

To simplify the algebra, consider an EPR-Bohm-Bell experiment which uses two identical spin one-half massive particles. The two particle system is initially prepared in a condition given by $W(1,2; s_1, s_2; \epsilon) = \nu(\hat{P}_{\Psi(1,2; s_1, s_2); \epsilon})$. The singlet state $\hat{P}_{\Psi(1,2, s_1, s_2)}$ projects onto the unit vector $\Psi(1, 2; s_1, s_2) = 1/2(\phi_L(1) \otimes \phi_R(2) + \phi_R(1) \otimes \phi_L(2)) \otimes (|+s_1\rangle \otimes |-s_2\rangle + |-s_1\rangle \otimes |+s_2\rangle)$ (5)

where $L$ and $R$ label opposite directions along the $z$-axis and $|\pm s_j\rangle$ represents particle $j$’s spin up (down) polarisation state along a direction $\vec{B}$ orthogonal to the $z$-axis. Partial tracing over the spatial Hilbert space produces the standard singlet spin state, over $\mathbb{C}^4$ it gives the pure state $\hat{P}_{\Psi(1,2)}$ which is of the same form as that used for spinless particles. Therefore before interacting with the magnetic field, a particle of mass $m$ moves along the qr-number $z$-axis to the right to extent $\hat{W}_R$ with approximate speed $v_\epsilon = \frac{\epsilon}{m}$ and to the left to extent $\hat{W}_L$ also with approximate speed $v_\epsilon = \frac{\epsilon}{m}$. In this way that the centre of mass coordinate remains close to the origin. The particles labeled 1 and 2 remain in contact; to the extent $\hat{W}_R$ they have the same qr-number trajectory to the right and to the extent $\hat{W}_L$ they have the same qr-number trajectory to the left.

If the time that the particles spend in the magnetic fields is short and if $\epsilon$ is small enough then after the passage through the magnetic fields and before reaching the detectors, the two particle system is in the condition $\nu(\hat{P}_{\Psi_{s_1, s_2}(1,2); \epsilon})$, an $\epsilon$-neighbourhood of $\hat{P}_{\Psi_{s_1, s_2}(1,2)}$ that projects onto the vector $\Psi^{s_1, s_2}(1, 2) = 1/2(\phi^+_L(1) \otimes \phi^-_R(2) \otimes |+ s_L(1)\rangle \otimes |s_R(2)\rangle + \phi^-_L(1) \otimes \phi^+_R(2) \otimes |s_R(1)\rangle \otimes |+ s_L(2)\rangle)$ (6)

where $\phi^\pm_K(j), K = L, R$ is the wave function of particle $j$ propagating to the upper (lower) left or right parts of the plane spanned by $\vec{e}_3$ and $\vec{u}_K$. $\pm s_K(j)$ denotes particle $j$’s spin up/down in the $\vec{u}_K$ direction.

Tracing over spin we obtain the reduced state $\hat{P}_{\Psi(1,2)} = 1/4(\hat{P}_{\phi^+_L(1) \otimes \phi^-_R(1)} + \hat{P}_{\phi^-_L(1) \otimes \phi^+_R(1)} + \hat{P}_{\phi^+_L(1) \otimes \phi^-_R(2)} + \hat{P}_{\phi^-_L(1) \otimes \phi^+_R(2)})$. (7)

If the wave functions are assumed to be non-overlapping and smooth then the single particle reduced state for the $j$th particle is $\rho(j) = 1/4(\hat{P}_{\phi^+_L(j) \otimes \phi^-_R(j)} + \hat{P}_{\phi^-_L(j) \otimes \phi^+_R(j)} + \hat{P}_{\phi^+_L(j) \otimes \phi^-_R(2)} + \hat{P}_{\phi^-_L(j) \otimes \phi^+_R(2)})$. The reduced condition of the $j$th particle is $\hat{W}_0(j) = \nu(\rho(j), \epsilon)$ which for small $\epsilon$ is well approximated by

$\frac{1}{4}(\nu(\hat{P}_{\phi^+_L(j), \epsilon}^+ \otimes \nu(\hat{P}_{\phi^-_L(j), \epsilon}^+ \otimes \nu(\hat{P}_{\phi^+_R(j), \epsilon}^+ \otimes \nu(\hat{P}_{\phi^-_R(j), \epsilon}^+)$. (8)

