1. (a) Do you think it is possible to have a vector space with exactly two vectors in it? Explain.
(b) Do you think that it is possible for a vector space to have two different zero vectors? That is, is it possible to have two different vectors \( \mathbf{0}_1 \) and \( \mathbf{0}_2 \) such that they both satisfy the vector space axioms?

2. Let \( f = \cos^2 x \) and \( g = \sin^2 x \). Which of the following lie in the space spanned by \( f \) and \( g \)?
   (a) \( \cos 2x \)  
   (b) \( 3 + x^2 \)  
   (c) \( 1 \)  
   (d) \( \sin x \)  
   (e) \( 0 \)

3. Let \( P_2 \) be the vector space of polynomials of degree less than or equal to 2. Determine whether or not the following polynomials span \( P_2 \).
   \[ p_1 = 1 - x + 2x^2, \quad p_2 = 3 + x, \quad p_3 = 5 - x + 4x^2, \quad p_4 = -2 - 2x + 2x^2 \]

4. Which of the following sets of vectors \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \) in \( \mathbb{R}^n \) are subspaces of \( \mathbb{R}^n \)?
   (a) All \( \mathbf{x} \) such that \( x_1 \geq 0 \).
   (b) All \( \mathbf{x} \) such that \( x_2 \) is rational.

5. Let the set \( F \) be all functions of the form \( f : \mathbb{R} \to \mathbb{R} \) with the usual meanings of addition of functions and multiplication by scalars. Which of the following sets are subspaces of \( F \)?
   (a) The set of functions \( g \) such that \( g(x^2) = g(x)^2 \).
   (b) The set of functions \( g \) such that \( g(0) = g(1) \).
   (c) The set of all polynomials of degree exactly equal to 2.
   (d) The set of all functions of the form \( k_1 + k_2 \sin x \), where \( k_1 \) and \( k_2 \) are real numbers.

6. Show that the solution vectors of a consistent nonhomogeneous system of \( m \) linear equations in \( n \) unknowns do not form a subspace of \( \mathbb{R}^n \).

7. Is the vector \((3, -1, 0, -1)\) in the space spanned by the vectors \((2, -1, 3, 2), (-1, 1, 1, -3), \) and \((1, 1, 9, -5)\)?

8. Explain why it is not possible for \( \sum_{n=1}^{\infty} \frac{1}{5 + 2^{-n}} \) to be convergent.

9. Suppose that a series \( \sum a_n \) has positive terms and its partial sums satisfy the inequality \( s_n \leq 1000 \) for all \( n \). Explain why \( \sum a_n \) must be convergent.

10. Does \( \sum_{n=0}^{\infty} ne^{-n^2} \) converge or diverge? Consider the partial sums, and the laws of logarithms.

11. In this problem, you will justify the integral test. Suppose \( a \geq 0 \) and \( f(x) \) is a decreasing positive function, defined for all \( x \geq a \). Let \( a_n = f(n) \).
   (a) Suppose \( \int_{a}^{\infty} f(x) \, dx \) converges. By considering rectangles under the graph of \( f \), show that \( \sum a_n \) converges.
   (b) Suppose \( \int_{a}^{\infty} f(x) \, dx \) diverges. By considering rectangles above the graph of \( f \), show that \( \sum a_n \) diverges.

12. Does \( \sum_{n=0}^{\infty} ne^{-n^2} \) converge or diverge?
13. You will show that the sequence

\[ t_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \]

has a limit, which is called the Euler constant, \( \gamma \). Incidentally, it is not known whether \( \gamma \) is rational or irrational.

(a) Figure 1 shows a graph of \( y = f(x) = 1/x \). To the figure, add the left hand Riemann sums from \( x = 1 \) to \( x = n \). Interpret \( t_n \) as an area to show that \( t_n > 0 \) for all \( n \).

(b) Interpret

\[ t_n - t_{n+1} = (\log(n+1) - \log n) - \frac{1}{n+1} \]

as a difference of areas to show that \( t_n - t_{n+1} > 0 \).

(c) What can you conclude?

![Figure 1: Graph of \( y = f(x) = 1/x \)]