Let $\hat{W}_s^a(j) = \nu(\hat{P}_{\phi^+_K(j), \epsilon})$ for $s = +, -, K = R, L$, $j = 1, 2$ then since we can choose the wavefunctions such that $\hat{W}_s^a(1) = \hat{W}_s^a(2)$, for each pair $(K, s)$ there is a particle of mass $m$, internal energy 0 and spin $\frac{1}{2}$, whose qr-number position is

$\bar{x}_Q(\hat{W}_s^a) = \frac{1}{2} \sum_{j=1}^{2} \bar{x}(j)_Q(\hat{W}_s^a(j))$. (9)

The resulting qr-number trajectories are described in the following

**Proposition 7.** After passing through the magnetic fields a massive spin $1/2$ particle has a qr-number trajectory that splits into parts, one goes to the right along the $e_3$-axis and up/down along $\tilde{u}_R$ to the extent $\hat{W}_R$ and the other goes to the left along the $e_3$-axis and up/down along $\tilde{u}_L$ to the extent $\hat{W}_L$. as is seen in the formulae

$\bar{x}_Q(\hat{W}_s^a(\tau + t) \approx (z_R + v_\epsilon t)\vec{e}_3 + \alpha^\pm t\vec{u}_K$. (10)
where $z_K$ is the $z$-coordinate of the SG device on the $K$ side, $v_z$ is the speed of the particle, $\alpha = \mu br\vec{u}_K \cdot \vec{e}_z(W^z_K) \approx \mu b r \frac{\gamma}{2} \hbar$ with $\mu$ the gyro-magnetic coupling constant of the particle, $b$ the (strong) gradient of the magnetic field along $\vec{u}$, $\tau$ the time of transit inside the Stern-Gerlach device and $t$ is the time since the particle left the device.

Because the total spin of the two particle system is zero the only possible trajectories must either exist to extent $\vec{W}^+_R \cup \vec{W}^-_L$ or to extent $\vec{W}^+_R \cup \vec{W}^-_R$. Therefore the centre of mass of the two particle system, $(\vec{x}_C)_{Q}(W_0(1,2)) = \frac{1}{2}[\vec{x}_Q(W_0(1)) + \vec{x}_Q(W_0(2))] = \frac{1}{2}[\vec{x}_Q(\vec{W}^+_R \cup \vec{W}^-_L) + \vec{x}_Q(\vec{W}^+_R \cup \vec{W}^-_R)]$ stays near the origin since $z_L = - z_R$ and $\vec{v}_K \cdot \vec{e}_z(W^z_K) = - \frac{\gamma}{2} \hbar$. The centre of mass momentum $(\vec{p}_C)_{Q}(W_0(1,2)) = \vec{p}_Q(W_0(1)) + \vec{p}_Q(W_0(2)) = \vec{p}_Q(\vec{W}^+_L \cup \vec{W}^-_L(1)) + \vec{p}_Q(\vec{W}^-_L \cup \vec{W}^+_R).

11. Conclusions

The qr-number picture answers the questions: A How does the outcome of the measurement of the spin at $L$ depend upon the setting for the spin measurement at $R$ when the distance between $L$ and $R$ prohibits any interaction between them? The qr-number distance between the particles that carry the spin is zero, so they can always interact. If the setting of the SG apparatus is changed at $R$ then both particles will experience the changed magnetic field because both are present at $R$ to extent $\vec{W}^+_R$ or $\vec{W}^-_R$.

B How are the properties of the two particle system related to those of the one particle subsystems? The two particle qr-number trajectory has two components, in the first, to extent $\vec{W}^+_R \cup \vec{W}^-_L$, each particle goes both up to the left and down to the right, in the second, to extent $\vec{W}^-_L \cup \vec{W}^+_R$, each particle goes both down to the left and up to the right.

At the measurement by a detector that was placed beyond the region of the magnetic field, the qr-number position of a particle is registered when the appropriate position quality is $\epsilon$ sharp collimated in a region and hence given as an approximate standard real number. In the qr-number picture measurement is a physical process that gives the standard result that correlates the observed position of the particle with its spin. Clearly the interaction between the two measuring apparatuses appears to be non-local because the SG devises are separated by spacelike distances in standard Euclidean space but the qr-number distance between the quantum particles is zero so that the particles are never separated in qr-number space.

We claim that these experiments strongly suggest that the spatial continuum of a quantum system should be described as a qr-number continuum rather than the classical real number continuum. It is well known that topos theory gives an alternative foundation for mathematics, the results of this paper indicate how topos theory may help to solve some of the conceptual problems in quantum theory.
